Electron-electron scattering. III. Helicity cross sections for electron-electron scattering*

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The helicity amplitudes for electron-electron scattering to order e^4 are calculated by using the five invariant amplitudes presented in a previous paper. As applications, we rederive the unpolarized cross section given by Polovin and the spin-momentum correlation result obtained by Barut and Fronsdal.

I. INTRODUCTION

In the first paper[1] of this series (hereafter referred to as I), the five invariant amplitudes to order e^4 for electron-electron scattering were calculated by using the causal techniques of source theory. As applications of these amplitudes, we derived in the second paper[2] (referred to as II) the helicity amplitudes for electron-positron scattering, the implied unpolarized differential cross section,[3] and the spin-momentum correlation[4] for a polarized target positron. As a final application of the results of the preceding papers, we will here consider the corresponding electron-electron scattering case. Since the procedure is identical to that of II, we will simply give the results. With the development of colliding beams, a detailed study of electron-electron scattering becomes possible. In particular, it is now feasible to perform experiments with polarized electrons for which the results of this paper are directly relevant.[5]

The basic equation is the scattering amplitude for like-charged particles. In terms of the results of I, the fourth-order amplitude is [cf. Eq. (I74)]

$$\langle 1_{p_1\sigma_1}1_{p_1'\sigma_1'}|1_{p_2\sigma_2}1_{p_2'\sigma_2'}\rangle = 8i\alpha^2(2\pi)^4\delta(p_1+p_{1'}-p_2-p_{2'})(d\omega_{p_1}d\omega_{p_{1'}}d\omega_{p_2}d\omega_{p_{2'}})^{1/2}\sum_{i=1}^{5}[M_1^i\Gamma_i(12';1'2)-M_2^i\Gamma_i(12;1'2')], \quad (1)$$

where[1,2]

$$\Gamma_i(ab;cd)\equiv u_a^*\gamma^0\Gamma_i u_b u_c^*\gamma^0\Gamma_i u_d\,, \quad (2)$$

and

$$M_1^i=\int dx\,dy\frac{h_i}{\sqrt{\Delta}}\frac{1}{u+y}\left(\frac{1}{s+x+4}+\frac{\theta_i}{t+x+4}\right)+\int dy\left(\frac{y-4\lambda^2}{y}\right)^{1/2}\frac{\chi_i}{(y-4)^2}\frac{1}{u+y}(1+\theta_i)$$
$$+\int dx\frac{\bar{\chi}_i}{[x(x+4)]^{1/2}}\left(\frac{1}{s+x+4}+\frac{\theta_i}{t+x+4}\right)+\int dx\frac{2D_i(u,s,t)}{u+x+4}, \quad (3)$$

$$M_2^i=M_1^i(t\leftrightarrow u)\,. \quad (4)$$

Here

$$\theta_i=\begin{cases}-1, & i=1,2\\ +1, & i=3,4,5\end{cases} \quad (5)$$

and the various weight functions are given in I. The kinematical relationships are

$$m^2s=(p_2+p_{2'})^2\,,\quad m^2t=(p_1-p_2)^2\,,\quad m^2u=(p_1-p_{2'})^2\,,$$
$$s+t+u+4=0\,. \quad (6)$$

Note that the $M_{1,2}^i$'s here are related to those for electron-positron scattering, aside from a factor of θ_i, by $s\leftrightarrow u$, as expected by crossing symmetry.

In Sec. II, the helicity amplitudes are presented in terms of the invariant amplitudes. Their explicit forms, their relation to the cross sections, and the soft-photon contributions are given in Sec. III. The spin-momentum correlation for the case where the spin of only one of the electrons is detected is derived in Sec. IV.

