How to construct a relativistic SU(6) classification symmetry group*

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A relation between the algebra of $\mathrm{SU}(6)_{W\,\mathrm{currents}}$ and that of $\mathrm{SU}(6)_{W,\mathrm{strong}}$ is given for the quark model with arbitrary interaction as quantized on a null plane. A unitary transformation is constructed which connects the generators of the algebras in such a way that the generators W_i^α of $\mathrm{SU}(6)_{W,\mathrm{strong}}$ classify states independent of momentum. The W_i^α are shown to have the simple angular momentum properties expected in the naive quark model. For the free-quark model, the results reduce essentially to those of Melosh, although no problems involved in the use of wave packets arise here.

I. INTRODUCTION

In the free-quark model with degenerate masses all 35 null-plane charges F_i^α commute with the Hamiltonian even though they are not generated by conserved currents. Although the F_i^α are not completely Lorentz invariant, they do generate an SU(6) algebra which is invariant under the subgroup of the Poincaré group[1] E(2)×D generated by $\vec{E}_\perp$, J_3, and K_3. As usual, the $\mathrm{SU}(6)_{W,\,\mathrm{currents}}$ invariance of the Hamiltonian implies that the F_i^α classify the states into degenerate SU(6) multiplets. The fact that the stability group of $\mathrm{SU}(6)_{W,\,\mathrm{currents}}$ is so large implies that the transformation properties of states with respect to the F_i^α are momentum independent, since E(2)×D can connect any two values of momentum of a given particle. The $\mathrm{SU}(6)_{W,\,\mathrm{currents}}$ symmetry is therefore not restricted to rest or collinear states only. In the free-quark model with nondegenerate masses the Hamiltonian is no longer SU(6) invariant, but the F_i^α still classify states into mass-nondegenerate multiplets of SU(6), and this classification scheme is still momentum independent. Finally, we may expect this to remain valid in the general interacting-quark model. It is likely, however, that the physical states—the eigenstates of the Hamiltonian—will not transform in any especially simple form under the action of the F_i^α. This is quite well known by now, and has been discussed from a variety of viewpoints.[2-4]

Historically, a principal reason that we consider an $\mathrm{SU}(6)_{W,\,\mathrm{strong}}$ approximate symmetry at all is because the observed particles and resonances seem to fall into supermultiplets of different spin.[5] This may be a dynamical accident. The actual algebraic framework of the theory of broken $\mathrm{SU}(6)_{W,\,\mathrm{strong}}$ symmetry derives by analogy from that of $\mathrm{SU}(6)_{W,\,\mathrm{currents}}$, whose operators are well determined in terms of physically measurable operators in weak and electromagnetic transitions. The obvious question is whether there is any relation between the two algebras.

Since commutation of the generators of a "symmetry" group with the Hamiltonian is not a sufficient criterion, in the null-plane formalism,[6,7] to imply mass degeneracy (even approximately), one is forced to search for a means of defining the generators of $\mathrm{SU}(6)_{W,\,\mathrm{strong}}$ other than requiring such commutation. In any event, since $\mathrm{SU}(6)_{W,\,\mathrm{strong}}$ is certainly not an exact symmetry for physical states,[8] such a definition would leave something to be desired.

It has been a frequently expressed[9,10] hope that some spin criterion would lead to a unique definition of the W_i^α. In the free-quark model Melosh[2] has shown that a spin criterion essentially determines the W_i^α, although it has been an open question as to whether one can use a similar criterion to define the W_i^α precisely. For interacting theories, nothing has been known.

Melosh has speculated that the unitary transformation which relates current and constituent quarks may have the form of a product of an operator which satisfies his spin criterion (and is therefore interaction independent) and a second operator which commutes with $\vec{\mathfrak{J}}$ and which contains all the pair states and exotic representations resulting from interactions. Unfortunately, he has no way of determining this second interaction-dependent factor.

The angular condition is formulated without restrictions on the masses of states—in fact without reference to the presence or nature of any interaction—so the obvious question is whether some (angular?) requirement of similar properties can be used to define the W_i^α unambiguously.

Recall that $\mathrm{SU}(6)_{W,\,\mathrm{strong}}$ finds its origins[5] in the old nonrelativistic SU(6) and in the attempt to avoid the inconsistencies that arose when SU(6) was applied to collinear processes. At first, therefore, $\mathrm{SU}(6)_{W,\,\mathrm{strong}}$ was thought of as a collinear vertex symmetry. However, it was found to be a broken symmetry even so restricted; the phenomenological successes of the work of Melosh[2] and his followers[11] have been in the study of the violations of $SU(6)_{W,\,\text{strong}}$ symmetry for decay and other collinear amplitudes. So, in fact, $SU(6)_{W,\,\text{strong}}$ is not a vertex symmetry, but rather a classification symmetry; but then one has no reason to confine it to collinear states.

