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High-energy potential scattering of Dirac particles

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A general method for studying the high-energy potential scattering of Dirac particles is presented. The failure of the usual eikonal picture for a pseudoscalar potential is attributed to competing pair-production processes.

The study of the motion of ultrarelativistic particles in an external field has been of great interest owing to the recent development of intense electromagnetic fields in lasers and the availability of energetic electrons in accelerators. With the aid of functional methods, the problems of high-energy potential scattering also serve as a first step in understanding high-energy quantum field theories. In nonrelativistic quantum mechanics, this subject had been well analyzed many years ago.[1] Since the physical concept of relativistic wave functions is different from that of single-particle wave functions, a direct generalization of the Glauber approximation to relativistic quantum mechanics must be looked at with certain reservations. In the search for a high-energy solution, one is, however, guided by the following empirical fact: Longitudinal velocities of the colliding particles seem to be separately conserved. Moreover, a very fast particle behaves like a classical particle because its de Broglie wavelength is small. Thus, at extremely high energies, the single-particle wave function $\psi(x, t)$ has a narrow spread in the longitudinal velocity

$$P_3H^{-1}\psi(x, t) \simeq \langle P_3H^{-1}\rangle\, \psi(x, t) \simeq \psi(x, t), \tag{1}$$

where P_3 (H) denotes the longitudinal momentum (energy) operator and $\langle P_3H^{-1}\rangle$ the classical value of the longitudinal velocity which is practically equal to unity ($\hbar = c = 1$) along the entire trajectory. Equation (1) cannot be made use of automatically in relativistic physics. Because of the possibility of real and/or virtual pair production from the potential, the single-particle interpretation of relativistic wave functions runs into theoretical difficulties. It is a well-established fact[2] that a Dirac particle with positive frequency before scattering can go over, after scattering, into a superposition of states of positive and of negative frequency. For a Klein-Gordon wave function, the situation is even more complicated. Here one may be faced with the occurrence of complex frequencies.[3]

In this paper, we wish to derive the high-energy Dirac solution[4] in an arbitrary, time-dependent potential. Although the method is applicable to all types of potentials, we shall give explicit results concerning the cases of vector coupling (electromagnetic interaction) and of pseudoscalar coupling only. For the vector coupling, pair production amplitudes vanish at high energies. Thus one is essentially dealing with the single-particle problem. The high-energy approximation assumes the familiar eikonal form. For the pseudoscalar coupling, virtual pair production amplitudes cannot be neglected in the computation of the scattering amplitude. Consequently, the high-energy approximation is no longer eikonal, but the exponentiation[5] still holds. The Dirac equation in a potential can be written

$$\left(i\,\frac{\partial}{\partial t} - H_0\right)\psi(x, t) = \beta V\psi(x, t), \quad H_0 \equiv \vec{\alpha}\cdot\vec{\mathrm{P}} + \beta m, \quad \vec{\mathrm{P}} \equiv -i\vec{\nabla}, \tag{2}$$

where $V(x, t; E)$ is a 4×4 matrix function of space-time coordinates and possibly of the energy E of the incoming particle. In the general case, the solution of (2), which contains only positive frequency at $t = -\infty$, involves both signs of the frequency as $t \to +\infty$. We write

$$\psi \equiv \psi_{\text{pos}} + \psi_{\text{neg}} \equiv e^{i(p_3x_3 - Et)}\phi_+ + e^{i(p_3x_3 + Et)}\phi_-, \tag{3}$$

where the 4-component spinors $\phi_\pm$ are functions of space-time coordinates, of the parameters of the incoming particle $(\vec{\mathrm{p}}, E, m)$, and of the parameters characterizing the potential (coupling constant, effective range). The boundary conditions are

$$\phi_-(t = -\infty) = 0, \tag{4a}$$

$$\phi_+(t = -\infty) = e^{ip_\perp x_\perp}u_i, \tag{4b}$$

where the 4-spinor u_i describes the spin state of the incoming particle. The scattering amplitude of the single-particle problem is determined by $\phi_+(t=+\infty)$ and the amplitude for pair production[6] is related to $\phi_-(t=+\infty)$. Thus the high-energy approximation previously discussed for the single-particle wave function should apply only to ψ_{pos}. We are only interested in those potentials such that the longitudinal motion of the particle (ψ_{pos}) is uniform and rectilinear as $E\to\infty$. It is not difficult to find that this is indeed the case if

$$\frac{V}{E} \ll 1 \quad \text{for } E\to\infty\,. \tag{5}$$

We have

$$\begin{aligned} i\dot{x}_3 &= [x_3, H]_- = i\alpha_3\,, \\ i\dot{\alpha}_3 &= [\alpha_3, H]_- = 2\alpha_3 H - 2P_3 - \{\alpha_3, \beta V\}_+\,. \end{aligned} \tag{6}$$

