Two-particle scattering amplitudes with spin-independent Poincaré-transformation properties

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(Received 5 September 1974)

For systems of massive particles with arbitrary spins we define complete sets of invariant c.m. helicity amplitudes. We construct the associated partial-wave amplitudes and give a detailed derivation of the constraint equations which they must satisfy if scattering processes are to be invariant under space-time translations and proper homogeneous Lorentz transformations. We then indicate how these amplitudes are related to those of Jacob and Wick and Feldman and Matthews. In appendixes we collect together the definitions and transformation properties of various two-particle states and scattering amplitudes, and derive some useful kinematical transformation formulas.

I. INTRODUCTION

We have shown elsewhere[1] that for scattering systems involving sets of r particles (k) it is possible to define complete sets of invariant scattering amplitudes which are parametrized by the eigenvalues $\lambda_{(k)}$ of spin-component observables[2] $\hat{S}_{(k)}{}^{q(k)}$ provided $r \geq 3$. The new observables $\hat{S}_{(k)}{}^{q(k)}$ are defined by[3]

$$\hat{S}_{(k)}{}^{q(k)} = [\Delta(\hat{p}_{(k)}, \hat{q}_{(k)})]^{-1} \hat{W}_{(k)} \cdot \hat{q}_{(k)} , \tag{1}$$

where $\hat{q}_{(k)}$ is a four-vector function of the single-particle momenta $\hat{p}_{(k)}$ for $k = 1, 2, \ldots, r$, and $\hat{W}_{(k)}$ is the Pauli-Lubanski spin of the particle (k),

$$\hat{W}_{(k)\mu} = -\tfrac{1}{2} \epsilon_{\mu}{}^{\nu\rho\sigma} \hat{p}_{(k)\nu} \hat{J}_{(k)\rho\sigma} . \tag{2}$$

For a two-particle system these Poincaré-invariant observables coincide up to a sign with the c.m. helicity[4] observables of Feldman and Matthews,

$$\begin{aligned} {}^2\hat{S}_{(k)} &= \hat{S}_{(k)}{}^{p[2]} \\ &= [\Delta(\hat{p}_{[2]}, \hat{p}_{(k)})]^{-1} \hat{W}_{(k)} \cdot \hat{p}_{(k)} , \end{aligned} \tag{3}$$

where the total momentum observable $p_{[2]}$ is defined by

$$\hat{p}_{[2]} = \hat{p}_{(1)} + \hat{p}_{(2)} . \tag{4}$$

In this paper we shall construct various types of two-particle c.m. helicity amplitudes. We shall investigate some of their properties and shall show how they are related to the helicity amplitudes of Jacob and Wick[5,6] and c.m. helicity amplitudes of Feldman and Matthews.[4,7] In order to follow our notation we suggest the reader refer to Appendix A of Ref. 1 and to Appendix A of this paper, where, for convenience, we have listed the definitions and properties of several two-particle states.

In Sec. II we construct a complete set of c.m. helicity scattering amplitudes which, apart from a momentum-independent phase factor, coincide in the c.m. frame with the helicity amplitudes of Jacob and Wick.[5] These amplitudes, contrary to the suggestion of Feldman and Matthews,[4] are not Poincaré-invariant. They are functions of single-particle momenta which change under constant velocity transformations of the reference frame. It is possible to define a complete set of c.m. helicity amplitudes which are invariant and coincide with Jacob-Wick-type amplitudes in that special c.m. frame which is conventionally chosen for the Reggeization of partial-wave amplitudes. We construct some explicitly in Sec. III and relate them to the conventional frame-dependent scattering amplitudes of field theory. We then define the associated partial-wave amplitudes and examine some of their properties.

In Sec. IV we discuss the relation between our amplitudes and those of other authors,[4-6] and in Sec. V we summarize the results obtained in Secs. II-IV.

In Appendixes A and B we collect together the definitions and properties of various two-particle states and scattering amplitudes, and in Appendix C we derive some kinematical transformation formulas.

II. CENTER-OF-MASS HELICITY AMPLITUDES

We define a set of c.m. helicity states which coincide with Jacob-Wick-type helicity states when the total three-momentum $\vec{p}$ is zero. We then show explicitly that the associated scattering amplitudes are not invariant.

A. Center-of-mass helicity states

We have shown elsewhere[2] that two-particle c.m. helicity states $|\vec{p}_{(1)}, \vec{p}_{(2)} : \lambda_{(1)}, \lambda_{(2)}]_+$ may be defined in terms of two-particle standard rest states $|\vec{0}, \vec{0} : \lambda_{(1)}, \lambda_{(2)}\rangle$ by Eq. (A1),

$$|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}]_+ = \hat{H}_{(1)+}(p_{(1)};p)\hat{H}_{(2)+}(p_{(2)},p)|\vec{0},\vec{0}:\lambda_{(1)},\lambda_{(2)}\rangle, \quad (5)$$

where p denotes the eigenvalue of the total momentum observable $\hat{p}_{[2]}$, Eq. (4). These states are parametrized by the eigenvalues $\vec{p}_{(1)}$, $\vec{p}_{(2)}$, $\lambda_{(1)}$, and $\lambda_{(2)}$ of observables $\hat{\underline{p}}_{(1)}$, $\hat{\underline{p}}_{(2)}$, $\hat{S}_{(1)}{}^{p[2]}$, and $\hat{S}_{(2)}{}^{p[2]}$, Eq. (3), respectively. Like the ordinary helicity states $|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}\}_+$ of Appendix A they are not well defined when the c.m. momentum $\vec{P}_{(1)}$ (A9) lies in the 3-direction. For this reason we prefer to consider a different set of c.m. helicity states $|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}]_\pm$, which, like the Jacob-Wick-type helicity states $|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}\}_\pm$ are well defined unless momentum $\vec{p}_{(1)}$ lies in the negative 3-direction,

$$|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}]_\pm = \hat{H}_{(1)+}(p_{(1)};p)\hat{H}_{(2)-}(p_{(2)};p)(-1)^{\sigma(2)-\lambda(2)} \times|\vec{0},\vec{0};\lambda_{(1)},-\lambda_{(2)}\rangle\ . \quad (6)$$

The new states are related to the old ones by a phase factor which depends upon the c.m. momentum $\vec{P}_{(2)}$ (A9),

$$|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}]_\pm = \exp[-2i\lambda_{(2)}\phi(P_{(2)})]|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}]_+, \quad (7)$$

and coincide in the c.m. frame with the Jacob-Wick-type helicity states $|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}\}_\pm$. This is possible because, by definition, the single-particle helicities and c.m. helicities coincide in the c.m. frame.

B. Frame-dependent center-of-mass helicity amplitudes

We sandwich a scattering operator $\hat{S}$ between complete sets of initial-particle states $|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}]_\pm$ and final-particle states $|\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}]_\pm$ (6) to obtain our c.m. helicity amplitudes,

$$S^\pm[\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}] = {}_\pm[\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\hat{S}|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}]_\pm. \quad (8)$$

By construction they must coincide with Jacob-Wick-type helicity amplitudes

$$S^\pm\{\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}\}$$

in the c.m. frame.

The generators $\hat{p}_\mu$ of space-time translations $\hat{a}$ and generators $\hat{J}_{\mu\nu}$ of homogeneous Lorentz transformations $\hat{\Lambda}$ of the scattering system as a whole are of the form

$$\hat{p}_\mu=\hat{p}_{(1)\mu}+\hat{p}_{(2)\mu}+\hat{p}_{(\bar{1})\mu}+\hat{p}_{(\bar{2})\mu}+\cdots, \quad (9)$$

$$\hat{J}_{\mu\nu}=\hat{J}_{(1)\mu\nu}+\hat{J}_{(2)\mu\nu}+\hat{J}_{(\bar{1})\mu\nu}+\hat{J}_{(\bar{2})\mu\nu}+\cdots, \quad (10)$$

and the associated homogeneous Lorentz transformation and translation operators $\hat{\Lambda}$ and $\hat{a}$ are given by

$$\hat{\Lambda}=\hat{R}(\alpha,\beta,\gamma)\hat{Z}(\delta)\hat{R}(0,\beta',\gamma'), \quad (11)$$

$$\hat{a}=e^{ia\cdot\hat{p}}\ . \quad (12)$$

For scattering theories which are invariant under space-time translations, rotations, and constant-velocity transformations, the scattering operator $\hat{S}$ must satisfy the equations

$$\hat{a}\hat{S}\hat{a}^{-1}=\hat{S}\ , \quad (13)$$

$$\hat{\Lambda}\hat{S}\hat{\Lambda}^{-1}=\hat{S}\ . \quad (14)$$

It then follows from Eq. (13) and the transformation property of our two-particle states under translations Eq. (A13) that the scattering amplitudes

$$S^\pm[\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}]$$

vanish unless energy and momentum are conserved. We may thus relate them to a set of reduced amplitudes,

$$S'^\pm[\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}],$$

which are functions of only eight independent momentum components,

$$S^\pm[\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}]=\delta^4(p-\bar{p})S'^\pm[\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}]\ , \quad (15)$$

where momenta p and $\bar{p}$ are defined by Eqs. (B3). Under the action of homogeneous Lorentz transformations $\hat{\Lambda}$ the c.m. helicity states $|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}]_\pm$ satisfy Eq. (A14). The Lorentz invariance of the scattering operator $\hat{S}$ described by Eq. (14) then implies that the c.m. helicity amplitudes

$$S'^\pm[\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}]$$

must satisfy the constraint equation

$$S'^\pm[\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}]=\exp[i\phi_{(\bar{1})}(\lambda_{(\bar{1})}-\lambda_{(\bar{2})})-i\phi_{(1)}(\lambda_{(1)}-\lambda_{(2)})] \times S'^\pm[\vec{p}^\dagger_{(\bar{1})},\vec{p}^\dagger_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\vec{p}^\dagger_{(1)},\vec{p}^\dagger_{(2)}:\lambda_{(1)},\lambda_{(2)}]\ , \quad (16)$$

where the transformed momenta are defined by Eq. (A15) and the phases $\phi_{(k)}$ are given by

$$\exp(-i\hat{J}_3\phi_{(k)})=\hat{H}_+(\hat{L}(\hat{\Lambda}:p):p_{(k)}) \ . \tag{17}$$

Since these frame-dependent phases are generally nonzero the c.m. helicity amplitudes

$$S'^{\pm}[\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}]$$

cannot be functions of Poincaré scalars alone.

