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s- and t-channel unitarity equations in the coherent-state representation*

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Using the coherent-state representation we write down a set of relations that must be satisfied by scattering amplitudes having full s-channel and elastic t-channel unitarity. While general solutions to this set of equations have not been found, we discuss several models that illustrate certain aspects of the problem. In particular we show that the "Born term" in the t-channel equation must itself satisfy full s-channel unitarity. Some properties of functional integration are discussed.

I. INTRODUCTION

One of the most elusive aspects of the theory of multiparticle production at high energies is the implementation of basic constraints that will limit the arbitrariness in the approximations needed to describe the dynamics of strongly interacting systems. It was natural to think that the unitarity condition should drastically reduce the number of acceptable models; however, the work of several authors in the past years[1-4] has shown that models satisfying s-channel unitarity can yield any asymptotic behavior and particle distributions not in violation of the Froissart and related bounds. It has thus become abundantly clear that some additional constraints must be introduced in order to further restrict the scattering amplitude. Since it is well known that these models violate t-channel unitarity, it seems natural to try to build models in which some degree of t-channel unitarity is incorporated. There are a number of qualitative features of t-channel unitarity which are in fact lacking in the previous models. For instance, without t-channel unitarity, triple-Pomeron couplings must vanish at $t=0$ and factorization can be only approximately satisfied.

The main problem that must be faced when considering a program of this kind is the need to keep the formalism as general as possible so multiparticle unitarity can be imposed at any stage of the calculations. In other words, one should work with the S operator rather than with its matrix elements between definite states. This implies finding a convenient formalism for the scattering operator that allows for the simultaneous imposition of the requirements of unitarity in one channel and the solution of an operator equation in the other channel. We will turn for this to the coherent-state representation discussed in Ref. 3. As we will show, it is possible to express all operators as functional integrals over a complete set of unitary operators with appropriate weight functions. It is then possible to insure unitarity by imposing constraints on these weight functions. Unfortunately, these constraints are highly nonlinear and we have not yet found a complete solution. We will, however, illustrate in a few examples several aspects of these constraints and will derive a few general features of the complete solution.

We begin in Sec. II by introducing the operator formalism and the coherent-state representation and discuss the kinematic approximations. Chief among these is the assumption that only protons and pions exist and that the protons carry essentially all the energy even when the total energy is small. This is in the same spirit as in the earlier eikonal models. We then introduce the unitarity relations and derive the conditions on the weight functions. In Sec. III, some examples are discussed which illustrate certain aspects of the constraints and in Sec. IV the full unitarity constraints are discussed. Section V contains further discussion with examples and conclusions and finally, the Appendix is devoted to a discussion of functional integration techniques with particular emphasis on how one defines the measure of such integrals.

II. FORMALISM

The problem we wish to consider is the construction of a scattering amplitude satisfying full multiparticle s-channel unitarity and, in addition, some degree of elastic t-channel unitarity. In the spirit of eikonal-type models we will allow for two types of particles, namely distinguishable "protons" and indistinguishable "pions," neglecting for the most part all internal quantum numbers.[5] All s-channel states will be assumed to contain two protons and any number of pions with the protons carrying essentially all the energy. This might not be a good approximation when used in the t-channel unitarity equations, because we need then the amplitude at low energies. However, we will continue using it since it is the simplest form for which we could find solutions of the equations.

Let us begin by defining a scattering operator T which vanishes except between states of the aforementioned type. By evaluating the matrix element of this operator between two-proton states, we obtain an operator in the pion space $T(s,\vec{b})$ where $(s,\vec{b})$ are defined by the two protons. In view of our kinematic assumptions, s is approximately the total energy squared. In the impact-parameter representation, this $T(s,\vec{b})$ is related to the S matrix by

$$T(s,\vec{b}) = 2is[1 - S(s,\vec{b})], \tag{1}$$

where

$$s = (p_1 + p_2)^2 = m^2 e^Y \tag{2}$$

and b is the impact parameter of the two protons. Again, with our approximation, b is conjugate to $t = (p_1 - p_1')^2$, which is approximately the total momentum transfer squared between the protons. It should be pointed out that we do not make these approximations because we feel that they accurately describe the data, but because they greatly simplify the calculations. We also define

$$T(s,t) = \int_0^\infty b\,db\,J_0(b\Delta)\,T(s,b), \tag{3}$$

where

$$t = (p_1' - p_1)^2 = (p_2' - p_2)^2 = -\Delta^2. \tag{4}$$

With these assumptions, the unitary relation for the S matrix is diagonal in impact-parameter space,

$$S(s,\vec{b})S^\dagger(s,\vec{b}) = 1 \tag{5}$$

and hence,

$$\frac{1}{2i}[T(s,\vec{b}) - T^\dagger(s,\vec{b})] = \frac{1}{4s}\,T(s,\vec{b})\,T^\dagger(s,\vec{b}). \tag{6}$$

s-channel unitarity can be guaranteed by defining a Hermitian operator $\chi(s,\vec{b})$ such that

$$S(s,\vec{b}) = e^{i\chi(s,\vec{b})}. \tag{7}$$

Turning now to t-channel unitarity, we consider at fixed t, rather than fixed b, a sum of operator diagrams (Fig. 1) in the s-channel physical region. Using the usual Feynman rules, we obtain

$$T(s,t,m^2) = V(s,t,m^2) - i\int \frac{d^4k}{(2\pi)^4}\,\frac{V(s_1,t,m^2,k^2,m^2,(k-q)^2)}{(k^2 - m^2 + i\epsilon)[(k-q)^2 - m^2 + i\epsilon]}\,T(s_2,t,m^2,k^2,m^2,(k-q)^2), \tag{8}$$

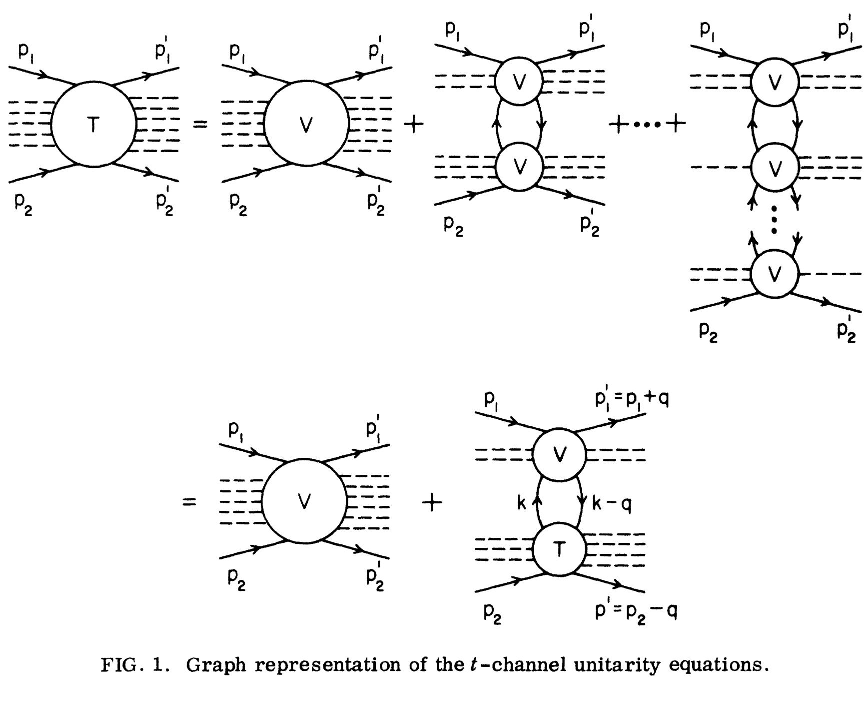


FIG. 1. Graph representation of the t-channel unitarity equations.

where we are ignoring the mass difference between the protons and pions and we do not allow for direct coupling between the external pions and the internal protons. Let us compute the elastic matrix element of this equation. If we assume, following Gribov,[6] that only small values of $k^2<0$ contribute to the leading behavior, the energy arguments of V and T in the integral are defined over nonoverlapping regions of rapidity space. Thus we find

$$\langle 0|T(s,t)|0\rangle = \langle 0|V(s,t)|0\rangle - i\int \frac{d^4k}{(2\pi)^4} \frac{\langle 0|V(\cdots)|0\rangle\langle 0|T(\cdots)|0\rangle}{(k^2-m^2+i\epsilon)[(k-q)^2-m^2+i\epsilon]}, \tag{9}$$

where $|0\rangle$ denotes a state of 2 protons and no pions. This is just the Bethe-Salpeter equation which, in the t-channel physical region, guarantees elastic t-channel unitarity provided $\langle 0|V|0\rangle$ is real below the first inelastic threshold.

