Comments and Addenda

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Comments on the absence of spontaneous symmetry breaking in low dimensions

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We present some comments intended to bring out simply and more explicitly the physics behind the rigorously proven results on the absence of continuous symmetry breaking in (1 + 1) dimensions at $T=0$ and (2 + 1) dimensions for $T>0$. It is also pointed out within a model that a massless boson is present, in a sense defined in the text, even though symmetry breaking disappears.

The connection, in the absence of long-range interactions, between the impossibility of broken continuous symmetry and dimensionality has been established with some generality and rigor for both zero and finite temperatures.[1-3] We summarize this connection in Table I.

In this note we make some simple pedagogical remarks which we hope will bring out more explicitly the physics behind these already proven results. We claim neither new results nor mathematical rigor. Let us work in a familiar model, and begin with one space dimension and one time dimension and at zero temperature.

Consider two scalar fields $\sigma(x,t)$ and $\pi(x,t)$ with a Lagrangian

$$\mathcal{L}(x)=\tfrac{1}{2}(\partial_\mu\sigma)^2+\tfrac{1}{2}(\partial_\mu\pi)^2-U(\pi^2+\sigma^2)\,, \tag{1}$$

where $U(\rho^2)$ has a minimum at $\rho=\rho_0$. An example is the familiar form $U(\rho^2)=-\frac{1}{2}m^2\rho^2+\frac{1}{4}g^2\rho^4$. The continuous symmetry in question is, of course, the symmetry of $\mathcal{L}(x)$ under rotations in the (σ,π) plane.

Spontaneously broken symmetry at $T=0$ refers to the nonvanishing vacuum expectation value of σ and/or π. More precisely, first restrict the system to a finite volume L and add a term

$$\mathcal{L}'=\lambda^2\sigma \tag{2}$$

to the Lagrangian where λ is small, real, and nonzero. One says that there is symmetry breaking if

$$\langle\sigma\rangle\equiv\lim_{\lambda\to 0}\lim_{L\to\infty}\langle 0|\sigma(x)|0\rangle \neq 0\,. \tag{3}$$

The term $\lambda^2\sigma$ in (2) is not unique. Any other form which picks out a preferred direction in the (σ,π) plane by giving it an infinitesimally lower energy density will suffice. But the order of limits in (3) is essential, for the result would necessarily vanish if the order were reversed.

If the fields were classical, then (i) the term $\frac{1}{2}[(\partial_\mu\pi)^2+(\partial_\mu\sigma)^2]$ would tend to make the vector $[\sigma(x),\pi(x)]$ constant at all points in space-time, (ii) $U(\sigma^2+\pi^2)$ would tend to make this constant vector have length ρ_0, and (iii) the small external field λ in (2) would tend to fix the direction of this vector along σ. For a classical field, the lowest-energy configuration is precisely $\sigma(x,t)=\rho_0$ and $\pi(x,t)=0$, and there is symmetry breaking even in one space dimension.

For quantum fields, the kinetic terms $(\partial\sigma/dt)^2$ and $(d\pi/dt)^2$ lead to zero-point motion, and the ground state has a wave function that will clearly spread to other field configurations as well. This will happen in any number of dimensions. But if the quantum spread is sufficiently large, then all memory of the classically preferred direction may be lost. As a result, $\langle\sigma\rangle$ could vanish along with $\langle\pi\rangle$, and there would be no symmetry breaking. This is what happens in one space dimension and one time dimension, as seen below.

Let us define polar fields by $\sigma=\rho\cos\theta$ and $\pi=\rho\sin\theta$, and rewrite the Lagrangian as

$$\mathcal{L}=\tfrac{1}{2}\rho^2(\partial_\mu\theta)^2+\tfrac{1}{2}(\partial_\mu\rho)^2-U(\rho^2)-\tfrac{1}{2}\lambda^2\theta^2\,, \qquad (4)$$

where the infinitesimal external field term $-\frac{1}{2}\lambda^2\theta^2$ is slightly different (for reasons of solubility) from (2), but is an equally good candidate for picking the direction of σ. We are interested in the spread of the ground-state wave function of this system in $\theta(x)$.

To get a qualitative answer, let us replace $\rho^2(x)$ by an average c-number value $\rho_0^{\,2}$, and consider just the behavior of the angular field $\theta(x)$. In other words, consider the simpler Lagrangian

$$\mathcal{L}_\theta=\tfrac{1}{2}\rho_0^{\,2}(\partial_\mu\theta)^2-\tfrac{1}{2}\lambda^2\theta^2\,. \qquad (5)$$

