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Landau's hydrodynamic model of particle production and electron-positron annihilation into hadrons*

Fred Cooper and Graham Frye
Belfer Graduate School of Science, Yeshiva University, New York, New York 10033

Edmond Schonberg
Courant Institute of Mathematical Sciences, New York University, New York, New York 10012
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The hydrodynamic model of Landau is formulated in very general terms and applied to the determination of average energy of secondaries, and single-particle inclusive distributions of secondaries. Attention is focused on the relationship between various dynamical assumptions and the equation of state assumed for the fluid motion. In the regimes of scaling and approximate scaling, we solve analytically the hydrodynamic motion of the fluid for both $pp \to \pi + X$ and $e^+ + e^- \to \pi + X$. For the annihilation process we solve the hydrodynamic equations numerically and discuss the validity of the scaling approximations. Explicit comparison is made between two dynamical models, the ultrarelativistic model (ideal-gas model) and the hadronic spectrum model.

I. INTRODUCTION

The hydrodynamic model of Landau[1,2] has had considerable success in explaining the features of single-particle production in hadron-hadron collisions.[2-4] The model is semiclassical, and can predict particle multiplicities and multiplicity distributions (dN/d^3p) with only very general assumptions, consistent with a wide variety of more fundamental theories. It is clearly desirable to explore new reactions where the hydrodynamic treatment of hadronic matter might be valid. Leptonic annihilation is just such a process. Since we feel it is also necessary to update previous reviews of the Landau model[5,6] we will try to be as precise and complete as possible in explaining the assumptions, flexibility, and application of Landau's approach. The basic conclusions of the detailed analysis to be presented here, relevant to present SPEAR accelerator energies, have been published earlier.[7] A special effort is made here to show the connection between scaling approximations, and more exact numeral solutions. We will find that the inclusive single-particle distribution $(E/N)\, dN/d^3p$ for $e^+ + e^- \to \pi + X$ is essentially determined by $\langle E_\pi \rangle$ and is almost independent of the choice of the equation of state assumed for the prehadronic matter.

The paper is divided into ten sections. In Sec. II we discuss the assumptions of the model. Section III discusses the initial state of the hadronic fluid and the equations governing the expansion. Section IV concerns itself with dynamical reasons for assuming different equations of state for the hadronic fluid. In Sec. V we discuss the hydrodynamic equations governing the fluid motion, and show how to obtain the distribution of energy and number of particles in terms of the fluid variables at breakup. Section VI deals with the question of how one obtains the single-particle inclusive distribution by assuming ideal-Bose-gas dynamics for the pions at fluid breakup. Section VII shows how to use the thermodynamic relations to obtain the multiplicity and average energy of secondaries. Section VIII discusses how to obtain analytic solutions when energies are high enough so that scaling laws hold. Connection with the usual Feynman scaling is discussed here. Section IX deals with partial scale breaking; i.e., how one obtains solutions when one has only approximate scaling laws. This section discusses how the scaling result of a flat rapidity distribution becomes modified to one of "Gaussian" shape. Section X concerns itself with numerical calculations that allow "exact" results for given initial data. These numerical calculations show tremendous deviation from the scaling results for e^+-e^- annihilation at present accelerator energies.

We relegate to appendices relevant formulas from thermodynamics, statistical mechanics, and transport theory.

II. THE MODEL

The hydrodynamical model is based on the following assumptions.

(a) In a high-energy collision (or annihilation process) a large amount of energy is pumped into a localized region of space. The volume of this region is assumed initially to be much smaller than that needed by N free hadrons, which is of the order $N\frac{4}{3}(1/m_\pi)^3 = NV_\pi$.

(b) Because of the strong interactions, relevant relaxation times are very short and the created prehadronic matter can be thought of as being in local statistical equilibrium. Phrased another way, mean free paths of the quanta involved are assumed much smaller than the characteristic lengths. Thus the system can be treated as a relativistic fluid whose collective motions are governed by the laws of relativistic hydrodynamics.

(c) The relevant variables for describing the collective behavior are the averaged thermodynamic field quantities such as energy density, pressure, entropy, and temperature density. The underlying dynamics then specifies, in this language, the form of the equation of state $p=p(\epsilon)$, where p, ϵ are the macroscopic pressure and energy density, respectively. Several choices of hadronic equations of state will be discussed below. The motion of the fluid will be described by the relativistic collective velocity field $u^\mu(x)$.

(d) The rest of the dynamics (apart from the equation of state) is embodied in the initial and final boundary conditions on the hydrodynamic equations. The initial condition on the equations is to specify the initial temperature distribution of the fluid. Thus we must make certain assumptions about the initial size of the system, and determine the initial temperature distribution from the center-of-mass energy via some knowledge of the relevant dynamics.

(e) Because of the pressure the system expands and cools. When the energy density reduces to that of one pion/hadronic volume, then the number of particles becomes a well-defined quantity. We then say that the fluid "breaks up" into quasifree final particles. We will try to make this criterion more specific for different equations of state.

(f) The breakup criterion defines a space-time surface which is an isotherm $kT(x,t)\approx m_\pi c^2$. Along this surface N is well defined, and we can determine the distribution of energy and number of particles as a function of the collective velocities.

If we further assume that the residual pion dynamics at break up is governed by an ideal-Bose-gas distribution function in the local rest frame, then we can directly determine the momentum-space distribution from transport theory.[8]

III. INITIAL CONDITIONS

The initial data for the hydrodynamic equations requires knowledge of V_0, the volume of the hadronic matter at the time when local statistical equilibrium is set up. For proton-proton collisions one might imagine a period of turbulence and passage of shock waves before the system starts to expand (see Refs. 1, 2, and 9). In a proton-proton collision, a natural volume is present, the Lorentz-contracted disk of matter comprising the proton in the c.m. frame. Thus it is expected that V_0 is not very different from $\pi m_\pi^{-3}\gamma$, where $\gamma=2M/E_{\text{c.m.}}$ is the Lorentz contraction in the center-of-mass frame. For e^+-e^- annihilation, knowledge of V_0 is equivalent to knowing the size of the produced electromagnetic field as far as its hadronic content, or knowing what is the size of the system when enough virtual quanta have been produced to consider it in local statistical equilibrium. This question is of course related to the question of what the cross section is for e^+-e^- annihilation into hadrons. For pp collisions the contracted proton assumption was equivalent to saying the cross section is approximately geometrical. Not having any deep wisdom on the subject of the e^+-e^- cross section we will assume only that the system is spherically symmetrical and the expansion is isotropic in the center-of-mass frame (preliminary data on the distribution of pions are consistent with this)[10] and let $V_0=\frac{4}{3}\pi r_0^3$, with r_0 a free parameter. By fitting the multiplicity data with different models we will find that r_0 is always of hadronic size (~fermi).

If the system undergoes a *large* expansion, then it is irrelevant how the initial energy is distributed over V_0 and it is sufficient to assume that the initial energy density is just constant over V_0, i.e.,

$$\epsilon_0=E_{\text{c.m.}}/V_0\,. \tag{1}$$

For hadronic collisions one probably wants to replace $E_{\text{c.m.}}$ by that fraction of the energy going into particle *production* (i.e., subtract the energy of the leading particles in pp collisions). We will find that as far as $(E/N)(dN/d^3p)$ is concerned, it does not depend on r_0 as a result of scaling laws satisfied by the hydrodynamic equations (i.e., r_0 determines N but not the shape of the distribution).

Once the initial state of local equilibrium is obtained we assume that the prematter expands according to the laws of hydrodynamics.

The relativistic equations of fluid motion are the generalizations of the Euler equation and are

$$\partial_\mu T^{\mu\nu}=0\,, \tag{2}$$

where $T^{\mu\nu}(x)$ is the energy-momentum tensor, or

$$T^k_{i;k}=0 \tag{2'}$$

(where the semicolon means covariant derivatives). (In what follows spherical coordinates will be denoted by latin letters.) For an ideal fluid (no viscosity) the form of the energy-momentum tensor $T^{\mu\nu}$ in a local rest frame is[1,5]

$$\begin{pmatrix} \epsilon & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix} , \tag{3}$$

where ϵ is the energy density in the local rest frame and p is the pressure (see Appendix D). Boosting to an arbitrary frame gives

$$T^{\mu\nu}(x) = (\epsilon + p)\, u^\mu u^\nu - p g^{\mu\nu} , \tag{4}$$

where $u^\mu(x) = \gamma(x)(1, \vec{v}(x))$ is the 4-velocity describing the collective motion.

IV. HADRONIC EQUATIONS OF STATE

To solve the hydrodynamic equations (2) we need to know $p = p(\epsilon)$ (equation of state); thus in this section we will discuss several equations of state found in the literature. (The reader can add his own choice.) Further discussion is found in appendixes, and in the references.

(a) Classical ultrarelativistic fluid (see Ref. 8). For a classical relativistic fluid containing particles of mass m described by a single-particle distribution function $g(x,p)$ we have from Appendix D that (barred variables denote the comoving frame)

$$\begin{aligned} \epsilon(\bar{x}) &= \int \bar{E} g(\bar{x},\bar{p})\, d^3\bar{p} , \\ p(\bar{x}) &= \int (\bar{p}^2/3\bar{E})\, g(\bar{x},\bar{p})\, d^3\bar{p} , \\ T^\mu{}_\mu(\bar{x}) &= \int (m^2/\bar{E})\, g(\bar{x},\bar{p})\, d^3\bar{p} = \epsilon - 3p . \end{aligned} \tag{5}$$

We see that as $\langle 1/E \rangle \to 0$ the equation of state becomes $p = \frac{1}{3}\epsilon$ and in general $p \leq \frac{1}{3}\epsilon$.

If $g(x,p)$ is the distribution function of an ideal Bose or Fermi gas, then for $T \geq m_\pi$,

$$\tfrac{1}{3}\epsilon \geq p \geq 0.29\epsilon . \tag{6}$$

(b) Resonance models of interactions. At present energies we might assume that important dynamics can be approximated by knowing the spectra of resonances. This is the Beth-Uhlenbeck[11] approximation to the partition function espoused by Hagedorn[12] and Shuryak.[13,4] Then

$$\ln Z = \int \rho(m/m_\pi) \ln[1 \pm \exp(E/T)] \frac{d^3p\, V}{(2\pi)^3}\, d(m/m_\pi) . \tag{7}$$

For $\rho(m) = b m^a$ the statistical mechanics can be worked out (see Appendix C) and we find

$$p = c_0{}^2 \epsilon , \quad \text{with} \quad c_0{}^2 = (a+4)^{-1} . \tag{8}$$

The spectrum of particles for $M \leq 3$ GeV is well fitted by

$$\rho(M/m_\pi) = \tfrac{1}{2}(M/m_\pi)^2 . \tag{9}$$

(c) Bag model of hadrons.[14] An MIT collaboration has put forth the interesting idea that the hadron itself is a free field (or weakly interacting field) in a bag. This leads to the following thermodynamic equations (see Appendix A):

$$\begin{aligned} \epsilon &= \frac{m_\pi}{V_\pi}[\lambda(T/m_\pi)^4 + B] \\ p &= \frac{1}{3}\epsilon - \frac{4}{3}\frac{m_\pi}{V_\pi} B , \\ s &= \frac{4}{3}\frac{\lambda}{V_\pi}\left(\frac{T}{m_\pi}\right)^3 . \end{aligned} \tag{10}$$

At equilibrium $p_c \equiv 0$, thus $B = \frac{1}{3}\lambda(T_c/m_\pi)^4$ and $\epsilon_c = m_\pi/V_\pi$, so that $(T_c/m_\pi) = (3/4\lambda)^{1/4}$. λ is determined by the statistical mechanics of the free-field theory. In this model the hydrodynamic expansion continues until $p \equiv 0$, at which point real pions appear.

Notice that in the bag model, at breakup $p = 0$ and $\epsilon = m_\pi/V_\pi$. Thus $T(x)$ is only a measure of the microscopic-field energy density; i.e., at breakup $T(x)$ is describing the internal distribution of "partons," *not* a thermal distribution of real particles. Thus for this model $p_\pi \equiv m_\pi \sinh\eta$, where η is the fluid rapidity. That is, the distribution of particles will be completely given by the hydrodynamics. Thus the bag model allows a connection with the *microscopic* physics describing the actual structure of the hadron; i.e., $T^{\mu\nu}(x)$ describes the microscopic parton physics. For a related discussion concerning the "parton" model see Hwa.[15]

In all these models $\partial p/\partial\epsilon \equiv c_0{}^2$ [(the speed of sound)^2] is a constant. Thus if we solve the hydrodynamic equations for arbitrary constant $c_0{}^2$ in terms of the temperature $T(x,t)$, we will have included models (a)–(c). To include Hagedorn's model one would let $\rho(m) = b m^a e^{cm}$ and work out the relevant (complicated) equation of state. For $m \leq 3$ GeV a power law describes the spectra adequately.

V. HYDRODYNAMICAL EQUATIONS

For a spherically symmetric expansion appropriate to annihilation, Eq. (2′) leads to the two equations

$$\begin{aligned} \frac{\partial T^0_0}{\partial t} - \frac{\partial T^r_0}{\partial r} + \frac{2}{r} T^r_0 &= 0 , \\ \frac{\partial T^r_r}{\partial r} - \frac{\partial T^r_0}{\partial t} + \frac{2}{r}(T^r_r + p) &= 0 . \end{aligned} \tag{11}$$

For a one-dimensional expansion in the longitudinal direction, appropriate to the early phases of the expansion following a purely hadronic collision, one has instead

$$\frac{\partial T_0^0}{\partial t}-\frac{\partial T_0^x}{\partial x}=0\,,\quad \frac{\partial T_x^x}{\partial x}-\frac{\partial T_0^x}{\partial t}=0\,. \tag{11'}$$

In what follows we will consider only the radial case. The one-dimensional case is obtained by deleting the $2/r$ terms.