It is generally the case that if scattering amplitudes are defined to be matrix elements of a scattering operator $\hat{S}$ between independent two-particle c.m. helicity states, they will not be invariant. Each amplitude will take the form of a Poincaré scalar function multiplied by a frame-dependent phase. In the case which we have considered we used the two independent four-momenta $p_{(1)}$ and $p_{(2)}$ to define the c.m. helicity state operators $\hat{H}_+(p_{(1)};p)$ and $\hat{H}_-(p_{(2)};p)$. The Lorentz frames of reference defined by these operators are consequently only defined up to a rotation about a 3-axis and the state phases become frame-dependent.

We shall show in Sec. III that, if one allows the definitions of initial-particle states to involve final-particle momenta and vice versa, one may then construct sets of invariant two-particle c.m. helicity amplitudes.

III. INVARIANT CENTER-OF-MASS HELICITY AMPLITUDES

The S matrix associated with the scattering of two particles (1) and (2) into two particles $(\bar{1})$ and $(\bar{2})$ may be a function of four single-particle momenta, three of which are usually linearly independent. Let us define a four-vector f by

$$f=p_{(\bar{1})}+p_{(2)} \ . \tag{18}$$

Away from the physical region boundary this vector will be linearly independent of the total four-momentum p and linearly independent of each single-particle momentum. We may then associate with each particle three linearly independent four-momenta and can use the multiparticle techniques developed elsewhere[1] to construct two-particle states with associated invariant scattering amplitudes.

A. Center-of-mass helicity states

We define our c.m. helicity states $|\vec{p}_{(1)},\vec{p}_{(2)}:f:\lambda_{(1)},\lambda_{(2)}]_{\pm}$ in terms of the standard rest states $|\vec{0},\vec{0}:\lambda_{(1)},\lambda_{(2)}\rangle$ of Eq. (A11) by

$$|\vec{p}_{(1)},\vec{p}_{(2)}:f:\lambda_{(1)},\lambda_{(2)}]_{\pm}=\hat{H}_{(1)+}(p_{(1)};p,f)\hat{H}_{(2)-}(p_{(2)};p,f)\times(-1)^{\sigma_{(2)}-\lambda_{(2)}}|\vec{0},\vec{0}:\lambda_{(1)},-\lambda_{(2)}\rangle \ , \tag{19}$$

where the homogeneous Lorentz transformation operators $\hat{H}_+(p_{(1)};p,f)$ and $\hat{H}_-(p_{(2)};p,f)$ are given in Appendix A. It should be noted that we have chosen the operators $\hat{H}_+(p;q,f)$ and $\hat{H}_-(p;q,f)e^{i\pi\hat{J}_2}$ to replace the multiparticle state operators $\hat{L}(p;q,f)$ of Eq. (A6) which have the same 4×4 matrix representation as $L(p;q,f)$ because they are closely related to the operators $\hat{H}_+(p;q)$ and $\hat{H}_-(p;q)e^{i\pi\hat{J}_2}$ in terms of which we defined the frame-dependent c.m. helicity states (6).

We may use Eqs. (A4) and (A5) to rewrite Eq. (19) in the form

$$|\vec{p}_{(1)},\vec{p}_{(2)}:f:\lambda_{(1)},\lambda_{(2)}]_{\pm}=\hat{L}_{[2]}(p;p_{(2)},f)\hat{Z}_{(1)}(P_{(1)})\hat{Z}_{(2)}^{-1}(P_{(2)})(-1)^{\sigma_{(2)}-\lambda_{(2)}}|\vec{0},\vec{0}:\lambda_{(1)},-\lambda_{(2)}\rangle \ , \tag{20}$$

where the c.m. momenta $P_{(1)}$ and $P_{(2)}$ are defined by Eq. (A9). As in the multiparticle case[1] we now modify the overall signs of initial and final two-particle states so that they have similar Lorentz transformation properties,

$$|\vec{p}_{(1)},\vec{p}_{(2)}:f:\lambda_{(1)},\lambda_{(2)}]_{p(2)f\pm}=|\vec{p}_{(1)},\vec{p}_{(2)}:f:\lambda_{(1)},\lambda_{(2)}]_{\pm} \ , \tag{21}$$

$$|\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:f:\lambda_{(\bar{1})},\lambda_{(\bar{2})}]_{p_{(2)}f\pm}=[l(\hat{L}^{-1}(p;p_{(2)},f):p;p_{(\bar{2})},f)]^{2(\sigma_{(\bar{1})}+\sigma_{(\bar{2})})}|\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:f:\lambda_{(\bar{1})},\lambda_{(\bar{2})}]_{\pm} \ , \tag{22}$$

where the sign function $l(\hat{L}^{-1}(p;p_{(2)},f):p;p_{(\bar{2})}f)$ is given by Eq. (A26). These c.m. helicity states have been so constructed that they coincide with the frame-dependent c.m. helicity states $|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}]_{\pm}$ and $|\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}]_{\pm}$ and with the helicity states $|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}\}_{\pm}$ and $|\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}\}_{\pm}$ in that c.m. frame in which the three-vector $\vec{p}_{(1)}\times\vec{p}_{(\bar{1})}$ lies in the positive 2-direction and momentum $\vec{p}_{(1)}$ lies in the positive 3-direction, when

$$L(p;p_{(2)},f)=I \ . \tag{23}$$

They satisfy the space-time translation equation (A13) and have the homogeneous Lorentz transformation properties (A14),

$$\hat{\Lambda}|\,\vec{p}_{(1)},\vec{p}_{(2)}:f:\lambda_{(1)},\lambda_{(2)}]_{p_{(2)}f^{\pm}}=[l(\hat{\Lambda}:p;p_{(2)},f)]^{2(\sigma_{(1)}+\sigma_{(2)})}|\,\vec{p}^{\dagger}_{(1)},\vec{p}^{\dagger}_{(2)};f^{\dagger}:\lambda_{(1)},\lambda_{(2)}]_{p_{(2)}f^{\pm}} \tag{24}$$

and

$$\hat{\Lambda}|\,\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:f:\lambda_{(\bar{1})},\lambda_{(\bar{2})}]_{p_{(2)}f^{\pm}}=[l(\hat{\Lambda}:p;p_{(2)},f)]^{2(\sigma_{(\bar{1})}+\sigma_{(\bar{2})})}|\,\vec{p}^{\dagger}_{(\bar{1})},\vec{p}^{\dagger}_{(\bar{2})}:f^{\dagger}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}]_{p_{(2)}f^{\pm}}, \tag{25}$$

where the transformed momenta are given by Eq. (A15) and the sign function $l(\hat{\Lambda}:p;\,p_{(2)},f)$ is defined by Eq. (A26).

B. Center-of-mass helicity amplitudes

We define a set of two-particle c.m. helicity amplitudes

$$S^{p_{(2)}f^{\pm}}[\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\,\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}]$$

in terms of the c.m. helicity states (21), (22):

$$S^{p_{(2)}f^{\pm}}[\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\,\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}]={}_{p_{(2)}f^{\pm}}[\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:f:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\,\hat{S}|\,\vec{p}_{(1)},\vec{p}_{(2)}:f:\lambda_{(1)},\lambda_{(2)}]_{p_{(2)}f^{\pm}}\,. \tag{26}$$

By construction they coincide in the special frame defined by Eq. (25) with the frame-dependent c.m. helicity amplitudes

$$S^{\pm}[\,\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\,\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}]$$

of Eq. (8) and with the Jacob-Wick–type helicity amplitudes

$$S^{\pm}\{\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\,\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}\}\,.$$

For theories which are invariant under space-time translations the translation property (A13) of the states (21) and (22) implies that S-matrix elements vanish unless energy and momentum are conserved. We may thus introduce a set of reduced amplitudes

$$S'^{p_{(2)}f^{\pm}}[\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\,\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}]$$

which are functions of only eight independent momentum components,

$$S^{p_{(2)}f^{\pm}}[\vec{p}_{(\bar{1})},p_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}]=\delta^4(p-\bar{p})S'^{p_{(2)}f^{\pm}}[\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\,\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}]\,. \tag{27}$$

We may also use the homogeneous Lorentz transformation properties (24) and (25) of the c.m. helicity states (21) and (2) and the invariance of the scattering operator $\hat{S}$ described by Eq. (14) to show that the reduced amplitudes satisfy a trivial constraint equation,

$$S'^{p_{(2)}f^{\pm}}[\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}:|\,\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}]=[l(\hat{\Lambda}:p;p_{(2)},f)]^{2(\sigma_{(1)}+\sigma_{(2)}+\sigma_{(\bar{1})}+\sigma_{(\bar{2})})}$$
$$\times S'^{p_{(2)}f^{\pm}}[\vec{p}^{\dagger}_{(\bar{1})},\vec{p}^{\dagger}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\,\vec{p}^{\dagger}_{(1)},\vec{p}^{\dagger}_{(2)}:\lambda_{(1)},\lambda_{(2)}], \tag{28}$$

where the transformed momenta are given by Eq. (A15). Consider first of all the effect of a transformation $\hat{\Lambda}$ of the form $\exp(2\pi i J_3)$. Equation (28) then takes the form

$$S'^{p_{(2)}f^{\pm}}[\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\,\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}]$$
$$=(-1)^{2(\sigma_{(1)}+\sigma_{(2)}+\sigma_{(\bar{1})}+\sigma_{(\bar{2})})}S'^{p_{(2)}f^{\pm}}[\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\,\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}], \tag{29}$$

and if S-matrix elements are not to vanish, the sum of all individual particle spins must be an integer. In this case for an arbitrary transformation $\hat{\Lambda}$ Eq. (28) becomes a spin-independent constraint equation,

$$S'^{p_{(2)}f^{\pm}}[\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\,\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}]=S'^{p_{(2)}f^{\pm}}[\vec{p}^{\dagger}_{(\bar{1})},\vec{p}^{\dagger}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\,\vec{p}^{\dagger}_{(1)},\vec{p}^{\dagger}_{(2)}:\lambda_{(1)},\lambda_{(2)}]\,. \tag{30}$$

The amplitudes

$$S'^{p_{(2)}f^{\pm}}[\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\,\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}]$$

are evidently functions of Poincaré scalars alone. In order to see this explicitly one need only replace the transformation $\hat{\Lambda}$ of Eq. (30) by $L^{-1}(p;p_{(2)},f)$. According to Eq. (A15), momenta $\vec{p}^{\dagger}_{(\bar{1})}$, $\vec{p}^{\dagger}_{(\bar{2})}$, $\vec{p}^{\dagger}_{(2)}$, and $\vec{p}^{\dagger}_{(2)}$ will then be replaced by $\vec{P}^{\star}_{(\bar{1})}$, $\vec{P}^{\star}_{(\bar{2})}$, $\vec{P}^{\star}_{(1)}$, and $\vec{P}^{\star}_{(2)}$ of Eq. (A10), and it follows from the discussion in Appendix C of Ref. 1 that each momentum component will be a function of scalar momentum products alone.