Returning to Eq. (8), in order to obtain a manageable equation, we assume that the off-shell dependence of $\langle 0|V|0\rangle$ and $\langle 0|T|0\rangle$ can be factorized[6] (or simply neglected)

$$T(s,t,m^2,k^2,m^2,(k-q)^2) = T(s,t,m^2)f_T(m^2,k^2,m^2,(k-q)^2).$$

Using the fact that the dominant contribution will be from the region $s_1s_2=s$, i.e., no rapidity gap across the propagators, Eq. (8) becomes approximately

$$T_{0Y}(s,t) = V_{0Y}(s,t) - ig(t)\int_0^Y dy\, V_{0y}(e^y,t)\, T_{yY}(e^{Y-y},t), \tag{10}$$

where

$$g(t) \approx \frac{1}{(2\pi)^4}\int \frac{d^2\kappa f_V(m^2,\kappa^2,m^2,(\kappa-Q)^2) f_T(m^2,\kappa^2,m^2,(\kappa-Q)^2)}{(\kappa^2-m^2+i\epsilon)[(\kappa-Q)^2-m^2+i\epsilon]} \tag{11}$$

is a real function of t for $t<4m^2$. The operators are not translationally invariant in rapidity space; the subscripts on the operators denote the range of rapidities over which they are defined. For instance, $V_{0y}(e^y,t)$ creates and annihilates any number of pions in the rapidity range $0\leq y_i\leq y$. The dependence on the t-channel elastic thresholds is contained in the factor $g(t)$.

We want first to find the restrictions on $V(s,t)$ that will guarantee that T satisfies full s-channel unitarity, and second to determine whether Eq. (10) places any restrictions on the general form of T.

In order to proceed, we need to express the operators in terms of some set of basic states. We will choose, as a complete set of states, the coherent states discussed in Ref. 3. Briefly, in this context, a coherent state is a state of two protons and an arbitrary number of pions defined as an eigenstate of the pion annihilation operator.

$$a(q_i)|\Pi(q_i)\rangle = \Pi(q_i)|\Pi(q_i)\rangle, \tag{12}$$

$$q_i = (y_i, \vec{q}_{\perp i}). \tag{13}$$

The proton momenta will not be explicitly displayed. Introducing the displacement operator

$$D(\Pi) = \exp\left\{\int dq[\Pi(q)a^\dagger(q) - \Pi^*(q)a(q)]\right\}, \tag{14}$$

$$D^\dagger(\Pi) = D^{-1}(\Pi) = D(-\Pi), \tag{15}$$

where $dq \equiv dy\, d^2q_\perp$, we find

$$|\Pi\rangle = D(\Pi)|0\rangle, \tag{16}$$

$$a(q)|0\rangle = 0. \tag{17}$$

These states are complete but not orthogonal:

$$\langle \Pi'|\Pi\rangle = \exp\left\{-\tfrac{1}{2}\int dq[|\Pi'(q)|^2 + |\Pi(q)|^2 - 2\Pi'^*(q)\Pi(q)]\right\}. \tag{18}$$

The average number of pions in such a state is

$$\bar{n} = \left\langle \Pi\left|\int dq\, a^\dagger(q)a(q)\right|\Pi\right\rangle = \int dq|\Pi(q)|^2 \tag{19}$$

and the probability density of finding n pions with momenta q_i is

$$|\langle 0|a(q_1)\cdots a(q_n)|\Pi\rangle|^2 = |\Pi(q_1)|^2|\Pi(q_2)|^2\cdots|\Pi(q_n)|^2 e^{-\bar{n}}; \tag{20}$$

thus, the probability of finding n pions with any momenta is

$$P_n = \frac{\bar{n}^n}{n!}e^{-\bar{n}}. \tag{21}$$

In general, the eigenfunctions $\Pi(q)$ can be complex, but as was shown in Ref. 4, complex eigenfunctions lead to ghosts and thus we will assume $\Pi(q)$ to be real.

Because the states are complete (in fact, overcomplete) we can express any operator in terms of these

$$T_{0Y}(s, t)=\int \delta\Pi_1\, \delta\Pi_2 |\Pi_1\rangle\langle\Pi_2|\, \tau_{0Y}(\Pi_1, \Pi_2, t)\,, \tag{22}$$

where $\delta\Pi_i$ denote functional integrals over all possible functions $\Pi_i(q)$. Equation (22) is the most general form of the expansion. For a wide class of operators, however, a simpler expression results from the completeness of the operators $D(\Pi)$, namely

$$T_{0Y}(s, t)=\int \delta\Pi\, D_{0Y}(\Pi)\, \tau_{0Y}(\Pi, t) \tag{23}$$

with the inversion

$$\tau_{0Y}(\Pi, t)=\mathrm{Tr}[D_{0Y}(-\Pi)\, T_{0Y}(s, t)]$$
$$=\int \delta\Pi'\langle\Pi'|D_{0Y}(-\Pi)\, T_{0Y}(s, t)|\Pi'\rangle\,. \tag{24}$$

The condition under which these relations hold is that the Hilbert-Schmidt norm of the operator

$$\mathrm{Tr}[TT^\dagger]=\int \delta\Pi\langle\Pi|TT^\dagger|\Pi\rangle$$
$$=\sum_n \langle n|TT^\dagger|n\rangle \tag{25}$$

be finite. It seems reasonable to suppose that this is true for T but it is clear that it is not true for S nor is it necessarily true for χ.

We will, for the remainder of this paper, concentrate on the representation Eq. (23).

Let us now put Eq. (23) and a similar representation for $V(s, t)$ into Eq. (10):

$$\int \delta\Pi\, D_{0Y}(\Pi)\, \tau_{0Y}(\Pi, t)=\int \delta\Pi\, D_{0Y}(\Pi)\, \nu_{0Y}(\Pi, t)-ig(t)\int_0^Y dy \int\delta\Pi_1 \int\delta\Pi_2\, D_{0y}(\Pi_1)\, D_{yY}(\Pi_2)\, \nu_{0y}(\Pi_1)\, \tau_{yY}(\Pi_2)\,. \tag{26}$$

Because Π_1 and Π_2 are defined over nonoverlapping regions of y, we have

$$\int\delta\Pi_1\int\delta\Pi_2=\int\delta\Pi \tag{27}$$

and

$$D_{0y}(\Pi_1)\, D_{yY}(\Pi_2)=D_{0Y}(\Pi)\,, \tag{28}$$

where

$$\Pi(q')=\begin{cases}\Pi_1(q), & 0\le y'\le y\\ \Pi_2(q), & y\le y'\le Y.\end{cases} \tag{29}$$

It is apparent that without additional restrictions, the set of functions $\Pi(q')$ will include arbitrarily discontinuous as well as continuous functions. One way to avoid such functions is to impose some additional boundary conditions on $\Pi(q)$. Since we know from the kinematics that all functions $\Pi(q)$ must vanish at $y=0, Y$, it might be reasonable to require that $\Pi(q)$ be continuous at any boundary. Thus in a product of the form $\nu_{0y}(\Pi_1)\,\tau_{yY}(\Pi_2)$, we would require $\Pi_1(0)=0$, $\Pi_1(y)=\Pi_2(y)$, and $\Pi_2(Y)=0$. This actually is reasonable from a dynamical point of view since from Refs. 3 and 4 we know that the presence of short-range correlations suppresses all contributions to the scattering amplitude for which the derivative of the pion field is not small or at least finite.

The set of functions $\Pi(q)$ is thus all continuous functions with a piecewise continuous derivative. One could obviously require continuous derivatives as well if it became necessary. For now, however, we will just require continuous $\Pi(q)$. Equation (26) becomes

$$\int \delta\Pi\, D_{0Y}(\Pi)\, \tau_{0Y}(\Pi, t)=\int \delta\Pi\, D_{0Y}(\Pi)\, \nu_{0Y}(\Pi, t)-ig(t)\int \delta\Pi\, D_{0Y}(\Pi)\int_0^Y dy\, \nu_{0y}(\Pi, t)\, \tau_{yY}(\Pi, t)\,, \tag{30}$$

where it is to be understood that inside the integral, the functionals $\nu_{0y}(\Pi)$ and $\tau_{yY}(\Pi)$ are to be evaluated for that part of the function $\Pi(y)$ $(0\le y\le Y)$ lying between the limits indicated by the subscripts. For example, if $\Pi(y)=ay$ $(0\le y\le Y)$, then $\nu_{0y}(\Pi, t)$ is evaluated for $\Pi_1(y')=ay'$ $(0\le y'\le y)$, and $\tau_{yY}(\Pi, t)$ is evaluated for $\Pi_2(y')=ay'$ $(y\le y'\le Y)$. The integral over y thus acts as a sum of partitions on the interval $0\le y\le Y$. Obviously, except for the special case that $\Pi(y)$ is constant, the functionals are not translationally invariant in the subscripts, i.e., $\tau_{0y}(\Pi) \neq \tau_{\eta, y+\eta}(\Pi)$.

