Testing the low-energy theorems of broken chiral and conformal symmetries

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We test the low-energy theorems of broken chiral and conformal invariance by use of the experimental $\pi\pi$ phases up to $m_{\pi\pi} = 1$ GeV. Our method differs considerably from a previous treatment by Renner and Staunton. Nevertheless, in the $(3, \bar{3}) \oplus (\bar{3}, 3)$ model considered by these authors we recover their result that no consistent solution exists. The same result is obtained for the (8, 8). In a mixed model a consistent solution is found. This model has been introduced previously by the authors in order to be able to cope with a large $\pi\pi$ scattering length $a_0^{(0)}$. Our assumptions include that δ is a c number and that u has dimension two, in agreement with the low-energy theorems we use.

I. INTRODUCTION

The low-energy theorems (LET's) following from broken chiral and conformal symmetries[1-4] require for a test knowledge of the Green's functions of the form

$$\Delta_{\Theta A}(q^2) = \int d^4x\, e^{iq\cdot x}\langle T(\Theta(x)A(0))\rangle$$

at $q^2 = 0$. Here Θ denotes the trace of a certain energy-momentum tensor and A stands for any one of the local scalar operators of the model of chiral-symmetry breaking. One furthermore requires $\Delta'_{\Theta\sigma}(0)$ to be known for a test with σ the pion σ-commutator term defined in Eq. (7).

The explicit form of the LET's to be tested depends on the model of chiral-symmetry breaking. Thus knowledge of the $\Delta_{\Theta A}(0)$ and $\Delta'_{\Theta\sigma}(0)$ will imply a test of the model of chiral-symmetry breaking.

The idea that there is a Goldstone boson of dilation-symmetry breaking (the ϵ) which may be used to compute the required vacuum expectation values (VEV's) has frequently been considered in the literature.[1-4] Despite its being theoretically very attractive, the experimental situation does not justify this picture.[5] In any case, the uncertainties are very large since the LET's if saturated by the ϵ require knowledge of m_ϵ and $\Gamma_{\epsilon\to\pi\pi}$. Both are experimentally not well defined and at any rate poorly known.

In this situation, attempts have been made[6,7] to make direct use of the $\pi\pi$ phases in order to saturate the VEV's required for a test of the $(3, \bar{3}) \oplus (\bar{3}, 3)$ model of chiral-symmetry breaking.[8] We modify and extend these attempts in the present paper. Even though our modification of the original Renner-Staunton (RS) method (Ref. 6) is considerable (see the following section for details), we still find the result of RS that no consistent solution of the LET's of broken chiral and conformal symmetry exists in the $(3, \bar{3}) \oplus (\bar{3}, 3)$ model with a c number δ. As proposed by RS, we are using the experimental $\pi\pi$ phases below $m_{\pi\pi} = 1$ GeV (see the Appendix for our treatment of the phases). The above result is also obtained for the (8, 8) (Ref. 9) with a c number δ. This should not come as a surprise since consideration of the meson-nucleon σ terms already implies presence of a q number δ in these irreducible models.[10]

We mention at this point that we assume that the dimension l_u of the chiral-symmetry-breaking Hamiltonian density is two, in agreement with the assumed validity of the low-energy theorems.

As a further main point of our paper we show that in the mixed model of Ref. 11 a consistent solution *does* exist. We have introduced this model in order to be able to cope with a possibly large scattering length $a_0^{(0)}$. The result mentioned above was anticipated in Ref. 11. There we also performed the conventional test of the LET's by saturation with an ϵ intermediate state. The result was favorable just as[3,4] the results of such a test in the $(3, \bar{3}) \oplus (\bar{3}, 3)$ [due to large errors, nothing convincing can now be said on[12] the pure (8, 8)]. Thus, the results of ϵ saturation and $\pi\pi$-phase saturation disagree in the case of the $(3, \bar{3}) \oplus (\bar{3}, 3)$, whereas they do agree in the mixed model.

Summarizing, we find that there is no consistent solution of the LET's of broken chiral and conformal symmetries in the case of the $(3, \bar{3}) \oplus (\bar{3}, 3)$ and the (8, 8), whereas such a solution exists in the case of the mixed model of Ref. 11.

II. THE LOW-ENERGY THEOREMS OF BROKEN CONFORMAL INVARIANCE, THE OMNÈS FUNCTION METHOD, AND TREATMENT OF POLYNOMIALS

We shall begin this section by enumerating the low-energy theorems of broken scale invariance.[1-4]

We shall assume that the chiral-invariant, scale-noninvariant part δ of the Hamiltonian density is a c number. Furthermore, the chiral-noninvariant Hamiltonian density u is assumed to have dimension l_u. This then implies for the trace Θ of the energy-momentum tensor the result

$$\Theta(x) = (4 - l_u)u(x) - \text{VEV}\,.$$

Furthermore, we have the Kleinert-Weisz result[4]:

$$i\langle [D, \Theta]\rangle_0 \equiv \Delta_{\Theta\Theta}(0) = l_u(4 - l_u)\langle u\rangle_0\,, \tag{1}$$

which connects the object $\Delta_{\Theta\Theta}(0)$, with which we shall most immediately be concerned, with $\langle u\rangle_0$. Here D_μ is the dilation current and we have defined

$$\Theta = \partial_\mu D^\mu\,,$$
$$D = \int d^3x\, D^0(x)\,, \tag{2}$$
$$\Delta_{\Theta\Theta}(q^2) = -i\int d^4x\, e^{iq\cdot x}\langle T(\Theta(x)\Theta(0))\rangle_0\,.$$