We have thus constructed explicitly a set of two-particle amplitudes which are parametrized by the eigenvalues of Poincaré-invariant observables alone. With the aid of the defining Eqs. (26) and (27) and those of the associated c.m. helicity states (19), (21), and (22), we may relate these invariant amplitudes to any of the frame-dependent two-particle amplitudes listed later in Table III. In particular, in the case of standard amplitudes

$$S'\langle \vec{p}_{(\bar{1})}, \vec{p}_{(\bar{2})}: \lambda_{(\bar{1})}, \lambda_{(\bar{2})} | \vec{p}_{(1)}, \vec{p}_{(2)}: \lambda_{(1)}, \lambda_{(2)}\rangle$$

we have the invariant amplitude expansion

$$\begin{aligned} S'\langle \vec{p}_{(\bar{1})}, \vec{p}_{(\bar{2})}: \lambda_{(\bar{1})}, \lambda_{(\bar{2})} | \vec{p}_{(1)}, \vec{p}_{(2)}: \lambda_{(1)}, \lambda_{(2)}\rangle &= \sum_{\lambda'_{(\bar{1})}\lambda'_{(\bar{2})}\lambda'_{(1)}\lambda'_{(2)}} D^{\sigma(\bar{1})*}_{\lambda'_{(\bar{1})}\lambda_{(\bar{1})}}(\hat{H}_+^{-1}(p_{(\bar{1})}; p, f)\hat{L}(p_{(\bar{1})}))\, D^{\sigma(\bar{2})*}_{\lambda'_{(\bar{2})}\lambda_{(\bar{2})}}(\hat{H}_-^{-1}(p_{(\bar{2})}; p, f)\hat{L}(p_{(\bar{2})})) \\ &\quad\times D^{\sigma(1)}_{\lambda'_{(1)}\lambda_{(1)}}(\hat{H}_+^{-1}(p_{(1)}; p, f)\hat{L}(p_{(1)}))\, D^{\sigma(2)}_{\lambda'_{(2)}\lambda_{(2)}}(\hat{H}_-^{-1}(p_{(2)}; p, f)\hat{L}(p_{(2)})) \\ &\quad\times [l(L^{-1}(p; p_{(2)}, f): p; p_{(\bar{2})}, f)]^{2(\sigma(\bar{1})+\sigma(\bar{2}))}(-1)^{\sigma(2)+\sigma(\bar{2})-\lambda'_{(2)}-\lambda'_{(\bar{2})}} \\ &\quad\times S'^{pf\pm}[\vec{p}_{(\bar{1})}, \vec{p}_{(\bar{2})}: \lambda'_{(\bar{1})}, -\lambda'_{(\bar{2})} | \vec{p}_{(1)}, \vec{p}_{(2)}: \lambda'_{(1)}, -\lambda'_{(2)}]. \qquad (31) \end{aligned}$$

Since the expansion coefficients are unitary we can invert this equation directly and express each invariant amplitude as a sum of standard amplitudes with frame-dependent coefficients.

Although two-particle q-spin operators $\hat{S}_{(k)}{}^{q(k)}(1)$ are uniquely determined up to a sign to coincide with the Feldman-Matthews c.m. helicity operators of Eq. (3), invariant two-particle helicity amplitudes are not unique. They may differ by Poincaré scalar phases which are functions of the four-vector f. We have examined here the case where the vector f is of the form (18). We could alternatively have taken it to be given by

$$f_\mu \to g_\mu = \epsilon_\mu{}^{\nu\rho\sigma} p_{(1)\nu} p_{(2)\rho} p_{(\bar{2})\sigma}, \qquad (32)$$

in which case the associated amplitudes

$$S'^{p(2)g\pm}[\vec{p}_{(\bar{1})}, \vec{p}_{(\bar{2})}: \lambda_{(\bar{1})}, \lambda_{(\bar{2})} | \vec{p}_{(1)}, \vec{p}_{(2)}: \lambda_{(1)}, \lambda_{(2)}]$$

are related to the amplitudes

$$S'^{p(2)f\pm}[\vec{p}_{(\bar{1})}, \vec{p}_{(\bar{2})}: \lambda_{(\bar{1})}, \lambda_{(\bar{2})} | \vec{p}_{(1)}, \vec{p}_{(2)}: \lambda_{(1)}, \lambda_{(2)}]$$

by

$$\begin{aligned} S'^{p(2)g\pm}[\vec{p}_{(\bar{1})}, \vec{p}_{(\bar{2})}: \lambda_{(\bar{1})}, \lambda_{(\bar{2})} | \vec{p}_{(1)}, \vec{p}_{(2)}: \lambda_{(1)}, \lambda_{(2)}] &= \exp[-i\tfrac{1}{2}\pi(\lambda_{(1)} + \lambda_{(2)} - \lambda_{(\bar{1})} - \lambda_{(\bar{2})})]\, S'^{p(2)f\pm}[\vec{p}_{(\bar{1})}, \vec{p}_{(\bar{2})}: \lambda_{(\bar{1})}, \lambda_{(\bar{2})} | \vec{p}_{(1)}, \vec{p}_{(2)}: \lambda_{(1)}, \lambda_{(2)}]. \qquad (33) \end{aligned}$$

The new amplitudes

$$S'^{p(2)g\pm}[\vec{p}_{(\bar{1})}, \vec{p}_{(\bar{2})}: \lambda_{(\bar{1})}, \lambda_{(\bar{2})} | \vec{p}_{(1)}, \vec{p}_{(2)}: \lambda_{(1)}, \lambda_{(2)}]$$

evidently coincide with the helicity amplitudes

$$S'^{\pm}\{\vec{p}_{(\bar{1})}, \vec{p}_{(\bar{2})}: \lambda_{(\bar{1})}, \lambda_{(\bar{2})} | \vec{p}_{(1)}, \vec{p}_{(2)}: \lambda_{(1)}, \lambda_{(2)}\}$$

in that c.m. frame in which the vector $\vec{p}_{(1)} \times \vec{p}_{(\bar{1})}$ lies in the positive 1-direction and vector $\vec{p}_{(1)}$ lies along the positive 3-axis, i.e.,

$$L(p; p_{(2)}, g) = I. \qquad (34)$$

Invariant two-particle c.m. helicity amplitudes are only well defined by Eq. (26) within the physical region boundary when[3] $\Delta(p_{(1)}, p_{(2)}, p_{(\bar{1})}) \neq 0$. On the boundary there is a phase ambiguity and at threshold, like the frame-dependent c.m. helicity amplitudes, they are not defined at all. This is not surprising because at threshold the matrix elements of c.m. helicity operators between momentum eigenstates are not all well defined.[2,8]

C. Partial-wave decomposition of two-particle amplitudes

We first of all introduce amplitudes of type II which are matrix elements of the scattering operator between two-particle states defined by a relation analogous to Eq. (A19),

$$S^{p_{(2)}f\pm}[\bar{s}:\bar{p};\bar{\phi},\bar{\theta}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\ s:p;\phi,\theta:\lambda_{(1)},\lambda_{(2)}]={}_{p_{(2)}f\pm}[\bar{s}:\bar{p};\bar{\phi},\bar{\theta}:\psi:\lambda_{(\bar{1})}\lambda_{(\bar{2})}|\hat{S}|s:p;\phi,\theta:\psi:\lambda_{(1)},\lambda_{(2)}]_{p_{(2)}f\pm}\,. \tag{35}$$

The space-time translation properties (13) of the scattering operator $\hat{S}$ and of these type-II states, (A21), then enable us to define a set of reduced amplitudes

$$S'^{p_{(2)}f\pm}[\bar{\phi},\bar{\theta}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\ s,\vec{p}|\ \phi,\theta:\lambda_{(1)},\lambda_{(2)}],$$

where

$$S^{p_{(2)}f\pm}[\bar{s}:\bar{p};\bar{\phi},\bar{\theta}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\ s:p;\phi,\theta:\lambda_{(1)},\lambda_{(2)}]=\delta^4(\bar{p}-p)S'^{p_{(2)}f\pm}[\bar{\phi},\bar{\theta}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\ s,p;\ \phi,\theta:\lambda_{(1)},\lambda_{(2)}]\ . \tag{36}$$

It follows from Eq. (A23), which relates type-I and type-II two-particle c.m. helicity states, that the corresponding amplitudes satisfy the equation

$$S'^{p_{(2)}f\pm}[\bar{\phi},\bar{\theta}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\,s,\vec{p}\,|\,\phi,\theta:\lambda_{(1)},\lambda_{(2)}]=\tfrac{1}{2}\pi s^{-1}[\Delta(s;1,2)\Delta(s;\bar{1},\bar{2})]S'^{p_{(2)}f\pm}[\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\,\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}], \tag{37}$$

which suggests that type-II amplitudes are invariant. Indeed, one may use the homogeneous Lorentz transformation properties (A25) of the type-II states of (A23) to show directly that the reduced type-II amplitudes are independent of the total three-momentum $\vec{p}$ and are functions of a single angle Θ,

$$\begin{aligned}S'^{p_{(2)}f\pm}[\bar{\phi},\bar{\theta}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\,s,\vec{p}\,|\,\phi,\theta:\lambda_{(1)},\lambda_{(2)}]&=S'^{p_{(2)}f\pm}[0,\Theta:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\,s,\vec{0}\,|\,0,0:\lambda_{(1)},\lambda_{(2)}]\\&\equiv S'^{p_{(2)}f\pm}[\lambda_{(\bar{1})},\lambda_{(\bar{2})}|s,\Theta\,|\lambda_{(1)},\lambda_{(2)}].\end{aligned} \tag{38}$$

We find

$$R(0,\Theta,0)=L^{-1}(p;p_{(2)},p_{(\bar{1})})L(p;p_{(\bar{2})},p_{(2)}), \tag{39}$$

and explicit expressions for the angle Θ as a function of momentum components are given in Appendix C.