We now come to an important point. We can only proceed if the measures defining the three functional integrals of Eq. (30) are the same. Since the measure of a functional integral is determined by the functional form of the integrand this is not a trivial point. Some examples will be given later to illustrate this property (see Appendix). Assuming that the measures are the same, we can combine terms to obtain

$$\int \delta\Pi\, D_{0Y}(\Pi) \left[\tau_{0Y}(\Pi, t) - \nu_{0Y}(\Pi, t) + ig(t) \int_0^Y dy\, \nu_{0y}(\Pi, t)\, \tau_{yY}(\Pi, t)\right] = 0\,, \tag{31}$$

which implies, using the completeness of the $D_{0Y}(\Pi)$ or the inversion Eq. (24),

$$\tau_{0Y}(\Pi, t) = \nu_{0Y}(\Pi, t) - ig(t) \int_0^Y dy\, \nu_{0y}(\Pi, t)\, \tau_{yY}(\Pi, t)\,. \tag{32}$$

This equation is supposed to be satisfied for any function $\Pi(y)$. There are, in fact, nontrivial solutions to this equation for at least some forms of $\nu_{0Y}(\Pi)$ as we will show. But first, let us find the restrictions imposed on $\tau_{0Y}(\Pi, t)$ by s-channel unitarity. Equation (6) yields

$$\frac{1}{2i}[\tau_{0Y}(\Pi, \vec{b}) - \tau^*_{0Y}(\Pi, \vec{b})] = \frac{1}{4s} \int \delta\Pi_1\, \tau_{0Y}(\Pi_1, \vec{b})\, \tau^*_{0Y}(\Pi_1 - \Pi, \vec{b})\,. \tag{33}$$

Rather than use this equation directly it will often be more convenient to use a form of T which guarantees unitarity, from Eqs. (1), (3), and (7):

$$T(s, t) = \frac{2s}{i} \int_0^\infty b\, db\, J_0(b\Delta)(e^{i\chi(s, \vec{b})} - 1)\,. \tag{34}$$

In order to cast this in the form of Eq. (23), we write, keeping in mind the restrictions on the norm of χ,

$$\chi(s, \vec{b}) = \int \delta\Pi'\, D(\Pi')\, \chi(\Pi', \vec{b})\,, \tag{35}$$

where

$$\chi = \chi^\dagger \text{ implies } \chi(\Pi', \vec{b}) = \chi^*(-\Pi', \vec{b}) \tag{36}$$

and thus

$$e^{i\chi(s, \vec{b})} - 1 = \sum_{n=1}^{\infty} \frac{i^n}{n!} \int \delta\Pi_1 \cdots \delta\Pi_n\, \chi(\Pi_1, \vec{b}) \cdots \chi(\Pi_n, \vec{b}) \times D(\Pi_1) D(\Pi_2) \cdots D(\Pi_n). \tag{37}$$

Using the reality of the functions $\Pi_i(q)$

$$D(\Pi_1) D(\Pi_2) \cdots D(\Pi_n) = D(\textstyle\sum_{i=1}^n \Pi_i) \tag{38}$$

and Eq. (37) becomes

$$e^{i\chi(s, \vec{b})} - 1 = \int \delta\Pi\, D(\Pi) \sum_{n=1}^{\infty} \frac{i^n}{n!} \int \delta\Pi_1 \cdots \delta\Pi_n\, \chi(\Pi_1, \vec{b}) \cdots \chi(\Pi_n, \vec{b})\, \delta_F(\textstyle\sum_{i=1}^n \Pi_i - \Pi)\,, \tag{39}$$

which involves functional convolutions of $\chi(\Pi_i, \vec{b})$.

Using the representation, (for real Π_i)

$$\delta_F(\textstyle\sum_{i=1}^n \Pi_i - \Pi) = \mathrm{Tr}[D(-\Pi) D(\sum_{i=1}^n \Pi_i)] = \displaystyle\int \delta\Pi' \exp\left[-2i \int (\textstyle\sum_{i=1}^n \Pi_i - \Pi)\, \Pi' dq\right]. \tag{40}$$

The nth-order convolution becomes

$$\int \delta\Pi_1 \cdots \delta\Pi_n\, \chi(\Pi_1, \vec{b}) \cdots \chi(\Pi_n, \vec{b}) \int \delta\eta \exp\left[-2i \int dq (\textstyle\sum \Pi_i - \Pi)\, \eta\right] = \displaystyle\int \delta\eta \exp\left[2i \int dq\, \Pi(q)\, \eta(q)\right] \tilde{\chi}^n(\eta, \vec{b})\,, \tag{41}$$

where

$$\tilde{\chi}(\eta, \vec{b}) = \int \delta\Pi_i\, \chi(\Pi_i, \vec{b}) \exp\left[-2i \int dq\, \Pi_i(q)\, \eta(q)\right] \tag{42}$$

is the functional Fourier transform of $\chi(\Pi_i, \vec{b})$. The restriction Eq. (36) implies

$$\tilde{\chi}(\eta, \vec{b}) = \tilde{\chi}^*(\eta, \vec{b}). \tag{43}$$

Thus,

$$e^{i\chi(s,\vec{b})}-1=\int\delta\Pi\, D_{0Y}(\Pi)\int\delta\eta\exp\left[2i\int dq\,\Pi(q)\eta(q)\right]\times[e^{i\tilde{\chi}(\eta,\vec{b})}-1]. \tag{44}$$

Notice that again we have interchanged the order of sums and functional integrals which can only be done if the measures of the various terms are the same.

Combining Eq. (44) with Eqs. (23) and (34), we can read off the form of $\tau_{0Y}(\Pi,t)$ which guarantees unitarity:

$$\tau_{0Y}(\Pi,t)=\frac{2s}{i}\int_0^\infty b\,db\,J_0(b\Delta)\int\delta\eta\exp\left[2i\int dq\,\Pi(q)\eta(q)\right]\times[e^{i\tilde{\chi}(\eta,\vec{b})}-1] \tag{45a}$$

or, if we define $\tilde{\tau}_{0Y}(\eta,t)$ as the Fourier transform of $\tau_{0Y}(\Pi,t)$,

$$\tilde{\tau}_{0Y}(\eta,t)=\frac{2s}{i}\int_0^\infty b\,db\,J_0(b\Delta)[e^{i\tilde{\chi}_{0Y}(\eta,\vec{b})}-1]. \tag{45b}$$

Finally, let us return again to Eq. (32) using Fourier-transform representations for τ and ν,

$$\int\delta\eta\exp\left(2i\int_{0Y}\Pi\eta\,dq\right)\tilde{\tau}_{0Y}(\eta,t)=\int d\eta\exp\left(2i\int_{0Y}\Pi\eta\,dq\right)\tilde{\nu}_{0Y}(\eta,t)$$
$$-ig(t)\int_0^Y dy\int\delta\eta_2\int\delta\eta_1\exp\left(2i\int_{0y}\Pi\eta_1dq\right)\exp\left(2i\int_{yY}\Pi\eta_2dq\right)\tilde{\nu}_{0y}(\eta_1,t)\,\tilde{\tau}_{yY}(\eta_2,t). \tag{46}$$

Again

$$\int\delta\eta_2\,\delta\eta_1=\int\delta\eta, \tag{47}$$

$$\exp\left(2i\int_{0y}\Pi\eta_1dq\right)\exp\left(2i\int_{yY}\Pi\eta_2dq\right)=\exp\left(2i\int_{0Y}\Pi\eta\,dq\right), \tag{48}$$

where

$$\eta(q)=\begin{cases}\eta_1(q), & 0\le y'\le y\\ \eta_2(q), & y\le y'\le Y\end{cases} \tag{49}$$

and thus

$$\tilde{\tau}_{0Y}(\eta,t)=\tilde{\nu}_{0Y}(\eta,t)-ig(t)\int_0^Y dy\,\tilde{\nu}_{0y}(\eta,t)\,\tilde{\tau}_{yY}(\eta,t). \tag{50}$$

Equations (45) and (50) [or (32)] will be the basis for the remainder of our discussion.

Finally, we will make one further approximation; namely, we will henceforth neglect the transverse-momentum $q_\perp$ dependence in $\Pi(q)$. This is not a serious approximation since we have already eliminated the dependence on $q_\perp$ from the overall energy-momentum conservation. The only place this dependence occurs is in integrals of the form $\int_{0Y}dq\,f_{0Y}[\Pi(q)]$ which will now become $\int_0^Y dy\,f_{0Y}[\Pi(y)]$. Clearly, one cannot learn much about the $q_\perp$ dependence using the approximations of eikonal-type models.