We shall also use for the local scalar operator $A(x)$ the definition

$$\Delta_{\Theta A}(q^2) = -i\int d^4x\, e^{iq\cdot x}\langle T(\Theta(x)A(0))\rangle_0\,. \tag{3}$$

If A has the dimension d_A, it follows that

$$\Delta_{\Theta A}(0) = d_A\langle A\rangle_0\,. \tag{4}$$

Thirdly, we have the theorem[4]

$$f_\pi^2\frac{d}{dt}F(0,0,t)\bigg|_{t=0} = \frac{d}{dt}\Delta_{\Theta\sigma}(t)\bigg|_{t=0} + f_\pi^2\,, \tag{5}$$

where

$$F(m_\pi^2, m_\pi^2, t) = \langle\pi|\Theta|\pi\rangle\,. \tag{6}$$

Equation (5) is actually valid for zero-mass pions. In a pole model, bringing (5) on-shell customarily introduces[6] extra terms of order (m_π^2/m_ϵ^2), where m_ϵ is the mass of the "dilation." We shall follow RS[6] in assuming that (5) is valid as it stands with the pions on-shell. This means that we shall have to show that our solutions are stable against small variations. The operator σ is defined by the equal-time commutator

$$\frac{i}{3}\sum_{j=1}^{3}\left[\int A_0^j(\vec{x},0)d^3x\,, \partial^\mu A_\mu^j(0)\right] = \sigma(0) \tag{7}$$

in any model.

The saturation of these equations with an ϵ pole has been widely discussed.[3,4] RS[6] examined the use instead of the measured $\pi\pi$ phase shifts. In detail, all spectral functions occurring in objects of the general form of Eq. (3) are assumed to be well approximated by the $\pi\pi$ contributions, and one uses an Omnès function of the measured phases to determine this contribution, up to the inclusion of polynomials.

Thus, for any scalar and isoscalar operator A with $\langle\pi|A|\pi\rangle(0) \neq 0$ one writes

$$\langle\pi|A|\pi\rangle(t) = \langle\pi|A|\pi\rangle(0)P_A(t)\Omega(t)\,, \tag{8}$$

where $P_A(t)$ is a polynomial normalized to unity at $t=0$, and $\Omega(t)$ is the Omnès function

$$\Omega(t) = \exp\left[\frac{t}{\pi}\int_{4m_\pi^2}^{\infty}\frac{\delta(s)}{s(s-t)}ds\right]\,, \tag{9}$$

with $\delta(s)$ the $I=0$, s-wave $\pi\pi$ phase shifts. It should be noticed that $\Omega(t)$ is universal, i.e., independent of the operator A. RS suggested that one should use linear expressions for the polynomials $P_A(t)$. Higher-order polynomials lead to bad high-energy behavior in various integrals, making the results sensitive to cutoff; linear forms can, however, be tolerated and, as we shall show presently, are indeed necessary. With all this we are in agreement. We disagree, on the other hand, with the suggestion of their second paper[6] that $\langle\pi|u_8|\pi\rangle(t)$ [where $u_8(x)$ is the $I=0$, $Y=0$ scalar in the $(3,\bar{3})\oplus(\bar{3},3)$ representation] might not need a polynomial. Indeed, we will argue the exact opposite: that the form factors of SU(3) nonsinglet operators *must* have a polynomial, and of a well-defined kind.

We shall demonstrate these two points using the $(3,\bar{3})\oplus(\bar{3},3)$ model, though this is an unimportant restriction. First, to show that there have to be some polynomials, assume that there are none; we shall then see that this is not possible for both $A = u_8$ and $A = u_0$. Thus,

$$\begin{aligned}\langle\pi|u_8|\pi\rangle(t) &= \langle\pi|u_8|\pi\rangle(0)\Omega(t)\,,\\ \langle\pi|u_0|\pi\rangle(t) &= \langle\pi|u_0|\pi\rangle(0)\Omega(t)\,.\end{aligned} \tag{10}$$

Hence,

$$\frac{\Delta_{\Theta u_8}}{\Delta_{\Theta u_0}} = \frac{\langle\pi|u_8|\pi\rangle(0)}{\langle\pi|u_0|\pi\rangle(0)}\,. \tag{11}$$

But according to (4) the left-hand side of (11) is also equal to $\langle u_8\rangle_0/\langle u_0\rangle_0$, which implies

$$\frac{\langle\pi|u_8|\pi\rangle(0)}{\langle\pi|u_0|\pi\rangle(0)} = \frac{\langle u_8\rangle_0}{\langle u_0\rangle_0}\,, \tag{12}$$

if Eqs. (10) are to be correct. Now SU(3) tells us that the right-hand side of (12) is small, while the left-hand side is not. Indeed, Ademollo and Gatto and the virial theorem give

$$\begin{aligned}\langle\pi|cu_8|\pi\rangle &= m_\pi^2 - \langle m^2\rangle_{AV}\,,\\ \langle\pi|u_0|\pi\rangle &= [(l_u-2)m_\pi^2 + (4-l_u)\langle m^2\rangle_{AV}]/(4-l_u)\,.\end{aligned} \tag{13}$$

Thus, we see that (12) is incorrect, and therefore, that assumptions (10) are, in general, untenable.