It is convenient to use the thermodynamic relations to rewrite these equations in a more useful form. From Appendix A we have

$$\epsilon = Ts - p\,,\quad d\epsilon = T ds\,,\quad dp = s dT\,.$$

Projecting Eq. (4) along the 4-velocity

$$u_\mu \partial_\nu T^{\mu\nu}=0$$

we find

$$(su^\mu)_{,\mu}=0\,,\quad \text{or } (su^i)_{;i}=0\,, \tag{12}$$

which expresses local entropy conservation. Thus we have the global conservation law of entropy,

$$S=\int su^\mu\, d\sigma_\mu\,, \tag{13}$$

as well as energy and momentum conservation,

$$E_{\text{c.m.}}=\int T^{0\mu}\, d\sigma_\mu\,. \tag{14}$$

Another differential equation is obtained by projecting Eq. (4) along a direction perpendicular to the 4-velocity:

$$T^k_{i;k}-u_i u^k T^l_{k;l}=0\,, \tag{15}$$

or

$$u^k[(Tu_i)_{,k}-(Tu_k)_{,i}]=0\,.$$

Thus in the case of (1+1)-dimensional flow (r, t or x, t) we have

$$\partial_1(Tu_0)=\partial_0(Tu_1)\,. \tag{16}$$

This equation tells us that the flow is potential flow

$$Tu_i \equiv \partial_i \varphi\,.$$

For equations of state where $dp/d\epsilon = c_0^2 =$ constant we have

$$(s/s_0)=(T/T_0)^{1/c_0^2}\,, \tag{16'}$$

and using entropy conservation we obtain

$$\Box^2\varphi+\frac{(1-c_0^2)}{2c_0^2}\,\partial^\mu\varphi\partial_\mu \ln(\partial_\lambda\varphi\partial^\lambda\varphi)=0\,, \tag{17}$$

which is derivable from the Lagrangian

$$\mathcal{L}=g(\partial_\mu\varphi\partial^\mu\varphi)^n\,,\quad n=\tfrac{1}{2}(1+c_0^{-2})\,. \tag{18}$$

We can make a Legendre transformation of the potential equation in the case of one-dimensional longitudinal motion. This leads to Khalatnikov's exact solution of the one-dimensional problem.[16] Equation (18) led Milekhin[17] to speculate on an analogous (1+1)-dimensional field theory of self-interacting scalar mesons.

In the spherical-expansion case, the introduction of the potential does not simplify the problem (due to $1/r$ terms in the D'Alembertian).

Introducing liquid rapidity as a variable allows us to write Eqs. (12) and (16) in suggestive forms. Using Eq. (16′) and letting $u^0=\cosh\eta$, $u^1=\sinh\eta$, so that $v=\tanh\eta$, we can write the entropy equation (12) as

$$\tanh\eta\,\partial_r \ln T+\partial_t \ln T + c_0^2(\tanh\eta\,\partial_t\eta+\partial_r\eta)+\frac{2c_0^2}{r}\tanh\eta=0\,. \tag{19}$$

The potential equation becomes

$$\tanh\eta\,\partial_t \ln T+\partial_r \ln T+\tanh\eta\,\partial_r\eta+\partial_t\eta=0\,. \tag{20}$$

It is the inhomogeneous $2/r$ term in the entropy equation, which was not present in the one-dimensional case, that prevents simple analytic solution. For $c_0^2=\frac{1}{3}$ cooling is very rapid and most particles are at radii of the order of r_0, so that this term cannot be neglected.

Considerable insight into these equations can be gained by determining the characteristic surfaces of the motion[18] (the sound cone takes the place of the light cone in determining domains of influence for a fluid). Denoting the characteristic directions by C_+ and C_- and parameterizing the characteristics by α and β, we find along C_+

$$\frac{r_\alpha}{t_\alpha}=\frac{\tanh\eta+c_0}{1+c_0\tanh\eta}\,, \tag{21a}$$

and along C_-

$$\frac{r_\beta}{t_\beta}=\frac{\tanh\eta-c_0}{1-c_0\tanh\eta}\,, \tag{21b}$$

where the subscripts α and β denote differentiation along the corresponding characteristics. Equations (21) express the well-known fact that the slope of the characteristic is obtained from the relativistic addition law of adding the local fluid velocity of sound. Using Eqs. (19) and (20) we obtain the characteristic equations

$$(T_\sigma/T\pm c_0\eta_0)(\sinh\eta\pm c_0\cosh\eta)+2c_0^2(r_\sigma/r)\sinh\eta=0\,, \tag{22}$$

where $\sigma=\alpha$ on C_+ and β on C_-, with the corresponding signs applying. In the one-dimensional problem there is no $2/r$ term and the characteristic equation can be integrated to give

$$\ln T = -c_0\eta + f(\beta) \text{ on } C_+ ,$$
$$\ln T = +c_0\eta + g(\alpha) \text{ on } C_- , \tag{23}$$

with β (α) constant along a particular C_+ (C_-). For $r>0$ and t very small $\sinh\eta \approx 0$ and the three-dimensional solution goes over to the progressive wave part of the one-dimensional solution.[19] We will use this fact later in our numerical calculations. We also note that the Eqs. (21) and (22) are invariant under

$$r \to \lambda r, \quad t \to \lambda t, \quad T \to \lambda' T. \tag{24}$$

This invariance will be exploited later.

The program is to solve the hydrodynamic equations for $\epsilon(x,t)$ and $v(x,t)$ using the initial data

$$\epsilon_0 = E_{\text{c.m.}}/V_0, \quad v=0 \text{ at } t=0.$$

When the energy density reaches a critical value $\epsilon_c(x,t) \sim m_\pi/V_\pi$, or equivalently $T_c(x,t)\sim m_\pi$, real pions evaporate from the fluid. $\epsilon(x,t)=\epsilon_c$ (a constant) defines a surface σ which can be parameterized by the fluid velocity $v=\tanh\eta$; i.e., $r=r(\eta)$, $t=t(\eta)$ along σ. Thus at breakup one can use the conservation laws to get the energy and entropy distribution as a function of fluid rapidity. For a spherical expansion we get

$$\frac{dE}{d\eta} = T^{0\nu}\left(\frac{\partial\sigma_\nu}{\partial\eta}\right)$$
$$= 4\pi r^2(\eta)\left\{[(\epsilon_c+p_c)(u^0)^2 - p]\frac{dr}{d\eta} - (\epsilon_c+p_c)u^0u^r\frac{dt}{d\eta}\right\}, \tag{25}$$

$$\frac{dS}{d\eta} = s_c u^\nu\left(\frac{\partial\sigma_\nu}{\partial\eta}\right)$$
$$= 4\pi r^2(\eta)s_c\left(u^0\frac{dr}{d\eta} - u^r\frac{dt}{d\eta}\right). \tag{26}$$

Via the thermodynamic relations (A6)–(A8) (see Appendix A) s_c, ϵ_c, and p_c are all constant along the surface of condensation $T(x,t)=T_c$. Thus for $u^0=\gamma \gg 1$ the entropy distribution and energy distribution are proportional at breakup, i.e.,

$$\frac{dS}{d\eta} \approx \left(\frac{s_c}{\epsilon_c+p_c}\right)\frac{1}{u_0}\frac{dE}{d\eta} = \frac{1}{T_c u_0}\frac{dE}{d\eta} \approx \frac{1}{m_\pi\cosh\eta}\frac{dE}{d\eta}.$$

If we assume that at breakup the residual dynamics are that of an ideal-Bose-gas distribution in a comoving frame of reference, then (see Sec. VI)

$$E\frac{dN}{d^3p} = \int_\sigma g(\overline{E}(x), T(x))\,p^\mu\, d\sigma_\mu , \tag{27}$$

where $p^\mu=(E,\vec{p})$ is the particle momentum in the c.m. frame, $\overline{E}(x)=p_\mu u^\mu(x)$ is its energy in the comoving frame, and

$$g(\overline{E},T) = g_\pi(2\pi)^{-3}[\exp(\overline{E}/T)-1]^{-1}.$$

From this we get directly

$$\frac{dN}{d\eta} = n_c u^\nu\left(\frac{\partial\sigma_\nu}{\partial\eta}\right) = \frac{n_c}{s_c}\frac{dS}{d\eta}, \tag{28}$$

where

$$n_c = \int g(\overline{E}, T_c)\,d^3\overline{p}.$$

In the case that the thermal motion is unimportant compared to the fluid motion, then one has the approximate relation

$$p_\pi \approx m_\pi \sinh\eta,$$

which is equivalent to y(particle rapidity)$\approx\eta$. Then one can calculate the approximate rapidity distribution from Eq. (28). However, if at breakup $p\equiv 0$, as in the bag model, there are no residual dynamics and the pions at breakup are free. Then as an identity, $\epsilon_c=m_\pi/V_\pi$, $p_c=0$, and $\eta=y$ = particle rapidity. In that case

$$\frac{dN}{d\eta} \equiv \frac{1}{m_\pi\cosh\eta}\frac{dE}{d\eta}$$
$$= \frac{4\pi r^2(\eta)}{V_\pi}\left(\cosh\eta\frac{dr}{d\eta} - \sinh\eta\frac{dt}{d\eta}\right)$$
$$= \frac{dN}{dy}. \tag{28$'$}$$

In the case of pp scattering the fluid rapidity distribution is quite broad[2,3] and at high energies one can neglect the thermal motions (corresponding to $kT\approx m_\pi$) to first approximation (although they are necessary to explain the transverse distribution). For the annihilation process, at c.m. energies of 3–10 GeV, fluid rapidities are less than 1, thus $m_\pi \sinh\eta$ is smaller than typical thermal fluctuations (~300 MeV) (see Fig. 2) and it is necessary to include a model for the fluctuation. In Sec. VI we will show how to determine directly the momentum distribution of secondaries, assuming that the relevant dynamics at breakup is that of an ideal Bose gas of pions in a comoving frame.

VI. SINGLE-PARTICLE DISTRIBUTION[8]

The hydrodynamic description is in terms of the macroscopic variables $\epsilon(x)$, $p(x)$, where x is an average position of a cell containing enough quanta for local statistical equilibrium to be meaningful. In the early stages these quanta might be the virtual quanta (quarks, partons, what have you) relevant to high-temperature dynamics. At breakup we will assume that the relevant quanta are the real pions that are produced. Given a distribution function $g(x,p)$ for the quanta one determines the energy-momentum tensor via the relation (see Appendixes D and E for details in classical and quantum physics)

$$T^{\mu\nu}(x) \equiv \int g(x,p) p^\mu p^\nu \frac{d^3p}{E} . \qquad (29)$$

This shows $g(x,p)$ is a Lorentz-invariant distribution function. One also determines the number of real quanta of mass m crossing a surface $d\sigma_\mu$ by

$$dN = \int g(x,p) p^\mu d\sigma_\mu \theta(p_0)\delta(p^2 - m^2) d^4p . \qquad (30)$$

This expression counts the number of world lines contained in the surface $d\sigma^\mu$.

The collective velocity $u^\mu(x, t)$ is defined by

$$n(x) u^\mu(x) = \int g(x,p) p^\mu \, d^3p/E . \qquad (31)$$

Thus going into a comoving frame (denoted by barred variables)

$$n(\bar{x}) = \int g(\bar{x},\bar{p}) d^3p , \qquad (32)$$

i.e., $n(\bar{x})$ is the local density of matter and is a Lorentz scalar. In the comoving frame one has [if $g(\bar{x},\bar{p})$ is isotropic in $\bar{p}$]

$$T^{\mu\nu}(\bar{x}) = \mathrm{diag}(\epsilon, p, p, p) ,$$

where

$$\epsilon(\bar{x}) = \int g(\bar{x},\bar{p}) \bar{p}_0 d^3\bar{p} ,$$

$$p(\bar{x}) = \tfrac{1}{3} \int \bar{p}^2 g(\bar{x},\bar{p}) d^3\bar{p}/\bar{E} .$$

If we assume that at breakup the distribution function describing pions in the local rest frame is that of an ideal Bose gas with $T(x,t) = T_c$, then

$$g(\bar{x},\bar{p}) = g(x,p)$$

$$= \frac{g_\pi}{(2\pi)^3} \{\exp[p^\mu u_\mu(x)/T_c] - 1\}^{-1} . \qquad (32')$$

From Eq. (30) we have

$$E \frac{dN}{d^3p} = \int g(x,p) p^\mu \, d\sigma_\mu ,$$

which for e^+-e^- annihilation (spherical symmetry) gives

$$E \frac{dN}{d^3p} = \frac{g_\pi}{(2\pi)^2} \int_\sigma [\exp(\bar{E}/T_c) - 1]^{-1} d\eta \, d\cos\theta \, r^2(\eta) \times \left(E \frac{dr}{d\eta} - p\cos\theta \frac{dt}{d\eta}\right) , \qquad (33)$$

where $\bar{E} = E\cosh\eta - p\sinh\eta\cos\theta$ and σ is the same surface as in Eq. (25). In the above, we have utilized the isotropy assumption and chosen p as the z axis of the fluid.

For $pp \to \pi + X$ it is usually assumed that the fluid velocity is mostly longitudinal because of the larger pressure gradient in that direction. Ignoring transverse fluid motion gives[20]

$$p^\mu d\sigma_\mu = \pi a_0^2 (p^0 dx - p_\parallel dt) ,$$

$$p_\parallel = \mu_T \sinh y ,$$

$$p_0 = \mu_T \cosh y ,$$

$$\mu_T = (p_\perp^2 + m_\pi^2)^{1/2} ,$$

or

$$\frac{1}{\pi} \frac{dN}{dy\, dp_\perp^2} \approx \frac{g_\pi}{(2\pi)^2} \pi a_0^2 \int_\sigma \frac{\mu_T \cosh y \frac{dx}{d\eta} - \mu_T \sinh y \frac{dt}{d\eta}}{\exp[\mu_T \cosh(y-\eta)/T_c] - 1} d\eta . \qquad (34)$$

VII. MULTIPLICITY AND AVERAGE ENERGY OF SECONDARIES

Knowledge of the equation of state and the initial and final conditions of the fluid, coupled with entropy conservation [Eq. (12)] allows a determination of the multiplicity as a function of λ, r_0, and $E_{\mathrm{c.m.}}$. A discussion of the relevant thermodynamic equations is found in Appendix A.