As the longitudinal motion becomes extremely fast, both H and P_3 of ψ_{pos} are approximately conserved quantities,

$$\dot{H} \simeq 0, \quad \dot{P}_3 \simeq 0\,. \tag{7}$$

The anticommutator $\{\alpha_3, \beta V\}_+$ is clearly of the same magnitude as V. Hence, with the aid of (5) and (7), Eq. (6) becomes

$$i\ddot{\alpha}_3 \simeq 2\dot{\alpha}_3 H\,, \quad \dot{\alpha}_3 \simeq \dot{\alpha}_3^0 e^{-2iHt}\,, \tag{8}$$

where

$$\dot{\alpha}_3^0 \equiv \dot{\alpha}_3(t=0)\,.$$

From (6) and (7) we obtain

$$\alpha_3 \simeq P_3 H^{-1} \simeq 1 \tag{9}$$

and

$$x_3 \simeq P_3 H^{-1} t + x_3^0\,,$$

which is precisely what we set out to establish. It is important to remember that the high-energy condition (9) applies only to ψ_{pos}, i.e.,

$$\begin{aligned} \alpha_3 \psi_{\text{pos}} &\simeq \langle P_3 H^{-1}\rangle \psi_{\text{pos}} \\ &\simeq \psi_{\text{pos}}\,, \end{aligned}$$

or equivalently from (3),

$$\alpha_3 \phi_+ \simeq \phi_+\,. \tag{10a}$$

As for ψ_{neg}, we have no knowledge of what its high-energy approximation should be except that $\psi_{\text{pos}} + \psi_{\text{neg}}$ is a solution of (2). Substituting (3) into (2) and taking account of (5) and (10a) we obtain

$$\alpha_3 \phi_- \simeq -\phi_-\,. \tag{10b}$$

In the representation

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix},$$

both (10a) and (10b) lead to

$$\phi_\pm = \begin{pmatrix} \psi_\pm \\ \pm\sigma_3 \psi_\pm \end{pmatrix}, \tag{11}$$

where $\psi_\pm$ are the 2-component spinors (appropriately normalized) and σ_i $(i=1,2,3)$ are the usual Pauli matrices. The Dirac equation (2) determines $\psi_\pm$ in terms of the input parameters. Substituting (3) and (11) into the left-hand side of (2) we get

$$\left(i\frac{\partial}{\partial t} - H_0\right)\psi = e^{i(p_3x_3 - Et)}A + e^{i(p_3x_3 + Et)}B\,, \tag{12}$$

where

$$A \equiv \begin{pmatrix} [i\partial_+ - (\vec{\sigma}\times\vec{\nabla})_3 - m]\,\psi_+ \\ \sigma_3[i\partial_+ + (\vec{\sigma}\times\vec{\nabla})_3 + m]\,\psi_+ \end{pmatrix},$$

$$B \equiv \begin{pmatrix} [i\partial_- + (\vec{\sigma}\times\vec{\nabla})_3 - m]\,\psi_- \\ \sigma_3[-i\partial_- + (\vec{\sigma}\times\vec{\nabla})_3 - m]\,\psi_- \end{pmatrix},$$

$$\partial_\pm \equiv \frac{\partial}{\partial t} \pm \frac{\partial}{\partial x_3}\,,$$

$$x_\pm \equiv \tfrac{1}{2}(t \pm x_3)\,,$$

$$(\vec{\sigma}\times\vec{\nabla})_3 \equiv \sigma_1\partial_2 - \sigma_2\partial_1\,.$$

The right-hand side of (2) depends on the potential. Let us first consider the case of vector coupling where

$$\beta V = -\beta\gamma^\mu A_\mu \equiv -\beta\hat{A}\,.$$

The right-hand side of (2) becomes

$$\beta V\psi = e^{i(p_3x_3 - Et)}C_V + e^{i(p_3x_3 + Et)}D_V\,,$$

$$C_V \equiv -\begin{pmatrix} [A_+ + i(\vec{\sigma}\times\vec{A})_3]\,\psi_+ \\ [\vec{\sigma}\cdot\vec{A} + \sigma_3 A_0]\psi_+ \end{pmatrix},$$

$$D_V \equiv -\begin{pmatrix} [A_- - i(\vec{\sigma}\times\vec{A})_3]\,\psi_- \\ [\vec{\sigma}\cdot\vec{A} - \sigma_3 A_0]\,\psi_- \end{pmatrix}, \tag{13}$$

where

$$A_\pm \equiv A_0 \pm A_3\,.$$

Equating (12) with (13) we get

$$(i\partial_+ + A_+)\psi_+ = \exp(2iEt)[m - i(\vec{\sigma}\times\vec{\pi})_3]\,\psi_-\,, \tag{14}$$

