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Spin tests and amplitude analysis for the reaction $0+\frac{1}{2}\rightarrow 0+J$ †

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A complete set of equations is found which relate the spin J of a fermion resonance, certain bilinear forms in the statistical tensors t_L^M (with L even), and the degree of target polarization for the reaction $0+\frac{1}{2}\rightarrow 0+J$ on a polarized target. For the case of production on an unpolarized target, the helicity amplitudes, to within a four-parameter ambiguity, are explicitly constructed from the even-L t_L^M and a complete set of trilinear constraints among the latter is obtained.

I. INTRODUCTION

The problem of determining the reaction amplitudes in two-body and quasi-two-body processes has recently received much theoretical and experimental attention.[1] Let the N independent amplitudes for a given reaction be denoted by ϕ_i; the observable quantities are then certain linear combinations of $\phi_i\phi_j^*$. The task is to express the amplitudes in terms of the observables, i.e., to solve a set of simultaneous bilinear equations. Although solutions to this problem are known for many special cases, a general algorithm for obtaining the solution is not available.

A related problem is that of determining the spin of one of the particles produced in the reaction. By this we mean a determination using only the quantities directly observable in the reaction, with no additional theoretical or phenomenological input. Here there also exists a large body of knowledge on dynamics-independent spin tests, but further progress remains possible.[2]

In this article we concentrate on the reaction

$$0+\tfrac{1}{2}(\vec{\mathrm{P}})\rightarrow 0+B^*, \qquad (1)$$

where B^* is a baryon resonance of spin J which, we assume, decays via a parity-conserving interaction into $p+\pi$, and where $\vec{\mathrm{P}}$ denotes the spin-$\frac{1}{2}$ target polarization vector. We suppose the reaction is studied at fixed quasi-two-body kinematics (s, t, and Φ, the azimuthal angle, fixed). By measuring the decay angular distribution of the resonance in its rest frame, $W(\theta,\phi)$, the experimental moments $\langle Y_L^M\rangle$ can be obtained via[3]

$$\langle Y_L^M\rangle=\int d\Omega\, W(\theta,\phi)\, Y_L^M(\theta,\phi) \qquad (2)$$

for L even, those with L odd being zero (in the absence of interfering background). The problems solved in this paper are the following:

(i) For the case where J is not known and a polarized target is available, we find a complete set of constraints relating the experimental moments and the degree of target polarization. These constraints depend on the spin assignment of the resonance, and should provide a strong rejection for incorrect hypotheses.

(ii) For the case of an unpolarized target where J is known, we prescribe an algorithm for extracting the reaction amplitudes from the $\langle Y_L^M\rangle$ to within a four-parameter ambiguity. In addition, a complete set of homogeneous trilinear constraints among the $\langle Y_L^M\rangle$ are found which must be satisfied if an amplitude analysis is to be feasible.

In Sec. II, we obtain the tests of the spin assignment by studying the simple case of a pure-state resonance. The amplitude analysis for an unpolarized target is discussed in Sec. III, while our conclusions and possible extensions of our results to other reactions are presented in Sec. IV.

II. SPIN ASSIGNMENT TESTS

The problem is to obtain tests of the spin assignment using only the measurable quantities $\langle Y_L^M\rangle$ for L even. These quantities are functions of the dynamical variables s, t as well as the target polarization vector $\vec{\mathrm{P}}$. The standard theory of decay angular distributions and spin density matrices says that the statistical tensors t_L^M are obtained from the $\langle Y_L^M\rangle$ via[3]

$$t_L^M=f(J,L)\langle Y_L^M\rangle, \qquad (3)$$

where $f(J,L)$ is a known function of its arguments. In all that follows we shall work only with the even-L t_L^M, but it should be kept in mind that these are obtained from the experimental $\langle Y_L^M\rangle$ simply by making a hypothesis for J. For each hypothesis, one gets a different set of t_L^M; the task is to find tests which confirm or reject the hypotheses. These dynamics-independent tests are provided by the constraints among the t_L^M. That constraints should exist is indicated by the fact that there are only $(2J+1)$ complex amplitudes or $4J+2$ real quantities, whereas there are $(J+\frac{1}{2})^2$ real even-L t_L^M for an unpolarized target and $2J(J+\frac{1}{2})$ real numbers among the even-L t_L^M with a polarized target. Thus, even for small J, there are more observables than amplitudes. In order to find the constraints, we take advantage of the possibility of varying the amplitudes while holding the even-L t_L^M fixed. For reaction (1) the ambiguity in the amplitudes corresponds to a four-parameter transformation, as was pointed out by Gizbert-Studnicki, Golemo, and Zalewski[4] for $J=\frac{3}{2}$, and by Doncel, Michel, and Minnaert[1] for arbitrary spin. This ambiguity will be used to facilitate the expression of the amplitudes in terms of the even-L t_L^M.