Secondly, having shown that some form factors at least must have polynomials, we shall argue that form factors of the kind $\langle \pi | B | \pi \rangle (t)$, with B an SU(3) nonsinglet operator, must be of the form

$$\langle \pi | B | \pi \rangle (t) = \langle \pi | B | \pi \rangle (0)(1 - t/m_\epsilon^2)\Omega(t) , \tag{14}$$

where m_ϵ approximates the position of the ϵ-S^* bump. The point is the following. The ϵ meson of broken scale invariance (for which we take the ϵ-S^* bump in our saturation scheme) is assumed to be an SU(3)-singlet object and, although it is not well-established experimentally, one can see its bump if one calculates and plots the Omnès function $\Omega(t)$. See Fig. 1 for this. The representation (8) of form factors then directly implies that one *always* sees this bump, unless the polynomial conspires to eliminate it, and, since the ϵ-S^* is assumed to be an SU(3) singlet, this the polynomial must do whenever the operator A is *not* a singlet operator. It is according to this requirement, then, that the polynomial in (14) has been chosen.

This result is immediately connected to the fact that the vacuum is approximately an SU(3) scalar. Using ϵ dominance for simplicity, one easily sees[10] that (if A has a dimension) $\langle A \rangle_0 = 0$ is equivalent to $\langle 0 | A | \epsilon \rangle = 0$. Thus, to have $\langle u_8 \rangle_0 \approx 0$, $\langle S_8 \rangle_0 \approx 0$, and $\langle \hat{S}_{27} \rangle \approx 0$ in our $\pi\pi$-phase saturation scheme, we have to choose the polynomial in such a way that the SU(3)-nonscalar operators decouple from the otherwise dominant ϵ-S^* bump (and do not generate a dominant contribution from s large around 2 GeV^2).

Form factors of the form (14) will be used for $\langle \pi | u_8 | \pi \rangle$ of the $(3,\bar{3}) \oplus (\bar{3},3)$, for example, as well as for the form factors $\langle \pi | S_8 | \pi \rangle$ and $\langle \pi | \hat{S}_{27} | \pi \rangle$ of the (8, 8).

We shall now make some general comments about the calculations of the next section. The LET's of broken chiral summetry provide us with an estimate of $\langle u \rangle_0$ in the $(3,\bar{3}) \oplus (\bar{3},3)$, the (8, 8), or the mixed model of Ref. 11. Equation (1) connects this with $\Delta_{\Theta\Theta}(0)$, which is given in turn through the constant β in

$$F(t) \equiv \langle \pi | \Theta | \pi \rangle (t) = (1 + \beta t) 2 m_\pi^2 \Omega(t) , \tag{15}$$

since

$$\Delta_{\Theta\Theta}(t) = \int_{4m_\pi^2}^\infty \frac{\rho(s) ds}{t - s} \tag{16}$$

with

$$\rho(s) \geq \rho^{(2\pi)}(s) = \frac{3}{32\pi^2}\left(\frac{s - 4m_\pi^2}{s}\right)^{1/2} | F(s) |^2 . \tag{17}$$

We will be using the approximation $\rho(s) = \rho^{(2\pi)}(s)$.

Our method will consist of finding β from the chiral estimate for $\langle u \rangle_0$, and then seeing whether the LET's of broken scale invariance can also be satisfied with this value.

We will write for an operator A with $\langle \pi | A | \pi \rangle (0) \neq 0$ in Eq. (8)

$$P_A(t) = (1 + \gamma_A t) . \tag{18}$$

For $A = \Theta$ we have $\beta = \gamma_\Theta$. We also know $\langle \pi | A | \pi \rangle (0)$ in this case from general arguments:

$$\langle \pi | \Theta | \pi \rangle (0) = 2m_\pi^2 . \tag{19}$$

For operators having $\langle \pi | A | \pi \rangle (0) = 0$ we will write

$$\langle \pi | A | \pi \rangle (t) = \gamma_A t \Omega(t) , \tag{20}$$

where now $\gamma_A = \langle \pi | A | \pi \rangle '(0)$.

The expressions we have introduced suffice for us to write an expression for $\Delta_{\Theta A}(t)$ in the approximation by intermediate states containing two pions:

$$\Delta_{\Theta A}(t) = \int_{4m_\pi^2}^\infty \frac{ds}{t - s} \frac{3}{32\pi^2}\left(\frac{s - 4m_\pi^2}{s}\right)^{1/2} \times (1 + \beta s)(1 + \gamma_A s) 2m_\pi^2 \langle \pi | A | \pi \rangle (0) | \Omega(s) |^2 . \tag{21}$$

In writing (21), we have restricted ourselves to the case $\langle \pi | A | \pi \rangle (0) \neq 0$. For an operator A violating this condition, an obvious replacement must be made in the integrand.