III. TWO EXAMPLES

Before discussing the combined Eqs. (32), (45), and (50), we will study two models which, while they do not satisfy the full set of equations, do illustrate certain features of the problem. First, let us show that nontrivial solutions of the t-channel equations do, in fact, exist.

For any given $\nu_{y_1y_2}(\Pi,t)$, one can construct the series

$$\nu_{0Y}-ig(t)\int_0^Y dy\,\nu_{0y}\,\nu_{yY}+[-ig(t)]^2\int_0^Y dy_1\,dy_2\,\nu_{0y_1}\,\nu_{y_1y_2}\nu_{y_2Y}+\cdots, \tag{51a}$$

which, if it converges, is the solution of Eq. (32). For this to be an acceptable solution, however, we must impose the further restriction that when this series appears in a functional integral, the measure of each term is the same. Thus $\nu_{y_1y_2}(\Pi,t)$ is not totally arbitrary even without imposing s-channel unitarity. In the same manner, one can construct the series for $\tilde{\nu}_{y_1y_2}(\eta,t)$:

$$\tilde{\nu}_{0y}-ig(t)\int_0^Y dy\,\tilde{\nu}_{0y}\,\tilde{\nu}_{yY}+[-ig(t)]^2\int_0^Y dy_1\,dy_2\,\tilde{\nu}_{0y_1}\,\tilde{\nu}_{y_1y_2}\,\tilde{\nu}_{y_2Y}+\cdots, \tag{51b}$$

which is the solution of Eq. (50) again if it converges and if each term has the same measure.

To show that closed form solutions exist, suppose $\nu_{y_1y_2}(\Pi, t)$ has the property

$$\nu_{y_1y_2}\,\nu_{y_2y_3}=\nu_{y_1y_3}, \tag{52}$$

$$\nu_{y_1y_1}=1. \tag{53}$$

It easily follows that $\nu_{y_1y_2}$ can be written in the form

$$\nu_{y_1y_2}(\Pi, t)=\exp\left[-\int_{y_1}^{y_2}F(\Pi, y, t)\,dy\right], \tag{54}$$

where $F(\Pi, y, t)$ is an arbitrary functional of Π which does not depend **explicitly on y_1 or y_2**. This $\nu_{y_1y_2}$ is of interest because it has been shown in Refs. 3 and 4 that arbitrary short-range correlation models can be written in this form. From Eq. (51), we find

$$\tau_{0Y}(\Pi, t)=\nu_{0Y}(\Pi, t)\left\{1-ig(t)\,Y+[-ig(t)]^2\,\frac{Y^2}{2!}+\cdots\right\}$$

$$=\exp\left\{-\int_0^Y[F(\Pi, y, t)+ig(t)]\,dy\right\}. \tag{55}$$

Notice that the measure of each term is controlled by the same function, namely $\nu_{0Y}(\Pi, t)$, and hence the subsidiary condition is satisfied.

Another solution for τ_{0Y} can obviously be found by imposing the combination property on $\tilde{\nu}_{y_1y_2}(\eta, t)$ rather than $\nu_{y_1y_2}(\Pi, t)$. Thus if

$$\tilde{\nu}_{y_1y_2}\,\tilde{\nu}_{y_2y_3}=\tilde{\nu}_{y_1y_3}=\exp\left[-\int_{y_1}^{y_3}G(\eta, y, t)\,dy\right]$$

$$(\tilde{\nu}_{y_iy_i}=1), \tag{56}$$

then

$$\tilde{\tau}_{0Y}(\eta, t)=\exp\left[-\int_0^Y[G(\eta, y, t)+ig(t)]\,dy\right] \tag{57}$$

with

$$\tau_{0y}(\Pi)=\int\delta\eta\exp\left[2i\int_0^Y\Pi\eta\,dy\right]$$

$$\times\exp\left[-\int_0^Y[G(\eta, y, t)+ig(t)]\,dy\right]. \tag{58}$$

To see what effect this has on the amplitudes, from Eqs. (23) and (55) we find

$$T_{0Y}=\int\delta\Pi\,D_{0Y}(\Pi)\exp\left[-\int_0^Y(F+ig)\,dy\right]$$

$$=e^{-ig(t)Y}\int\delta\Pi\,D_{0Y}(\Pi)\exp\left\{-\int_0^yF[\Pi, y, t]\,dy\right\}. \tag{59}$$

Thus, all amplitudes are multiplied by a common phase but are otherwise unchanged. In particular, the positions of Regge poles generated by the functional integration (Refs. 3 and 4) and the **asymptotic behavior of the various cross sections are** unaffected. The question of whether or not this T can satisfy full s-channel unitarity will be taken up later.

As a second example, let us consider a model in which the asymptotic behavior is affected by the t-channel sum. In Ref. 3, a simple model was discussed which satisfies full s-channel unitarity. (Later it will be shown that a necessary condition on ν_{0Y} is that it satisfy full s-channel unitarity.) For this model

$$\chi(s, b)=\frac{\Lambda(b)}{2}[D(\Pi_0)+D(-\Pi_0)], \tag{60}$$

where $\Pi_0(y)$ is an arbitrary, but fixed, function of y. In Ref. 3 it was shown that this model was interesting because it yields constant total and elastic cross sections, bootstraps the input pole, and produces only long-range correlations. Suppose that we assume

$$V(s, t)=\frac{2s}{i}\int_0^\infty b\,db\,J_0(b\Delta)(e^{i\chi(s,\vec{b})}-1); \tag{61}$$

then from Eq. (10)

$$T(s, t)=\frac{2s}{i}\int_0^\infty b\,db\,J_0(b\Delta)(e^{i\chi}-1)-ig(t)\frac{4s}{i^2}\int_0^\infty b_1\,db_1\,J_0(b_1\Delta)\int_0^\infty b_2\,db_2\,J_0(b_2\Delta)\int_0^Y dy(e^{i\chi_{0y}(\vec{b}_1)}-1)(e^{i\chi_{yY}(\vec{b}_2)}-1)+\cdots. \tag{62}$$

The first term is just the amplitude discussed in Ref. 3. The elastic amplitude is

$$\frac{2s}{i}\int_0^\infty b\,db\,J_0(b\Delta)\langle 0|\,e^{i\chi(s,\vec{b})}-1|\,0\rangle$$

$$=2s\int_0^\infty b\,db\,J_0(b\Delta)[\langle 0|\chi|0\rangle+\tfrac{1}{2}i\langle 0|\chi^2|0\rangle+\cdots]. \tag{63}$$

The Born term is

$$2e^{(1-\lambda/2)Y}\int_0^\infty b\,db\,J_0(b\Delta)\,\Lambda(b), \tag{64}$$

where we have assumed that $\Pi_0(y)$, while not necessarily constant, is such that to leading order in Y

$$\int_0^Y \Pi_0^2(y)\, dy = \lambda Y\,. \tag{65}$$

Thus this term which is purely real has a pole at $l = 1 - \lambda/2$. The next order is

$$\frac{ie^Y}{2}(1 + e^{-2\lambda Y}) \int_0^\infty b\, db\, J_0(b\Delta)\, \Lambda^2(b)\,, \tag{66}$$

which yields a pole at $l = 1$ with a daughter at $l = 1 - 2\lambda$. One can sum this to all orders with the result that

$$\langle 0| V_{0Y}|0\rangle = \frac{2s}{i} \int_0^\infty b\, db\, J_0(b\Delta) \Big[J_0(\Lambda) - 1 + 2 \sum_{n=1}^{\infty} i^n J_n(\Lambda) e^{-n^2 \lambda Y/2} \Big] \tag{67}$$

and thus in the full amplitude, the leading singularity is still at $l = 1$ with daughters at $l = 1 - n^2\lambda/2$. Notice that the poles' positions are all fixed (independent of t) and that the residues no longer factorize.