II. HELICITY AMPLITUDES

In applying Eq. (1) to a particular helicity state, we encounter two basic structures. The first of these is

$$F(11';22')=\sum_{i=1}^{5}M_2^i\Gamma_i(12;1'2'), \qquad (7)$$

which is identical to that of II. The result, as given in Eq. (II3), is (again dropping the subscript 2)

$$\begin{aligned}
F(++;++)&=\frac{1}{2}sM^1+\left(\frac{s+2}{2}-\frac{t}{s+4}\right)M^2-\frac{u}{s+4}M^4-\frac{1}{2}sM^5,\\
F(--;++)&=\frac{1}{2}(s-u)M^1+\frac{t}{s+4}M^2-\frac{1}{4}tM^3+\frac{1}{2}\frac{s+2}{s+4}tM^4+\frac{1}{2}(t+4)M^5, \qquad (8)\\
F(-+;-+)&=-\frac{1}{2}uM^1-\frac{1}{2}\frac{s+2}{s+4}uM^2-\frac{u}{s+4}M^4-\frac{1}{2}uM^5,\\
F(+-;-+)&=-\frac{t}{s+4}M^2-\frac{1}{4}tM^3+\frac{t}{s+4}M^4,\\
F(+-;++)&=-\frac{1}{2}\frac{(-stu)^{1/2}}{s+4}(M^2-M^4).
\end{aligned}$$

The second structure is

$$\tilde{F}(11';22')=\sum_{i=1}^{5}M_1^i\Gamma_i(12';1'2). \qquad (9)$$

This can be obtained from Eq. (7) by the use of a Fierz transformation[2,6]

$$\Gamma_i(12';1'2)=\lambda_{ij}\Gamma_j(12;1'2'), \qquad (10)$$

with

$$\lambda_{ij}=\frac{1}{4}\begin{bmatrix}-2&0&-8&0&2\\0&-2&-8&-6&2\\-1&0&0&0&-1\\2&-2&0&2&0\\2&0&-8&0&-2\end{bmatrix}. \qquad (11)$$

We then find

$$\begin{aligned}
\tilde{F}(++;++)&=-F(++;++)[u\leftrightarrow t],\\
\tilde{F}(--;++)&=-F(--;++)[u\leftrightarrow t],\\
\tilde{F}(-+;-+)&=-F(+-;-+)[u\leftrightarrow t], \qquad (12)\\
\tilde{F}(+-;-+)&=-F(-+;-+)[u\leftrightarrow t],\\
\tilde{F}(+-;++)&=F(+-;++)[u\leftrightarrow t].
\end{aligned}$$

III. CROSS SECTIONS

The helicity amplitudes are defined in terms of the scattering amplitudes by

$$\langle 1_{p_1\sigma_1}1_{p_{1'}\sigma_{1'}}|1_{p_2\sigma_2}1_{p_{2'}\sigma_{2'}}\rangle =8\pi i\alpha(2\pi)^4\delta(p_1+p_{1'}-p_2-p_{2'})\times(d\omega_{p_1}\cdots d\omega_{p_{2'}})^{1/2}f(\sigma_1\sigma_{1'};\sigma_2\sigma_{2'}), \qquad (13)$$

with

$$f(\cdots)=f^{(2)}(\cdots)+\frac{\alpha}{\pi}f^{(4)}(\cdots). \qquad (14)$$

Here the lowest-order helicity amplitudes are given by the lowest-order interaction, Eq. (I63). They are

$$\begin{aligned}
f^{(2)}(++;++)&=-\frac{s+2}{t}-\frac{s+2}{u}+\frac{4}{s+4},\\
f^{(2)}(--;++)&=-\frac{4}{s+4},\\
f^{(2)}(-+;-+)&=-\frac{s}{s+4}-\frac{s+2}{t}, \qquad (15)\\
f^{(2)}(+-;-+)&=-\frac{s}{s+4}-\frac{s+2}{u},\\
f^{(2)}(+-;++)&=\frac{(-stu)^{1/2}}{s+4}\left(\frac{1}{t}-\frac{1}{u}\right).
\end{aligned}$$

The fourth-order helicity amplitudes are obtained by using Eqs. (8) and (12) in conjunction with Eq. (1). The necessary spectral integrals are given in Appendix C of II. The final results are

$$\begin{aligned}
f^{(4)}(++;++)=&f^{(2)}(++;++)\hat{K}\ln\lambda\\
&+\left[\frac{1}{t}\left(s+2-\frac{2t}{s+4}\right)[(u+2)M(u)-(s+2)M(s)]\ln t\right.\\
&+\left(\frac{s+2}{s+4}\frac{4}{t+4}-\frac{1}{2}\frac{s}{s+4}\right)\ln t+\left(2+\frac{1}{2}\frac{s+2}{s+4}t\right)M(t)+M(s)\\
&+\left(\frac{s+2}{s+4}\frac{8}{t+4}-4\frac{s+3}{s+4}-\frac{s^2+5s+2}{s(s+4)}t-\frac{1}{2}\frac{s+2}{s(s+4)}t^2\right)G(t)\\
&\left.-D(t,s)-\left(s+3+\frac{1}{2}\frac{3s^2+14s+4}{s(s+4)}t+\frac{1}{2}\frac{s+2}{s(s+4)}t^2\right)D(t,u)+\frac{1}{2}\left(\frac{s+2}{t}-\frac{2}{s+4}\right)\mathcal{L}(t)+(t\leftrightarrow u)\right], \qquad (16)
\end{aligned}$$