Let us consider a pure state of spin J described completely by $(2J+1)$ complex numbers u_m, $-J\leqslant m\leqslant J$. The statistical tensors t_L^M are defined by

$$t_L^M=\sum_{m,m'} C_{mMm'}^{JLJ} u_m u_{m'}^* . \tag{4}$$

If one knows only the even-L t_L^M, one may partially invert this relation,

$$(2J+1)^{-1}\sum_{L\text{ even}}(2L+1)C_{mMm'}^{JLJ}t_L^M = \tfrac{1}{2}\left[u_m u_{m'}^* + (-1)^{m-m'}u_{-m'}u_{-m}^*\right], \tag{5}$$

with $C_{mMm'}^{JLJ}$ being a Clebsch-Gordan coefficient.

If with the spinor

$$\begin{pmatrix} u_m \\ u_{-m} \end{pmatrix}$$

we associate the 2×2 matrix

$$U_m=\begin{pmatrix} u_m & (-1)^{m-1/2}u_{-m}^* \\ (-1)^{m-1/2}u_{-m} & -u_m^* \end{pmatrix}, \tag{6}$$

it is clear that the right-hand side of Eq. (5) is some element of the matrix product

$$U_m U_{m'}^+$$

for $m, m'\geqslant 0$. If we make the substitution

$$U_k \to U_k V \quad \text{for all } k\geqslant 0 ,$$

where V is any matrix in SU(2), then under this implicit transformation of the u_m, all even-L t_L^M are preserved. One important consequence of this is that for any matrix U in the form of Eq. (6), one can always find a V such that

$$UV=\begin{pmatrix} i(-\det U)^{1/2} & 0 \\ 0 & i(-\det U)^{1/2}\end{pmatrix}. \tag{7}$$

This means that the three-parameter ambiguity inherent in V can be used to set $u_{-m}=\mathrm{Re}\,u_m=0$ for one value of m, say r.[5] The amplitude analysis using the even-L t_L^M then follows quite simply. First choose $m=m'=r$ in Eq. (5) to obtain

$$u_r = i\left[\frac{2}{2J+1}\sum_{L\text{ even}}(2L+1)C_{r0r}^{JLJ}t_L^0\right]^{1/2}. \tag{8}$$

Next choosing $m'=r$, $m\neq(r,-r)$, we find from Eq. (5)

$$u_m=\left[\frac{2}{2J+1}\sum_{L\text{ even}}(2L+1)\sum_m C_{mMr}^{JLJ}t_L^M\right]\Big/ u_r^* . \tag{9}$$

We now have expressed the "oriented" amplitudes completely in terms of the even-L t_L^M, but only a subset of these have in fact been used. It is now possible to express the remaining even-L t_L^M in terms of the oriented amplitudes by choosing $m,m'\neq(r,-r)$ in Eq. (5) to find

$$\sum_{\substack{L_1L_2\\ \text{even}}}(2L_1+1)(2L_2+1)\sum_{M_1M_2}t_{L_1}^{M_1}t_{L_2}^{M_2}\left(C_{rM_1r}^{JL_1J}C_{mM_2m'}^{JL_2J}-C_{mM_1r}^{JL_1J}C_{rM_2m'}^{JL_2J}-C_{mM_1-r}^{JL_1J}C_{-rM_2m'}^{JL_2J}\right)=0 . \tag{10}$$

Note that in this equation there is no sum over r, and the sums over M_1 and M_2 contain only one term. If one lets m take on all positive values from $\frac{1}{2}$ to J, except r, and m' take on all values from $-J$ to m, except r, $-m$, and $-r$, one obtains a set of $(J-\frac{1}{2})(2J-2)$ bilinear constraints among the even-L t_L^M. By replacing the even-L t_L^M in Eq. (10) by their expressions in terms of $u_m u_{m'}^*$, one sees that the constraints are identically satisfied by any set of amplitudes u_m, and hence are independent of the dynamics.