Like type-I amplitudes, the reduced type-II amplitudes

$$S'^{p_{(2)}f\pm}[\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\,s,\Theta\,|\lambda_{(1)},\lambda_{(2)}]$$

are well defined away from the physical region boundary where $\Theta\neq 0,\pi$.

We now sandwich the scattering operator $\hat{S}$ between the c.m. helicity states of Eq. (A27) to obtain a set of type-III scattering amplitudes,

$$\begin{aligned}S[\bar{s},\bar{\sigma}:\bar{p},\bar{\lambda}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\,s,\sigma:p,\lambda:\lambda_{(1)},\lambda_{(2)}]\\=[\bar{s},\bar{\sigma}:\bar{p},\bar{\lambda}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\,\hat{S}\,|\,s,\sigma:p,\lambda:\lambda_{(1)},\lambda_{(2)}].\end{aligned} \tag{40}$$

The translation properties (A28) of the type-III states of Eq. (A27) and translational invariance (13) of the scattering operator $\hat{S}$ enable us to define a reduced S matrix $S'[\bar{\sigma}:\bar{\lambda}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\,s,\vec{p}\,|\,\sigma:\lambda:\lambda_{(1)},\lambda_{(2)}]$:

$$\begin{aligned}S[\bar{s},\bar{\sigma}:\bar{p},\bar{\lambda}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\,s,\sigma:p,\lambda:\lambda_{(1)},\lambda_{(2)}]\\=\delta^4(\bar{p}-\bar{p})\\\times S'[\bar{\sigma}:\bar{\lambda}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\,s,\vec{p}\,|\,\sigma:\lambda:\lambda_{(1)},\lambda_{(2)}].\end{aligned} \tag{41}$$

The Lorentz transformation properties Eq. (A29) of the states defined by Eq. (A27) and Lorentz invariance of the scattering operator $\hat{S}$ described by Eq. (14) then enable us to show that our reduced amplitudes satisfy the constraint equation

$$S'[\bar{\sigma}:\bar{\lambda}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\,s,\vec{p}\,|\,\sigma:\lambda:\lambda_{(1)},\lambda_{(2)}]=\sum_{\bar{\lambda}'\lambda'}D^{\bar{\sigma}*}_{\bar{\lambda}'\bar{\lambda}}(\hat{L}(\hat{\Lambda}:p))D^{\sigma}_{\lambda'\lambda}(\hat{L}(\hat{\Lambda}:p))S'[\bar{\sigma}:\bar{\lambda}':\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\,s,p^{\dagger}\,|\,\sigma:\lambda':\lambda_{(1)},\lambda_{(2)}]. \tag{42}$$

Let us first of all take the Lorentz transformation $\hat{\Lambda}$ to be of the form $\exp(2\pi iJ_3)$. We then have the relation

$$S'[\bar{\sigma},\bar{\lambda}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\,s,\vec{p}\,|\,\sigma:\lambda:\lambda_{(1)},\lambda_{(2)}]=(-1)^{2(\sigma+\bar{\sigma})}S'[\bar{\sigma},\bar{\lambda}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\,s,\vec{p}\,|\,\sigma:\lambda:\lambda_{(1)},\lambda_{(2)}], \tag{43}$$

and our amplitudes must vanish unless the sum of initial and final spins is an integer. If we now take $\hat{\Lambda}$ to be of the form $\hat{L}(p')\hat{L}^{-1}(p)$ for any momentum p' we find

$$S'[\bar{\sigma},\bar{\lambda}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\,s,\vec{p}\,|\,\sigma,\lambda:\lambda_{(1)},\lambda_{(2)}]=S'[\bar{\sigma},\bar{\lambda}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\,s,\vec{p}'\,|\,\sigma,\lambda:\lambda_{(1)},\lambda_{(2)}]\,. \tag{44}$$

The amplitudes must be independent of the total three-momentum $\vec{p}$. Finally, if we take the Lorentz transformation $\hat{\Lambda}$ to be $\hat{L}(p)\hat{R}(\alpha,\beta,\gamma)\hat{L}^{-1}(p)$ and integrate over the angles α, β, and γ we obtain the spin-dependent constraint equation

$$S'[\bar{\sigma},\bar{\lambda}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}\,|\,s,\vec{p}\,|\,\sigma,\lambda:\lambda_{(1)},\lambda_{(2)}]=\delta_{\bar{\lambda}\lambda}\,\delta_{\bar{\sigma}\sigma}\sum_{\lambda'}S'[\bar{\sigma},\lambda':\lambda_{(\bar{1})},\lambda_{(\bar{2})}\,|\,s,\vec{p}\,|\,\sigma,\lambda':\lambda_{(1)},\lambda_{(2)}]\,. \tag{45}$$

It follows that the amplitudes of Eq. (41) are diagonal in parameters σ and λ and independent of the value of the spin component λ. These observations enable us to define new reduced amplitudes of type III $S'[\lambda_{(\bar{1})},\lambda_{(\bar{2})}\,|\,s,\sigma\,|\,\lambda_{(1)},\lambda_{(2)}]$, which are explicitly functions of only six parameters,

$$S'[\bar{\sigma};\,\bar{\lambda}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}\,|\,s,\vec{p}\,|\,\sigma:\lambda:\lambda_{(1)},\lambda_{(2)}]=\delta_{\bar{\sigma}\sigma}\delta_{\bar{\lambda}\lambda}S'[\lambda_{(\bar{1})},\lambda_{(\bar{2})}\,|\,s,\sigma\,|\,\lambda_{(1)},\lambda_{(2)}]\,. \tag{46}$$

Such amplitudes are well defined throughout the physical region and on the physical region boundary away from threshold, whereas the type-I and type-II amplitudes defined by Eqs. (26) and (35) were only well defined within the physical region.

We may use Eq. (A27) relating type-III states to type-II states to expand type-III amplitudes in a series of type-II amplitudes

$$S'^{p_{(2)}f\,\pm}[\lambda_{(\bar{1})},\lambda_{(\bar{2})}\,|\,s,\Theta\,|\,\lambda_{(1)},\lambda_{(2)}]=\sum_{\sigma}\,[2\sigma+1]\,d^{\sigma}_{\bar{\mu}\mu}(\Theta)S'[\lambda_{(\bar{1})},\lambda_{(\bar{2})}\,|\,s,\sigma\,|\,\lambda_{(1)},\lambda_{(2)}]\,,\qquad \Theta\in(0,\pi) \tag{47}$$

where

$$d^{\sigma}_{\bar{\mu}\mu}(\Theta)=D^{\sigma}_{\bar{\mu}\mu}(\hat{R}(0,\Theta,0)) \tag{48}$$

and

$$\bar{\mu}=\lambda_{(\bar{1})}-\lambda_{(\bar{2})},\quad \mu=\lambda_{(1)}-\lambda_{(2)}\,. \tag{49}$$

We see that amplitudes of type III are simply partial-wave amplitudes associated with the invariant amplitudes of types I and II. One may now use Eq. (37) relating amplitudes of types I and II, and equations of the type (B6) relating various two-particle scattering amplitudes to obtain a partial-wave decomposition of any two-particle S matrix listed in Table V. In particular the Jacob-Wick-type helicity amplitudes

$$S'^{\pm}\{\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\,\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}\}$$

satisfy the equation

$$\begin{aligned}S'^{\pm}\{\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}\,|\,\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}\}&=2s\pi^{-1}[\Delta(s;1,2)\Delta(s;\bar{1},\bar{2})]^{-1/2}\\&\times\sum_{\sigma\lambda'_{(\bar{1})}\lambda'_{(\bar{2})}\lambda'_{(1)}\lambda'_{(2)}}[2\sigma+1][l(\hat{L}^{-1}(p;\,p_{(2)},f):p;\,p_{(\bar{2})},f)]^{2\sigma}\\&\times D^{\sigma_{(\bar{1})}*}_{\lambda'_{(\bar{1})}\lambda_{(1)}}(\hat{H}_{+}(\hat{L}^{-1}(p):p_{(\bar{1})}))\,D^{\sigma_{(\bar{2})}}_{\lambda'_{(\bar{2})}\lambda_{(\bar{2})}}(\hat{H}_{-}(\hat{L}^{-1}(p):p_{(\bar{2})}))D^{\sigma_{(1)}}_{\lambda'_{(1)}\lambda_{(1)}}(\hat{H}_{+}(\hat{L}^{-1}(p):p_{(1)}))\\&\times D^{\sigma_{(2)}*}_{\lambda'_{(2)}\lambda_{(2)}}(\hat{H}_{-}(\hat{L}^{-1}(p):p_{(2)}))\,e^{-i[\phi_{(1)}(\lambda_{(1)}-\lambda_{(2)})-\phi_{(\bar{1})}(\lambda_{(\bar{1})}-\lambda_{(\bar{2})})]}d^{\sigma}_{\bar{\mu}\mu}(\Theta)S'[\lambda'_{(\bar{1})},\lambda'_{(\bar{2})}|s,\sigma|\lambda'_{(\bar{1})},\lambda'_{(\bar{2})}]\,,\end{aligned} \tag{50}$$

where the phases $\phi_{(1)}$ and $\phi_{(\bar{1})}$ are given by

$$\hat{R}_3(\phi_{(1)})=\hat{R}_3(R_{+}^{-1}(P_{(1)})P_{(\bar{1})})=\hat{R}_3(R_{+}^{-1}(P_{(1)})F) \tag{51}$$

and

$$\hat{R}_3(\phi_{(\bar{1})})=\hat{R}_3(R_{+}^{-1}(P_{(\bar{1})})P_{(2)})=\hat{R}_3(R_{+}^{-1}(P_{(\bar{1})})F)\,. \tag{52}$$