In order to count the number of independent relations present in various models, it is important to realize the linearity of the expression in (21):

$$\Delta_{\Theta(a_1 A_1 + a_2 A_2)} = a_1 \Delta_{\Theta A_1} + a_2 \Delta_{\Theta A_2} . \tag{22}$$

Using (21), $\Delta_{\Theta A}(0)$ and $\Delta'_{\Theta\sigma}(0)$ may be computed in terms of γ_A, β, γ_σ, and the integrals

$$I_{-1} \equiv \frac{3}{32\pi^2}\int_{4m_\pi^2}^{\Lambda^2}\left(\frac{s - 4m_\pi^2}{s}\right)^{1/2} \frac{|\Omega(s)|^2}{s^2} ds ,$$

$$I_0 \equiv \frac{3}{32\pi^2}\int_{4m_\pi^2}^{\Lambda^2}\left(\frac{s - 4m_\pi^2}{s}\right)^{1/2} \frac{|\Omega(s)|^2}{s} ds ,$$

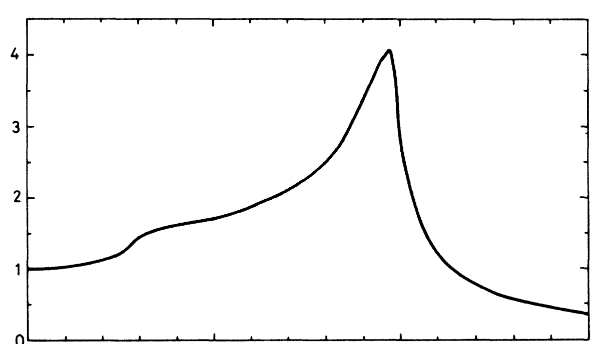

FIG. 1. The absolute value $|\Omega|$ of the Omnès function as computed from Fig. 2.

$$I_1 \equiv \frac{3}{32\pi^2}\int_{4m_\pi^2}^{\Lambda^2}\left(\frac{s-4m_\pi^2}{s}\right)^{1/2}|\Omega|^2 ds\,,$$

$$I_2 \equiv \frac{3}{32\pi^2}\int_{4m_\pi^2}^{\Lambda^2}\left(\frac{s-4m_\pi^2}{s}\right)^{1/2} s|\Omega(s)|^2 ds\,. \qquad (23)$$

The values of these integrals are given and discussed in the Appendix.

We will calculate to lowest order in chiral-symmetry breaking. That is, we will assume

$$\langle u_8\rangle_0=\langle S_8\rangle_0=\langle \hat{S}_{27}\rangle_0=0 \qquad (24)$$

in all three models. As a consistency check, if computed from (21), these numbers must come out small.

We are now in a position to make explicit the LET's to be tested in terms of the integrals I in Eqs. (23). Firstly, one has for the VEV ($f_\pi^2=0.44m_\pi^2$) and for any A with $\langle\pi|A|\pi\rangle(0)\neq 0$ (assuming β and γ_A to be given in units of GeV^{-2})

$$\frac{\Delta_{\Theta A}}{4m_\pi^2 f_\pi^2}=-\frac{\langle\pi|A|\pi\rangle(0)}{2m_\pi^2}[0.217+0.132(\beta+\gamma_A)+0.108\beta\gamma_A]=\begin{cases}2\dfrac{\langle A\rangle_0}{4m_\pi^2 f_\pi^2} & \text{if } A \text{ has dimension 2}\\[2ex] 4\dfrac{\langle u\rangle_0}{4m_\pi^2 f_\pi^2} & \text{if } A=\Theta\,.\end{cases} \qquad (25)$$

In the models considered, the VEV of u is known as

$$\frac{\langle u\rangle_0}{m_\pi^2 f_\pi^2}=\begin{cases}-13 & (3,\bar 3)\oplus(\bar 3,3)\\ -6 & (8,8)\\ -21 & (3,\bar 3)\oplus(\bar 3,3)\oplus(8,8) \text{ for } m_\pi a_0^{(0)}=1\\ -17.3 & (3,\bar 3)\oplus(\bar 3,3)\oplus(8,8) \text{ for } m_\pi a_0^{(0)}=0.6\,.\end{cases} \qquad (26)$$

The above results can be found in the literature.[3,6,11-13]

Thus, by use of (25), the values in (26) may be translated into values of β for the respective models:

$$\beta[\text{GeV}^{-2}]=\begin{cases}9.73 \text{ or } -12.17 & (3,\bar 3)\oplus(\bar 3,3)\\ 6.20 \text{ or } -8.65 & (8,8)\\ 12.78 \text{ or } -15.22 & (3,\bar 3)\oplus(\bar 3,3)\oplus(8,8) \text{ for } m_\pi a_0^{(0)}=1\\ 11.49 \text{ or } -13.94 & (3,\bar 3)\oplus(\bar 3,3)\oplus(8,8) \text{ for } m_\pi a_0^{(0)}=0.6\,.\end{cases} \qquad (27)$$

The reader should notice that for $\gamma_A=\beta$ Eq. (25) reduces to a quadratic equation for β. Hence there are two possible values of β for any $\langle u\rangle_0$ in Eq. (27). We shall henceforth consider β as known in the respective models.

Before turning explicitly to particular models of chiral-symmetry breaking, it is worthwhile to make Eq. (5) more explicit. We have

$$F'(0)=2m_\pi^2[\beta+\Omega'(0)] \qquad (28)$$

from Eq. (15). Thus we need $\Omega'(0)$. This follows numerically as

$$\Omega'(0)=2.54 \text{ GeV}^{-2}\,. \qquad (29)$$

By use of (21) for $A=\sigma$ an expression for the slope $\Delta'_{\Theta\sigma}(0)$ follows in terms of β, γ_σ, $\langle\pi|\sigma|\pi\rangle(0)$, and the integrals in (23). Putting everything together, we arrive at

$$22.97-\beta=\frac{\langle\pi|\sigma|\pi\rangle(0)}{m_\pi^2}[0.57+0.217(\beta+\gamma_\sigma)+0.132\beta\gamma_\sigma]\,. \qquad (30)$$

In the above, β and γ_σ are in GeV^{-2}.