The initial condition on the fluid is

$$\epsilon_0 = E_{\mathrm{c.m.}}/V_0 = \lambda \frac{m_\pi}{V_\pi} \left(\frac{T_0}{m_\pi}\right)^{1+c_0^{-2}} + B \frac{m_\pi}{V_\pi} , \qquad (35)$$

where we have used $dp/d\epsilon = c_0^2 = \text{constant}$; λ is proportional to the degeneracy of states in the underlying dynamics (see Appendix B), and B is a constant of integration that we will use to adjust the initial condition to the final state, described by the ideal-Bose-gas equation of state.

The initial entropy, which by Eq. (12) is equal to the final entropy, is related to T_0 and V_0 by

$$S = s_0 V_0 = \frac{\lambda}{V_\pi} (1 + c_0^2) \left(\frac{T_0}{m_\pi}\right)^{1/c_0^2} V_0 . \qquad (36)$$

We will assume that all the produced hadrons are pions, and when $c_0^2 \neq \frac{1}{3}$ we will choose T_c so that the underlying dynamics goes smoothly into the ideal-Bose-gas dynamics. If one is interested in ratios of particles at $T = m_\pi c^2$ before they decay into pions, one can use the statistical mechanical formula in Appendix C to find them, following the approach of Hagedorn.[14] Here when we use the hadron spectrum as dynamics, we will let the temperature cool slightly below $T=m_\pi$ so that we have only pions.

With that assumption, we get for N_π

$$N_\pi=\frac{n(T_c)}{s(T_c)}S$$
$$=\frac{n(T_c)}{s(T_c)}(1+c_0{}^2)\left(\frac{E_{\rm c.m.}}{m_\pi}-B\frac{V_0}{V_\pi}\right)^{(1+c_0{}^2)^{-1}}$$
$$\times\left(\frac{\lambda V_0}{V_\pi}\right)^{c_0{}^2(1+c_0{}^2)^{-1}}. \tag{36$'$}$$

For all cases studied the B term can be neglected, and for all practical purposes

$$N_\pi=\frac{n_c}{s_c}(1+c_0{}^2)\left(\frac{E_{\rm c.m.}}{m_\pi}\right)^{(1+c_0{}^2)^{-1}}\left(\frac{\lambda V_0}{V_\pi}\right)^{c_0{}^2(1+c_0{}^2)^{-1}}, \tag{37}$$

which for $c_0{}^2=\frac{1}{3}$, $T_c=m_\pi$ gives

$$N_\pi=\frac{1}{3}\left(\frac{E_{\rm c.m.}}{m_\pi}\right)^{3/4}\left(\frac{\lambda V_0}{V_\pi}\right)^{1/4}, \tag{38}$$

a result previously obtained by Shuryak,[21] and Carruthers and Minh.[22] For $pp\to\pi+X$, where $V_0=\pi a^3M/E_{\rm c.m.}$, Eq. (37) leads to the result

$$N_\pi\propto E_{\rm c.m.}{}^a,\quad \frac{1-c_0{}^2}{1+c_0{}^2}=a.$$

For $e^++e^-\to\pi+X$ the energy dependence of V_0 is not known *a priori*. For $\langle E_\pi\rangle$ we obtain

$$\langle E_\pi\rangle=\frac{s_c}{n_c}(1+c_0{}^2)^{-1}m_\pi\left[\left(\frac{T}{m_\pi}\right)+\frac{B}{\lambda}\left(\frac{T}{m_\pi}\right)^{-1/c_0{}^2}\right]$$
$$\approx m_\pi\frac{s_c}{n_c}(1+c_0{}^2)^{-1}\left(\frac{E_{\rm c.m.}}{\lambda m_\pi}\frac{V_\pi}{V_0}\right)^{c_0{}^2(1+c_0{}^2)^{-1}}. \tag{39}$$

We observe that $\langle E_\pi\rangle$ is energy-dependent unless $c_0{}^2=0$ (pure thermodynamic model without hydrodynamic expansion) or $V_0\propto E_{\rm c.m.}$ (which seems unlikely).

This makes it quite easy to distinguish this model from the statistical models of Pomeranchuk[23] and Satz.[24] In those models one gets fixed $\langle E_\pi\rangle\approx 420-500$ depending on the critical temperature.

We note that for $c_0{}^2=\frac{1}{3}$, $\lambda=3.57$ (ideal Bose gas), $r_0=1.75a_0$, one gets the relation

$$\langle E_\pi\rangle=3m_\pi\left(\frac{E_{\rm c.m.}}{2\ {\rm GeV}}\right)^{1/4}, \tag{39$'$}$$

which agrees with the preliminary SPEAR[10] results for $e^+-e^-\to$ hadrons that show $\langle E_\pi\rangle=470$, 540 MeV at $E_{\rm c.m.}=3, 5$ GeV, respectively. It is interesting that this equation of state ($c_0{}^2=\frac{1}{3}$) leads to the same r_0 for $E_{\rm c.m.}=3$ and 5 GeV, since the cross section also appears constant (see Ref. 7).

If instead we use the mass-spectrum dynamics of Eq. (7), $\rho(m/m_\pi)=\frac{1}{2}(m/m_\pi)^2$ leads to $\lambda=16.5$, and $E/N=470$ at 3 GeV yields $r_0=1.76$, whereas $E/N=540$ at 5 GeV yields $r_0=1.57$. Thus one can get agreement with the $\langle E_\pi\rangle$ measurement at SPEAR with resonance dynamics if we allow for a slight decrease in r_0 with energy. Notice, however, both models give an r_0 of the order of one pion Compton wave length (i.e., typical hadronic size unLorentz contracted). If the annihilation cross section remains constant and $c_0{}^2=\frac{1}{3}$, then we expect r_0 to remain constant and would expect that the average energy of secondaries would continue to grow as a power.

We see the absolute normalization of N depends on λ (which counts the number of different particles in the dynamics) and r_0 (which is *a priori* unknown in this approach). However, the shape of the single-particle distribution depends only on T_0 and thus on E/N via Eq. (39).

For $c_0{}^2=\frac{1}{3}$, $s(T_c)=4.2$, $T_c=m_\pi$, and Eq. (39) becomes

$$\langle E_\pi\rangle=3m_\pi(T/m_\pi).$$

We see that unless $\langle E_\pi\rangle\geq 420$ MeV, there is no hydrodynamic motion, since we assumed $T_0\geq m_\pi$ for the fluid to exist (i.e., unless $T_0\geq m_\pi$ the mean free paths are too great to allow the pions to interact as a fluid).

For the imperfect-gas model one has a spectrum of particles at $T=m_\pi$, and the critical temperature is determined by the temperature at which ϵ and p smoothly go over to the ideal-gas formula. For the observed mass spectra, $\rho(m/m_\pi)=\frac{1}{2}(m/m_\pi)^2$, and one has from Eq. (C4) $\lambda=16.5$, $c_0{}^2=\frac{1}{6}$, and a value of B of -0.049 is needed for a smooth transition. One then finds $T_c=0.58m_\pi$ and $n_c/s_c=0.208$. This gives for $\langle E_\pi\rangle$

$$\langle E_\pi\rangle=\frac{\epsilon_0}{s_0}\frac{s_c}{n_c}\approx 4.1\left(\frac{T_0}{T_c}\right). \tag{40}$$

Thus for $c_0{}^2=\frac{1}{6}$ the model makes sense for $T_0\geq T_c$ or $E_\pi\geq 350$ MeV.

If Eq. (39′) continues to hold at all energies, we can fulfill this equation in both models, with values of r_0 given in Table I for various $E_{\rm c.m.}$.

VIII. SCALING SOLUTIONS OF THE HYDRODYNAMIC EQUATIONS: CONNECTION WITH "FEYNMAN SCALING"

If the initial size of the system is much smaller than the breakup size, one might hope we can ignore the natural scale (which in pp collisions $\sim 1/E$, since $V_0 \sim V_\pi M/E_{\rm c.m.}$), and try solutions of the form

$$v = r/t \;\; (x/t) \;\; \text{for } r, t \gg r_0 . \tag{41}$$

If $v \equiv r/t$, we can find exact solutions to the equations of motion [Eqs. (19) and (20)].

Introducing

$$\alpha = \tfrac{1}{2}\ln\left(\frac{t+r}{t-r}\right), \qquad \beta = \tfrac{1}{2}\ln\left(\frac{t^2-r^2}{r_0^2}\right) = \ln\left(\frac{\tau}{r_0}\right) \tag{42}$$

so that

$$r = \tau\sinh\alpha = r_0 e^\beta \sinh\alpha, \qquad t = \tau\cosh\alpha = r_0 e^\beta \cosh\alpha, \tag{42'}$$

where τ is the proper time if $v \equiv r/t$, and r_0 is an arbitrary parameter, we get the exact equations

TABLE I. Comparison of parameters of the two models, evaluated for equal $\langle E_\pi \rangle$. These parameters correspond to values of $E_{\rm c.m.}$ and r_0 found in the last two columns if $\langle E_\pi \rangle = 3m_\pi(\frac{1}{2}E_{\rm c.m.})^{1/4}$. The upper entry corresponds to $c_0^2 = \frac{1}{3}$, $\lambda = 3.57$, $B = 0.11$, $T_c/m_\pi = 1$, $s_c/n_c = 4.2$. The lower entry corresponds to $c_0^2 = \frac{1}{6}$, $\lambda = 16.5$, $B = -0.026$, $T_c/m_\pi = 0.58$, $s_c/n_c = 4.8$.

$\langle E_\pi \rangle$ (MeV)	T_0/m_π	$\langle \gamma \rangle$	$E_{\rm c.m.}$ (GeV)	r_0/m_π^{-1}
466	1.028	1.001	3	1.73
	0.8145	1.22		1.76
530	1.181	1.025	5	1.72
	0.921	1.37		1.56
627	1.41	1.12	10	1.71
	1.089	1.61		1.33
805	1.82	1.38	25	1.66
	1.397	2.06		1.01
938	2.12	1.59	50	1.70
	1.628	2.4		0.89

$$\partial_\alpha(\ln T)\sinh(\eta-\alpha) + \partial_\beta(\ln T)\cosh(\eta-\alpha) + c_0^2[\eta_\alpha\cosh(\eta-\alpha) + \eta_\beta\sinh(\eta-\alpha)] + 2c_0^2\frac{\sinh\eta}{\sinh\alpha} = 0, \tag{43}$$

where $\eta_\alpha = \partial_\alpha\eta$, and

$$\eta_\alpha\sinh(\eta-\alpha) + \eta_\beta\cosh(\eta-\alpha) + (\ln T)_\alpha\cosh(\eta-\alpha) + (\ln T)_\beta\sinh(\eta-\alpha) = 0, \tag{44}$$

where in the x, t problem there is no $\sinh\eta/\sinh\alpha$ term.

The choice of variables α, β are obvious for scaling solutions. If $v \equiv r/t$, then $\eta = \tanh^{-1}v = \alpha$ and Eqs. (19) and (20) become

$$\partial_\beta \ln T + \lambda c_0^2 = 0, \tag{45a}$$

$$\partial_\alpha \ln T = 0, \tag{45b}$$

where

$$\lambda = \begin{cases} 1, & \text{one-dimensional } pp \text{ case} \\ 3, & \text{spherical expansion} \end{cases} .$$

These equations integrate immediately to

$$T/T_0 = e^{-\lambda c_0^2 \beta}, \quad \text{or } \epsilon/\epsilon_0 = e^{-\lambda(1+c_0^2)\beta} . \tag{46}$$

Also $\tau = t(1-v^2)^{1/2}$ and

$$T/T_0 = (\tau/r_0)^{-\lambda c_0^2}, \qquad \epsilon/\epsilon_0 = (\tau/r_0)^{-\lambda(1+c_0^2)} = \left(\frac{t^2-r^2}{r_0^2}\right)^{-\lambda(1+c_0^2)/2} . \tag{47}$$

Thus at the critical surface where ϵ = constant we find that the proper time is constant, i.e., all the pieces of the fluid take the same proper time to break up if $v \equiv r/t$.

The pions evaporate out of the fluid along the surface $T(x,t) = T_c$, which is also a surface of ϵ, p, and s being constant. In a one-dimensional expansion ($pp \to \pi + X$) Eq. (28) becomes

$$\frac{dN}{d\eta} = \pi a^2 n_c\left(\cosh\eta\frac{dx}{d\eta} - \sinh\eta\frac{dt}{d\eta}\right)_\sigma . \tag{28''}$$

Parameterizing σ by η and using Eqs. (42) and (42′) we get

$$\frac{dN}{d\eta} = \pi a^2 n_c\left[\frac{\partial\alpha}{\partial\eta}\tau\cosh(\eta-\alpha) + \frac{\partial\tau}{\partial\eta}\sinh(\eta-\alpha)\right]_\sigma ; \tag{28'''}$$

thus when $\eta \equiv \alpha$ we get

$$\frac{dN}{d\eta} = \pi a^2 n_c \tau(\eta) . \tag{48}$$

At breakup $\epsilon = \epsilon_c$, thus via Eq. (47)

$$\tau = \tau_c = r_0\left(\frac{\epsilon_0}{\epsilon_c}\right)^{1/(1+c_0^2)} . \tag{49}$$

For $pp \to \pi + X$, $V_0 \approx \pi a^3 M/E$, $r_0 \approx aM/E$, $\epsilon_c \approx m_\pi/a^3$, $a = m_\pi^{-1}$, and in the approximation $v \equiv x/t$

$$\frac{dN}{d\eta} \propto \left(\frac{M}{m_\pi}\right)^{1/(1+c_0^2)}\left(\frac{E_{\rm c.m.}}{M}\right)^{(1-c_0^2)/(1+c_0^2)} . \tag{50}$$

Thus in that approximation one gets a flat rapidity distribution with the height going as

$$E_{\text{c.m.}}^{(1-c_0^2)/(1+c_0^2)} .$$

Thus for $c_0^2=\frac{1}{3}$ one gets $E_{\text{c.m.}}^{1/2}$ ($=E_{\text{lab}}^{1/4}$) growth in the height of the plateau. To get a nonincreasing height one needs $c_0^2 \to 1$. $c_0^2\to 1$ can be obtained if the hadronic process is one-dimensional in momentum space as well as in x space. That is, if the distribution $g(x,p)$ is of the form $\tilde{g}(x,p)\delta(p_\perp^2)$, then in the ultrarelativistic limit $p\approx\epsilon$ (see Appendix D). However, $c_0^2\to 1$ is also true for free pions produced by a classical source.