Let us now consider the second term of Eq. (62). The elastic matrix element is

$$ig(t)\, 4s \int_0^\infty b_1\, db_1\, J_0(b_1\Delta) \int_0^\infty b_2\, db_2\, J_0(b_2\Delta) \times \int_0^Y dy \langle 0| e^{i\chi_{0y}} - 1|0\rangle \langle 0| e^{i\chi_{yY}} - 1|\rangle\,. \tag{68}$$

From this term, we can isolate, for example, the bare triple-Pomeron coupling. As we just observed, the bare Pomeron results from terms of order χ^2. Thus, let us consider the product of the second-order factor in one term and the fourth-order factor in the other term inside the integrand:

$$T_{3P}(s,t) = ig(t)\, 4s \int_0^\infty b_1\, db_1\, J_0(b,\Delta) \int_0^\infty b_2\, db_2\, J_0(b_2\Delta) \int_0^Y dy \frac{\langle 0| (i\chi_{0y})^2 |0\rangle}{2!} \frac{\langle 0| (i\chi_{yY})^4 |0\rangle}{4!} \tag{69}$$

$$= -\frac{ig(t)4s}{(2!4!)} \int_0^Y dy \langle 0| [D_{0y}(\Pi_0) + D_{0y}(-\Pi_0)]^2 |0\rangle \langle 0| [D_{yY}(\Pi_0) + D_{yY}(-\Pi_0)]^4 |0\rangle \times \int_0^\infty b_1\, db_1\, J_0(b,\Delta)\, \Lambda^2(b_1) \int b_2\, db_2\, J_0(b_2\Delta)\, \Lambda^4(b_2)\,. \tag{70}$$

The term of $O(\chi^2)$ was discussed above. The leading contribution is just $\langle 0| 2D(0)|0\rangle = 2$. The term of $O(\chi^4)$ can be similarly evaluated and to leading order, $\langle 0| [D_{yY}(\Pi_0) + D_{yY}(-\Pi_0)]^4 |0\rangle = 6$.

Notice that there are contributions which resemble the usual sort of picture in which the Pomerons are created one at a time [Fig. 2(a)] plus terms in which the intermediate states of the two Pomerons are simultaneously on-shell [Fig. 2(b), 2(c)]. In this way, we account for the complete discontinuity of the two-Pomeron contributions.

The triple-Pomeron contribution is thus

$$T_{3P}(s,t) = \frac{-ig(t) s \ln s}{32} \int_0^\infty b_1\, db_1\, J_0(b_1\Delta)\, \Lambda^2(b_1) \times \int_0^\infty b_2\, db_2\, J_2(b_2\Delta)\, \Lambda^4(b_2)\,. \tag{71}$$

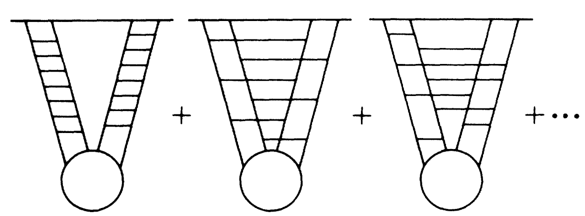
An extra factor of 2 results from adding the contribution with the two expansions in powers of χ interchanged. We obtain the usual $s \ln s$ behavior associated with the triple-Pomeron coupling. Notice that this term is pure imaginary with a negative sign.[6-9] Notice also that all terms of $O(g(t))$ behave as $s \ln s$ to leading order so that the triple-Pomeron cannot be isolated by its asymptotic behavior alone.

In the next order, we can, for example, calculate the two-Pomeron bubble graph (Fig. 3):

$$T_{2PB} = -i \frac{g^2(t) s \ln^2 s}{2^8} \left[\int_0^\infty b\, db\, J_0(b\Delta)\, \Lambda^2(b) \right]^2 \times \int_0^\infty b\, db\, J_0(b\Delta)\, \Lambda^4(b)\,. \tag{72}$$

Let us now compute the complete t-channel sum. The nth term in the sum has the form

$$T_n = [-ig(t)]^n \int_0^Y dy_1\, dy_2 \cdots dy_n \langle 0| V_{0y_1}|0\rangle \times \langle 0| V_{y_1 y_2}|0\rangle \cdots \langle 0| V_{y_n Y}|0\rangle\,. \tag{73}$$



(a) (b) (c)

FIG. 2. χ^4 **contributions** to the triple-Pomeron vertex.

From Eq. (67) we observe that because we have computed the elastic matrix elements before summing, $\langle 0| V_{y_i y_j}(s,t)|0\rangle$ is a function only of $y_j - y_i$ rather than y_i and y_j separately. Because of this, we can use the convolution theorem. Defining

$$f(j) \equiv \int_0^\infty dy\, e^{-jy}\langle 0| V(y,t)|0\rangle \tag{74}$$

$$= \frac{2}{i}\int_0^\infty b\,db\,J_0(b\Delta)\left[\frac{J_0(\Lambda(b))-1}{j-1} + 2\sum_{n=1}^{\infty}\frac{i^n J_n(\Lambda(b))}{j-(1-\frac{1}{2}n^2\lambda)}\right], \tag{75}$$

then

$$T_n = [-ig(t)]^n \frac{1}{(2\pi i)}\int_{c-i\infty}^{c+i\infty} dj\, e^{jY} f^{n+1}(j), \tag{76}$$

and finally,

$$\langle 0| T(s,t)|0\rangle = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dj\, e^{jY}\left[\frac{f(j)}{1+ig(t)f(j)}\right]. \tag{77}$$

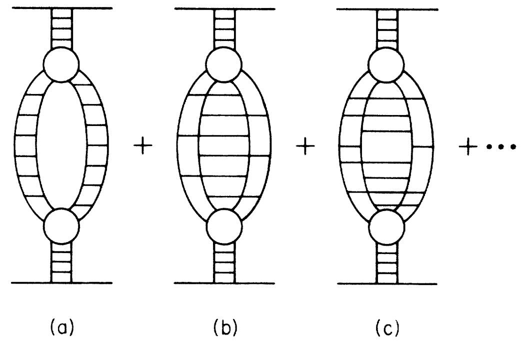


FIG. 3. χ^4 contributions to the two-Pomeron bubble graph.

Previously, the poles appeared as singularities in $f(j)$ [Eq. (75)] but after performing the t-channel sum, these poles no longer are present because of a cancellation between the numerator and denominator. The singularities that are present appear as zeros of the denominator

$$j-1-2g(t)\int_0^\infty b\,db\,J_0(b\Delta)\left\{1-J_0(\Lambda(b))-2(j-1)\sum_{n=1}^{\infty}\frac{i^n J_n(\Lambda(b))}{j-(1-\frac{1}{2}n^2\lambda)}\right\} \tag{78}$$

or

$$j = 1 + \frac{2g(t)\int_0^\infty b\,db\,J_0(b\Delta)[1-J_0(\Lambda(b))]}{1+4g(t)\int_0^\infty b\,db\,J_0(b\Delta)\sum_{n=1}^{\infty}\{i^n J_n(\Lambda(b))/[j-(1-\frac{1}{2}n^2\lambda)]\}}. \tag{79}$$

This latter expression is still an equation for j. We see now that the positions of the singularities are functions of t both as a result of $g(t)$ and the dependence on $J_0(b\Delta)$. In general, we cannot explicitly solve this expression but we can get an approximation to the amplitude by setting $j=1$ in the denominator of Eq. (79). Then

$$\langle 0| T(s,t)|0\rangle \approx 2i\exp\left\{Y\left[1+\frac{2g(t)\int_0^\infty b\,db\,J_0(b\Delta)[1-J_0(\Lambda(b))]}{1+4g(t)\int_0^\infty b\,db\,J_0(b\Delta)(2/\lambda)\sum[i^n J_n(\Lambda(b))/n^2]}\right]\right\}$$
$$\times\frac{\int_0^\infty b\,db\,J_0(b\Delta)[1-J_0(\Lambda(b)]}{\left[1+4g(t)\int_0^\infty b\,db\,J_0(b\Delta)(2/\lambda)\sum[i^n J_n(\Lambda(b))/n^2]\right]^2}. \tag{80}$$

Unless either $g(0)=0$ or $\int_0^\infty b\,db[1-J_0(\Lambda(b))]=0$, this amplitude will obviously violate s-channel unitarity because the leading singularity will lie to the right of $j=1$. The latter condition is ruled out because the entire amplitude vanishes at $t=0$ if this is true. The former is not ruled out but there does not appear to be any particular reason why $g(0)$ should vanish. If, however, it did vanish, from Eq. (71) we see that it implies a vanishing triple-Pomeron coupling. In fact, it says that all renormalization contributions vanish at $t=0$. Of course $g(0)=0$ by itself does not guarantee s-channel unitarity since the unitarity condition is a constraint on $T(s,b)$ which involves all values of t.

This is not a simple matter to check. Computing even just the elastic amplitude $\langle 0|T(s,t)|0\rangle$ is obviously difficult.