The remaining steps are most conveniently carried out in the explicit models of chiral-symmetry breaking. This is done in the next sections.

III. THE $(3,\bar 3)\oplus(\bar 3,3)$

We will check Eq. (30) by making use of the fact that all parameters in this relation are fixed by our assumptions. Firstly, we have in the present model

$$\langle\pi|\sigma|\pi\rangle(0)=m_\pi^2\,. \qquad (31)$$

This relation can be derived for $l_u=2$ by a variety of methods. We refer to the literature for this.

Secondly, in the present model one has

$$\Theta=2(u_0+cu_8)-\text{VEV}\,. \qquad (32)$$

Since also σ can be written as a linear combination of u_0 and u_8, one may derive a linear relation between Θ, u_8, and σ:

$$\sigma-\text{VEV}=\frac{\sqrt2+c}{3}\left[\frac{1}{\sqrt2}\Theta+(1-c\sqrt2)u_8\right]. \qquad (33)$$

For c we have the expression of Ref. 8:

$$c+\sqrt{2}=\frac{3\sqrt{2}\,m_\pi^2}{2m_K^2+m_\pi^2}. \tag{34}$$

We are interested in

$$\langle\pi|\sigma|\pi\rangle(t)\equiv\langle\pi|\sigma-\mathrm{VEV}|\pi\rangle(t).$$

This may be obtained from (33) since we know

$$\langle\pi|u_8|\pi\rangle(t)=\frac{m_\pi^2}{c+\sqrt{2}}(1-t/m_\epsilon^2)\Omega(t) \tag{35}$$

together with Eq. (15) for $\langle\pi|\Theta|\pi\rangle(t)$. Equation (35) follows from (19), (31), and (33) together with the assumed decoupling of u_8 from the ϵ-S^* bump. Thus, with β in units of GeV^{-2}, we arrive at

$$\langle\pi|\sigma|\pi\rangle(t)=m_\pi^2\left(1+\frac{\beta-13.56}{13}t\right)\Omega(t). \tag{36}$$

Thus, we have

$$\gamma_\sigma=\frac{1}{13}(\beta-13.56)$$

and Eq. (31) to be used in (30).

In order to check the low-energy theorem we will use the two possible values of β in (27). With $\beta=9.73\ \mathrm{GeV}^{-2}$ we find 13.14 and 2.24 for the left-hand side and right-hand side of (30), respectively. With $\beta=-12.17\ \mathrm{GeV}^{-2}$ the numbers are 35.67 and 0.68, respectively. This disagreement is strong. It should be noted that the smallness of the right-hand side is *not* the result of a cancellation. Thus we conclude that in the $(3,\bar{3})\oplus(\bar{3},3)$ the sum rules connecting chiral- and dilational-symmetry breakings are not fulfilled by the experimental $\pi\pi$ phase up to 1 GeV for $l_u=2$. This assumes the ϵ-S^* bump to be an SU(3) singlet decoupled from u_8. Our result is stable against small variations of the mass of this bump.

We should like to add that because of the inconsistency already obtained, it is irrelevant if $\langle u_8\rangle_0$, is actually "small," as has to be expected.

IV. (8,8)

The method to be employed here is basically the same as in the previous section. We have first of all

$$\begin{aligned}\Theta&=2u-\mathrm{VEV}\\&=2\left(z\frac{B}{2\sqrt{5}}S_0+\sqrt{3}\,\frac{B}{2\sqrt{5}}S_8\right)-\mathrm{VEV}.\end{aligned} \tag{37}$$

In the present model, σ is a linear combination of S_0, S_8, and $\hat{S}_{27}$:

$$\begin{aligned}\sigma=&\tfrac{3}{2}(z+\tfrac{1}{2})\frac{B}{2\sqrt{5}}S_0+\frac{6\sqrt{3}}{5}(\tfrac{3}{2}+z)\frac{B}{2\sqrt{5}}S_8\\&+\tfrac{1}{10}(z-\tfrac{7}{2})\frac{B}{2\sqrt{5}}\hat{S}_{27}.\end{aligned} \tag{38}$$

The soft-pion limit (which is in agreement with $l_u=2$) yields the values of

$$\langle\pi|\sigma|\pi\rangle(0),$$

$$\left\langle\pi\left|\frac{B}{2\sqrt{5}}S_8\right|\pi\right\rangle(0),$$

and

$$\left\langle\pi\left|\frac{B}{2\sqrt{5}}\hat{S}_{27}\right|\pi\right\rangle(0)$$

in the customary fashion:

$$\langle\pi|\sigma|\pi\rangle(0)=-\tfrac{61}{9}m_\pi^2, \tag{39}$$

$$\left\langle\pi\left|\frac{B}{2\sqrt{5}}S_8\right|\pi\right\rangle(0)=-\frac{2}{3\sqrt{3}}(m_K^2-m_\pi^2), \tag{40}$$

and

$$\left\langle\pi\left|\frac{B}{2\sqrt{5}}\hat{S}_{27}\right|\pi\right\rangle(0)=-\tfrac{4}{27}(m_K^2-m_\pi^2). \tag{41}$$