If in the constraints, Eq. (10), one replaces the index r by another value, s, one obtains another complete set of constraints which are equivalent to the first set, even though they are not obtainable by linear manipulations of the former. One way of eliminating the dependence on r is to multiply Eq. (10) by $C_{mNm'}^{JKJ}$, with K even, and sum over m, m', and r. The result is[6]

$$\sum_{\substack{L_1L_2\\ \text{even}}}G(L_1,L_2,K)\sum_{M_1,M_2}t_{L_1}^{M_1}t_{L_2}^{M_2}C_{M_1M_2N}^{L_1L_2K}=0 , \tag{11}$$

where the quantity $G(L_1,L_2,K)$ is given by

$$G(L_1, L_2, K) = (2L_1+1)(2L_2+1)\{(2J+1)\delta_{L_1 0}\delta_{L_2 K} - 2[(2J+1)(2K+1)]^{1/2} W(JL_1JL_2;JK)\}, \quad (12)$$

with W being a Racah coefficient. This relation is again an identity for all even K from 0 to $(2J-1)$, $-K \leq N \leq K$, but the price paid in eliminating r, m, m' is that these equations are not functionally independent.

In order to use these pure-state conditions we employ the Eberhard-Good theorem,[7] which asserts that in reaction (1) the final state baryon will be in a pure state if the initial target polarization is 100%, i.e., when $|\vec{P}|=1$, independent of the direction of $\vec{P}$. Since the constraint equations, provided one uses $(d\sigma/d\Omega)t_L^M$, the moments weighted by the differential cross section, are bilinear, the right-hand side is an inhomogeneous bilinear form in $\vec{P}$. This may always be written as[8]

$$\text{right-hand side} = B + C(1-P^2) + P\sum_{m=-1}^{1} Y_1^m(\beta,\alpha)D_m + P^2\sum_{n=-2}^{2} Y_2^n(\beta,\alpha)E_n, \quad (13)$$

where $P=|\vec{P}|$, and β, α are the spherical polar angles of the vector $\vec{P}$. In order for this form to vanish when $|\vec{P}|=1$ for all directions of $\vec{P}$, one must have B, D_m, and E_n all zero, while C may be nonzero. Thus for each constraint equation (10) or (11), if the spin hypothesis is correct, the left-hand side must behave as $C^r_{mm'}(1-P^2)$ for Eq. (10) and $C^K_N(1-P^2)$ for Eq. (11). If the spin hypothesis is incorrect, one will in general observe nonzero B, D_m, and E_n; hence each constraint equation permits, in principle, a strong rejection of the false spin assignments. The unknown quantities $C^r_{mm'}$ or C^K_N are real numbers which depend upon the dynamical variables s, t but not upon the polarization $\vec{P}$. Let us note also that if there exists a phase coherence among the helicity amplitudes for reaction (1) the quantities C are all zero, since the even-L t_L^M would all be independent of $\vec{P}$.[9]

III. UNPOLARIZED TARGET AMPLITUDE ANALYSIS AND TRILINEAR CONSTRAINTS

If reaction (1) is observed on an unpolarized target, the relation between the even-L t_L^M and the helicity amplitudes $f_{\lambda\frac{1}{2}}$ is

$$t_L^M = \tfrac{1}{2}\,\text{Re}\left\{\sum_{\lambda\lambda'}\left[f_{\lambda\frac{1}{2}}f^*_{\lambda'\frac{1}{2}} + (-1)^{\lambda-\lambda'}f_{-\lambda\frac{1}{2}}f^*_{-\lambda'\frac{1}{2}}\right]C^{JLJ}_{\lambda M\lambda'}\right\}. \quad (14)$$

Note that this differs from Eq. (4) mainly in that the real part of the bilinear form enters. One way to find the ambiguity in the amplitudes which corresponds to fixed even-L t_L^M is to associate with the pair of amplitudes $(f_{\lambda\frac{1}{2}}, f_{-\lambda\frac{1}{2}})$ the 2×2 matrix

$$F_\lambda = \begin{pmatrix} f_{\lambda\frac{1}{2}} & (-1)^{\lambda-1/2}f^*_{-\lambda\frac{1}{2}} \\ (-1)^{\lambda-1/2}f_{-\lambda\frac{1}{2}} & -f^*_{\lambda\frac{1}{2}} \end{pmatrix}. \quad (15)$$