The scaling solution does not take into account global energy conservation (i.e., $\int T^{0\nu}d\sigma_\nu=\infty$ if $v\equiv r/t$) and the fact that $E_{\text{c.m.}}$ is finite does *two* things. First, it gives a maximum rapidity given by

$$\eta_{\max}=(1/c_0)\ln(T_0/T_c)\,, \tag{51}$$

that is, the usual logarithmic growth of the length of the rapidity plot. Second, for the solution that corresponds to the correct boundary condition on ϵ_0, v is not identically x/t and the isotherms deviate from $\tau=\text{constant}$. The solution at very high energies for small η resembles a Gaussian in rapidity and is not "flat." Thus correction to $x\equiv vt$ (partons are "free") changes a flat rapidity distribution to a "Gaussian." How the height varies with energy depends on the equation of state of the hadronic matter at formation. In Sec. IX we discuss how to improve upon $v=r/t$ by assuming $v=Kr/t$, with K slowly varying.

For the spherically symmetric expansion (assumed valid for $e^+ + e^- \to$ hadrons) one has

$$\begin{aligned}\frac{dN}{d\eta}&=n_c 4\pi r^2\left(\cosh\eta\,\frac{dr}{d\eta}-\sinh\eta\,\frac{dt}{d\eta}\right)_\sigma\\&=n_c 4\pi\tau^2\sinh^2\eta\left[\frac{\partial\alpha}{\partial\eta}\,\tau\cosh(\eta-\alpha)\right.\\&\qquad\left.+\frac{\partial\tau}{\partial\eta}\sinh(\eta-\alpha)\right]_\sigma\,,\end{aligned} \tag{52}$$

and the scaling solution $\eta=\alpha$ leads to [using (47)]

$$\begin{aligned}\frac{dN}{d\eta}&=n_c 4\pi\tau^3\sinh^2\eta\\&=n_c 4\pi r_0^3\left(\frac{\epsilon_0}{\epsilon_c}\right)^{1/(1+c_0^2)}\sinh^2\eta\,.\end{aligned} \tag{53}$$

Thus

$$\frac{1}{4\pi m_\pi^2\sinh^2\eta}\,\frac{dN}{d\eta}\propto(E_{\text{c.m.}})^{1/(1+c_0^2)}\,(r_0)^{3c_0^2/(1+c_0^2)}\,. \tag{54}$$

If we can neglect thermal motion, then we expect this will resemble $E\,dN/d^3p$ if there is a scaling region. Later we will see that for $c_0^2=\frac{1}{6}$ such a region does exist. Thus $v=x/t$, r/t lead to plateaus in invariant phase space in one and three dimensions, respectively.

Although these distributions violate global energy conservation, some of the qualitative properties of these solutions as described above are to be preserved when global energy conservation obtains. In the one-dimensional case, this is due to the fact that, at least for part of the expansion, the scaling formula (41) is approximately valid. In the following section we shall describe a better solution to the one-dimensional problem based on that approximation.

For the three-dimensional case, for $c_0^2=\frac{1}{3}$ Eq. (41) is *never* valid, at least for $E_{\text{c.m.}}\le 100$ GeV, as the numerical solution shows. However, the determining factor in the distribution is the spherical nature of the expansion: Most of the entropy is concentrated at large radii, and therefore in regions of large rapidity. The numerical solution bears this out. For $c_0^2=\frac{1}{6}$ and large T_0, part of the critical isotherm is similar to a hyperboloid, and one gets a behavior similar to Eq. (54) for small η. For small c_0^2 cooling is slow and $r\gg r_0$ holds on the critical isotherms. This is not true for $c_0^2=\frac{1}{3}$, $E_{\text{c.m.}}<100$ GeV, where cooling is very rapid.

IX. QUASISCALING SOLUTIONS

In order to improve on the scaling formula (41), Landau assumed[1] that for part of the expansion at least, one could write*

$$v=g\,\frac{r}{t}\,, \tag{55}$$

where g is a slowly varying function of r and t, such that $\partial g/\partial r$, $\partial g/\partial t$ can be neglected.

In terms of light-cone variables (41) we write the equivalent relation[13,4]

$$\eta=\alpha+f$$

and (56)

$$\frac{\partial f}{\partial\alpha}\,,\ \frac{\partial f}{\partial\beta}\approx 0\,.$$

Using (56) in our equations of motion (43) and (44) we can eliminate f. We find

$$\tanh f=\frac{\partial\varphi/\partial\beta+\lambda(1+c_0^2)}{\partial\varphi/\partial\alpha+(\lambda-1)(1+c_0^2)\coth\alpha}\,,$$

$$\varphi=\ln(\epsilon/\epsilon_0) \tag{57}$$

and

$$\tanh f=\frac{K\partial\varphi/\partial\alpha}{1+K\partial\varphi/\partial\beta}\,,$$

where $K=c_0^2/(1+c_0^2)$ and λ is defined as in Eq. (45). This leads to the following equation for the energy density:

$$K\left(\frac{\partial\varphi}{\partial\alpha}\right)^2+(\lambda-1)c_0^2\frac{\partial\varphi}{\partial\alpha}\coth\alpha$$
$$=K\left(\frac{\partial\varphi}{\partial\beta}\right)^2+\frac{\partial\varphi}{\partial\beta}(1+\lambda c_0^2)+\lambda(1+c_0^2). \quad (58)$$

This equation has the general solution

$$\varphi=A\beta+\varphi_1(\alpha,A)+C, \quad (59a)$$

where

$$\varphi_1(\alpha,A)=\alpha\left(A^2+\frac{1+c_0^2}{K}(A+1)\right)^{1/2} \quad (59b)$$

in the one-dimensional case,[4] and a complicated function of $\sinh\alpha$, which we shall not write down, in the spherical case [see, however, Eq. (58′) below]. The two-parameter system (59) is the general solution of (58). The particular solution that matches a given boundary condition is obtained by letting $C=C(A)$ and finding the envelope of the resulting one-parameter family:

$$\frac{\partial\varphi}{\partial A}=\beta+\frac{\partial\varphi_1}{\partial A}+\frac{dC}{dA}=0. \quad (60)$$

In the one-dimensional case, the initial conditions are defined at constant time: $T(x,t_0)=T_0$ for small t_0. Assuming that the expansion is slow (which turns out to be correct, as shown by the numerical solution) we can say that $T\approx T_0$ for small α, β. Then $T\sim T_0$, $\partial\varphi/\partial A\sim 1$, and $dC/dA\sim 1$. For large α, β, i.e., at breakup, C can therefore be neglected altogether. Equation (60) allows us to find A as a function of α and β and finally to determine φ. We find

$$A\approx\frac{1}{2}\frac{(1-c_0^4)}{c_0^2}\frac{\beta}{(\beta^2-\alpha^2)^{1/2}}-\frac{1}{2}\frac{(1+c_0^2)^2}{c_0^2}\beta,$$

so (61)

$$\varphi=\ln(\epsilon/\epsilon_0)$$
$$=\frac{1}{2}\frac{(1-c_0^4)}{c_0^2}(\beta^2-\alpha^2)^{1/2}-\frac{1}{2}\frac{(1+c_0^2)^2}{c_0^2}\beta.$$

Notice that for $\alpha=0$, this agrees with the scaling solution Eq. (47). The critical surface $T=T_c$ or, equivalently,

$$\epsilon(x,t)=\epsilon_c$$

has the equation

$$\beta=\frac{-K'L-[L^2-(K'^2-1)\alpha^2]^{1/2}}{K'^2-1}$$
$$=\ln(\tau/r_0), \quad (62)$$

where

$$K'=\frac{1+c_0^2}{1-c_0^2},\quad L=\frac{2c_0^2}{1-c_0^4}\ln\left(\frac{\epsilon_c}{\epsilon_0}\right).$$

To determine the particle distribution, it is sufficient to use the scaling value for $\eta=\alpha$ and Eq. (62) for β; we find from Eq. (48)

$$\frac{dN}{d\eta}\approx\pi a^2 n_c\tau(\alpha)=\pi a^2 r_0 e^{\beta}n_c,$$

with

$$n_c\approx\frac{1}{V_\pi},\quad r_0\approx\frac{2m}{E_{\text{c.m.}}}a.$$

The original result of Landau[1] and Milekhin[25] is recovered when we expand the exponent in powers of L, for $L\gg\alpha$:

$$\frac{dN}{d\eta}\approx\pi a^2 n_c\left(\frac{\epsilon_0}{\epsilon_c}\right)^{1/(1+c_0^2)}r_0 e^{-\alpha^2/2|L|}. \quad (63)$$

We notice that this improved result has the same normalization at $\alpha=0$ as the exact scaling result. However, the plateau is replaced by a "Gaussian" in rapidity when $L\gg\alpha$, and we also notice from Eq. (62) that this solution is valid only for

$$\alpha\le\frac{1}{c_0}\ln(T_0/T_c)$$

since one can rewrite Eq. (62) as

$$\beta=(2c_0^2)^{-1}[-\chi(1+c_0^2)-(1-c_0^2)(\chi^2-c_0^2\alpha^2)^{1/2}],$$

where $\chi=\ln(T_c/T_0)$.

In the one-dimensional case, one can use the potential equation and make a Legendre transformation and solve the problem exactly. This has been done by Khalatnikov. The stated result for $c_0^2=\frac{1}{3}$ is[16,25]

$$\frac{dN}{d\eta}=\pi a^2 n_c r_0 e^{-2x}$$
$$\times\left[I_0((x^2-\tfrac{1}{3}\eta^2)^{1/2})-\frac{\partial}{\partial x}I_0((x^2-\tfrac{1}{3}\eta^2)^{1/2})\right],$$

where (64)

$$x=\ln(T_c/T_0)$$
$$=\frac{c_0^2}{1+c_0^2}\ln\left(\frac{\epsilon_c}{\epsilon_0}\right).$$

I_0 is a Bessel function. Using $I_0(z)\approx e^z(2\pi z)^{-1/2}$ for large z, one gets Eq. (63). For the exact solution we notice

$$\eta_{\max}=\frac{1}{c_0}\ln\left(\frac{T_0}{T_c}\right),$$

which shows that the length of the rapidity axis grows logarithmically with $E_{\text{c.m.}}$.

For the spherical case, if we further make the assumption $\tanh\alpha\approx 1$, then we get the following equation for large α:

$$K\left(\frac{\partial\varphi}{\partial\alpha}\right)^2+2c_0^2\frac{\partial\varphi}{\partial\beta}=K\left(\frac{\partial\varphi}{\partial\beta}\right)^2+(1+3c_0^2)\frac{\partial\varphi}{\partial\beta}+3(1+c_0^2)\,, \tag{58'}$$

for which the general solution is

$$\varphi=A\beta+\varphi_1(\alpha,A)+C\,,$$

$$\varphi_1(\alpha,A)=\left[-(1+c_0^2)+\left(A^2+\frac{(1+3c_0^2)}{K}A+\frac{3(1+c_0^2)}{K}+\frac{c_0^4}{K^2}\right)^{1/2}\right]\alpha\,. \tag{65}$$

Since this only valid for large α we cannot extrapolate back safely to the original (α,β small).

If we blithely find the envelope by assuming that we can again ignore dC/dA, then the envelope satisfies

$$\beta+\frac{\partial\varphi_1}{\partial A}=0\,,$$

yielding

$$A=-\frac{\lambda'}{2}+\frac{\beta}{2}\left(\frac{\lambda'^2-4\lambda}{\beta^2-\alpha^2}\right)^{1/2}\,, \tag{66}$$

with

$$\lambda=c_0^2\frac{(3+c_0^2)}{K^2}\,,\quad \lambda'=\frac{1+3c_0^2}{K}\,,\quad K=\frac{c_0^2}{1+c_0^2}\,,$$

or

$$\varphi=\ln(\epsilon/\epsilon_0)=-(1+c_0^2)\alpha-\tfrac{1}{2}\lambda'\beta+\frac{1}{2K}[(1-5c_0^2)(1-c_0^2)]^{1/2}(\beta^2-\alpha^2)^{1/2}\,. \tag{67}$$

Thus we find that for $\beta>\alpha$ this solution only makes sense for $c_0^2<\frac{1}{5}$. This is probably due to the fact that the smaller c_0^2 is, the slower the cooling process and the more likely one can find a region where φ is relatively near its initial value and $v=gr/t$ is reasonably valid. One has for $\eta\approx\alpha$, using this result,

$$\frac{dN}{d\eta}\approx n_c 4\pi\tau^3\sinh^2\eta=n_c 4\pi r_0^3 e^{3\beta}\sinh^2\eta\,,$$

where

$$\beta=-[4c_0^2(1+3c_0^2)]^{-1}\left\{\frac{(1+3c_0^2)}{[(1-5c_0^2)(1-c_0^2)]^{1/2}}\phi+[\phi^2-4c_0^2(1+3c_0^2)\alpha^2]^{1/2}\right\}$$

and

$$\phi=2c_0^2(1+c_0^2)^{-1}(1-5c_0^2)^{-1/2}(1-c_0^2)^{-1/2}[\ln(\epsilon_c/\epsilon_0)+(1+c_0^2)\alpha]\,. \tag{68}$$

Since this result is only valid for $c_0^2\leq\frac{1}{5}$, if at all, we turn in Sec. X to numerical solutions.