In our special c.m. frame (23) the expansion takes the familiar Regge-theory form

$$S'^{\pm}\{\vec{P}^{\star}_{(\bar{1})},\vec{P}^{\star}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}\,|\,\vec{P}^{\star}_{(1)},\vec{P}^{\star}_{(2)}:\lambda_{(1)},\lambda_{(2)}\}=2s\pi^{-1}[\Delta(s;\bar{1},\bar{2})\Delta(s,1,2)]^{-1/2}\times\sum_{\sigma}[2\sigma+1]\,d^{\sigma}_{\bar{\mu}\mu}(\Theta)\times S'[\lambda_{(\bar{1})},\lambda_{(\bar{2})}\,|\,s,\sigma\,|\,\lambda_{(1)},\lambda_{(2)}]\,, \tag{53}$$

where the starred momenta are defined by Eq. (A10). Indeed we may take advantage of this fact to rewrite Eq. (50) in the simpler form

$$S'[\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}]$$
$$=2s\pi^{-1}[\Delta(s;1,2)\Delta(s;1,2)]^{-1/2}\sum_{\sigma\lambda'_{(\bar{1})}\lambda'_{(\bar{2})}\lambda'_{(1)}\lambda'_{(2)}\lambda'}[2\sigma+1]D^{\sigma(\bar{1})*}_{\lambda'_{(\bar{1})}\lambda_{(\bar{1})}}(\hat{H}_+(\hat{\Lambda}:p_{(1)}))D^{\sigma(\bar{2})}_{\lambda'_{(\bar{2})}\lambda_{(\bar{2})}}(\hat{H}_-(\hat{\Lambda}:p_{(2)}))$$
$$\times D^{\sigma(1)}_{\lambda'_{(1)}\lambda_{(1)}}(\hat{H}_+(\hat{\Lambda}:p_{(1)}))D^{\sigma(2)*}_{\lambda'_{(2)}\lambda_{(2)}}(\hat{H}_-(\hat{\Lambda}:p_{(2)}))$$
$$\times d^{\sigma}_{\bar{\mu}\mu}(\Theta)S'[\lambda'_{(\bar{1})},\lambda'_{(\bar{2})}|s,\sigma|\lambda_{(1)},\lambda_{(2)}], \quad (54)$$

where $\hat{\Lambda}=\hat{L}^{-1}(p;p_{(2)},f)$. We should like to stress the point that the helicity amplitude expansion [Eq. (53)] is only valid in the special Lorentz frame defined by Eq. (23), whereas the invariant c.m. helicity amplitude expansion [Eq. (47)] is valid in all Lorentz frames.

One may use the orthogonality and completeness properties of rotation group [SU(2)] representation functions or the defining equation (A22) of type-III states to express the partial-wave amplitudes $S'[\lambda_{(\bar{1})},\lambda_{(\bar{2})}|s,\sigma|\lambda_{(1)},\lambda_{(2)}]$ as integrals over type-II invariant amplitudes $S'^{p_{(2)}f\pm}[\lambda_{(\bar{1})},\lambda_{(\bar{2})}|s,\Theta|\lambda_{(1)},\lambda_{(2)}]$:

$$S'[\lambda_{(\bar{1})},\lambda_{(\bar{2})}|s,\sigma|\lambda_{(1)},\lambda_{(2)}]$$
$$=\tfrac{1}{2}\int_0^{\pi}d^{\sigma*}_{\bar{\mu}\mu}(\Theta)S'^{p_{(2)}f\pm}[\lambda_{(\bar{1})},\lambda_{(\bar{2})}|s,\Theta|\lambda_{(1)},\lambda_{(2)}]$$
$$\times\sin\Theta\, d\Theta\,. \quad (55)$$

The integrand is not well defined when the angle Θ is zero or π. However, since this constitutes a set of points of measure zero the integral is unaffected and our partial-wave amplitudes are well defined away from threshold. In another paper[9] we shall see that it is possible to construct a complete set of c.m. orbital angular momentum partial-wave amplitudes which, unlike these c.m. helicity amplitudes, are well defined everywhere on the physical-region boundary.

IV. COMPARISON WITH CONVENTIONAL AMPLITUDES

The first major step in constructing an equation of the form (47) was taken by Jacob and Wick,[5] who determined the partial-wave expansion of a set of helicity amplitudes

$$\langle\theta\phi\lambda_{(\bar{1})}\lambda_{(\bar{2})}|\hat{S}(s^{1/2})|00\lambda_{(1)}\lambda_{(2)}\rangle$$

in terms of a set of partial-wave helicity amplitudes

$$\langle\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\hat{S}^{\sigma}(s^{1/2})|\lambda_{(1)},\lambda_{(2)}\rangle$$

in the c.m. frame. One can show that in the *special* frame defined by Eq. (23) their amplitudes are related to the c.m. helicity amplitudes of Eq. (47) by

$$\langle\Theta 0\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\hat{S}(s^{1/2})|00\lambda_{(1)},\lambda_{(2)}\rangle$$
$$=4\pi^{-1}(-1)^{[\sigma_{(2)}+\sigma_{(\bar{2})}-\lambda_{(2)}-\lambda_{(\bar{2})}]}$$
$$\times S'^{p_{(2)}f\pm}[\lambda_{(\bar{1})},\lambda_{(\bar{2})}|s,\Theta|\lambda_{(1)},\lambda_{(2)}]\,, \quad (56)$$

$$\langle\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\hat{S}^{\sigma}(s^{1/2})|\lambda_{(1)},\lambda_{(2)}\rangle$$
$$=(-1)^{[\sigma_{(2)}+\sigma_{(\bar{2})}-\lambda_{(2)}-\lambda_{(\bar{2})}]}$$
$$\times S'[\lambda_{(\bar{1})},\lambda_{(\bar{2})}|s,\sigma|\lambda_{(1)},\lambda_{(2)}]\,. \quad (57)$$

However, since they only define helicity states and partial-wave helicity states in the c.m. frame, their helicity amplitudes and partial-wave helicity amplitudes are only defined in the c.m. frame.

In his paper on three-particle states Wick[6] defines precisely a set of helicity states $|p_{(1)}\lambda_{(1)};p_{(2)}\lambda_{(2)}\rangle$, which are parametrized by the eigenvalues $\lambda_{(1)}$ and $\lambda_{(2)}$ of the helicity operators $\hat{S}_{(1)}$ and $\hat{S}_{(2)}$, and a set of partial-wave *c.m. helicity* states $|p;\sigma\lambda:\lambda_{(1)},\lambda_{(2)}\rangle$, which are parametrized by the eigenvalues λ, $\lambda_{(1)}$, and $\lambda_{(2)}$ of the helicity operator $\hat{S}$ and the *c.m. helicity* operators $\hat{S}_{(1)}{}^{p[2]}$ and $\hat{S}_{(2)}{}^{p[2]}$ defined by Eq. (3), respectively. His helicity states are related to those of Appendix A by

$$|p_{(1)}\lambda_{(1)}:p_{(2)}\lambda_{(2)}\rangle=\begin{cases}\zeta|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}\}, & \theta(p_{(1)}),\ \theta(p_{(2)})\neq 0,\pi\\ \zeta|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}\}_{\pm}, & \vec{p}=\vec{0},\ \theta(p_{(1)})=0,\ \text{etc.},\end{cases} \quad (58)$$

where

$$\zeta=(-1)^{2[\sigma_{(1)}\epsilon(p_{(1)})+\sigma_{(2)}\epsilon(p_{(2)})]} \quad (59)$$

and

$$\epsilon(p)=\text{int.pt.}[\phi(p)/\pi]\,. \quad (60)$$

His partial-wave c.m. helicity states are related to the standard type-III partial-wave states of Eq. (A27) by

$$|p;\sigma\lambda:\lambda_{(1)},\lambda_{(2)}\rangle$$
$$=(-1)^{2\epsilon(p)}\sum_{\lambda'}D^{\sigma}_{\lambda'\lambda}(\hat{R}(p))|s,\sigma:p,\lambda':\lambda_{(1)},\lambda_{(2)}]\,. \quad (61)$$

We see that the corresponding Wick partial-wave expansion would be an expansion of helicity amplitudes in terms of partial-wave *c.m. helicity* amplitudes.

Feldman and Matthews have made the suggestion that a frame-independent equation of the form (47) could be obtained if one replaced the helicity amplitudes which coincide with Jacob-Wick helicity amplitudes in the c.m. frame. Although their Wick-type partial-wave amplitudes would be invariant, we have shown explicitly in Sec. II that the c.m. helicity amplitudes would be frame-dependent. On the other hand we have seen that it is possible to construct c.m. helicity amplitudes

$$S'^{p_{(2)}f\pm}[\vec{p}_{(\bar{1})}, \vec{p}_{(\bar{2})} : \lambda_{(\bar{1})}, \lambda_{(\bar{2})} | \vec{p}_{(1)}, \vec{p}_{(2)} : \lambda_{(1)}, \lambda_{(2)}]$$

by Eq. (26) which *are* invariant provided we only demand that they coincide with Jacob-Wick–type amplitudes in the special c.m. frame of Eq. (23). We suggest that these amplitudes should be identified with the amplitudes

$$\langle \bar{P}, \tfrac{1}{2}(p_{(\bar{1})} - p_{(\bar{2})}), \lambda_{\bar{1}}, \lambda_{\bar{2}} \,|\, \hat{S} \,|\, P, \tfrac{1}{2}(p_{(1)} - p_{(2)}), \lambda_1, \lambda_2 \rangle$$

of Feldman and Matthews.