The right-hand side of Eq. (14) then corresponds to elements of the matrix product

$$\text{Re}(F_\lambda F^+_{\lambda'}).$$

It may then be shown that under the substitution

$$F_\lambda \to e^{-i(\Psi/2)\sigma_y}F_\lambda V e^{i(\Psi/2)\sigma_y} \quad \text{for all } \lambda \geq 0,$$

the even-L t_L^M are preserved, where V denotes an arbitrary SU(2) matrix and Ψ is an arbitrary angle. This is just the expression for the helicity amplitudes of the ambiguity in the transversity amplitudes found in Ref. 4, except that ours is for arbitrary J. One may then make use of the four parameters Ψ, V in order to solve for the oriented amplitudes, just as in Sec. II. For two arbitrary values of helicity ρ, τ, one can always find Ψ, V such that, after the transformation,

$$F_\rho = \begin{pmatrix} i(-\det F_\rho)^{1/2} & 0 \\ 0 & i(-\det F_\rho)^{1/2} \end{pmatrix}, \quad (16a)$$

$$F_\tau = \begin{pmatrix} i\,\text{Im}f_{\tau\frac{1}{2}} & (-1)^{\tau-1/2}f^*_{-\tau\frac{1}{2}} \\ (-1)^{\tau-1/2}f_{-\tau\frac{1}{2}} & i\,\text{Im}f_{\tau\frac{1}{2}} \end{pmatrix}. \quad (16b)$$

That is, the four parameters may be chosen such that

$$f_{-\rho\frac{1}{2}} = \text{Re}f_{\rho\frac{1}{2}} = \text{Re}f_{\tau\frac{1}{2}} = 0.$$

Equation (14) may be partially inverted, to give

$$\text{Re}(f_{\lambda\frac{1}{2}}f^*_{\lambda'\frac{1}{2}} + (-1)^{\lambda-\lambda'}f_{-\lambda\frac{1}{2}}f^*_{-\lambda'\frac{1}{2}}) = \frac{2}{(2J+1)}\sum_{L\ \text{even}}(2L+1)\sum_M t_L^M C^{JLJ}_{\lambda M\lambda'}. \quad (17)$$

If in Eq. (17) we choose $\lambda=\lambda'=\rho$,

$$f_{\rho\frac{1}{2}} = i\left[\frac{2}{2J+1}\sum_{L\ \text{even}}(2L+1)t_L^0 C^{JLJ}_{\rho 0\rho}\right]^{1/2}. \quad (18)$$

Choosing next $\lambda'=\rho$, $\lambda\neq(\rho, -\rho)$, we obtain the imaginary parts of all other amplitudes:

$$\text{Im}f_{\lambda\frac{1}{2}} = \frac{2}{2J+1}\sum_{L\ \text{even}}(2L+1)\sum_M t_L^M C^{JLJ}_{\lambda M\rho}\Bigg]\Bigg/\ \text{Im}f_{\rho\frac{1}{2}}. \quad (19)$$

A two-step procedure then gives the real parts, once the imaginary parts are known. First we set $\lambda=\lambda'=\tau$ in Eq. (17):

$$(\mathrm{Re}f_{-\tau\frac{1}{2}})^2=\left[\frac{2}{2J+1}\sum_{L\ \mathrm{even}}(2L+1)\,t_L^0 C^{JLJ}_{\rho0\rho}\right]-(\mathrm{Im}f_{\tau\frac{1}{2}})^2-(\mathrm{Im}f_{-\tau\frac{1}{2}})^2\,. \tag{20}$$

Positivity requires that the right-hand side be nonnegative, and one may always choose the positive root, since the negative root would correspond only to increasing the arbitrary angle Ψ by π. Finally one obtains the remaining real parts by choosing $\lambda'=\tau$, $\lambda\neq(\rho,-\rho,\tau,-\tau)$ in Eq. (17):

$$\mathrm{Re}f_{-\lambda\frac{1}{2}}=[(-1)^{\lambda-\tau}/\mathrm{Re}f_{-\tau\frac{1}{2}}]\left\{\left[\frac{2}{2J+1}\sum_{L\ \mathrm{even}}(2L+1)\sum_M t_L^M C^{JLJ}_{\lambda M\tau}\right]-\mathrm{Im}f_{\lambda\frac{1}{2}}\mathrm{Im}f_{\tau\frac{1}{2}}-(-1)^{\lambda-\tau}\mathrm{Im}f_{-\lambda\frac{1}{2}}\mathrm{Im}f_{-\tau\frac{1}{2}}\right\}. \tag{21}$$