X. NUMERICAL SOLUTIONS

In order to solve numerically the equations of motion, it is best to use the method of characteristics,[18] and make use of Eqs. (21) and (22). The initial conditions are specified at $t=0$:

$$T(r,t=0)=T_0\,,\quad 0<r\leq r_0$$
$$T(r,0)=0\,,\quad r>r_0 \tag{69a}$$

and

$$\eta(r,0)=0\,.$$

With these boundary conditions, the surface $t=0$ is nowhere tangent to a characteristic, and the problem is fully determined. From the values of η at time t_0, one determines the slopes of the characteristics along that surface from Eq. (21) and uses Eqs. (22) to determine η and T at time $t_0+\Delta t$.

The motion of the edge of the fluid has to be treated with certain care. In the one-dimensional case, an exact solution is known for the expansion of the fluid, initially at rest, into vacuum. The motion of the leading edge is a progressive wave, in which η and T are related by

$$\ln(T/T_0)=-c_0\eta\,.$$

In the three-dimensional case no such analytic solution is known. However, it is easy to show that for a very short time after the beginning of the spherical expansion the inhomogeneous term in the equation of motion can be neglected, and Eq. (23) holds.

Our boundary conditions can then be chosen so that no discontinuity appears in the temperature distribution at some time after the beginning of the expansion. We chose this as $t=0$, and set

$$T(r,0)=T_0\,,\quad \eta=0\,,$$

for

$$0 < r \leq r_0 - \epsilon \,,$$

and

$$T(r,0) = T_0(r_0 - r)/\epsilon \,, \qquad \eta = -(1/c_0)\ln(T/T_0)\,, \tag{69b}$$

for

$$r_0 - \epsilon \leq r \leq r_0 \,,$$

where $\epsilon \ll r_0$. Our results were insensitive to the choice of ϵ as long as $\epsilon < 0.2r_0$. The choice of a linear drop of T [Eq. (69b)] is to some extent arbitrary. T could be any smooth monotonically decreasing function of r. Again, the results are insensitive to the specific form chosen. The three-dimensional expansion has qualitative features similar to those of the one-dimensional problem:

(a) During an initial period from $t=0$ to $t \approx r_0/c_0$, the initial disturbance at the edge propagates inwards and sets the fluid in motion. At the same time the edge of the fluid moves outward at a speed $v \approx 1$. In the region of the leading edge the isotherms, including the critical isotherm, are timelike and begin as straight lines. This is similar to the progressive wave region of the one-dimensional problem.[19]

(b) For $t > r_0/c_0$ the whole fluid is in motion, and because of the three-dimensional nature of the expansion, it cools very rapidly. By the time the initial sphere has expanded to a few times its original size, the fluid is completely cool, and the final particles have "evaporated" from it. The critical isotherm becomes spacelike after some time $t > r_0/c_0$, and for very high initial temperatures starts looking like a hyperbola. This is similar to the behavior of the fluid in the "nontrivial" region of the one-dimensional problem. However, in the one-dimensional problem the expansion takes much longer, and the isotherm in the nontrivial region is closer to a hyperbola whose asymptote is the light cone.

We notice that the boundary conditions at $t=0$ [Eq. (69b)] are invariant under $r \to \lambda r$, $\epsilon \to \lambda\epsilon$, and $r_0 \to \lambda r_0$.

Since the characteristic equations (21) and (22) are also invariant under $r \to \lambda r$, $t \to \lambda r$, it is sufficient to use $r_0 = 1$ and measure r and t in units of r_0.

Thus

$$\frac{dS}{d\eta} = 4\pi r_0^3 s_c \left(\cosh\eta \frac{d\tilde{r}}{d\eta} - \sinh\eta \frac{d\tilde{t}}{d\eta} \right) \tilde{r}^2(\eta)\,, \tag{70}$$

where

$$\tilde{r} = r/r_0 \,, \quad \tilde{t} = t/r_0 \,.$$

We notice

$$\frac{dN/d\eta}{N} = \frac{dS/d\eta}{S} = \frac{3s_c}{s_0}\left(\cosh\eta \frac{d\tilde{r}}{d\eta} - \sinh\eta \frac{d\tilde{t}}{d\eta} \right)\tilde{r}^2(\eta) = 3\left(\frac{T_c}{T_0}\right)^{1/c_0^2}\left(\cosh\eta \frac{d\tilde{r}}{d\eta} - \sinh\eta \frac{d\tilde{t}}{d\eta} \right)\tilde{r}^2(\eta) \tag{71}$$

depends only on T_0/T_c and c_0^2 via the scaling properties of the differential equations (21) and (22).

Using finite difference methods we calculate the expression Eq. (71) along the critical isotherm $T(x,t) = T_c$. We display the critical isotherm $T_c = m_\pi$ for $c_0^2 = \frac{1}{3}$ for various T_0 in Fig. 1(a). We have, at present SPEAR energies of $E_{\text{c.m.}} = 3$ and 5, via Eqs. (39) that $T_0 = 1.03$ and 1.18, respectively. Notice that for $T_0 < 2m_\pi$ there is no hint of the isotherm looking like $\tau = \text{constant}$. For $T_0 = 4m_\pi$ we see the beginnings of such behavior. However, even at $T_0 = 4m_\pi$ the outer edge of the isotherm is far from a hyperboloid, and that is where most of the entropy lies.

For $c_0^2 = \frac{1}{6}$, cooling is slower [see Fig. 1(b)] and for $T/T_c = 3.8$ for $r < 3r_0$ the isotherm is almost a hyperbola with $v \approx \frac{3}{2}(r/t)$. Thus for small c_0^2, cooling is slow enough so that scaling will eventually set in and control the pions not coming from the outer shell.

Since Eq. (71) depends only on T_c/T_0 and c_0^2, the experimental value of $\langle E_\pi \rangle$ determines $(dN/d\eta)/N$ via Eq. (39). To obtain $dN/d\eta$ one uses Eq. (37) to determine N from $E_{\text{c.m.}}$.

In the curves that follow, we will consider the ultrarelativistic model, with $c_0^2 = \frac{1}{3}$, and the hadron spectrum model, where $c_0^2 = \frac{1}{6}$. Values of T_0 are determined from $\langle E_\pi \rangle$ using the values of λ, B, and s_c/n_c found in Table I.

In Figs. 2(a) and 2(b) we plot $(dN/d\eta)/N$ for different $\langle E_\pi \rangle$ for the two models. We notice that this function is sharply peaked. Since

$$\langle \gamma \rangle = \int_\sigma u^0 u^\mu \, d\sigma_\mu \Big/ \int_\sigma u^\mu \, d\sigma_\mu = \frac{1}{1+c_0^2}\left[\frac{T_0}{T_c} + c_0^2 \left(\frac{T_c}{T_0}\right)^{a-1} \right], \quad a = 1 + c_0^{-2}\,, \tag{72}$$

we have

$$\cosh\eta_{\text{peak}} \approx \langle \gamma \rangle \quad \text{or} \quad \eta_{\text{peak}} \approx \cosh^{-1}\langle \gamma \rangle \,. \tag{73}$$

We also have that since the maximum η is obtained at $t \approx 0$, it is determined by the solution of the one-dimensional problem. Thus

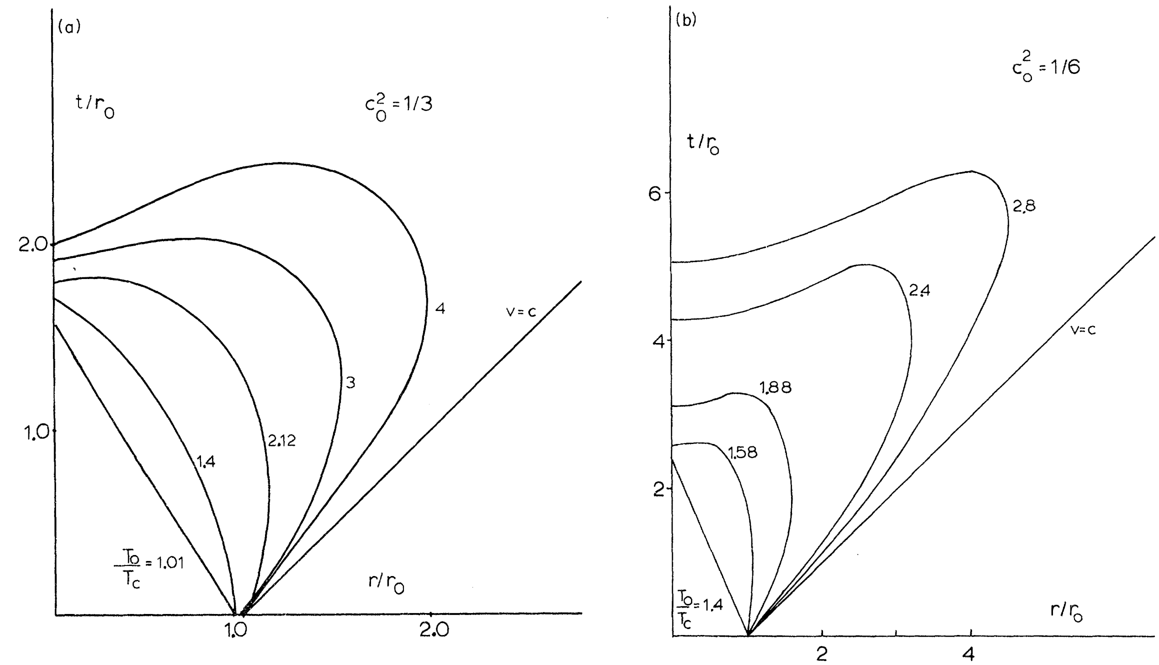

FIG. 1. (a) The isotherms $T = T_c$ for $c_0^2 = \frac{1}{3}$. The fluid is at rest for the area beneath the isotherm for $T_0 = T_c$. Series of curves are for increasing T_0/T_c. (b) Same as (a) for $c_0^2 = \frac{1}{6}$.

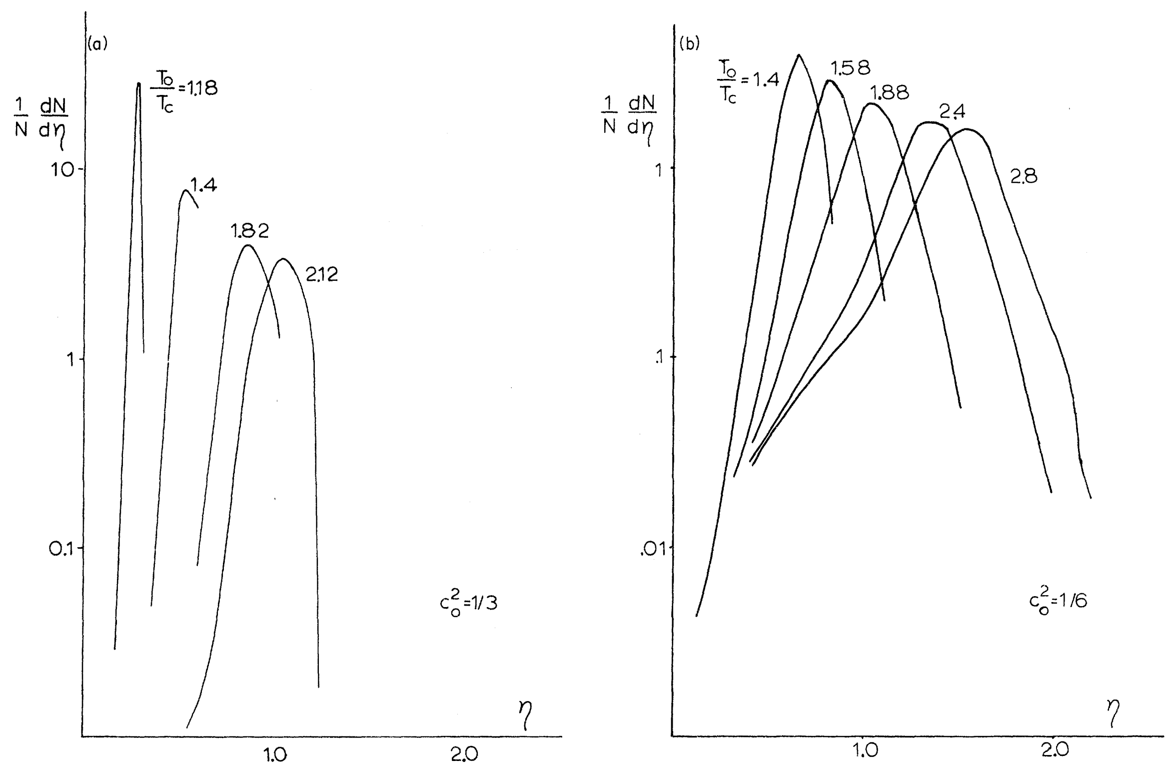

FIG. 2. (a) $(dN/d\eta)/N$ vs η for $c_0^2 = \frac{1}{3}$. Values of T_0/m_π used are those of Table I. (b) $(dN/d\eta)/N$ vs η for $c_0^2 = \frac{1}{6}$.

$$\eta_{\max} = \frac{1}{c_0} \ln\left(\frac{T_0}{T_c}\right) , \tag{74}$$

whereas for large $\langle \gamma \rangle$ Eq. (73) approaches

$$\eta_{\text{peak}} \to \ln\left(\frac{T_0}{T_c}\right) + \ln\left(\frac{2}{1+c_0^2}\right) . \tag{73'}$$

More interesting, is the quantity

$$\frac{1}{4\pi} (m_\pi^2 \sinh^2\eta)^{-1} \frac{dN/d\eta}{N} \tag{74'}$$

whose deviation from a constant measures the "violation" of the scaling relation $v = r/t$, $(dN/d\eta) \propto \sinh^2\eta\, \tau^3$, where τ is constant along an isotherm. For dynamical theories where at break-up the pressure is zero, this quantity is just the invariant distribution. In any event to the extent that

$$p_\pi \approx m_\pi \sinh\eta ,$$

one has

$$E \frac{dN/d^3p}{N} \approx \frac{1}{4\pi} (m_\pi^2 \sinh^2\eta)^{-1} \frac{dN/d\eta}{N} . \tag{74''}$$

This last relation [Eq. (74″)] is expected to hold in general *only* if the hydrodynamic distribution is much broader than thermal fluctuations. Figure 5(f) shows a comparison of the two distributions of Eq. (74″) for $\langle E_\pi \rangle = 938$ MeV. We see that for $c_0^2 = \frac{1}{6}$ the hydrodynamic distribution is approaching the particle distribution.