As $\lambda\to 0$, this is just the Lagrangian for a string stretched on the surface of a long cylinder of radius ρ_0. The range of $\theta(x)$ is from $-\infty$ to $+\infty$ to account for configurations where the string is wound any number of times around the cylinder at places. Equation (5) also corresponds to a free scalar field, whose solutions are trivial and are made up of a set of uncoupled oscillators, one for each normal mode k. Before the limits in (3) are taken, we can work in a large but finite volume L. Then the free scalar field in (5) is well defined. As usual,

$$\theta(x,t)=\frac{1}{\rho_0\sqrt{L}}\sum_k\frac{1}{(2\omega_k)^{1/2}}\left[a_k e^{i(\omega t-kx)}+a_k^\dagger e^{-i(\omega t-kx)}\right],$$

with $[a_k,a_{k'}^\dagger]=\delta_{kk'}$, $\omega_k\equiv(k^2+\lambda^2)^{1/2}$. The ground-state wave function of this system is

$$|0\rangle=\prod_k\psi_{0k}\,, \qquad (6)$$

where ψ_{0k} is just the ground-state wave function of the harmonic oscillator corresponding to mode k. In the representation where the field $\theta(x)$ is diagonal, ψ_{0k} is just a Gaussian in $(\theta_k+\theta_k^*)$. The strong difference from the classical case is evident in (6). The lowest-energy classical configuration was restricted to the $k=0$ mode, where it was further restricted to $\theta=0$. The quantum ground state not only has a spread in the $k=0$ mode but has all the infinitely many other modes present as well. The effect of this spread on fluctuations of $\theta(x)$ can be estimated from the second moment:

$$\lim_{L\to\infty}\langle 0|\,\theta(x)\theta(0)\,|0\rangle_{t=0}=\frac{1}{2\pi\rho_0^{\,2}}\int_0^\infty\frac{dk\,e^{-ikx}}{2(k^2+\lambda^2)^{1/2}}\equiv c(x)\,. \qquad (7)$$

It is clear that as $\lambda\to 0$, (7) has an infrared divergence in one space dimension. Thus, the fluctuations in θ are infinitely large, and all memory of the classical minimum at $\theta=0$ is wiped out. In fact,

$$\langle 0|\,e^{i\theta(x)}\,|0\rangle_{t=0}=\langle 0|\,e^{i\theta^+(x)}e^{-c(0)/2}e^{i\theta^-(x)}\,|0\rangle$$
$$=e^{-c(0)/2}\,, \qquad (8)$$

where $\theta^\pm(x)$ are the usual creation and annihilation decompositions of $\theta(x)$ whose commutator $[\theta^-(x),\theta^+(0)]_{t=0}=c(x)$ is again given by (7). Once again, because of the infrared divergence in $c(0)$ as $\lambda\to 0$, $\langle 0|e^{i\theta(x)}|0\rangle$ vanishes, and therefore so do $\langle\sigma(x)\rangle\propto\langle\cos\theta(x)\rangle$ and $\langle\pi(x)\rangle\propto\langle\sin\theta(x)\rangle$. Thus, there is no symmetry breaking. The spread of the ground-state wave function (6) in the variable $\theta(x)$ is infinitely large due to zero-point motion, so that all directions in the (σ,π) plane enjoy equal probability.

The existence of the infrared divergence in (7) for the two-point function of the θ field has a clear physical meaning. As L, the volume, increases, so does the probability of finding the string wound around the cylinder an arbitrarily large number of times. Thus, if the large fluctuations of the θ field are interpreted "modulo 2π," the model makes physical sense. No matter how large θ may become, $\sigma(x)$ and $\pi(x)$, proportional to $\cos\theta(x)$ and $\sin\theta(x)$, respectively, cannot become large. Their two-point functions are well defined, as can be seen explicitly:

$$\langle 0|\,\sigma(x)\sigma(0)\,|0\rangle$$
$$\propto\langle 0|\,(e^{i\theta(x)}+e^{-i\theta(x)})(e^{i\theta(0)}+e^{-i\theta(0)})\,|0\rangle\,. \qquad (9)$$

Splitting $\theta(x)=\theta^+(x)+\theta^-(x)$ as before, it is easy to see that

$$\langle 0|\,e^{i\theta(x)}e^{i\theta(0)}\,|0\rangle=e^{-[c(0)+c(x)]}\,, \qquad (10)$$

while

$$\langle 0|\,e^{-i\theta(x)}e^{i\theta(0)}\,|0\rangle=e^{-[c(0)-c(x)]}\,. \qquad (11)$$

Clearly, the infrared divergence is still present, as $\lambda\to 0$, in the exponent in (10), making (10) vanish, but it is canceled in the exponent of (11), making that term nonzero and finite. Hence,

TABLE I. Possibility of breaking continuous symmetry.

Space dimensionality	$T>0$	$T=0$
1	no (See Refs. 1 and 2)	no (See Ref. 3)
2	no (See Refs. 1 and 2)	yes
>2	yes	yes

$$\langle\sigma(x)\sigma(0)\rangle = \langle\pi(x)\pi(0)\rangle \propto e^{-[c(0)-c(x)]} = \exp\left[-\frac{1}{2\pi\rho_0^2}\int_0^\Lambda \frac{dk}{k}(1-e^{-ikx})\right] \tag{12}$$

is well defined.