It should be noticed that in this particular example, we avoided the use of functional integrals by dealing directly with the operator $\chi(s,b)$. It is instructive, however, to redo some of the previous calculations using the functional techniques discussed earlier. From Eq. (60)

$$\chi(\Pi, b) = \tfrac{1}{2}\Lambda(b)[\delta_F(\Pi-\Pi_0)+\delta_F(\Pi+\Pi_0)] \tag{81}$$

and thus

$$\begin{aligned}\tilde{\chi}(\eta,\vec{b}) &= \int \delta\Pi \exp\left(-2i\int_0^Y \eta\Pi dy\right)\chi(\Pi,b)\\ &= \tfrac{1}{2}\Lambda(b)\left[\exp\left(-2i\int_0^Y \eta\Pi_0 dy\right)+\exp\left(2i\int_0^Y \eta\Pi_0 dy\right)\right].\end{aligned} \tag{82}$$

From Eqs. (3), (44), and (82),

$$\begin{aligned}\langle 0|T(s,\vec{b})|0\rangle &= \frac{2s}{i}\int\delta\Pi\exp\left[-\tfrac{1}{2}\int_0^Y dy\,\Pi^2(y)\right]\int\delta\eta\exp\left(2i\int_0^Y\eta\Pi dy\right)[e^{i\tilde{\chi}(\eta,\vec{b})}-1]\\ &= \frac{2s}{i}\int\delta\eta\exp\left[-2\int_0^Y\eta^2(y)\,dy\right][e^{i\tilde{\chi}(\eta,\vec{b})}-1].\end{aligned} \tag{83}$$

Now, expanding the exponential

$$\langle 0|T(s,\vec{b})|0\rangle = \frac{2s}{i}\int\delta\eta\exp\left(-2\int_0^Y\eta^2\,dy\right)\left\{\left(\sum_{l,m=0}^{\infty}\frac{i^{l+m}}{l!m!}\left[\tfrac{1}{2}\Lambda(b)\right]^{l+m}\exp\left[2i(l-m)\int_0^Y\eta\Pi_0 dy\right]\right)-1\right\}. \tag{84}$$

We observe that every term in the series has the same measure since in each case, the most rapidly decreasing factor is the Gaussian factor which multiplies each term (see Appendix). Using the fact that

$$\int\delta\eta\exp\left(-2\int_0^Y\eta^2 dy\right)\exp\left[2i(l-m)\int_0^Y\eta\Pi_0 dy\right] = \exp\left[-\frac{(l-m)^2}{2}\int_0^Y\Pi_0{}^2 dy\right], \tag{85}$$

it is easy to show that the resulting amplitude agrees with the result quoted in Ref. 3.

Having examined two simple models, let us now continue with the formal development.

IV. COMBINED UNITARITY EQUATIONS

Having established that solutions to the t-channel equation do exist, let us now compare Eq. (51) with the form required by s-channel unitarity. We shall show explicitly that $V(s,t)$ itself must satisfy full s-channel unitarity. Considering first Eqs. (45) and (51b) we have

$$\frac{2s}{i}\int_0^\infty b\,db\,J_0(b\Delta)[e^{i\tilde{\chi}_{0Y}(\eta,\vec{b})}-1] = \tilde{\nu}_{0Y}-ig(t)\int_0^Y dy\,\tilde{\nu}_{0y}\,\tilde{\nu}_{yY}+\cdots. \tag{86}$$

Clearly, $\tilde{\chi}(\eta,\vec{b})$ must depend on $g(t)$ although because of the integral transform, it will manifest itself as some dependence on b. In order to study this dependence, let us introduce a coupling λ

$$g(t)\rightarrow\lambda g(t) \tag{87}$$

and consider an expansion of the left-hand side of Eq. (86) in powers of λ. Since $\tilde{\nu}_{y_1y_2}$ is, by assumption, independent of $g(t)$, the right-hand side is already such an expansion. Defining

$$\tilde{\chi}(\eta,\vec{b};\lambda)\equiv\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\tilde{\chi}^{(n)}(\eta,\vec{b}), \tag{88}$$

$$\tilde{\chi}^{(n)}(\eta,\vec{b}) = \frac{d^n}{d\lambda^n}\tilde{\chi}(\eta,\vec{b};\lambda)\bigg|_{\lambda=0}, \tag{89}$$

the left-hand side becomes

$$\frac{2s}{i}\int_0^\infty b\,db\,J_0(b\Delta)(e^{i\tilde{\chi}^{(0)}}\{1+\lambda(i\tilde{\chi}^{(1)})+\lambda^2[i\tilde{\chi}^{(2)}-(\tilde{\chi}^{(1)})^2]+\cdots\}-1). \tag{90}$$

Unitarity requires that $\tilde{\chi}(\eta,\vec{b};\lambda)$ be real. Suppose that the various $\tilde{\chi}^{(n)}$ were complex. Then for a given function $\eta(y)$, one could presumably find some value of $\lambda=\lambda_1$ such that the sum Eq. (88) was real. For some different function $\eta_2(y)$, another value of $\lambda=\lambda_2$ would be required. We know, however, that Eq. (86) must hold for any function $\eta(y)$ for a single value of λ; hence we conclude that all $\tilde{\chi}^{(n)}(\eta,\vec{b})$ must be real.

By comparing powers of λ we find

$$\frac{2s}{i}\int_0^\infty b\,db\,J_0(b\Delta)(e^{i\tilde{\chi}^{(0)}}-1)=\tilde{\nu}_{0Y}(\eta,t)\,, \tag{91a}$$

$$\frac{2s}{i}\int_0^\infty b\,db\,J_0(b\Delta)\,e^{i\tilde{\chi}^{(0)}}(i\tilde{\chi}^{(1)})=-ig(t)\int_0^Y dy\,\tilde{\nu}_{0y}\,\tilde{\nu}_{yY}\,, \tag{91b}$$

$$\frac{2s}{i}\int_0^\infty b\,db\,J_0(b\Delta)\,e^{i\tilde{\chi}^{(0)}}\tfrac{1}{2}[i\tilde{\chi}^{(2)}-(\tilde{\chi}^{(1)})^2]=[-ig(t)]^2\int_0^Y dy_1\,dy_2\,\tilde{\nu}_{0y_1}\,\tilde{\nu}_{y_1y_2}\,\tilde{\nu}_{y_2Y} \tag{91c}$$

etc. We are supposing that at least for some values of λ, the two series converge so that we can compare coefficients.

From (91a) we obtain the necessary but insufficient constraint that $\tilde{\nu}_{0Y}$ itself must satisfy full s-channel unitarity if this whole scheme is to be consistent. From (91b) and (91c)

$$\tilde{\chi}^{(1)}(\eta,\vec{b})=\frac{-i}{2s}e^{-i\tilde{\chi}^{(0)}}\int_0^\infty \Delta\,d\Delta\,J_0(b\Delta)g(t)\int_0^Y dy\,\tilde{\nu}_{0y}\,\tilde{\nu}_{yY}$$
$$=\frac{-\int_0^\infty \Delta\,d\Delta\,J_0(b\Delta)g(t)\int_0^Y dy\,\tilde{\nu}_{0y}\,\tilde{\nu}_{yY}}{\left[\int_0^\infty \Delta\,d\Delta\,J_0(b\Delta)\,\tilde{\nu}_{0Y}+2s/i\right]}\,, \tag{92a}$$

$$\tilde{\chi}^{(2)}(\eta,\vec{b})=-i\left[(\tilde{\chi}^{(1)})^2-\frac{i}{s}e^{-i\tilde{\chi}^{(0)}}\int_0^\infty \Delta\,d\Delta\,J_0(b\Delta)g^2(t)\int_0^Y dy_1\,dy_2\,\tilde{\nu}_{0y_1}\,\tilde{\nu}_{y_1y_2}\tilde{\nu}_{y_2Y}\right]\,, \tag{92b}$$

etc.

This infinite set of relations with the constraints $\tilde{\chi}^{(n)}=\tilde{\chi}^{(n)*}$ constitutes a set of necessary and sufficient conditions to yield amplitudes satisfying full s-channel and elastic t-channel unitarity.

The solution of this set of constraints appears formidable. For instance, from the form of these equations, it does not appear likely that these constraints will close, meaning that $\tilde{\chi}^{(n)}$ being real for $0\leq n\leq N$ will be sufficient to guarantee that $\tilde{\chi}^{(n)}$ be real for $n>N$.

It is a simple matter to formulate the corresponding set of relations for $\nu_{y_1y_2}$. We leave it as an exercise for the interested reader.

V. DISCUSSION AND CONCLUSIONS

In the previous section, we have given a set of relations which will yield amplitudes satisfying full s-channel and elastic t-channel unitarity. Unfortunately, we have not yet been able to find a solution to these equations. We have, however, learned something about possible solutions by considering examples which do not have both properties.

Let us first reconsider the solution to the t-channel equation discussed in Sec. III, namely Eq. (57). The set of relations Eq. (91) then implies that a $G(\eta,y,t)$ must be found such that the equality

$$\exp\left[-\int_0^Y G(\eta,y,t)\,dy\right]$$
$$=\frac{2s}{i}\int_0^\infty b\,db\,J_0(b\Delta)(e^{i\tilde{\chi}^{(0)}(\eta,\vec{b})}-1) \tag{93a}$$
$$=\frac{2s}{i}e^{i\lambda g(t)Y}\int_0^\infty b\,db\,J_0(b\Delta)(e^{i\tilde{\chi}(\eta,\vec{b},\lambda)}-1) \tag{93b}$$

is satisfied for arbitrary λ with $\tilde{\chi}^{(0)}$ and $\tilde{\chi}$ real. It is easy to show that no solution exists for the one-dimensional problem in which the distinction between t and b is neglected. A proof in the general case is less obvious because of the integrations and the fact that the Bessel function does not have a definite sign. We feel, however, that no solution is possible for models of this type because of the lack of long-range correlations.