We further know that

$$2z+1=-\frac{3m_\pi^2}{2(m_K^2-m_\pi^2)}. \tag{42}$$

In order to get a relation between β and γ_σ to be used to test (30) we may eliminate $BS_0/2\sqrt{5}$ from Eq. (38) in terms of Θ [Eq. (37)]. Except for VEV's, which are irrelevant here, we get

$$\begin{aligned}\sigma=&\frac{9}{4}\frac{m_\pi^2}{2m_K^2+m_\pi^2}\Theta\\&+\frac{3\sqrt{3}}{10}\frac{8m_K^4-25m_\pi^2m_K^2+8m_\pi^4}{(m_K^2-m_\pi^2)(2m_K^2+m_\pi^2)}\frac{B}{2\sqrt{5}}S_8\\&-\frac{1}{40}\frac{16m_K^2-13m_\pi^2}{m_K^2-m_\pi^2}\frac{B}{2\sqrt{5}}\hat{S}_{27}.\end{aligned} \tag{43}$$

Taking the pion matrix element of (43) we may express γ_σ by β using (39)–(43) and the fact that $\gamma_{S_8}=\gamma_{\hat{S}_{27}}=-1/m_\epsilon^2$. We arrive at

$$\gamma_\sigma=-0.027\beta-1.160, \tag{44}$$

if γ_σ and β are expressed in GeV^{-2}. For $\beta=6.21$ this leads approximately to 17 and -4 for the left-hand side and right-hand side of Eq. (30), respectively. The other possible value, $\beta=-8.64$, yields 31.62 and 9.06 for these numbers, respectively.

The discrepancy is large and thus our conclusions are analogous to the ones at the end of the last section.

V. THE $(3,\bar{3})\oplus(\bar{3},3)\oplus(8,8)$

In the present model, some modifications of the method employed in the previous two sections are necessary. We will see that there is one more parameter and, at the same time, one more relation to be checked.

We start by recalling that in Ref. 11 we had assumed that u was given by

$$\Theta = 2u - \text{VEV}$$
$$= 2\left(z \frac{B}{2\sqrt{5}} S_0 + \sqrt{3} \frac{B}{2\sqrt{5}} S_8 + u_0 - \sqrt{2} u_8\right) - \text{VEV}. \quad (45)$$

It follows from this Hamiltonian that σ is still given by Eq. (38):

$$\sigma = \tfrac{3}{2}(z + \tfrac{1}{2}) \frac{B}{2\sqrt{5}} S_0 + \frac{6\sqrt{3}}{5} (z + \tfrac{3}{2}) \frac{B}{2\sqrt{5}} S_8 + \tfrac{1}{10} (z - \tfrac{7}{2}) \frac{B}{2\sqrt{5}} \hat{S}_{27}. \quad (46)$$

Upon taking the VEV of (46) we get

$$-f_\pi^2 m_\pi^2 = \tfrac{3}{4}(1+2z)\left\langle \frac{B}{2\sqrt{5}} S_0 \right\rangle_0 \quad (47)$$

to lowest order in chiral-symmetry breaking. In the soft-pion limit, matrix elements of the form $\langle \pi | S^{\alpha\beta} | \pi\rangle(0)$ are given in terms of $\langle (B/2\sqrt{5})S_0\rangle_0$. Using (47), one arrives at

$$\left\langle \pi \left| \frac{B}{2\sqrt{5}} S_0 \right| \pi \right\rangle(0) = \frac{2m_\pi^2}{1+2z}, \quad (48)$$

$$\left\langle \pi \left| \frac{B}{2\sqrt{5}} S_8 \right| \pi \right\rangle(0) = \frac{1}{\sqrt{3}} \frac{m_\pi^2}{1+2z}, \quad (49)$$

and

$$\left\langle \pi \left| \frac{B}{2\sqrt{5}} \hat{S}_{27} \right| \pi \right\rangle(0) = \frac{2}{9} \frac{m_\pi^2}{1+2z}. \quad (50)$$

This results in

$$\langle \pi | \sigma | \pi\rangle(0) = \frac{1}{9} \frac{29+38z}{1+2z} m_\pi^2, \quad (51)$$

Since we know that $\gamma_{S_8} = \gamma_{\hat{S}_{27}} = -1/m_\epsilon^2$, we may write

$$\Omega^{-1}(t) \frac{\langle \pi|\sigma|\pi\rangle(t)}{\langle\pi|\sigma|\pi\rangle(0)} \equiv 1 + \gamma_\sigma t$$
$$= 1 + \frac{1}{2(29+38z)} \left[27(2z+1)\gamma_{S_0} - \frac{31+22z}{m_\epsilon^2}\right] t. \quad (52)$$

Thus we have derived an expression for γ_σ in terms of γ_{S_0} and z. For z we will take the two values corresponding to $m_\pi a_0^{(0)} = 1.0$ and 0.6, respectively. We have found

$$z = \begin{cases} -0.45 \text{ for } m_\pi a_0^{(0)} = 1.0 \\ -0.41 \text{ for } m_\pi a_0^{(0)} = 0.6. \end{cases} \quad (53)$$