$$(i\partial_- + A_-)\psi_- = \exp(-2iEt)[m + i(\vec{\sigma}\times\vec{\pi})_3]\,\psi_+\,, \tag{15}$$

where

$$\vec{\pi} \equiv -i\vec{\nabla} - \vec{A}\,.$$

From the boundary condition (4a), the solution of (15) can be formally written

$$\begin{aligned} \psi_- &= (i\partial_- + A_-)^{-1}\exp(2iEt)[m + i(\vec{\sigma}\times\vec{\pi})_3]\,\psi_+ \\ &= \exp(2iEt)(i\partial_- + A_- + 2E)^{-1}[m + i(\vec{\sigma}\times\vec{\pi})_3]\,\psi_+\,, \end{aligned}$$

and Eq. (14) becomes

$$(i\partial_+ + A_+ -[m-i(\vec{\sigma}\times\vec{\pi})_3]\,(i\partial_- + A_- + 2E)^{-1}[m+i(\vec{\sigma}\times\vec{\pi})_3])\psi_+ = 0\,.$$

As $E\to\infty$, the last term of the above equation can be approximated by

$$(2E)^{-1}[m-i(\vec{\sigma}\times\vec{\pi})_3]\,[m+i(\vec{\sigma}\times\vec{\pi})_3] = (2E)^{-1}[\pi_\perp{}^2 - F_{12}\sigma_3] \simeq O(1/E)\,,$$

$$F_{12} \equiv \partial_1 A_2 - \partial_2 A_1\,.$$

Thus neglecting terms of the order $1/E$ means excluding virtual-pair-production amplitudes from the potential. We have instead of (14)

$$(i\partial_+ + A_+)\psi_+ = 0\,. \qquad (16)$$

The solution of (16) which satisfies the boundary condition (4b) is

$$\psi_+ = \exp\left[i\int_{-\infty}^{0} ds\,\epsilon\cdot A(x_\mu + \epsilon_\mu s)\right] e^{ip_\perp x_\perp}\psi_{+i}\,,$$

$$\epsilon_\mu \equiv (0, 0, 1, 1)$$

or, in the 4-component covariant form,

$$\psi_{\text{pos}} \simeq \exp\left[i\int_0^{+\infty} ds\,p\cdot A(x_\mu - p_\mu s)\right] e^{ip\cdot x} u_i\,,$$

which yields the scattering amplitude[7]

$$t_{fi} \simeq -iE\int d^2x_\perp\,dx_- e^{ix\cdot(p_i - p_f)}\left\{\exp\left[i\int_{-\infty}^{+\infty} ds\,p\cdot A(x_u - p_u s)\right] - 1\right\}$$

and the Green's function $G_A(p, x)$ in an external electromagnetic field

$$G_A(p, x) \simeq G(p)\exp\left[2p\cdot\int dk A(k) e^{ik\cdot x}\frac{1}{2k\cdot p}\right], \qquad (17)$$

where $G(p)$ is the free Green's function. Comparing (17) with the exact expression

$$G_A(p, x) = G(p)\left[1 + \sum_{n=1}^{+\infty}\int dk_1\cdots dk_n \hat{A}(k_1)G(p+k_1)\cdots\hat{A}(k_n)G(p+k_1+\cdots+k_n)\exp[i(k_1+\cdots+k_n)\cdot x]\right], \qquad (18)$$

one easily sees that the high-energy approximation (17) is indeed the familiar Lévy-Sucher[8] eikonal approximation.

We now proceed to study the case of pseudoscalar coupling where

$$\beta V \equiv \beta\gamma_5 U\,.$$

The right-hand side of (2) becomes

$$\beta\gamma_5\psi = e^{i(p_3x_3 - Et)}C_P + e^{i(p_3x_3+Et)}D_P\,,$$

$$C_P \equiv \begin{pmatrix} U\sigma_3\psi_+ \\ -U\psi_+ \end{pmatrix},$$

$$D_P \equiv \begin{pmatrix} -U\sigma_3\psi_- \\ -U\psi_- \end{pmatrix}. \qquad (19)$$

Equating (19) with (12) we have

$$i\partial_+\psi_+ = e^{2iEt}[m - \sigma_3 U - (\vec{\sigma}\times\vec{\nabla})_3]\,\psi_-\,, \qquad (20)$$

$$i\partial_-\psi_- = e^{-2iEt}[m + \sigma_3 U + (\vec{\sigma}\times\vec{\nabla})_3]\,\psi_+\,. \qquad (21)$$