In order to correct for the lack of long-range correlations, it is natural to try a model in which we first introduce short-range correlations in $\tilde{\chi}$, for example,

$$\tilde{\chi}(\eta,\vec{b})=\exp\left\{-\int_0^Y[a(b)\eta^2+c(b)\dot{\eta}^2]\,dy\right\} \tag{94}$$

and then long-range correlations by exponentiating $\tilde{\chi}$

$$\tilde{\tau}(\eta,\vec{b}) = \frac{2s}{i}\,(e^{i\tilde{\chi}(\eta,\vec{b})}-1)\,. \qquad (95)$$

Unfortunately, while this model seems intuitively correct, it is shown in the Appendix that it does not survive the functional integrations. The problem is that the measure of the functional integration is coupled to the number of chains, i.e., the powers of $\tilde{\chi}$ in the expansion of $(e^{i\tilde{\chi}}-1)$. What is needed is to couple the measure to something that is independent of the number of chains. A hint towards a possible resolution of this difficulty was provided by Sugar[2] by showing that, within the context of a unitary multiperipheral model, a saturation of forces can be achieved by coupling the pions to the isospin carried by the chains. Since the total isospin exchanged is independent of the number of chains, this suggests that the correct procedure would be to construct a model in which the measure is controlled by the isospin rather than by the number of chains. Since the short-range correlations are introduced via the same factors which control the measure, namely the derivatives of $\Pi(y)$, it further suggests that the existence of short-range correlations is strongly connected with the existence of particles carrying isospin.

A third type of model we have not discussed here but we shall consider in the future is the case of "black-disk models." Part of the problem here arises from the fact that the operator χ in these models is an unbounded operator and therefore cannot be simply represented in the form of Eq. (23) but as in Eq. (22) increasing the difficulties in finding solutions to the t-channel unitarity equations.

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APPENDIX

In this appendix, we will discuss some properties of functional integrals pertinent to this work.[10] The general idea of such an integral is to compute the sum of a given functional $F[\Pi(q)]$ over all possible functions $\Pi(q)$. Thus the two principal problems in defining such an integral are first to characterize all possible functions $\Pi(q)$ in some manner and second, to specify the measure of the integral for a given representation of the functions $\Pi(q)$. The measure can be thought of as the volume of functional space one is to associate with a given $\Pi(q)$ when going from a continuous integral to a discrete sum. These points can best be illustrated by considering some examples.

Let us first consider the Gaussian integral,

$$I_1 = \int \delta\Pi \exp\left[-a\int_0^Y \Pi^2(y)\,dy\right]. \qquad (A1)$$

We will consider Π to be a function of only one variable, namely y. We now need to characterize the space of functions $\Pi(y)$. There are two convenient methods for doing this. The first involves dividing the interval 0, Y into N subintervals and setting $\Pi(y)=\Pi(y_i)$ within each subinterval. A particular function is thus represented by the set of points $\Pi(y_i)$ which as $N\to\infty$ should more and more closely describe the particular function. By allowing for arbitrary values of $\Pi(y_i)$, we can allow for arbitrary functions $\Pi(y)$. The second type of representation is the familiar normal-mode expansion

$$\Pi(y) = \sum_{n=0}^{\infty} \alpha_n f_n(y)\,, \qquad (A2)$$

where

$$\int_0^y dy\, f_n(y) f_{n'}(y) = \delta_{nn'}\,. \qquad (A3)$$

By allowing for arbitrary α_n, a wide class of functions can be characterized.

Let us now consider the integral (A1). Using the first type of representation,

$$\exp\left[-a\int_0^Y \Pi^2(y)\,dy\right] = \lim_{N\to\infty} \exp\left[-a\Delta y \sum_{i=1}^{N} \Pi^2(y_i)\right], \qquad (A4)$$

where $\Delta y = Y/N$. The differential $\delta\Pi$ becomes

$$\lim_{N\to\infty} \prod_{i=1}^{N} M_i \int_{-\infty}^{\infty} d\Pi(y_i)\,,$$

where by integrating over all possible values of $\Pi(y_i)$, we sum over all possible functions $\Pi(y)$. M_i, the measure associated with each $\Pi(y_i)$, will be determined by requiring the integral to be finite.

$$\int \delta\Pi \exp\left[-a\int_0^y \Pi^2(y)\,dy\right]$$

$$= \lim_{N\to\infty}\left(\prod_{i=1}^{N} M_i \int_{-\infty}^{\infty} d\Pi_i\right) \exp\left(-a\Delta y \sum_{i=1}^{N} \Pi^2\right) \qquad (A5)$$

$$= \lim_{N\to\infty} \prod_{i=1}^{\infty}\left(M_i \int_{-\infty}^{\infty} d\Pi_i\, e^{-a\Delta y \Pi_i^2}\right) \qquad (A6)$$

$$= \lim_{N\to\infty} \prod_{i=1}^{\infty}\left[M_i\left(\frac{\pi}{a\Delta y}\right)^{1/2}\right]. \qquad (A7)$$

If this product is to be finite it is necessary that $[M_i(\pi/a\Delta y)^{1/2}]$ approach unity as $N\to\infty$. This requires

$$M_i=\left(\frac{a\Delta y}{\pi}\right)^{1/2} \tag{A8}$$

Actually, there is an arbitrariness in this definition because a finite number of the M_i's can differ from the above value by a finite factor without affecting the boundedness of the infinite product. Since the points y_i are arbitrary and equivalent, however, it does not seem proper to change the M_i except perhaps at the end points. We will require Eq. (A8) for all M_i with the knowledge that in so doing, we are making a particular choice concerning the boundary conditions at $y=0, Y$. With this choice,

$$I_1=\int\delta\Pi\exp\left[-a\int\Pi^2(y)\,dy\right]=1\,. \tag{A9}$$

Next, let us compute the same integral using the normal-mode representation. In this case

$$\int_0^Y\Pi^2(y)\,dy=\sum_{n=0}^{\infty}\alpha_n^2 \tag{A10}$$

and

$$\int\delta\Pi=\prod_{i=0}^{\infty}M_i\int_{-\infty}^{\infty}d\alpha_i\,. \tag{A11}$$

Thus

$$I_1=\prod_{i=0}^{\infty}\left(M_i\int_{-\infty}^{\infty}d\alpha_i\right)\exp(-a\textstyle\sum_i\alpha_i^2) \tag{A12}$$

$$=\prod_{i=0}^{\infty}\left[M_i\left(\frac{\pi}{a}\right)^{1/2}\right]\,. \tag{A13}$$

In this case, I_1 finite requires

$$M_i=\left(\frac{a}{\pi}\right)^{1/2} \tag{A14}$$

for all but a finite number of M_i's. Again, choosing (A14) for all i, we find $I_1=1$ as before. In this case the arbitrariness is associated with a finite number of normal modes. By redefining the appropriate M_i's, we can lump all the arbitrary factors into M_0, for instance. Again, we see that the arbitrariness is somehow connected with boundary conditions. Notice that in both cases, we were unable to specify the measure until after we had performed the integral.

As a second example, consider the integral

$$I_2=\int\delta\Pi\exp\left\{-\int_0^Y[a\Pi^2(y)+b\mathring{\Pi}^2(y)]\,dy\right\}, \tag{A15}$$

where $\mathring{\Pi}(\pi)\equiv(d/dy)\Pi(y)$. Introducing the normal-mode representation, we find

$$\int_0^Y[a\Pi^2(y)+b\mathring{\Pi}^2(y)]\,dy=\sum_{n=0}^{\infty}(a\alpha_n^2+bk_n^2\alpha_n^2)\,, \tag{A16}$$

where $k_n=\pi n/Y$. The integral becomes

$$I_2=\prod_{i=0}^{\infty}\left(M_i\int_{-\infty}^{\infty}d\alpha_i\,e^{-[a+bk_i^2]\alpha_i^2}\right) \tag{A17}$$

$$=\prod_{i=0}^{\infty}M_i\left(\frac{\pi}{a+bk_i^2}\right)^{1/2}. \tag{A18}$$

In this case, because $k_i\to\infty$ as $i\to\infty$, we do not set $M_i=[(a+bk_i^2)/\pi]^{1/2}$, rather

$$M_i=\left(\frac{bk_i^2}{\pi}\right)^{1/2},\quad i\neq 0\,. \tag{A19}$$

Then

$$I_2=M_0\left(\frac{\pi}{a}\right)^{1/2}\prod_{i=1}^{\infty}\frac{1}{(1+a/bk_i^2)^{1/2}} \tag{A20}$$

$$=M_0\left(\frac{\pi}{a}\right)^{1/2}\left(\frac{a}{b}\right)^{1/4}\frac{y^{1/2}}{\{\sinh[(a/b)^{1/2}Y]\}^{1/2}}. \tag{A21}$$

Again the inevitable arbitrariness appears as a result of M_0 not being fixed. If we choose M_0 such that the integral over $d\alpha_0$ is unity, then $M_0=1/y^{1/2}$ and

$$I_2=\frac{1}{(ab)^{1/4}}\left(\frac{\pi}{\sinh[(a/b)^{1/2}Y]}\right)^{1/2}. \tag{A22}$$

The evaluation of this integral using the other representation for $\Pi(y)$ is given in Ref. 11 and will not be repeated here. The same result is obtained with a particular choice concerning the boundary conditions.