In the irreducible models of the previous sections it was possible to express γ_σ in terms of β. As is clearly seen from Eqs. (45) and (46), such a possibility does not exist in the mixed model. Instead, γ_σ and γ_{S_0} are linearly related by Eq. (52) or

$$\gamma_\sigma = \begin{cases} 0.113\gamma_{S_0} - 1.002 \text{ for } m_\pi a_0^{(0)} = 1.0 \\ 0.181\gamma_{S_0} - 0.925 \text{ for } m_\pi a_0^{(0)} = 0.6 \end{cases} \quad (54)$$

in the present model. Thus, Eq. (30) does not by itself provide a test of the model. We may rather use that relation in order to obtain γ_σ for the two possible values of β. This will yield γ_{S_0} from (54). A test of the model is thus provided by (25) for $A = (B/2\sqrt{5})S_0$. Namely, we have explicitly

$$0.45 = 0.132(\beta + \gamma_{S_0}) + 0.108\beta\gamma_{S_0}. \quad (55)$$

In performing the test as explained above, four possibilities have to be considered. For $m_\pi a_0^{(0)} = 1$ we have to consider $z = -0.45$ together with $\beta = 12.78$ and -15.22 GeV^{-2}; for $m_\pi a_0^{(0)} = 0.6$ we have to consider $z = -0.41$ and $\beta = 11.49$ and -13.94 GeV^{-2} [see Eq. (27)]. For $\beta = 12.78$, -15.22, 11.49, and -13.94 GeV^{-2} together with the appropriate value of z we get from (30) $\gamma_\sigma = -1.35$, -3.14, -0.97, and -4.26 GeV^{-2}. This yields $\gamma_{S_0} = -3.08$, -18.92, -2.49, and -18.43 GeV^{-2} by use of (54). Thus we compute for the right-hand side of (55) the values -2.97, -27, -1.9, and -23.

In comparing the left-hand side of (55)—i.e., 0.45—to the numbers here obtained, it appears that no solution is particularly good, though the positive values for β are preferable. Indeed, the positive values are not excluded for the following reasons. First, there is a strong dependence on small variations of the parameters in the region of interest. This is particularly clear if one looks at (54) and remembers that the γ_σ for positive β are around unity.

Secondly, as one of the consistency conditions, we may compute $\langle u_8\rangle_0$, $\langle S_8\rangle_0$, and $\langle \hat{S}_{27}\rangle_0$ by our saturation method and have to find "small" values. Even though our subtraction method [decoupling the SU(3)-nonscalar operation from the ϵ-S^* bump] generally will yield "small" values for these VEV's, this is not sufficient for them to be neglected in lowest order of chiral-symmetry breaking. Namely, in going from (46) to (47), we have neglected the VEV of the SU(3)-nonscalar part as compared to $B\langle S_0\rangle_0/2\sqrt{5}$. Because of the small factor $(z + \frac{1}{2})$ in front of $BS_0/2\sqrt{5}$, we have to be particularly careful. Indeed, taking the VEV of (46) and [rather than neglecting the VEV of the SU(3)-nonscalar part] computing everything by means of (25) we find

$$-1=\frac{3}{4}(1+2z)\frac{1}{f_\pi^2 m_\pi^2}\left\langle\frac{B}{2\sqrt{5}}S_0\right\rangle_0-\frac{1}{5}\frac{1}{1+2z}\left[6(z+\tfrac{3}{2})+\frac{1}{9}(z-\tfrac{7}{2})\right]\times\left[0.217+0.132(\beta-m_\epsilon{}^{-2})-0.108\beta m_\epsilon{}^{-2}\right] .$$

Numerically, the contribution of the terms we are supposed to neglect turn out to be *larger* than the term we should keep. This inconsistency and analogous ones can easily be avoided within the errors of the present approach. The key observation is the following: If β and m_ϵ are chosen such that

$$0=0.217+0.132(\beta-m_\epsilon{}^{-2})-0.108\beta m_\epsilon{}^{-2}, \qquad (56)$$

then [compare (25)] all VEV's of SU(3)-nonscalar operators will vanish in our saturation scheme.

The subtraction at $m_\epsilon{}^2$ was in fact chosen so as to satisfy two goals, namely (1) absence of the ϵ-S^* bump in $\langle\pi|A|\pi\rangle(t)$ for A an SU(3)-nonscalar and (2) $\langle A\rangle_0\approx 0$ for these A. The first condition is satisfied by any $m_\epsilon{}^{-2}\approx 1.0$ GeV^{-2}, whereas the required smallness of $\langle A\rangle_0$ requires (56) to be almost exactly satisfied. With $\beta=12.78$, -15.22, 11.49, and -13.94 GeV^{-2} we find $m_\epsilon{}^{-2}=1.26$, 1.19, 1.26, and 1.18 GeV^{-2} from (56). These are certainly values for $m_\epsilon{}^{-2}$ which are consistent with our first requirement on the subtraction.

As may be seen directly from the above coincidence of the values $m_\epsilon{}^{-2}$ for two very different sets of values of β, Eq. (56) cannot be used to determine β.