$$
\begin{aligned}
f^{(4)}(--;++) &= f^{(2)}(--;++)\hat{K}\ln\lambda \\
&+\bigg[\frac{2}{s+4}[(u+2)M(u)-(s+2)M(s)]\ln t-\frac{2t}{(t+4)(s+4)}\ln t \\
&+\left(\frac{t}{s+4}-1\right)M(t)-M(s)-t\left(\frac{4}{s+4}\frac{1}{t+4}+\frac{t+2}{s(s+4)}\right)G(t) \\
&+D(t,s)+\left(1-\frac{t(t+2)}{s(s+4)}\right)D(t,u)+\frac{1}{s+4}\mathcal{L}(t)+(t\leftrightarrow u)\bigg],
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
f^{(4)}(-+;-+) &= f^{(2)}(-+;-+)\hat{K}\ln\lambda \\
&-\frac{2}{s+4}[(t+2)M(t)-(s+2)M(s)]\ln u-\frac{s+2}{s+4}\frac{u}{t}[(u+2)M(u)-(s+2)M(s)]\ln t \\
&+\frac{2u}{(u+4)(s+4)}\ln u+\left(\frac{2}{t+4}-\frac{1}{2}\frac{s}{s+4}\frac{t}{t+4}\right)\ln t+\left(1-\frac{u}{s+4}\right)M(u) \\
&-\left(1-\frac{u}{s+4}\right)M(t)+\frac{1}{2}sM(s)+u\left(\frac{4}{s+4}\frac{1}{u+4}+\frac{u+2}{s(s+4)}\right)G(u) \\
&+\left(\frac{8}{t+4}\frac{s+2}{s+4}-\frac{1}{2}\frac{t(t+2)}{u}-4\frac{s+3}{s+4}-\frac{s^2+4s-2}{s(s+4)}t+\frac{t^2}{s(s+4)}\right)G(t) \\
&+\left(\tfrac{1}{2}s-\frac{(s+2)(s+4)}{2u}\right)D(t,s)+\frac{u(u+2)}{s(s+4)}D(u,t) \\
&-\left(1-\frac{t(t+2)}{s(s+4)}\right)D(t,u)-\frac{1}{s+4}\mathcal{L}(u)+\frac{1}{2}\left(\frac{s+2}{t}+\frac{s+2}{s+4}\right)\mathcal{L}(t),
\end{aligned} \tag{18}
$$

$$
f^{(4)}(+-;-+)=f^{(4)}(-+;-+)(u\leftrightarrow t), \tag{19}
$$

$$
\begin{aligned}
f^{(4)}(+-;++) &= f^{(2)}(+-;++)\hat{K}\ln\lambda \\
&-\frac{1}{2}\frac{(-stu)^{1/2}}{s+4}\bigg[\frac{2}{t}[(u+2)M(u)-(s+2)M(s)]\ln t-\frac{2}{t+4}\ln t+\left(1-\frac{s+4}{t}\right)M(t) \\
&+\left(\frac{t}{u}-\frac{t+6}{s}-\frac{4}{t+4}\right)G(t)-\frac{s+4}{u}D(t,s)+\frac{u-2}{s}D(t,u)+\frac{1}{t}\mathcal{L}(t)-(t\leftrightarrow u)\bigg].
\end{aligned} \tag{20}
$$