In Eqs. (18)–(21) we have a prescription for determining the "oriented" amplitudes for the reaction from a subset of the even-L t_L^M. The complete set of amplitudes consistent with these measurements is obtained by letting the matrix V range throughout SU(2), while Ψ runs from 0 to 2π. In order to fix three of these four parameters one would have to determine some odd-L t_L^M or some even-L t_L^M on a polarized target (one of the four parameters corresponds to an unobservable over-all phase). However, not all the even-L t_L^M have been used in obtaining these oriented amplitudes. Consistency requires that the remaining even-L t_L^M be given correctly by the amplitudes, hence constraints follow. If in Eq. (17) we choose λ and λ' arbitrarily, but not equal to any of $(\rho,-\rho,\tau,-\tau)$ (which is possible only for $J\geq\frac{5}{2}$), the left-hand side may be expressed in terms of even-L t_L^M. Multiplying by $(\mathrm{Re}f_{-\tau\frac{1}{2}})^2(\mathrm{Im}f_{\rho\frac{1}{2}})^2$, we obtain a homogeneous trilinear equation, which may be written as

$$\sum_{\substack{L_1L_2L_3\\ \mathrm{even}}}(2L_1+1)(2L_2+1)(2L_3+1)\sum_{M_1M_2M_3}t_{L_1}^{M_1}t_{L_2}^{M_2}t_{L_3}^{M_3}G(L_1,L_2,L_3,M_1,M_2,M_3,\lambda,\lambda',\rho,\tau)=0\,. \tag{22}$$

The numerical coefficient has the explicit form

$$\begin{aligned}G(L_1,L_2,L_3,M_1,M_2,M_3,\lambda,\lambda',\rho,\tau)=\;&C^{JL_1J}_{\lambda M_1\lambda'}(C^{JL_2J}_{\rho M_2\rho}C^{JL_3J}_{\tau M_3\tau}-C^{JL_2J}_{\tau M_2\rho}C^{JL_3J}_{\tau M_3\rho}-C^{JL_2J}_{-\tau M_2\rho}C^{JL_3J}_{-\tau M_3\rho})\\&-\{C^{JL_1J}_{\rho M_1\rho}[C^{JL_2J}_{\lambda M_2\tau}C^{JL_3J}_{\lambda' M_3\tau}-(-1)^{\lambda-\lambda'}C^{JL_2J}_{-\lambda M_2\tau}C^{JL_3J}_{-\lambda' M_3\tau}]+(\rho\leftrightarrow\tau)\}\\&+C^{JL_1J}_{\tau M_1\rho}\{[C^{JL_2J}_{\lambda M_2\tau}C^{JL_3J}_{\lambda' M_3\tau}-(-1)^{\lambda-\lambda'}C^{JL_2J}_{-\lambda M_2\tau}C^{JL_3J}_{-\lambda' M_3\rho}]+(\lambda\leftrightarrow\lambda')\}\\&-(-1)^{\lambda-\tau}C^{JL_1J}_{-\tau M_1\rho}\{[C^{JL_2J}_{-\lambda M_2\tau}C^{JL_3J}_{\lambda' M_3\rho}-(-1)^{\lambda-\lambda'}C^{JL_2J}_{\lambda M_2\tau}C^{JL_3J}_{-\lambda' M_3\rho}]-(\lambda\leftrightarrow-\lambda')\}\,.\end{aligned} \tag{23}$$