[A technical point: The number $c(0)$ used above has an ultraviolet divergence due to the momentum integral in Eq. (7). This is related to our cavalier replacement of the field operator $\rho^2(x)$ in the angular Lagrangian by an average c number ρ_0^2. Note that $c(0)$ also has a $1/\rho_0^2$ factor, where $\rho_0 \sim \langle\rho(x)\rho(x)\rangle$ again has an ultraviolet divergence. Since, in the original theory, neither $\langle\sigma(x)\sigma(0)\rangle$ not $\langle\pi(x)\pi(0)\rangle$ is expected to vanish for $x\neq 0$, these offsetting ultraviolet divergences presumably do not appear at all when the original model in (1) is solved correctly. However, our results are independent of these technical complications in the ultraviolet range, since the restoration or otherwise of symmetry is an infrared phenomenon. We therefore assume an ultraviolet cutoff Λ where necessary in our qualitative remarks.]

Next, let us elaborate a little on the statement that "there are no Goldstone bosons in two dimensions."[3] It is not as though the massless Goldstone boson (which refers in our example to excitations of the θ field when the space dimensionality is greater than 1) acquires a mass in (1+1) dimensions. Instead, its two-point function diverges at the infrared end, making a massless free scalar field theory mathematically ill-defined. Be that as it may, we can consider the situation with an arbitrarily large (but finite) volume L, and/or an arbitrarily small but finite λ. In that case, the theory should be well-defined, and excited states can be obtained from the ground state (6) by applying the operators $a_k^\dagger$. The energy of the lowest excited state, $(k_{\min}^2+\lambda^2)^{1/2}$, will become arbitrarily small. The system will resemble one containing a massless boson to high accuracy, despite the fact that symmetry breaking ($\langle\sigma\rangle$) is negligibly small.

This may be stated more precisely. It is believed that in (1+1) dimensions, a real field ϕ with a Lagrangian

$$\mathcal{L} = \tfrac{1}{2} :(\partial_\mu\phi)^2: - \tfrac{1}{2}m_0^2 :\phi^2: - \tfrac{1}{4}\lambda :\phi^4:$$

leads to two phases (symmetry breaking) for $\gamma = \lambda/m_0^2 > \gamma_c$, where γ_c is some finite number. The "mass gap," as a function of γ, would become zero at the point γ_c. For the case of the *complex* field, with a corresponding

$$\mathcal{L} = \tfrac{1}{2} :|\partial_\mu\phi|^2: - \tfrac{1}{2}m_0^2 :|\phi|^2: - \tfrac{1}{4}\lambda :(|\phi|^2)^2: ,$$

our heuristic considerations would lead us to conjecture that even though there can never be breaking of the continuous U(1) symmetry, or massless bosons in (1+1) dimension, nevertheless the mass gap would become zero at $\gamma \equiv \lambda/m_0^2 = \gamma_c$, *and* would *stay* at zero for $\gamma > \gamma_c$. However, a massless particle interpretation of this continuous spectrum for $\gamma > \gamma_c$ would not be viable.

In the case of space dimension $D>1$, it is evident that there is no infrared divergence in $\langle\theta(x)\theta(0)\rangle$. Quantum fluctuations in $\theta(x)$ are still present, but are finite. Therefore, there is still a bias in favor of the $\theta=0$ direction, and symmetry breaking exists. It can be easily checked that $\langle\cos\theta\rangle$ and hence $\langle\sigma\rangle$ are nonvanishing for $D>1$ at zero temperature.

At finite temperatures, one would expect the spread in configurations of $\theta(x)$ due to quantum effects to be further compounded by the demands of entropy. In other words, a system at finite T will contain a statistical mixture of not only the ground state (6), which, as we saw, already had a considerable spread in $\theta(x)$, but of excited states as well. It is not surprising then that symmetry breaking requires a higher space dimensionality than 2. More precisely, at finite T, considerations regarding the vacuum expectation value of any operator A are replaced by those of the thermodynamic expectation value

$$\langle A\rangle_T \equiv \frac{\mathrm{Tr}[e^{-\beta H}A]}{\mathrm{Tr}\,e^{-\beta H}}.$$

Thus, in the limit $\lambda\to 0$, $L\to\infty$, we have

$$\langle a_{\vec{k}}^\dagger a_{\vec{k}'}\rangle_T = \frac{\delta(\vec{k}-\vec{k}')}{e^{\beta k}-1}, \tag{13}$$

where

$$\beta \equiv 1/k_B T,$$

where k_B is the Boltzmann constant. Hence,

$$\langle\theta(x)\theta(0)\rangle_T = \frac{1}{(2\pi)^D}\int_0^\Lambda \frac{d^Dk}{2k}e^{-ikx}\left(1+\frac{2}{e^{\beta k}-1}\right). \tag{14}$$

The first term in (14) is the earlier $T=0$ result, but the second term has a stronger infrared behavior. It is clear that unless $D>2$, (14) will be infrared-divergent, producing infinitely large fluctuations in $\theta(x)$ and destroying symmetry breaking as before.

Needless to say, all these remarks apply only to continuous symmetry. Broken discrete symmetry *can* happen at $T=0$ in one space dimension, although it disappears for any finite T. An example of this in a renormalizable field theory has been discussed in detail elsewhere.[4]

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