V. GENERAL DISCUSSION AND CONCLUSIONS

In Sec. II we constructed a complete set of c.m. helicity states $|\vec{p}_{(1)}, \vec{p}_{(2)} : \lambda_{(1)}, \lambda_{(2)}]_\pm$, Eq. (6), which coincided in the c.m. frame with Jacob-Wick–type helicity states $|\vec{p}_{(1)}, \vec{p}_{(2)} : \lambda_{(1)}, \lambda_{(2)}\}_\pm$. We then showed that the corresponding c.m. helicity amplitudes

$$S'^{\pm}[\vec{p}_{(\bar{1})}, \vec{p}_{(\bar{2})} : \lambda_{(\bar{1})}, \lambda_{(\bar{2})} | \vec{p}_{(1)}, \vec{p}_{(2)} : \lambda_{(1)}, \lambda_{(2)}],$$

Eq. (8), were not invariant. In Sec. III we showed that if one allowed initial two-particle states to have phases determined by a final-particle momentum and vice versa, it was possible to construct c.m. helicity states $|\vec{p}_{(1)}, \vec{p}_{(2)} : f : \lambda_{(1)}, \lambda_{(2)}]_{p_{(2)}f\pm}$, Eq. (22), in such a way that the corresponding S-matrix elements

$$S'^{p_{(2)}f\pm}[\vec{p}_{(\bar{1})}, \vec{p}_{(\bar{2})} : \lambda_{(\bar{1})}, \lambda_{(\bar{2})} | \vec{p}_{(1)}, \vec{p}_{(2)} : \lambda_{(1)}, \lambda_{(2)}],$$

Eq. (26), were invariant. Moreover, in Eq. (31) we indicated how the standard two-particle amplitudes of field theory could be expanded in terms of such invariant amplitudes with frame-dependent coefficients.

We proceeded to define reduced two-particle c.m. helicity amplitudes,

$$S'^{p_{(2)}f\pm}[\lambda_{(\bar{1})}, \lambda_{(\bar{2})} \,|\, s, \Theta \,|\, \lambda_{(1)}, \lambda_{(2)}]$$

by Eq. (38) and

$$S'[\lambda_{(\bar{1})}, \lambda_{(\bar{2})} \,|\, s, \sigma \,|\, \lambda_{(1)}, \lambda_{(2)}]$$

by Eq. (46), which were related by the partial-wave equation (47):

$$S'^{p_{(2)}f\pm}[\lambda_{(\bar{1})}, \lambda_{(\bar{2})} \,|\, s, \Theta \,|\, \lambda_{(1)}, \lambda_{(2)}] = \sum_\sigma (2\sigma+1) d^\sigma_{\lambda_{(\bar{1})}-\lambda_{(\bar{2})}\,\lambda_{(1)}-\lambda_{(2)}}(\Theta) S'[\lambda_{(\bar{1})}, \lambda_{(\bar{2})} \,|\, s, \sigma \,|\, \lambda_{(1)}, \lambda_{(2)}], \quad \Theta \in (0, \pi). \tag{62}$$

In Eqs. (56) and (57) of Sec. IV we pointed out that, apart from a frame-independent phase, in the special c.m. frame (23) these c.m. helicity amplitudes of types II and III coincided with the helicity amplitudes of Jacob and Wick.[5] Moreover, we suggested that the type-I amplitudes of Eq. (26) be identified with the invariant amplitudes of Feldman and Matthews.[4]

In this paper we have made no reference to the analyticity, crossing, or unitarity properties of c.m. helicity amplitudes. We should however like to point out that because such amplitudes coincide with Jacob-Wick–type amplitudes in the special c.m. frame defined by Eq. (23), any analyticity, crossing, or unitarity formulas will closely resemble those which have already been derived for helicity amplitudes.[8,7,10,11]

ACKNOWLEDGMENT

We should like to thank Professor D. H. Martin for the use of facilities in the Physics department of Queen Mary College.

APPENDIX A: TWO-PARTICLE STATES

1. States of type I

One may use the formalism which we have developed in Appendix A of Ref. 1 to define various two-particle states in terms of standard rest states $|\vec{0}, \vec{0} : \lambda_{(1)}, \lambda_{(2)}\rangle$,

$$|\vec{p}_{(1)}, \vec{p}_{(2)} : \lambda_{(1)}, \lambda_{(2)}) = \hat{B}_{(1)}(p_{(1)}) \hat{B}_{(2)}(p_{(2)}) |\vec{0}, \vec{0} : \lambda_{(1)}, \lambda_{(2)}\rangle . \tag{A1}$$

The parameters $\lambda_{(1)}$ and $\lambda_{(2)}$ on the left-hand side of this equation are eigenvalues of some spin-component observables $\hat{\Sigma}_{(1)}$ and $\hat{\Sigma}_{(2)}$. We list the states together with the associated operators $\hat{\Sigma}_{(1)}$ and $\hat{B}(p_{(1)})$ in Table I (see Ref. 12). Since the properties of these states are discussed in Appendix A of Ref. 1 we shall not review them here.

For two-particle systems it proves convenient to construct states of mixed type. Before doing this we define the following momentum-dependent

TABLE I. Two-particle states.[a]

State	$\lvert\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)})$	$\hat{B}(p_{(1)})$		$\hat{\Sigma}_{(1)}$	Reference
Standard helicity	$\lvert\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}\rangle$	$\hat{L}(p_{(1)})$	[A11]	$\hat{S}_{(1)3}$	12
	$\lvert\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}\}_+$	$\hat{H}_+(p_{(1)})$	[A10]	$\hat{S}_{(1)}$	5, 6
	$\lvert\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}\}$	$\hat{H}(p_{(1)})$	[A9]	$\hat{S}_{(1)}$	6
c.m. helicity	$\lvert p_{(1)},p_{(2)}:\lambda_{(1)},\lambda_{(2)}]_+$	$\hat{H}_+(p_{(1)};p_{(2)})$	(A2)	$\hat{S}_{(1)}{}^{p[2]}$	2

[a] Numbers in square brackets refer to equations in Ref. 1.

Lorentz transformations for timelike momenta p, $p_{(k)}$, and $q_{(k)}$ (Ref. 3):

$$\hat{H}_+(q;p)=\hat{L}(p)\hat{H}_+(Q),\quad \Delta(p,q)\neq 0,\quad \theta(Q)\neq\pi \tag{A2}$$

$$\hat{H}_-(q;p)=\hat{L}(p)\hat{H}_-(Q),\quad \Delta(p,q)\neq 0,\quad \theta(Q)\neq 0 \tag{A3}$$

$$\hat{H}_+(q;p,f)=\hat{H}_+(q;p)\hat{R}_3(R_+^{-1}(Q)F),\quad \Delta(p,q,f)\neq 0,\ \theta(Q)\neq\pi \tag{A4}$$

$$\hat{H}_-(q;p,f)=\hat{H}_-(q;p)\hat{R}_3(R_-^{-1}(Q)F),\quad \Delta(p,q,f)\neq 0,\ \theta(Q)\neq 0 \tag{A5}$$

$$\hat{L}(p;q,f)=\hat{L}(p)\hat{R}_-(Q)\hat{R}_3(R_-^{-1}(Q)F),\quad \Delta(p,q,f)\neq 0,\ \theta(Q)\neq 0 \tag{A6}$$

$$\hat{H}_+(p;p_{(2)},f:p_{(k)};q_{(k)},f_{(k)})=\hat{L}(p;p_{(2)},f)\hat{H}_+(P^\star_{(k)};Q^\star_{(k)},F^\star_{(k)}), \tag{A7}$$

$$\hat{H}_-(p;p_{(2)},f:p_{(k)};q_{(k)},f_{(k)})=\hat{L}(p;p_{(2)},f)\hat{H}_-(P^\star_{(k)};Q^\star_{(k)},F^\star_{(k)}), \tag{A8}$$

where capital letters are used to denote c.m. momenta,

$$P_{(k)}=L^{-1}(p)p_{(k)},\ \text{etc.}, \tag{A9}$$

and starred letters denote the special c.m. momenta

$$P^\star_{(k)}=L^{-1}(p;p_{(2)},f)p_{(k)},\ \text{etc.} \tag{A10}$$

We define mixed two-particle states $\lvert\vec{p}_{(1)},\vec{p}_{(2)};\lambda_{(1)},\lambda_{(2)})_\pm$ in terms of standard rest states $\lvert\vec{0},\vec{0}:\lambda_{(1)},\lambda_{(2)}\rangle$,

$$\lvert\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)})_\pm=\hat{B}_{(1)+}(p_{(1)})\hat{B}_{(2)-}(p_{(2)})(-1)^{\sigma(2)-\lambda(2)}\lvert\vec{0},\vec{0}:\lambda_{(1)},-\lambda_{(2)}\rangle, \tag{A11}$$

and list them together with the corresponding operators $\hat{B}_+(p_{(1)})$ and $\hat{\Sigma}_{(1)}$ in Table II.

The overlap of any two states $\lvert\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)})_\pm$ and $\lvert\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)})^\star$ is given by

$$\begin{aligned}{}_\pm^\star(\vec{p}\,'_{(1)},\vec{p}\,'_{(2)}:\lambda'_{(1)},\lambda'_{(2)}\lvert\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)})_\pm&=4p_{(1)0}p_{(2)0}\delta(\vec{p}\,'_{(1)}-\vec{p}_{(1)})\delta(\vec{p}\,'_{(2)}-\vec{p}_{(2)})\\&\times D^{\sigma(1)}_{\lambda'_{(1)}\lambda_{(1)}}(\hat{B}_+^{\star-1}(p_{(1)})\hat{B}_+(p_{(1)}))D^{\sigma(2)*}_{\lambda'_{(2)}\lambda_{(2)}}(\hat{B}_-^{\star-1}(p_{(2)})\hat{B}_-(p_{(2)})),\end{aligned} \tag{A12}$$

where the functions $D^{\sigma}_{\lambda'\lambda}(\hat{R})$ are irreducible unitary rotation group [SU(2)] representations of rotations $\hat{R}$.