In Figs. 3(a) and 3(b) we plot (74) vs η. We notice that for $c_0^2 = \frac{1}{3}$ we never see a large scaling contribution. Most of the entropy is in the outer shell. However, for $c_0^2 = \frac{1}{6}$, where cooling is slower, for $T_0/T_c = 2.8$ we notice that for one unit of rapidity one sees a plateau of slow pions and then a peak of fast pions from the outer shell. The plateau comes from the small-r, spacelike portion of the critical isotherm. On that part of the isotherm, the relationship $v \approx \frac{3}{2}(r/t)$ is approximately true. Whenever the isotherms approach hyperbolas one expects quasiscaling.

Since $\langle E_\pi \rangle$ determines the shape of the hydrodynamic distribution, in Fig. 4 we plot $(1/4\pi) \times (m_\pi^2 \sinh^2\eta)^{-1}(dN/d\eta)/N$ for fixed $\langle E_\pi \rangle = 627$, 805, 938 for the two models defined in Table I. We plot vs $m_\pi \sinh\eta$, which would be p_π if thermal motion was ignorable. Notice that the $c_0^2 = \frac{1}{6}$ distribution is approaching a limiting form, whereas the $c_0^2 = \frac{1}{3}$ distributions are sharply peaked.

If we divide the single-particle distribution of Eq. (33) by N, then the resulting quantity

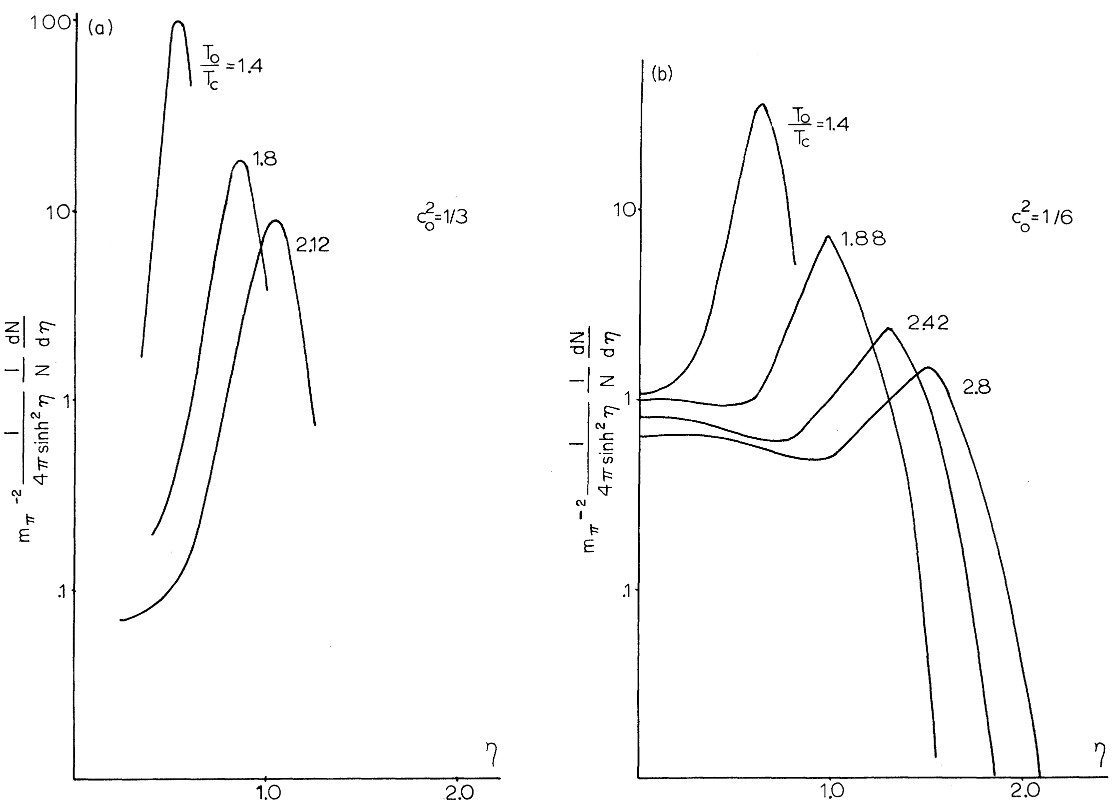

FIG. 3. (a) $(1/4\pi)(\sinh\eta)^{-2} m_\pi^{-2}(dN/d\eta)/N$ vs η for $c_0^2 = \frac{1}{3}$. (b) $(1/4\pi)(\sinh\eta)^{-2} m_\pi^{-2}(dN/d\eta)/N$ vs η for $c_0^2 = \frac{1}{6}$.

$$E(dN/d^3p)/N = \frac{3}{2}\frac{g_\pi}{(2\pi)^3}\, V_\pi\, \frac{s_c}{n_c}\left[\lambda(1+c_0^2)\left(\frac{T_0}{m_\pi}\right)^{1/c_0^2}\right]^{-1}\int \frac{\tilde{r}^2(\eta)\, d\cos\theta\, d\eta(E\, d\tilde{r}/d\eta - p\cos\theta\, d\tilde{t}/d\eta)}{\exp[(E\cosh\eta - p\sinh\eta\cos\theta)/T_c] - 1} \qquad (75)$$

is independent of r_0, and depends only on c_0^2 and T_0/T_c, and thus by Eq. (39) on $\langle E_\pi\rangle$.

If we assume that Eq. (39′) will continue to hold at high energies, then c.m. energies of 3, 5, 10, 25, 50 will correspond to $\langle E_\pi\rangle = 466$, 530, 627, 805, 938 as shown in Table I. In Figs. 5(a)–5(e) we plot Eq. (75) vs p at these values of $\langle E_\pi\rangle$. For $\langle E_\pi\rangle = 938$ [Fig. 5(f)] we also plot the purely hydrodynamic distribution Eq. (74). We notice that although the two different particle distributions are quite similar for the same $\langle E_\pi\rangle$, the purely hydrodynamic distributions are quite different; that of $c_0^2 = \frac{1}{3}$ is still sharply peaked, whereas that for $c_0^2 = \frac{1}{6}$ is starting to approach the real particle distribution, which includes the thermal fluctuations (i.e., the thermal fluctuations are becoming less important in the latter case).

We notice from Figs. 5(a)–5(f) that the distribution $E(dN/d^3p)/N$ gets broader compared to a Bose distribution whose asymptote is e^{-E/m_π}. If $dN/d\eta$ was truly a δ function, then we have for large p, $p \approx E$, $\gamma(E - pv) \approx e^{-\eta}E$; thus we expect the falloff of the distribution of Eq. (75) to go asymptotically as

$$\exp\left(-\frac{Ee^{-\eta_0}}{T_0}\right),$$

where $\cosh\eta_0 = \langle\gamma\rangle$. Since for a given $\langle E_\pi\rangle$, $e^{-\eta_0}/T_0$ is independent of c_0^2, the large-p behavior of $E\,dN/d^3p$ is independent of c_0^2 and just depends on $\langle E_\pi\rangle$. Since r_0 is a "free" parameter, determined at present by N, we find that for e^+e^- annihilation, the hydrodynamic model predicts similar results for all equations of state $p = c_0^2\epsilon$ if the process is a spherical expansion, as assumed here.

XI. CONCLUSIONS

In this paper we have discussed how one uses the hydrodynamic model to calculate single-particle inclusive distributions for simple initial geometries pertinent to head-on pp collisions and e^+e^- annihilation. We discussed how at high energies if the expansion phase is long, the final particles do not remember the initial data and scale parameters, and one enters a scaling regime which, for pp scattering, is similar to Feynman scaling. What remains to be shown is that this "macroscopic" picture can be derived from field theory when collective motions are suitably handled.

ACKNOWLEDGMENTS

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APPENDIX A: THERMODYNAMIC RELATIONS[5]

Prematter is assumed to consist of several composite hadronic species i. If the number of particles of type i is N_i, the Gibbs free energy is

$$\phi = \sum_i \mu_i N_i = E + pV - TS, \qquad (A1)$$

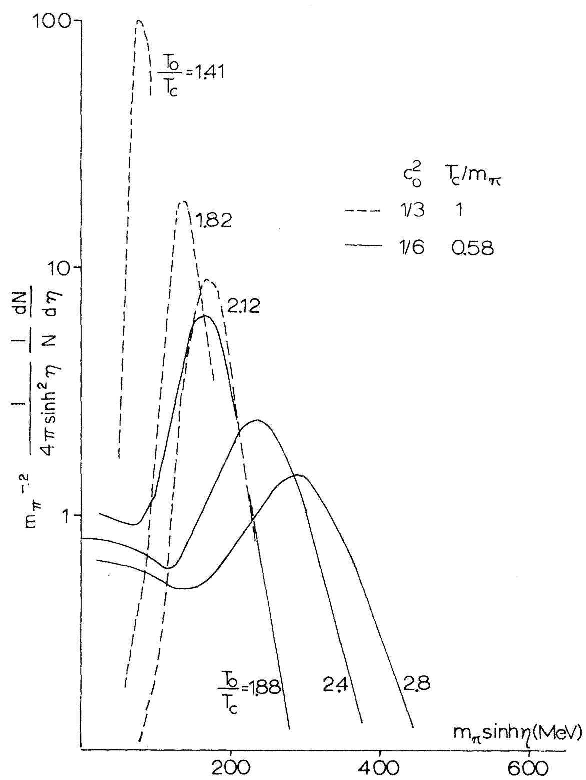


FIG. 4. A comparison of $(1/4\pi)(\sinh\eta)^{-2} m_\pi^{-2}(dN/d\eta)/N$ for the cases $c_0^2 = \frac{1}{3}$, $c_0^2 = \frac{1}{6}$ for identical values of $\langle E_\pi\rangle = 627$, 805, and 938 MeV.

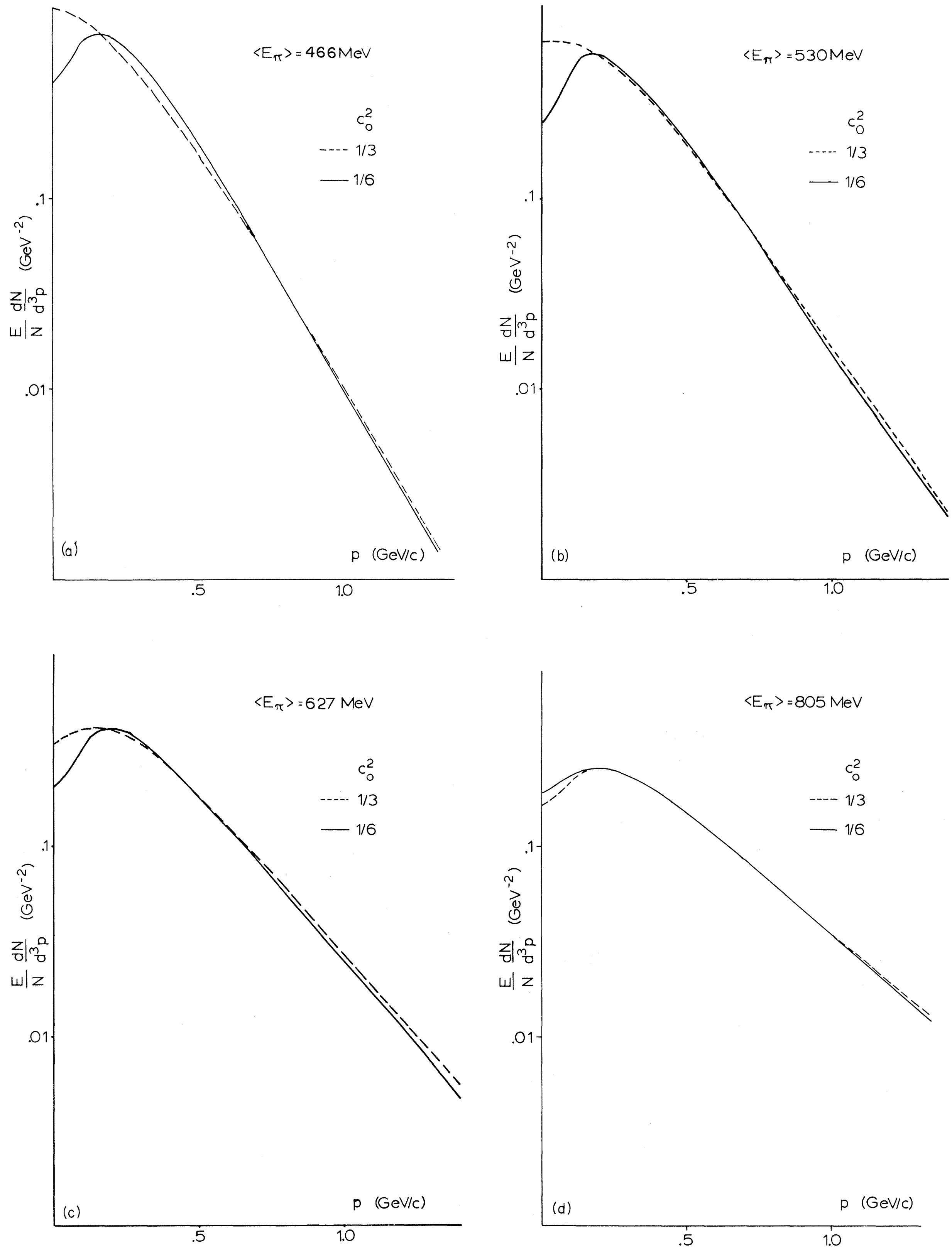
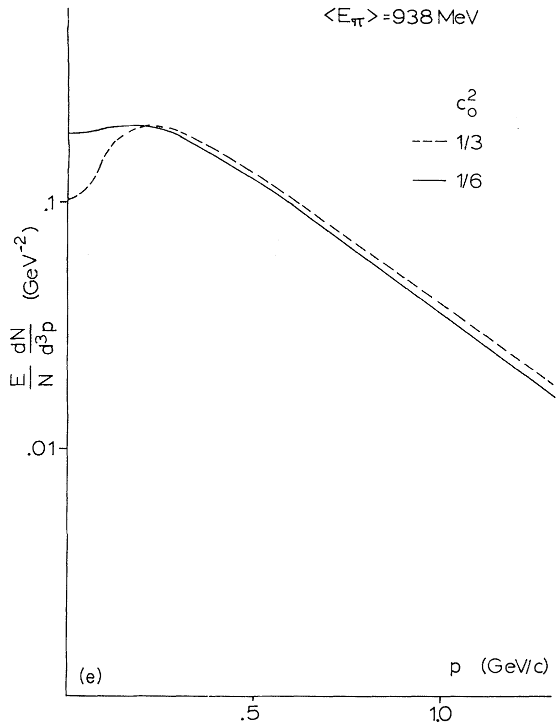


FIG. 5. *(Continued on following page)*.