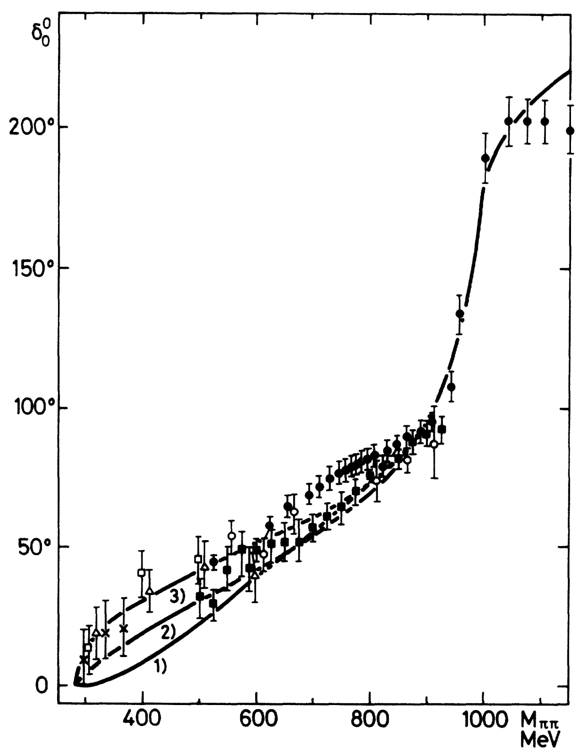

FIG. 2. The $\pi\pi$ phase shifts S_0^0 of Ref. 14.

Using the "new" values of $m_\epsilon{}^2$, Eq. (54) is changed to

$$\gamma_\sigma=\begin{cases}0.113\gamma_{S_0}-1.116 \text{ for } m_\pi a_0^{(0)}=1, & \beta>0\\ 0.113\gamma_{S_0}-1.051 \text{ for } m_\pi a_0^{(0)}=1, & \beta<0\\ 0.181\gamma_{S_0}-1.034 \text{ for } m_\pi a_0^{(0)}=.6, & \beta>0\\ 0.181\gamma_{S_0}-0.968 \text{ for } m_\pi a_0^{(0)}=.6, & \beta<0\end{cases} \qquad (57)$$

The values of γ_σ as computed from (30) remain unchanged. Thus, we have the following list of $\gamma_{S_0}=-2.069$, -18.39, $+0.367$, and -18.17 GeV^{-2}. We see clearly the large uncertainty in γ_{S_0} for positive β coming from the fact that γ_σ is for these β almost equal to the constant term in (57). Computing the right-hand side of (55) we find now -1.44, 25.78, 2.02, and 23.11. Clearly the negative values of β are excluded. We see also that in Eq. (55) for physical (positive) β and z (i.e., γ_{S_0}) there is a cancellation implying the strong change of the value predicted for $\langle BS_0/2\sqrt{5}\rangle_0$ by saturation. It is seen that between $m_\pi a_0^{(0)}=1$ and $m_\pi a_0^{(0)}=0.6$ there *is* a consistent solution of all low-energy theorems of broken chiral and dilation symmetry. Going back to Ref. 11 we find this consistent solution at $m_\pi a_0^{(0)}=0.81$, $z=-0.435$, $\beta=12.13$, $\gamma_\sigma=-1.213$, and $\gamma_{S_0}=-0.986$. Taking into account also ambiguities in the cutoff, we have the following main conclusions.

There is no consistent solution of the low-energy theorems of broken chiral and dilation symmetry in the $(3,\bar{3})\oplus(\bar{3},3)$ and the (8, 8), whereas such a solution exists for large values of $m_\pi a_0^{(0)}$ in our mixed model. It would, however, not be meaningful to go any further in the second case and to take seriously the values written above for which an exact solution obtains. Since the solution depends on small variations of the parameters, the final value is influenced by ambiguities such as, e.g., taking exactly the same subtraction for S_8 and $\hat{S}_{27}$ or the cutoff. We should go further once $m_\pi a_0^{(0)}$ really turns out to be large and becomes known with high accuracy.

APPENDIX

In this appendix we wish to discuss our treatment of the $\pi\pi$ phases for the calculation of the Omnès function Ω, as well as giving the values of the various weighted integrals of $|\Omega|^2$ which appear in the text, Eq. (23).

We calculated Ω from (9) using the $\pi\pi$ phase shifts of Basdevant *et al.*[14] Our Fig. 2 is a reproduction of their Fig. 1(a), and we actually chose to use their curve 2—putting the phase shift equal to π from $m_{\pi\pi}=1$ GeV onwards. Doing the integral on a computer gave the Omnès function of Fig. 1.

Given Ω, we then calculated the integrals I_{-1}–I_2 of (23), also on the computer. In order to display the dependence of these numbers on the cutoff we give their values for $\Lambda^2=4$ GeV^2 and $\Lambda^2=9$ GeV^2. A cutoff is necessary since $\Omega(s)\sim s^{-1}$ for high s, and I_2 is, consequently, divergent. We found

$$I_{-1}=0.249 \text{ for } \Lambda^2=4 \text{ or } 9 \text{ GeV}^2,$$
$$I_0=0.0954 \text{ for } \Lambda^2=4 \text{ GeV}^2,$$
$$=0.0955 \text{ for } \Lambda^2=9 \text{ GeV}^2,$$
$$I=0.579 \text{ for } \Lambda^2=4 \text{ GeV}^2,$$
$$=0.584 \text{ for } \Lambda^2=9 \text{ GeV}^2,$$
$$I_2=0.0474 \text{ for } \Lambda^2=4 \text{ GeV}^2,$$
$$=0.051 \text{ for } \Lambda^2=9 \text{ GeV}^2.$$

In our calculations we used the values for $\Lambda^2=4$ GeV^2.

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