The states transform in the following way under space-time translations $\hat{a}$ and homogeneous Lorentz transformations $\hat{\Lambda}$ of Eqs. (11) and (12):

$$\hat{a}\lvert\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)})_\pm=e^{ip\cdot a}\lvert\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)})_\pm, \tag{A13}$$

$$\hat{\Lambda}\lvert\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)})_\pm=\sum_{\lambda'_{(1)}\lambda'_{(2)}}D^{\sigma(1)}_{\lambda'_{(1)}\lambda_{(1)}}(\hat{B}_+(\hat{\Lambda}:p_{(1)}))D^{\sigma(2)*}_{\lambda'_{(2)}\lambda_{(2)}}(\hat{B}_-(\hat{\Lambda}:p_{(2)}))\lvert\vec{p}^{\,\dagger}_{(1)},\vec{p}^{\,\dagger}_{(2)}:\lambda'_{(1)},\lambda'_{(2)})_\pm, \tag{A14}$$

TABLE II. Mixed two-particle states.[a]

State	$\lvert\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)})_\pm$	$\hat{B}_+(p_{(1)})$		$\hat{\Sigma}_{(1)}$	Reference
Helicity	$\lvert\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}\}_\pm$	$\hat{H}_+(p_{(1)})$	[A10]	$\hat{S}_{(1)}$	5
c.m. helicity	$\lvert\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}]_\pm$	$\hat{H}_+(p_{(1)};p_{[2]})$	(A2)	$\hat{S}_{(1)}{}^{p[2]}$	2
	$\lvert\vec{p}_{(1)},\vec{p}_{(2)}:f:\lambda_{(1)},\lambda_{(2)}]_\pm$	$\hat{H}_+(p_{(1)};p_{[2]},f)$	(A4)	$\hat{S}_{(1)}{}^{p[2]}$	1
	$\lvert\vec{p}_{(1)},\vec{p}_{(2)}:f:\lambda_{(1)},\lambda_{(2)}]_{qf\pm}$	$\hat{H}(p_{[2]};q,f:p_{(1)};p_{[2]},f)$	(A7)	$S_{(1)}{}^{p[2]}$	1

[a] Numbers in square brackets refer to equations in Ref. 1.

where the transformed momenta are given by

$$p_{(1)}^{\dagger} = \Lambda p_{(1)}, \text{ etc.} \tag{A15}$$

and the Wigner rotations $\hat{B}_+(\hat{\Lambda}:p_{(1)})$ and $\hat{B}_-(\hat{\Lambda}:p_{(1)})$ are defined by

$$\hat{B}_+(\hat{\Lambda}:p_{(1)}) = \hat{B}_+^{-1}(\Lambda p_{(1)})\hat{\Lambda}\hat{B}(p_{(1)}), \tag{A16}$$

$$\hat{B}_-(\hat{\Lambda}:p_{(1)}) = \hat{B}_-^{-1}(\Lambda p_{(1)})\hat{\Lambda}\hat{B}(p_{(1)}). \tag{A17}$$

2. States of types II and III

States of type II are parametrized by the square of the total four-momentum s, the total three-momentum $\vec{p}$, and angles ϕ and θ which are associated with the c.m. momentum $P_{(1)}$ defined by Eq. (A9):

$$R(\phi, \theta, -\phi) = R_+(P_{(1)}). \tag{A18}$$

They are simply related to the corresponding states of type I,[3]

$$|s:\vec{p}; \phi, \theta: \lambda_{(1)}, \lambda_{(2)})_\pm = s^{-1/2}[\tfrac{1}{2}\pi\Delta(s,1,2)]^{1/2} \times |\vec{p}_{(1)}, \vec{p}_{(2)}: \lambda_{(1)}, \lambda_{(2)})_\pm, \tag{A19}$$

and have the simple normalization

$$_\pm(s':\vec{p}'; \phi', \theta': \lambda'_{(1)}, \lambda'_{(2)}|s:\vec{p}; \phi, \theta: \lambda_{(1)}, \lambda_{(2)})_\pm = \delta(s'-s)2p_0\delta(p'-p)4\pi\delta(\phi'-\phi)\delta(\cos\theta'-\cos\theta)\delta_{\lambda'_{(1)}\lambda_{(1)}}\delta_{\lambda'_{(2)}\lambda_{(2)}}. \tag{A20}$$

They transform in the following way under space-time translations $\hat{a}$ and homogeneous Lorentz transformations $\hat{\Lambda}$:

$$\hat{a}|s:\vec{p}; \phi, \theta: \lambda_{(1)}, \lambda_{(2)})_\pm = e^{ip\cdot a}|s:\vec{p}; \phi, \theta: \lambda_{(1)}, \lambda_{(2)})_\pm, \tag{A21}$$

$$\hat{\Lambda}|s:\vec{p}; \phi, \theta: \lambda_{(1)}, \lambda_{(2)})_\pm = \sum_{\lambda'_{(1)}\lambda'_{(2)}} D^{\sigma_{(1)}}_{\lambda'_{(1)}\lambda_{(1)}}(\hat{B}_+(\hat{\Lambda}:p_{(1)}))D^{\sigma_{(2)}*}_{\lambda'_{(2)}\lambda_{(2)}}(\hat{B}_-(\hat{\Lambda}:p_{(2)}))|s:\vec{p}^{\dagger}; \phi^{\dagger}, \theta^{\dagger}: \lambda'_{(1)}, \lambda'_{(2)})_\pm, \tag{A22}$$

where angles $\phi^\dagger$ and $\theta^\dagger$ are defined in terms of the transformed momenta of Eq. (A15) by Eq. (A18).

The special c.m. helicity states of type II depend on a phase angle ψ,

$$|s:\vec{p}; \phi, \theta: \psi: \lambda_{(1)}, \lambda_{(2)}]_{p_{(2)}f\pm} = s^{-1/2}[\tfrac{1}{2}\pi\Delta(s,1,2)]^{1/2}|\vec{p}_{(1)}, \vec{p}_{(2)}: f: \lambda_{(1)}, \lambda_{(2)}]_{p_{(2)}f\pm}, \tag{A23}$$

where

$$R(\phi, \theta, \psi-\phi) = L^{-1}(p)L(p; p_{(2)}, f). \tag{A24}$$

They satisfy the normalization equation (A20) and space-time translation equation (A21). They transform in the following way under homogeneous Lorentz transformations:

$$\hat{\Lambda}|s:\vec{p}; \phi, \theta: \psi: \lambda_{(1)}, \lambda_{(2)}]_{p_{(2)}f\pm} = [l(\hat{\Lambda}:p, p_{(2)}, f)]^{2(\sigma_{(1)}+\sigma_{(2)})}|s:\vec{p}^{\dagger}; \phi^\dagger, \theta^\dagger: \psi^\dagger: \lambda_{(1)}, \lambda_{(2)}]_{p_{(2)}f} \tag{A25}$$

where the sign function $l(\hat{\Lambda}:p, p_{(2)}, f)$ is defined by

$$l(\hat{\Lambda}:p, p_{(2)}, f)^{2f_3} = \hat{L}(\hat{\Lambda}:p, p_{(2)}, f) = \hat{L}^{-1}(p^\dagger, p_{(2)}^\dagger, f^\dagger)\hat{\Lambda}\hat{L}(p; p_{(2)}, f), \tag{A26}$$

and the angles $\phi^\dagger$, $\theta^\dagger$, and $\psi^\dagger$ are given in terms of momenta $p^\dagger$, $p_{(2)}^\dagger$, and $f^\dagger$ of Eq. (A15) by Eq. (A24).

We now define states of type III which are parametrized by the total effective spin or c.m. total angular momentum parameter σ and the third component of spin λ,

$$|s, \sigma: \vec{p}, \lambda: \lambda_{(1)}, \lambda_{(2)}] = \frac{(2\sigma+1)^{1/2}}{4\pi}\int_0^\pi\int_0^{2\pi} D^{\sigma*}_{\lambda\,\lambda_{(1)}-\lambda_{(2)}}(\hat{R}(\phi, \theta, -\phi))\,|s:\vec{p}; \phi, \theta: \lambda_{(1)}, \lambda_{(2)}]_\pm \sin\theta\, d\theta\, d\phi. \tag{A27}$$

These states transform like single-particle states under space-time translations $\hat{a}$ and homogeneous Lorentz transformations $\hat{\Lambda}$,

$$\hat{a}|s, \sigma: \vec{p}, \lambda: \lambda_{(1)}, \lambda_{(2)}] = e^{ip\cdot a}|s, \sigma: \vec{p}, \lambda: \lambda_{(1)}, \lambda_{(2)}], \tag{A28}$$

$$\hat{\Lambda}|s, \sigma: \vec{p}, \lambda: \lambda_{(1)}, \lambda_{(2)}] = \sum_{\lambda'} D^{\sigma}_{\lambda'\lambda}(\hat{L}(\hat{\Lambda}:p))|s, \sigma: \vec{p}, \lambda': \lambda_{(1)}, \lambda_{(2)}], \tag{A29}$$

and have the simple normalization

$$[s',\sigma':\vec{p}',\lambda':\lambda'_{(1)},\lambda'_{(2)}|s,\sigma:\vec{p},\lambda:\lambda_{(1)},\lambda_{(2)}] = \delta(s'-s)\delta_{\sigma'\sigma}2p_0\delta(\vec{p}'-\vec{p})\delta_{\lambda'\lambda}\delta_{\lambda'_{(1)}\lambda_{(1)}}\delta_{\lambda'_{(2)}\lambda_{(2)}}. \tag{A30}$$

APPENDIX B: SOME TWO-PARTICLE SCATTERING AMPLITUDES

We define scattering amplitudes to be matrix elements of a scattering operator $\hat{S}$ between two-particle states listed in Tables I and II,

$$S(\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)})=(\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\hat{S}|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}). \tag{B1}$$

In order to follow our notation one should refer to Table III in which we list various amplitudes together with the corresponding two-particle states.