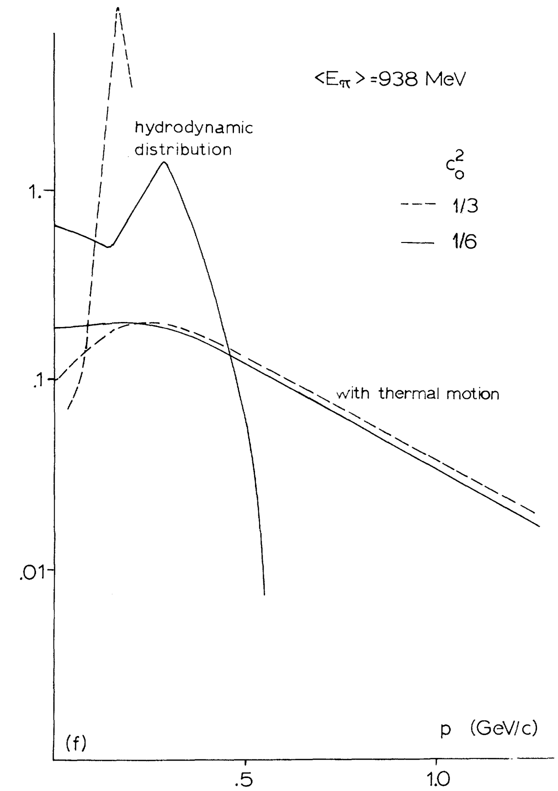


FIG. 5. (a)–(e) Comparison of $(E/N)dN/d^3p$ for the two models of Table I. (f) Comparison of $(E/N)dN/d^3p$ and $1/4\pi)m_\pi^{-2}(\sinh\eta)^{-2}(dN/d\eta)/N$ for $c_0^2=\frac{1}{3}$, $c_0^2=\frac{1}{6}$ at $\langle E_\pi\rangle=940$ MeV.

where $\mu_i=\mu_i(p,T)$ is the chemical potential of specie i. If there is no constraint on N_i, the equilibrium value of N_i is determined by

$$\partial\phi/\partial N_i=0\,,\quad\text{or } \mu_i=0\,. \tag{A2}$$

We assume Eq. (A2) holds for each i, i.e., perfect nonconservation of particle number. In terms of the densities $\epsilon=E/V$ and $s=S/V$ Eq. (A1) then becomes

$$\epsilon+p=Ts \tag{A3}$$

and the thermodynamic law $dE=-pdV+TdS$ yields

$$dp=s\,dT\,,\quad d\epsilon=Tds\,. \tag{A4}$$

The speed of sound is defined as

$$c_0^2=\frac{dp}{d\epsilon}=\frac{d\ln T}{d\ln s}\,. \tag{A5}$$

We assume c_0 is constant in order to have relatively simple hydrodynamic equations. Equation (A5) gives

$$s=\lambda(1+c_0^2)(T/T_\pi)^{c_0^{-2}}V_\pi^{-1}\,, \tag{A6}$$

where λ is a dimensionless integration constant and $V_\pi=\frac{4}{3}(m_\pi)^{-3}$. Equation (A4) gives

$$\epsilon=[\lambda(T/T_\pi)^{1+c_0^{-2}}+B]m_\pi c^2V_\pi^{-1}\,, \tag{A7}$$

$$p=[c_0^2\lambda(T/T_\pi)^{1+c_0^{-2}}-B]m_\pi c^2V_\pi^{-1}\,, \tag{A8}$$

where the "bag term" B is another integration constant.

An ideal Bose gas can be cast in the form of Eqs. (A7) and (A8), with $c_0^{-2}=3$, if we allow λ and B to be functions of the temperature, but then c_0 is no longer the speed of sound. We have

$$\lambda^{\text{ideal}}=(g_\pi/2\pi)T^{-4}\times\int_0^\infty p^2dp(E+p^2/3E)\times\{\exp[(E-\mu)/kT]-1\}^{-1}\,, \tag{A9}$$

$$B^{\text{ideal}}=(g_\pi/6\pi)\times\int_0^\infty p^2\,dp(M_\pi^2E)^{-1}\times\{\exp[(E-\mu)/kT]-1\}^{-1}\,. \tag{A10}$$

In the high-temperature (or blackbody) limit $m_\pi/T\to 0$ we have

$$\lambda^{\text{ideal}}\to(\tfrac{1}{45}2\pi^3g_\pi)\text{ for bosons}$$
$$\to(\tfrac{7}{180}2\pi^3g_f)\text{ for fermions,} \tag{A11}$$

$$B^{\text{ideal}} \to (\tfrac{1}{36}\pi g_\pi)(kT/m_\pi)^2 \text{ for bosons}$$

$$\to (\tfrac{1}{72}\pi g_f)(kT/m_\pi)^2 \text{ for fermions.} \qquad \text{(A12)}$$

The temperature dependences of λ^{ideal} and B^{ideal} are shown in Fig. 6.

In the very-high-temperature limit, we suppose that the hadronic fluid equation of states becomes the ultrarelativistic form with $c_0^{-2}=3$ and λ proportional to the number of types of quarks. Let us say the quark description becomes essential for $T > T_q$.

There is no reason to assume the prematter regime $T_c < T < T_q$ is either ideal or ultrarelativistic. At present accelerator energies the temperature range (T_c, T_0) is small, so the behavior of the system should be adequately described by two parameters λ and c_0. The λ enters as λV_0 and B/λ [see Eqs. (36′) and (39)]. Since B/λ is small, λ scales the initial volume V_0. For fixed V_0, T_0, and c_0, it is then nice that the multiplicity and total energy are both proportional to λ.

The effect of c_0 is to control the rate of the hydrodynamic expansion. For small c_0 the expansion is slow. We can see this in the equation of state

$$p = c_0^2\epsilon - (1+c_0^2)B\,. \qquad \text{(A13)}$$

If the whole energy density is devoted to driving the three-dimensional expansion $c_0^{-2}=3$. If some of the energy density is soaked up in internal processes of the fluid, such as forming composites or local rotational modes, it will not be available for driving the expansion and this effect is parameterized by c_0, with $c_0^{-2}>3$. The bag term B decreases the pressure too, if $B>0$, but its effect drops out of the hydrodynamic equations.

The final stage of the system is described by an ideal gas of pions. In order for the total energy and entropy of the Bose distribution to agree with that of the fluid, we want ϵ and p of the prematter Eqs. (A7) and (A8) to match continuously with the energy density and pressure of an ideal Bose gas at the condensation temperature T_c:

$$\epsilon(T_c) = \epsilon^{\text{ideal}}(T_c) \equiv \epsilon_c\,, \qquad \text{(A14)}$$

$$p(T_c) = p^{\text{ideal}}(T_c) \equiv p_c\,. \qquad \text{(A15)}$$

Solving Eqs. (A7), (A8), (A14), and (A15) for λ and B, we obtain

$$\lambda = (1+c_0^2)^{-1}(c_0^2\epsilon_c - p_c)(V_\pi/m_\pi)\,, \qquad \text{(A16)}$$

$$B = (1+c_0^2)^{-1}(T_\pi/T_c)^{1+c_0^{-2}}(\epsilon_c + p_c)(V_\pi/m_\pi)\,. \qquad \text{(A17)}$$

The contours $\lambda=$ constant and $B=$ constant are shown in the (T_c, c_0) plane in Fig. 7.

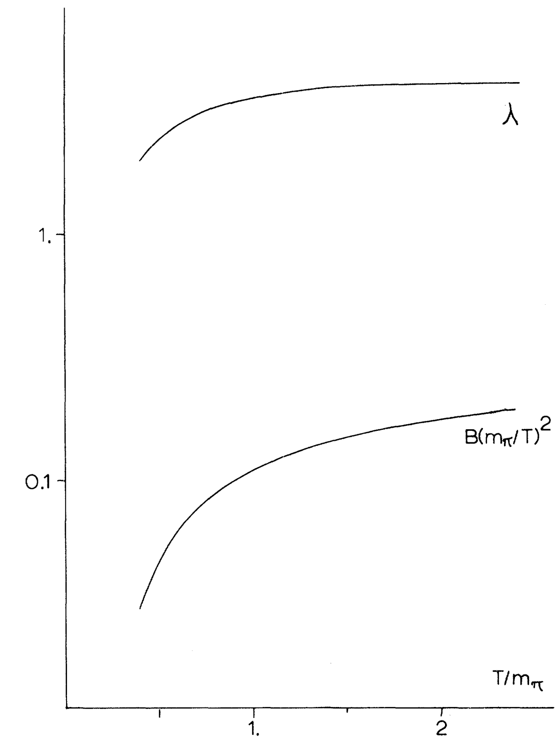


FIG. 6. Temperature dependence of λ_{ideal} and B_{ideal}.

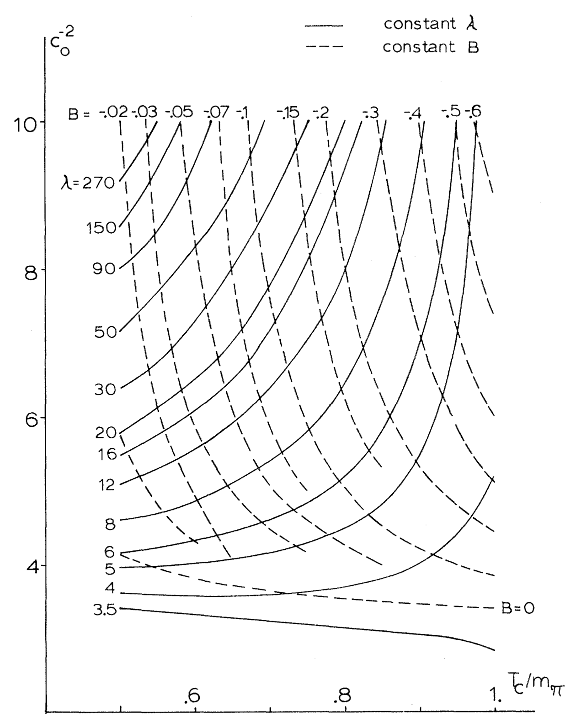


FIG. 7. The contours $\lambda=$constant, $B=$constant in the $(T_c\,,\ c_0^{-2})$ plane.

APPENDIX B: STATISTICAL MECHANICS OF IDEAL BOSE AND FERMI GASES

The partition function for an ideal Bose or Fermi gas is given by

$$\ln Z \approx -g \int \frac{d^3p}{(2\pi)^3} \ln\left[1 \mp \exp\left(\frac{\mu - E}{T}\right)\right] , \qquad \text{(B1)}$$

where μ = the chemical potential, g = the statistical weight $= (2s+1)(2I+1)$, and $E = (p^2 + m^2)^{1/2}$. The upper and lower signs correspond to bosons and fermions, respectively. In Eq. (B1) we have made the semiclassical approximation $\sum_i = \int d^3p\, V/(2\pi)^3$ and we have ignored the Bose-condensation contribution.