The solution of (21) which satisfies (4a) can be formally written

$$\psi_- = e^{-2iEt}(i\partial_- + 2E)^{-1}[m + \sigma_3 U + (\vec{\sigma}\times\vec{\nabla})_3]\,\psi_+$$
$$\simeq e^{-2iEt}(2E)^{-1}[m + \sigma_3 U + (\vec{\sigma}\times\vec{\nabla})_3]\,\psi_+\,,$$

and Eq. (20) becomes

$$i\partial_+\psi_+ \simeq (2E)^{-1}[m^2 - \nabla_\perp^2 - i\sigma_\perp\partial_\perp - U^2]\,\psi_+\,. \qquad (22)$$

The right-hand side of (22) can be simplified if we limit ourselves to the following configuration:

$$\frac{U}{E},\ \frac{P_\perp}{E} \ll 1\,,$$

$$\frac{P_\perp}{E} \ll \frac{U^2}{E}\,.$$

Then (20) becomes

$$i\partial_+\psi_+ \simeq -(2E)^{-1}U^2\psi_+\,. \qquad (23)$$

The above derivation shows clearly that one cannot neglect ψ_- in the determination of ψ_+. Since ψ_- has no counterpart in nonrelativistic physics, one expects that the usual eikonal approximation should fail as a high-energy approximation.

From (23) we have

$$\psi_{\text{pos}} \simeq \exp\left[\frac{i}{2E}\int_0^{+\infty} ds\,U^2(x_\mu - \epsilon_\mu s)\right] e^{ip\cdot x} u_i\,,$$

which yields the high-energy Green's function G_U in a pseudoscalar potential

$$G_U(p,x)\simeq G(p)\exp\left[i\int dk\,dk'\,U(k)U(k')e^{i(k+k')x}\,\frac{1}{2p\cdot(k+k')}\right]. \tag{24}$$

Comparing (24) with the exact expression of $G_U(p,x)$ which is obtained from (18) by replacing $\hat{A}$ by $\gamma_5 U$, we easily see that (24) is not the usual eikonal Green's function. As to the scattering amplitude, we shall show that exponentiation[5] still holds. Starting from

$$(m+\gamma\cdot P+\gamma_5 U)G_U=1 \tag{25}$$

and multiplying both sides with $m-\gamma\cdot P-\gamma_5 U$, we get

$$[m^2+P^2+\gamma_5(\gamma\cdot PU)-U^2]G_U=m-\gamma\cdot P-\gamma_5 U,$$

which becomes

$$(m^2+P^2-U^2)G_U\simeq m-\gamma\cdot P,$$

or equivalently,

$$\begin{aligned}&[m+\gamma\cdot P-(m-\gamma\cdot P)^{-1}U^2]G\simeq 1,\\ &[m+\gamma\cdot P-\beta(2E)^{-1}U^2]G\simeq 1.\end{aligned} \tag{26}$$

The last equation was obtained by limiting ourselves to the scattering problem. Comparing (25) with (26) we have

$$\beta\gamma^5 U\underset{E\to\infty}{\simeq}-(2E)^{-1}U^2,$$

which immediately leads to the high-energy scattering amplitude[9]

$$t_{fi}\simeq iE\int d^2x_\perp\,dx_-\,e^{ix\cdot(p_i-p_f)}\left\{\exp\left[\frac{i}{2E}\int_{-\infty}^{+\infty}ds\,U^2(x_\mu-\epsilon_\mu s)\right]-1\right\}.$$

Thus exponentiation still holds even though the eikonal approximation fails.

Although the results obtained in this work are not new,[7,9] our method of approximation is general and simple enough to justify yet another paper on high-energy potential scattering. The infinite-momentum variables are not introduced[9] *ad hoc*, but make their appearance as a result of the straight-line approximation on the single-particle component of the relativistic wave function. It is with this method that one is able to see clearly the relativistic role of particle production in high-energy potential scattering.

[1]R. J. Glauber, in *Lectures in Theoretical Physics*, edited by W. E. Brittin *et al*. (Interscience, New York, 1959), Vol. I.

[2]R. P. Feynman, Phys. Rev. 80, 440 (1950).

[3]Ya. B. Zel'dovich and V. S. Popov, Usp. Fiz. Nauk 105, 403 (1971) [Sov. Phys.—Usp. 14, 673 (1972)].

[4]See also L. I. Schiff, Phys. Rev. 103, 443 (1956).

[5]H. Cheng and T. T. Wu, Phys. Rev. D 3, 2397 (1971).

[6]B. Narozhnyi and A. J. Nikishov, Yad. Fiz. 11, 1072 (1970) [Sov. J. Nucl. Phys. 11, 596 (1970)].

[7]See also H. Cheng and T. T. Wu, Phys. Rev. D 1, 1069 (1970), for the special case of a static potential.

[8]M. Lévy and J. Sucher, Phys. Rev. 186, 1656 (1969).

[9]The same result had been obtained by C. E. Carlson and T. L. Neff, Phys. Rev. D 4, 532 (1971), with the infinite-momentum technique.