The following functions have been used in the above expressions for the helicity amplitudes [see Eqs. (II16), (II17), and (II18)]:

$$
\begin{aligned}
\hat{K} &= K(u\leftrightarrow s) \\
&= 2[(t+2)M(t)+(u+2)M(u)-(s+2)M(s)-1],
\end{aligned} \tag{21a}
$$

$$
\begin{aligned}
\mathcal{L}(x) &= 4[1-2\Phi_x\coth 2\Phi_x+\tfrac{1}{2}\Phi_x\tanh\Phi_x \\
&\quad -\tfrac{1}{2}(x+2)N(x)] \\
&\quad +2[(1-\tfrac{1}{3}\coth^2\Phi_x)(1-\Phi_x\coth\Phi_x)-\tfrac{1}{9}],
\end{aligned} \tag{21b}
$$

$$
\ln x=\ln|x|-\pi i\eta(-x), \tag{21c}
$$

$$
M(x)=\frac{\Phi_x}{\sinh 2\Phi_x}, \tag{21d}
$$

$$
\Phi_x=\tfrac{1}{2}\ln\left|\frac{[(x+4)/x]^{1/2}+1}{[(x+4)/x]^{1/2}-1}\right|-\tfrac{1}{2}\pi i\eta(-x), \tag{21e}
$$

$$
D(x,y)=M(y)\ln x+N(y), \tag{21f}
$$

$$
\begin{aligned}
N(x) &= \frac{1}{\sinh 2\Phi_x}[-\Phi_x^{\,2}-2\Phi_x\ln(1+e^{-2\Phi_x}) \\
&\quad +f(-e^{-2\Phi_x})+\tfrac{1}{12}\pi^2],
\end{aligned} \tag{21g}
$$

and

$$
G(x)=\frac{1}{\sinh 2\Phi_x}[f(e^{-2\Phi_x})+\Phi_x^{\,2}+\tfrac{1}{3}\pi^2]. \tag{21h}
$$

In the above, $f(x)$ is the Spence function[7]:

$$
f(x)=-\int_0^x\frac{dz}{z}\ln|1-z|. \tag{22}
$$

The helicity amplitudes yield the differential cross sections in the center-of-mass system as

$$
\left(\frac{d\sigma}{d\Omega}\right)_{\sigma_2\sigma_2'\to\sigma_1\sigma_1'}=-\frac{\alpha^2}{m^2 s}\left\{[f^{(2)}(\sigma_1\sigma_1';\sigma_2\sigma_2')]^2+\frac{2\alpha}{\pi}f^{(2)}(\sigma_1\sigma_1';\sigma_2\sigma_2')\,\mathrm{Re}f^{(4)}(\sigma_1\sigma_1';\sigma_2\sigma_2')\right\}. \tag{23}
$$

In particular, the unpolarized differential cross section is given by

$$\frac{d\sigma}{d\Omega} = -\frac{\alpha^2}{m^2 s}\left(\hat{U} + \frac{2\alpha}{\pi}\mathrm{Re}\hat{Y}\right), \tag{24}$$

where

$$\hat{U} = U(s \leftrightarrow u) = \frac{(s+2)^2}{t^2} - 4\frac{s+3}{s+4}\frac{1}{t} + 1 - 4\frac{s+3}{s+4}\frac{1}{u} + \frac{(s+2)^2}{u^2}, \tag{25}$$

which is the well-known result for Møller scattering,[8] and

$$\begin{aligned}\hat{Y} &= Y(s \leftrightarrow u) \\ &= \hat{U}\hat{K}\ln\lambda + \frac{1}{4}\Bigg\{\left[4\frac{(s+2)^2}{t^2} + 2\frac{s^2-12}{s+4}\frac{1}{t} - 2\frac{s^2+8s+12}{s+4}\frac{1}{u} + 2\right][-(u+2)M(u) + (s+2)M(s)]\ln t \\ &\quad - \left[\frac{u+2}{t} - \frac{2(u-4)(s+2)}{u(u+4)}\right]\ln u + \left[4\frac{s}{u} + 6 + \frac{(s+2)(s-6)}{t}\right]M(u) - \left[\frac{(s+2)(s+4)}{t} + s - 4\right]M(s) \\ &\quad + \left[8(s+2)\left(\frac{1}{t+4} + \frac{1}{t}\right) - \frac{s(s+6)}{u} + t + 3s + 6\right]G(t) - \left[\frac{2s(s+2)}{t} + s + 4 + \frac{4s}{u}\right]D(t,s) \\ &\quad + \left[\frac{2(s+4)(s+2)}{t} + s - \frac{4(s+2)}{u}\right]D(t,u) - \left[2\frac{(s+2)^2}{t^2} + \frac{s^2-12}{s+4}\frac{1}{t} - \frac{s^2+8s+12}{s+4}\frac{1}{u} + 1\right]\mathcal{L}(t) + (t \leftrightarrow u)\Bigg\}.\end{aligned} \tag{26}$$

Both of these results are the crossed form of the electron-positron case.