In terms of $\ln Z$, the usual thermodynamic variables are

$$\overline{E} = KT^2 \frac{\partial}{\partial T} \ln Z , \quad \overline{N} = KT \frac{\partial \ln Z}{\partial \mu} ,$$
$$p = \frac{KT \ln Z}{V} , \quad \frac{S}{K} = -\ln Z - T\frac{\partial}{\partial T} \ln Z . \qquad \text{(B2)}$$

In what follows we set $K = \hbar = c = 1$. Expanding the logarithm we get

$$\ln Z = V \frac{g}{2\pi^2 a_0^3} \left(\frac{T}{m_\pi}\right)^3 y^2 \sum_n e^{\mu n/T} K_2(ny) \frac{(\pm 1)^n}{n^2} , \qquad \text{(B3)}$$

where $y = m/T$, $K_2(ny)$ is the modified Bessel function of the second kind, and the plus sign (minus sign) corresponds to bosons (fermions). Introducing the densities $n = \overline{N}/V$, $s = S/V$, $\epsilon = \overline{E}/V$ we get via Eqs. (B2)

$$n = \frac{2g}{3\pi V_\pi} \left(\frac{T}{m_\pi}\right)^3 F(y) ,$$
$$s = \frac{2g}{3\pi V_\pi} \left(\frac{T}{m_\pi}\right)^3 G(y) ,$$
$$\epsilon = \frac{2g}{3\pi V_\pi} \left(\frac{T}{m_\pi}\right)^4 m_\pi \phi(y) , \qquad \text{(B4)}$$
$$p = \frac{KT \ln Z}{V} = Ts - \epsilon .$$

For $\mu = 0$ we have

$$F(y) = y^2 \sum_n (\pm 1)^n K_2(ny)/n ,$$
$$G(y) = y^2 \sum_n (\pm 1)^n [4K_2(ny) + nyK(ny)]/n^2 ,$$
$$\phi(y) = y^2 \sum_n \frac{(\pm 1)^n}{n^2} [3K_2(ny) + nyK_1(ny)] .$$

F, G, and ϕ are tabulated in the Landau-Belinkij article[1] and are very slowly varying functions of T for $T \geq M$.

Comparing (B4) with the thermodynamic equation

$$\epsilon = \lambda \frac{m_\pi}{V_\pi} T^4$$

we have

$$\lambda = \frac{2g}{3\pi} \phi(m/T) \quad \text{(see Fig. 6)}, \qquad \text{(B5)}$$

where $\phi(0) = 6.49$ (5.68) for bosons (fermions). Thus if we believe when $E_{\text{c.m.}} \to \infty$ that an asymptotically free field theory of quarks gives the relevant dynamics, then measuring $N_\pi(E_{\text{c.m.}})$ for $pp \to \pi + X$, where V_0 is known, should determine λ_0 and thus g, the number of quarks in the free-quark Lagrangian.

APPENDIX C: STATISTICAL MECHANICS OF AN IMPERFECT GAS

Following Hagedorn[12] and Shuryak,[13,4] one can assume that at intermediate energies the relevant dynamics is governed by the resonance spectra (Beth-Uhlenbeck approximation).[11] In the narrow-resonance approximation we replace all the phase shifts by a sum over resonances. In the continuum approximation this leads to the following partition function:

$$\ln Z = \int \frac{d^3p\, V}{(2\pi)^3} \rho(m/m_\pi) \ln\left[1 \mp \exp\left(\frac{\mu - E}{T}\right)\right] \times d(m/m_\pi) , \qquad \text{(C1)}$$

where the $\mp$ sign refers to boson or fermion resonances.

As Shuryak noted,[13,4] for the case

$$\rho(m/m_\pi) = b(m/m_\pi)^a , \qquad \text{(C2)}$$

the integral can be evaluated and one obtains

$$p = \frac{1}{4\pi^2} \frac{b}{a_0^3} m_\pi \left(\frac{T}{m_\pi}\right)^{4+a+1} \zeta(a+5) \times \frac{\Gamma(a+4)\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{2}a+2)} ,$$
$$\epsilon = (a+4)p , \quad s = \frac{\epsilon + p}{T} , \qquad \text{(C3)}$$
$$n = \frac{1}{a+5} \frac{\zeta(a+4)}{\zeta(a+5)} s ,$$
$$\zeta(z) = \sum (\pm 1)^n / n^z ,$$

where the plus sign is for bosons, $a_0 = m_\pi^{-1}$. Comparing with the thermodynamic relations, we see

$$c_0^2 = \frac{1}{a+4}\,,$$
$$\lambda(c_0^2) = \frac{b}{3\pi}\,\zeta\left(1+\frac{1}{c_0^2}\right) \times \frac{\Gamma(1/(c_0^2+1))\Gamma((1-3c_0^2)/2c_0^2)\Gamma(\frac{3}{2})}{\Gamma(1/2c_0^2)}\,. \tag{C4}$$

APPENDIX D: CONNECTION BETWEEN HYDRODYNAMIC VARIABLES AND CLASSICAL KINETIC THEORY[26]

In classical kinetic theory the single-particle distribution function $g(x,p)$ obeys the Boltzmann equation

$$p^\mu \partial_\mu g(x,p) = \Delta\Gamma(x,p)\,, \tag{D1}$$

where $\Delta\Gamma$ is the rate of change in g due to collisions. The stress-energy tensor is given by

$$T^{\mu\nu} = \int p^\mu p^\nu g(x,p) Dp\,, \tag{D2}$$

where

$$Dp = 2\theta(p_0)\delta(p^2-m^2)\,d^4p = \frac{d^3p}{p_0}\,,$$

and is conserved by virtue of energy-momentum conservation in individual collisions:

$$\partial_\nu T^{\mu\nu}(x) = \int p^\mu \Delta\Gamma Dp = 0\,. \tag{D3}$$

The collective 4-velocity $u^\mu(x) = \gamma(x)(1,\vec{v}(x))$ is defined by

$$n(x)u^\mu(x) = \int p^\mu g(x,p) Dp\,. \tag{D4}$$

$n(x)u^\mu(x)$ is the number-current density.

$$N(\sigma) = \int_\sigma n(x)u^\mu(x)\,d\sigma_\mu = \int Dp \int_\sigma g(x,p)p^\mu d\sigma_\mu \tag{D5}$$

is the net number of particles on σ, but in general

$$\partial_\mu(nu^\mu) \neq 0\,.$$

A comoving frame, denoted by barred variables, is defined by $u^\mu(\bar{x}) = (1,\vec{0})$. Thus

$$p^\mu = L^\mu_\nu(\vec{v})\bar{p}^\nu\,, \tag{D6}$$

with

$$L^\mu_0(\vec{v}) = u^\mu\,,$$
$$L^\mu_i = -\delta^{\mu 0}u_i + (1-\delta^{\mu 0})[\delta^\mu_i - (\gamma-1)u^\mu u_i/v^2\gamma^2]\,,$$

and we have

$$\bar{n}(\bar{x}) = n(x) = \int g(\bar{x},\bar{p})\,d^3\bar{p}\,. \tag{D7}$$

If we write $T^{\mu\nu}$ in terms of local rest-frame variables, we obtain

$$T^{\mu\nu}(x) = \int (\bar{E}u^\mu + L^\mu_i\bar{p}^i)(\bar{E}u^\nu + L^\nu_j\bar{p}^j)g(\bar{x},\bar{p})\,d^3\bar{p}/\bar{E}$$
$$= \int g(\bar{x},\bar{p})[\bar{E}u^\mu u^\nu + L^\mu_i L^\nu_j(\bar{p}^i\bar{p}^j/\bar{E}) + (\bar{p}^i L^\mu_i u^\nu + \bar{p}^j L^\nu_j u^\mu)]\,d^3\bar{p}\,. \tag{D8}$$

If $g(\bar{x},\bar{p})$ is isotropic in $\bar{p}$ (isotropy in the local frame), then

$$\int g(\bar{x},\bar{p})\bar{p}^i\bar{p}^j\frac{d^3\bar{p}}{\bar{E}} = \int g(x,p)\frac{\bar{p}^2}{3\bar{E}}\,d^3\bar{p}\,\delta_{ij}\,,$$
$$\int g(\bar{x},\bar{p})\bar{p}^i\frac{d^3p}{\bar{E}} = 0\,,$$

and we get (since $\sum_{i=1}^3 L^\mu_i L^\nu_i = u^\mu u^\nu - g^{\mu\nu}$)

$$T^{\mu\nu}(x) = (\epsilon+p)u^\mu u^\nu - pg^{\mu\nu}\,, \tag{D9}$$

where

$$\epsilon(x) = \epsilon(\bar{x}) = \int \bar{E}g(\bar{x},\bar{p})\,d^3\bar{p}\,,$$
$$p(x) = p(\bar{x}) = \int \frac{\bar{p}^2}{3\bar{E}}\,d^3\bar{p}\,g(\bar{x},\bar{p})\,,$$

and $\bar{T}^{\mu\nu}(\bar{x}) = \text{diagonal}\,(\epsilon,p,p,p)$. On the other hand, if the fluid is constrained to move in x space only in one direction (the x direction) and

$$g(\bar{x},\bar{p}) = \tilde{g}(\bar{x},\bar{p}_\parallel)\delta^2(\bar{p}_\perp^2) \tag{D10}$$

(that is, the dynamics is strongly damped in transverse-momentum space), then the fluid is in one space-one time dimension in both x and p space, and

$$\epsilon(x) = \int \bar{E}\tilde{g}(\bar{x},\bar{p})\,dp\,,$$
$$p(x) = \int \frac{\bar{p}^2}{\bar{E}}\,\tilde{g}(x,p)\,dp\,, \tag{D11}$$

with

$$\bar{T}^{\mu\nu}(\bar{x}) = \text{diagonal}\,(\epsilon,p,0,0)\,.$$

In the three-dimensional case

$$T^\mu_\mu(x) = \epsilon - 3p = m^2\left\langle\frac{1}{\bar{E}}\right\rangle\,, \tag{D12a}$$

and in the one-dimensional fluid

$$T^\mu_\mu(x) = \epsilon - p = m^2\left\langle\frac{1}{\bar{E}}\right\rangle\,. \tag{D12b}$$

As $\bar{E}\to\infty$, one has $p\to\frac{1}{3}\epsilon$ (ϵ) in the ultrarelativistic three-dimensional (one-dimensional) fluid.

APPENDIX E: HYDRODYNAMIC DESCRIPTION IN FIELD THEORY

Consider an ensemble of noninteracting pions described by a time-independent density matrix ρ. Define the pion Green's function

$$G(x_1x_2)=\langle T(\varphi(x_1)\varphi(x_2))\rangle = \mathrm{Tr}\rho T(\varphi(x_1)\varphi(x_2)) . \quad (E1)$$

Since $\varphi(x)$ obeys $(\Box^2+m^2)\varphi(x)=0$, one has

$$(\Box_1{}^2+m^2)G(x_1x_2)=-i\delta^4(x_1-x_2). \quad (E2)$$

We can introduce the functions $G^>(x_1x_2)$, $G^<(x_1x_2)$ by

$$G^>(x_1x_2)=\langle \varphi(x_1)\varphi(x_2)\rangle ,$$
$$G^<(x_1x_2)=\langle \varphi(x_2)\varphi(x_1)\rangle .$$

Thus

$$G(x_1x_2)=\theta(x_{10}-x_{20})G^>(x_1x_2) + \theta(x_{20}-x_{10})G^<(x_1x_2).$$

Introducing relative and center-of-mass coordinates by

$$r=x_1-x_2,\quad x=\tfrac{1}{2}(x_1+x_2) \quad (E3)$$

we define

$$G^<(x,p)=\frac{1}{(2\pi)^4}\int d^4r\,e^{ip\cdot r}\,G^<(x_1x_2). \quad (E4)$$

We shall see that the quantum analog of $g(x,p)$ can be defined as follows:

$$f(x,p)\equiv\frac{1}{(2\pi)^4}\,\theta(p_0)\int d^4r\,e^{ip\cdot r}\,G^<(x_1,x_2) = \theta(p_0)G^<(x,p). \quad (E5)$$

Decomposing $\varphi(x)$ into positive- and negative-frequency Fourier components,

$$\varphi(x)=\frac{1}{\sqrt{2}}\int\frac{d^4k}{(2\pi)^{3/2}}[a(k)e^{-ik\cdot x}+a^\dagger(k)e^{ik\cdot x}] \times\delta^+(k^2-m^2), \quad (E6)$$

where

$$\delta^+(k^2-m^2)=2\theta(k_0)\delta(k^2-m^2),$$
$$\varphi(x)\equiv\varphi^{(+)}(x)+\varphi^{(-)}(x),$$

one has

$$f(x,p)=\frac{1}{2(2\pi)^3}\int e^{iq\cdot x}\langle a^\dagger(p+\tfrac{1}{2}q)a(p-\tfrac{1}{2}q)\rangle \times\delta^+((p+\tfrac{1}{2}q)^2-m^2) \times\delta^+((p-\tfrac{1}{2}q)^2-m^2)\,d^4q$$
$$=\frac{1}{(2\pi)^4}\int d^4r\,e^{ip\cdot r} \times\langle\varphi^{(-)}(x-\tfrac{1}{2}r)\varphi^{(+)}(x+\tfrac{1}{2}r)\rangle . \quad (E7)$$

Since $(\Box^2+m^2)\varphi^\pm(x)=0$, $f(x,p)$ obeys the collisionless Boltzmann equation

$$p^\mu\partial_\mu f(x,p)=0,$$

whence

$$nu^\mu(x)=\int f(x,p)p^\mu\,d^4p \quad (E9)$$

and

$$T^{\mu\nu}(x)=\int f(x,p)p^\mu p^\nu\,d^4p \quad (E10)$$

are conserved. That is,

$$\partial_\mu[nu^\mu(x)]=0,\quad \partial_\mu T^{\mu\nu}(x)=0. \quad (E11)$$

$nu^\mu(x)$ can be written as

$$i\langle\varphi^{(-)}(x)\overleftrightarrow{\partial}_\mu\varphi^{(+)}(x)\rangle ,$$

showing that nu^μ is the number current and

$$\int nu^\mu(x)\,d\sigma_\mu=\langle N(\sigma)\rangle . \quad (E12)$$

Similarly $T^{\mu\nu}$ is the "classical" energy-momentum tensor

$$\int T^{00}(x)\,d^3x=\int f(x,p)(p^0)^2\,d^4p\,d^3x = \int\langle\omega_p a^\dagger(p)a(p)\rangle\frac{d^3p}{2\omega_p} = \langle E\rangle . \quad (E13)$$

Thus noninteracting pions can be described hydrodynamically via the pion Green's function, just as is found in nonrelativistic quantum statistical mechanics such as described in Kadanoff and Baym.[27] The interacting case will be discussed at some future date. $f(x,p)$ defined above is the relativistic version of the Wigner distribution function discussed in Ref. 27.

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