Note that the sums over M_1,M_2,M_3 in Eq. (22) are formal in that the only values of M_1, M_2, and M_3 which enter are determined by λ, λ', ρ, and τ, via Eq. (23). By letting λ acquire all values from $\frac{1}{2}$ to J, except ρ and τ, and letting λ' take on all values from $(-\lambda+1)$ to λ, except for $\rho,-\rho,\tau,-\tau$, one generates $(J-\frac{3}{2})^2$ independent trilinear constraints among the even-L t_L^M. Once again, a more elegant form in which the arbitrariness of ρ, τ, λ, and λ' is absent can be obtained by multiplying Eq. (22) by $C^{JKJ}_{\lambda N\lambda'}$, with K even, and summing over λ, λ', ρ, and τ. After some manipulations, one may write the result as

$$\begin{aligned}t_K^N&\left[(2J+1)(t_0^0)^2-2\sum_{L\ \mathrm{even}}(2L+1)\sum_M(t_L^M)^2\right]\\&-4t_0^0\left(\frac{2J+1}{2K+1}\right)^{1/2}\left[\sum_{\substack{L_1L_2\\ \mathrm{even}}}(2L_1+1)(2L_2+1)\,W(JL_1JL_2;JK)\sum_{M_1M_2}C^{L_1L_2K}_{M_1M_2N}t_{L_1}^{M_1}t_{L_2}^{M_2}\right]\\&+\frac{8}{(2K+1)^{1/2}}\sum_{\substack{L_1L_2L_3L\\ \mathrm{even}}}\left[(2L_1+1)(2L_2+1)(2L_3+1)(2L+1)^{1/2}W(JLJL_3;JK)\,W(JL_1JL_2;JL)\right.\\&\left.\times\left(\sum_{M_1M_2M_3M}C^{L_1L_2L}_{M_1M_2M}C^{LL_3K}_{MM_3N}t_{L_1}^{M_1}t_{L_2}^{M_2}t_{L_3}^{M_3}\right)\right]=0\,.\end{aligned} \tag{24}$$

These equations for $-K \leq N \leq K$ and $0 \leq K \leq (2J-1)$, K even, are not all independent, unlike the set represented by Eq. (23).

These trilinear constraint equations are essentially the expression, in terms of the even-L t_L^M, of the fact that the density matrix for reaction (1) is of rank two for fixed quasi-two-body kinematics. The rank-two condition implies that any 3×3 minor of the density matrix has zero determinant, thus yielding certain homogeneous trilinear forms in the t_L^M which vanish. In general, however, this rank method mixes up the odd-L and even-L t_L^M rather inextricably. The advantage of our result is that only the even-L t_L^M appear in the trilinear constraints; hence our relations involve only observable quantities.

IV. CONCLUSIONS

In this article we have obtained three distinct results concerning the reaction

$$0+\tfrac{1}{2}(\vec{\mathrm{P}}) \to 0+B^*,$$

where only the even-L part of the B^* spin-J density matrix is observed:

(1) a complete set of homogeneous bilinear relations in the even-L t_L^M which are proportional to $(1-|\vec{\mathrm{P}}|^2)$, and which should provide severe tests of the spin hypothesis when a polarized target is used;

(2) an algorithm for calculating the reaction amplitudes from the even-L t_L^M, when $\vec{\mathrm{P}}=0$, to within four parameters; and

(3) a complete set of trilinear homogeneous relations among the even-L t_L^M for $\vec{\mathrm{P}}=0$, $J \geq \frac{5}{2}$, which must vanish as a consequence of the rank-two condition.

Although the results we have derived are exact, they may be criticized on the grounds that the idealization employed, such as production of a single baryon resonance at fixed quasi-two-body kinematics, is just not realistic. If, in addition, the requirement of fixed quasi-two-body kinematics is loosened to consider $\langle t_L^M \rangle$ averaged over some interval in momentum transfer, the constraint equations need not hold. Finally there is the problem of the sensitivity of these results to experimental errors, given the complexity of the forms involved. We recognize these difficulties, but we feel that the present work still helps to fix the limits of amplitude analysis and constraint equations in the best of all possible cases, and that such knowledge may prove useful in the treatment of more realistic problems, in particular the superposition of two or more resonances.

Finally we wish to remark that our analysis of $0+\frac{1}{2}\to 0+J$ can be extended to partial-amplitude analyses and constraints for other reactions. Consider, for example, $s_1+s_2\to s_3+s_4$, where at least two particles have odd-half-integer spin, and let the amplitudes be ϕ_i. There then exists a value of J such that the amplitudes for the two reactions may be put in one-to-one correspondence. Knowing how to perform the amplitude analyses for spin-J production, we can find sets of $\mathrm{Re}(\phi_i\,\phi_j^*)$ which suffice to perform amplitude analysis for the more general reaction. The homogeneous trilinear constraints among the even-L t_L^M then become similar forms among various combinations of $\mathrm{Re}(\phi_i\,\phi_j^*)$.

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