In the rest of this appendix we shall only be concerned with amplitudes defined in terms of mixed two-particle states. The properties of the other amplitudes have been discussed in Appendix B of Ref. 1.

Each amplitude satisfies a space-time translation contraint equation of the form

$$S^{\pm}(\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)})=e^{i(\bar{p}-p)\cdot a}S^{\pm}(\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}) \tag{B2}$$

where

$$p=p_{(1)}+p_{(2)},\quad \bar{p}=p_{(\bar{1})}+p_{(\bar{2})}. \tag{B3}$$

We are thus able to factor out a δ function of energy-momentum conservation and define a set of reduced amplitudes $S'^{\pm}(\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)})$,

$$S^{\pm}(\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)})=\delta^4(\bar{p}-p)S'^{\pm}(\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}). \tag{B4}$$

They satisfy homogeneous Lorentz transformation constraint equations of the form

$$\begin{aligned}S'^{\pm}(\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)})=\sum_{\lambda'_{(\bar{1})}\lambda'_{(\bar{2})}\lambda'_{(1)}\lambda'_{(2)}}&D^{\sigma(\bar{1})*}_{\lambda'_{(\bar{1})}\lambda_{(\bar{1})}}(\hat{B}_+(\hat{\Lambda}:p_{(\bar{1})}))D^{\sigma(\bar{2})}_{\lambda'_{(\bar{2})}\lambda_{(\bar{2})}}(\hat{B}_-(\hat{\Lambda}:p_{(\bar{2})}))D^{\sigma(1)}_{\lambda'_{(1)}\lambda_{(1)}}(\hat{B}_+(\hat{\Lambda}:p_{(1)}))\\&\times D^{\sigma(2)*}_{\lambda'_{(2)}\lambda_{(2)}}(\hat{B}_-(\hat{\Lambda}:p_{(2)}))S'^{\pm}(\vec{p}^{+}_{(\bar{1})},\vec{p}^{+}_{(\bar{2})}:\lambda'_{(\bar{1})},\lambda'_{(\bar{2})}|\vec{p}^{+}_{(1)},\vec{p}^{+}_{(2)}:\lambda'_{(1)},\lambda'_{(2)}),\end{aligned} \tag{B5}$$

where the transformed momenta and generalized Wigner rotations are defined by Eqs. (A15), (A16), and (A17).

Since each set of scattering amplitudes is complete we may use Eq. (A12) connecting multiparticle states

TABLE III. Two-particle scattering amplitudes.

Amplitude	$\|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)})$	$S(\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}\|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)})$
Standard helicity	$\|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}\rangle$	$S\langle\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}\|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}\rangle$
	$\|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}\}_+$	$S^{+}\{\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}\|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}\}$
	$\|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}\}$	$S\{\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}\|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}\}$
	$\|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}\}_\pm$	$S^{\pm}\{\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}\|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}\}$
c.m. helicity	$\|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}]_+$	$S^{+}[\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}\|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}]$
	$\|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}]_\pm$	$S^{\pm}[\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}\|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}]$
	$\|\vec{p}_{(1)},\vec{p}_{(2)}:f:\lambda_{(1)},\lambda_{(2)}]_\pm$	$S^{\pm}[\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}\|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}]$
	$\|\vec{p}_{(1)},\vec{p}_{(2)}:f:\lambda_{(1)},\lambda_{(2)}]_{\alpha f\pm}$	$S^{\alpha f\pm}[\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}\|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}]$

to relate the amplitudes to each other:

$$S'^{\star\pm}(\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}) = \sum_{\lambda'_{(\bar{1})}\lambda'_{(\bar{2})}\lambda'_{(1)}\lambda'_{(2)}} D^{\sigma(\bar{1})*}_{\lambda'_{(\bar{1})}\lambda_{(\bar{1})}}(\hat{B}^{\star-1}_{+}(p_{(\bar{1})})\hat{B}_{+}(p_{(\bar{1})}))D^{\sigma(\bar{2})}_{\lambda'_{(\bar{2})}\lambda_{(\bar{2})}}(\hat{B}^{\star-1}_{-}(p_{(\bar{2})})\hat{B}_{-}(p_{(\bar{2})}))$$
$$\times D^{\sigma(1)}_{\lambda'_{(1)}\lambda_{(1)}}(\hat{B}^{\star-1}_{+}(p_{(1)})\hat{B}_{+}(p_{(1)}))D^{\sigma(2)*}_{\lambda'_{(2)}\lambda_{(2)}}(\hat{B}^{\star-1}_{-}(p_{(2)})\hat{B}_{-}(p_{(2)}))$$
$$\times S'^{\pm}(\vec{p}_{(\bar{1})},\vec{p}_{(\bar{2})}:\lambda_{(\bar{1})},\lambda_{(\bar{2})}|\vec{p}_{(1)},\vec{p}_{(2)}:\lambda_{(1)},\lambda_{(2)}). \quad \text{(B6)}$$

APPENDIX C: TWO-PARTICLE KINEMATICS

In Appendix C of Ref. 1 we define some kinematical Δ and Ω functions. For a timelike momentum $p_{(1)}$ we identify $\Delta(p_{(1)})$ with the mass

$$\Delta(p_{(1)}) = [p_{(1)}{}^2]^{1/2} = m_{(1)}. \quad \text{(C1)}$$

For two timelike momenta $p_{(1)}$ and $p_{(2)}$ we define the parameter s by

$$s = (p_{(1)} + p_{(2)})^2 \quad \text{(C2)}$$

and relate the Δ function $\Delta(p_{(1)}, p_{(2)})$ to the threshold-pseudothreshold function $\Delta(s{:}1, 2)$,

$$\Delta(p_{(1)}, p_{(2)}) = \tfrac{1}{2}\Delta(s;1,2) = \tfrac{1}{2}\{[s-(m_{(1)}-m_{(2)})^2] \times [s-(m_{(1)}+m_{(2)})^2]\}^{1/2}. \quad \text{(C3)}$$

Moreover, when momenta $p_{(\bar{1})}$ and $p_{(\bar{2})}$ are timelike and satisfy the equation

$$p_{(1)} + p_{(2)} = p_{(\bar{1})} + p_{(\bar{2})}, \quad \text{(C4)}$$

we may define Mandelstam variables t and u,

$$t = (p_{(1)} - p_{(\bar{1})})^2, \quad u = (p_{(1)} - p_{(\bar{2})})^2, \quad \text{(C5)}$$

and relate the Δ function $\Delta(p_{(1)}, p_{(2)}, p_{(\bar{1})})$ to the Kibble boundary function $\phi(s, t, u)$,

$$\Delta(p_{(1)}, p_{(2)}, p_{(\bar{1})}) = \tfrac{1}{4}\phi(s,t,u)$$
$$= \tfrac{1}{4}[stu - s(m_{(1)}{}^2 - m_{(\bar{1})}{}^2)(m_{(2)}{}^2 - m_{(\bar{2})}{}^2) - t(m_{(1)}{}^2 - m_{(2)}{}^2)(m_{(\bar{1})}{}^2 - m_{(\bar{2})}{}^2)$$
$$- (m_{(1)}{}^2 m_{(\bar{2})}{}^2 - m_{(2)}{}^2 m_{(\bar{1})}{}^2)(m_{(1)}{}^2 + m_{(\bar{2})}{}^2 - m_{(2)}{}^2 - m_{(\bar{1})}{}^2)]^{1/2}. \quad \text{(C6)}$$

Moreover, the Ω function may be expressed in the form

$$\Omega(p; p_{(2)}, p_{(\bar{2})}) = \tfrac{1}{4}[s(t-u) + (m_{(1)}{}^2 - m_{(2)}{}^2)(m_{(\bar{1})}{}^2 - m_{(\bar{2})}{}^2)]. \quad \text{(C7)}$$

In Sec. III we introduced a rotation of the form

$$R(\Phi, \Theta, \Psi - \Phi) = L^{-}(p; p_{(2)}, p_{(\bar{1})})L(p; p_{(\bar{2})}, p_{(2)}). \quad \text{(C8)}$$

It follows from this equation that angles Φ, Θ, and Ψ are given by

$$R(\Phi, \Theta, -\Phi) = R_{-}(L^{-1}(p; p_{(2)}, p_{(\bar{1})})p_{(\bar{2})}), \quad \text{(C9)}$$

$$R_3(\Phi - \Psi) = R_3(L^{-1}(p; p_{(\bar{2})}, p_{(2)})p_{(2)}). \quad \text{(C10)}$$

We now use Eqs. (C17) and (C18) of Ref. 2 to show that angles Φ and Ψ are zero and angle Θ is given by

$$\cos\Theta = \frac{-\Omega(p; p_{(2)}, p_{(\bar{2})})}{\Delta(p, p_{(2)})\Delta(p, p_{(\bar{2})})}, \qquad \sin\Theta = \frac{\Delta(p)\Delta(p, p_{(2)}, p_{(\bar{2})})}{\Delta(p, p_{(2)})\Delta(p, p_{(\bar{2})})}. \quad \text{(C11)}$$

With the aid of Eqs. (C1), (C3), (C6), and (C7) these formulas may be rewritten in the conventional form

$$\cos\Theta = \frac{s(t-u) + (m_{(1)}{}^2 - m_{(2)}{}^2)(m_{(\bar{1})}{}^2 - m_{(\bar{2})}{}^2)}{\Delta(s;1,2)\Delta(s;\bar{1},\bar{2})}, \quad \text{(C12)}$$

$$\sin\Theta = \frac{2s^{1/2}\phi(s,t,u)}{\Delta(s;1,2)\Delta(s;\bar{1},\bar{2})}. \quad \text{(C13)}$$

[1]G. L. Tindle, Phys. Rev. D 10, 3480 (1974).

[2]G. L. Tindle, Phys. Rev. D 3, 1468 (1971).

[3]Kinematical Δ and Ω functions are defined in Appendix C.

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