Notice that in this case, the measure with the exception of M_0, is determined by that part of the integrand which involves the derivative. We can, in fact, determine the measure by requiring

$$\int\delta\Pi\exp\left[-b\int\dot{\Pi}^2(y)dy\right]=1\,. \tag{A23}$$

It seems to be a general property of these integrals that the measure, with the exception of M_0, can always be determined by requiring the functional integral of the most rapidly decreasing factor to be unity. In the first example, this is just the requirement

$$\int\delta\Pi\exp\left(-a\int\Pi^2dy\right)=1\,, \tag{A24}$$

which agrees with what we found. Another example is

$$I_3 = \int \delta\Pi \exp\left\{-\int_0^y dy[a\Pi^2(y) + b\dot{\Pi}^2(y) + c\ddot{\Pi}^2(y)]\right\}. \tag{A25}$$

In this case, the correct measure is found by requiring

$$\int \delta\Pi \exp\left[-c\int \ddot{\Pi}^2(y)dy\right] = 1. \tag{A26}$$

Let us now turn to a somewhat different type of problem, namely, the Fourier transform. Consider

$$I_4[\eta(y)] = \int \delta\Pi \exp\left[2i\int_0^Y \eta(y)\Pi(y)dy\right]\exp\left[-a\int_0^Y \Pi^2(y)dy\right]$$

$$= \int \delta\Pi \exp\left[-a\int_0^Y (\Pi - i\eta/a)^2\, dy\right]\exp\left[-(1/a)\int_0^Y \eta^2 dy\right]. \tag{A27}$$

Changing variables to $\Pi' = \Pi - i\eta/a$, we find

$$I_4[\eta(y)] = \exp\left[-(1/a)\int_0^Y \eta^2(y)dy\right]\int \delta\Pi' \exp\left[-a\int_0^Y \Pi'^2(y)dy\right]$$

$$= \exp\left[-(1/a)\int_0^Y \eta^2(y)dy\right]. \tag{A28}$$

Similarly

$$I_5[\eta(y)] = \int \delta\Pi \exp\left[2i\int_0^Y \eta(y)\Pi(y)dy\right]\exp\left\{-\int_0^Y [a\Pi^2(y) + b\dot{\Pi}^2(y)]dy\right\}$$

$$= \prod_{i=0}^{\infty} M_i\left(\frac{\pi}{a+bk_i^2}\right)^{1/2}\exp\left[-\frac{\beta_i^2}{4(a+bk_i^2)}\right], \tag{A29}$$

where

$$\eta(y) = \sum_{i=0}^{\infty}\beta_n f_n(y).$$

In this case, the most rapidly decreasing factor involves the derivative so that M_i should be given by (A19). From (A27), it is clear that this choice leads to a finite result. Again, we have remaining the arbitrariness associated with M_0.

Finally, let us consider a third type of problem, namely the functional integral of a sum of terms. As an example, consider the integral

$$I_6 = \int \delta\Pi\left[\exp\left(-a\int \Pi^2 dy\right) + \exp\left(-b\int \Pi^2 dy\right)\right]; \quad b > a. \tag{A30}$$

We require a single measure for the complete integrand. Thus

$$I_6 = \prod_{i=0}^{\infty}\left(M_i\int_{-\infty}^{\infty} d\alpha_i\right)\left[\exp\left(-a\sum\alpha_i^2\right) + \exp\left(-b\sum\alpha_i^2\right)\right] \tag{A31}$$

$$= \prod_{i=0}^{\infty} M_i\left[\prod_{i=0}^{\infty}\left(\int_{-\infty}^{\infty} d\alpha_i\, e^{-a\alpha_i^2}\right) + \prod_{i=0}^{\infty}\left(\int_{-\infty}^{\infty} d\alpha_i\, e^{-b\alpha_i^2}\right)\right] \tag{A32}$$

$$= \lim_{N\to\infty}\prod_{i=0}^{N}(M_i)\left[\left(\frac{\pi}{a}\right)^{N/2} + \left(\frac{\pi}{b}\right)^{N/2}\right] \tag{A33}$$

$$= \lim_{N\to\infty}\prod_{i=1}^{N}(M_i)\left(\frac{\pi}{a}\right)^{N/2}\left[1 + \left(\frac{a}{b}\right)^{N/2}\right]. \tag{A34}$$

Clearly we must choose $M_i = (a/\pi)^{1/2}$ and since $\lim_{N\to\infty}(a/b)^{N/2} = 0$, $I_6 = 1$. Hence we find that in general,

$$\int\delta\Pi\,(F_1[\Pi] + F_2[\Pi]) \neq \int\delta\Pi\,F_1[\Pi] + \int\delta\Pi\,F_2[\Pi]\,. \qquad (A35)$$

In fact, (A33) becomes an equality only if the measures of the two terms are equal. In the example just discussed, we see that the integral is controlled by the least rapidly decreasing term.

As a second example, let us consider the integral

$$I_7 = \int \delta\Pi\{\exp[ie^{-\int_0^y(a\Pi^2+b\dot{\Pi}^2)dy}] - 1\}\,. \qquad (A36)$$

This integral is of interest because it is the type that results from assuming χ to be a short-range correlation chain. Again expanding in normal modes and using a series expansion for the exponential,

$$I_7 = \lim_{N\to\infty}\prod_{l=0}^{N}\left(M_l\int_{-\infty}^{\infty}d\alpha_l\right)\sum_{n=1}^{\infty}\frac{i^n}{n!}\exp\left[-n\sum_{l=0}^{\infty}(a+bk_l^2)\alpha_l^2\right] \qquad (A37)$$

$$= \lim_{N\to\infty}\prod_{l=0}^{N}M_l\sum_{n=1}^{\infty}\frac{i^n}{n!}\prod_{j=0}^{N}\left(\frac{\pi}{n(a+bk_j^2)}\right)^{1/2} \qquad (A38)$$

$$= \lim_{N\to\infty}\prod_{l=0}^{N}\left[M_l\left(\frac{\pi}{a+bk_l^2}\right)^{1/2}\right]\left(\sum_{n=1}^{\infty}\frac{i^n}{n!}\frac{1}{n^N}\right). \qquad (A39)$$

The measure must be independent of n since the value of the integral cannot depend on our decision to expand the exponential. We see then that

$$M_i = \left(\frac{bk_i^2}{\pi}\right)^{1/2} \qquad (A40)$$

in order that the integral be finite and in addition, only the first term in the series survives in the limit $N\to\infty$. Thus we find

$$I_7 = i\int\delta\Pi\exp\left[-\int_0^y(a\Pi^2+b\dot{\Pi}^2)dy\right] \qquad (A41)$$

$$= \frac{i}{(ab)^{1/4}}\left(\frac{\pi}{\sinh[(a/b)^{1/2}y]}\right)^{1/2}. \qquad (A42)$$

Again, it is the least rapidly decreasing term in the series which controls the integral. The important conclusion to be drawn from this example is that since s-channel unitarity requires contributions from all terms of the series, only those χ's will be acceptable for which χ^n has the same measure for all n. This is the case for the unitary model of Sec. III in which χ does not have short-range correlations and is apparently not the case for the type of short-range correlations discussed in Ref. 3.

One might like to introduce a convergence factor in order to be able to perform the integral in Eq. (A36), for example

$$1 = \int\delta\gamma\exp\left\{-\int_0^Y dy[\gamma(y) - \ddot{\Pi}(y)]^2\right\}. \qquad (A43)$$

By reversing the order of functional integrations a finite result, including contributions from all terms in the series, is obtained for the Π integration, but on performing the γ integration one again obtains the result stated below Eq. (A40).

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