Finally, the soft-photon contributions are essentially the same as those given in II except for some changes. We find in the center-of-mass system

$$\frac{d\sigma}{d\Omega}(\text{inelastic})_{\sigma_2\sigma_{2'} \to \sigma_1\sigma_{1'}} = \left[\frac{d\sigma}{d\Omega}(\text{elastic})_{\sigma_2\sigma_{2'} \to \sigma_1\sigma_{1'}}\right]\frac{2\alpha}{\pi}\hat{J}_0, \tag{27}$$

where

$$\begin{aligned}\hat{J}_0 &= 2\pi^2\int d\omega_k\left(-\frac{p_{2'}}{p_{2'}k} - \frac{p_2}{p_2k} + \frac{p_{1'}}{p_{1'}k} + \frac{p_1}{p_1k}\right)^2 \\ &= \hat{K}\ln(2\Delta E/m\lambda) + 2\,\mathrm{Re}[\Phi_s\tanh\Phi_s - N(s)\cosh2\Phi_s + \tfrac{1}{2}\pi^2\coth2\Phi_s] \\ &\quad + 2\left[\frac{H(\theta_t)\cosh2\Phi_t}{\sinh2\Phi_s} - N(t)\cosh2\Phi_t - \frac{2\Phi_t\,\mathrm{Re}\ln2\sinh\Phi_s}{\tanh2\Phi_t} + (t \to u)\right].\end{aligned} \tag{28}$$

Here ΔE is the minimum detectable energy and

$$H(\theta) = \frac{1}{\sin\frac{1}{2}\theta}\int_{\cos(\theta/2)}^{1}\frac{dx}{[x^2 - \cos^2(\frac{1}{2}\theta)]^{1/2}}\left(\frac{\ln[\frac{1}{2}(1+\beta x)]}{1-\beta x} - \frac{\ln[\frac{1}{2}(1-\beta x)]}{1+\beta x}\right), \tag{29}$$

$$\beta = [(s+4)/s]^{1/2}, \quad \theta_t = \theta, \quad \theta_u = \pi - \theta.$$

Our result is in agreement with that given by Polovin[3] except for the typographical errors mentioned in II.

IV. SPIN-MOMENTUM CORRELATIONS

As another application of the helicity amplitudes, we will consider the spin-momentum correlation, first calculated by Barut and Fronsdal.[9] This is the case where one of the incoming electrons is polarized and the spins of the other particles are not detected. The differential cross section is

$$\frac{d\sigma}{d\Omega} = (1 + \xi P)\left(\frac{d\sigma}{d\Omega}\right)_{\text{unpol}}, \tag{30}$$

where ξ is the degree of polarization. The calculation is as presented in II. We have

$$\hat{U}P = \alpha\frac{\vec{n}\cdot\vec{p}_2\times\vec{p}_1}{|\vec{p}_2\times\vec{p}_1|}(-stu)^{1/2}\hat{\mathfrak{a}}. \tag{31}$$

Here, $\vec{n}$ is the direction of polarization and

$$\hat{\mathcal{A}} = \frac{1}{\pi(-stu)^{1/2}} \mathrm{Im}\{f^{(2)}(5)[f^{(4)}(2) - f^{(4)}(4) - f^{(4)}(3) - f^{(4)}(1)] + f^{(4)}(5)[f^{(2)}(4) + f^{(2)}(3) + f^{(2)}(1) - f^{(2)}(2)]\}, \tag{32}$$

where the numbers in $f()$ label the helicity amplitudes in their order of appearance in Eq. (8). The result is

$$\hat{\mathcal{A}} = -\frac{1}{[s(s+4)]^{1/2}}\left[\frac{3}{2}\left(\frac{s+2}{u^2} + 2\frac{s+3}{ut}\right)\ln\left(-\frac{t}{s+4}\right) + \frac{1}{t} - (t \leftrightarrow u)\right]. \tag{33}$$

This is in agreement with that given by Barut and Fronsdal.

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