Granting this, we have two possible extreme positions that may alternatively be adopted: The symmetry of hadronic states is of the rest type (only J_3 leaves the W_i^α algebra exactly invariant), which is the conclusion of de Alwis and Stern,[3] or the symmetry of hadronic states is momentum independent, which is the attitude taken in the present work. Of course, positions intermediate between these two may be adopted, but they are more difficult to express in a precise manner.

In order to discuss a classification symmetry such as $SU(6)_W$, without any problems of the existence of operators when currents are not conserved or problems raised by theorems of the type proved by Coleman,[12] it is best to define all operators and commutators on a standard null plane. In this case, the good charges F_i^α, as defined by commutation with the stability group of the null plane, constructed by integrals of current densities over the null plane, annihilate the vacuum and obey an SU(6) algebra.[7] These current densities are bilinear forms in good fields, $\psi_+(x)$, which are projections of the quark field operators $\psi(x)$ onto the two-dimensional invariant subspace of the four-dimensional representation space; the $\psi_+(x)$ then satisfy covariant anticommutation relations on the null plane. On the null plane, however, two representations of the anticommutation relations may be unitarily equivalent and yet they may have different representations of the Poincaré group associated with them.[13] This is quite different from the situation in the spacelike formalism.[14] We thus search for another spinor field, which we call $\varphi_+(x;\text{ constituent})$, which satisfies the null-plane anticommutation relations, is unitarily related to $\psi_+(x)$, and for which the resultant charges W_i^α obtained by transforming the F_i^α may be used to classify states according to the SU(6) algebra that they generate in a momentum-independent manner, where the states for arbitrary momentum are obtained from rest states by means of the traditional Lorentz boosts generated by the operators K_i.

The form of this paper is organized as follows. In Sec. II we discuss the action of the operators of the Lorentz group on states and on field operators, then we define current quark and constituent quark bases, and finally we define good fields ψ_+. Armed with this formalism, we construct in Sec. III the transformation which relates to two quark bases, and in Sec. IV we use this result to compute the generators W_i^α of $SU(6)_{W,\,\text{strong}}$. Then we discuss their angular momentum properties and their behavior under the action of other elements of the Poincaré group. In Sec. V we discuss some applications to the phenomenology after investigating the algebraic structure of matrix elements of integrated currents and their moments; in particular, we study the magnetic moment operator for spin-$\frac{1}{2}$ particles. Finally, in Sec. VI we give a summary of our results.

II. BASIC FORMALISM

The free-field Fourier expansion for $\psi(x)$ and other fields in the theory must be modified in the presence of their interactions,[15] since the fields no longer have the space-time coordinate dependence given by solutions of the free-field equations. In fact, we do not know in general what this coordinate dependence will be. Nevertheless, some definite statements can still be made without our specifying the interaction. We work in the Schrödinger picture so we can evaluate all Heisenberg picture operators on the standard light plane $x^+=0$. Making a three-dimensional Fourier expansion of $\psi(x)$, we have

$$\psi(x^+=0,x^-,\vec{x}_\perp)=[2(2\pi)^3]^{-1/2}\int d^2p_\perp\int_0^\infty\frac{d\eta}{\eta}\sum_\lambda[b(p,\lambda;0)\,u(p,\lambda)\exp(-ip\cdot x)+d^\dagger(p,\lambda;0)\,u(-p,-\lambda)\exp(ip\cdot x)]\,. \tag{2.1}$$

The development of $\psi(x)$ in x^+ is determined by the generator of x^+ displacements, P^-, which takes the place of the Hamiltonian in the conventional formalism. Explicitly, we have

$$[P^-,\psi(x)]=\frac{1}{i}\,\frac{\partial\psi(x)}{\partial x^+}\ . \tag{2.2}$$

With no interaction, it is straightforward to show that the above expansion holds for arbitrary x^+ and that, moreover,

$$\begin{aligned} b_{\text{free}}(p,\lambda;x^+)&=\exp(-ip^-x^+)\,b_{\text{free}}(p,\lambda;0)\ ,\\ d^\dagger_{\text{free}}(p,\lambda,x^+)&=\exp(-ip^-x^+)\,d^\dagger_{\text{free}}(p,\lambda;0)\ . \end{aligned} \tag{2.3}$$

In the presence of interaction these simple results no longer obtain; we rather have

$$\begin{aligned} b(p,\lambda;x^+)&=e^{iP^-x^+}b(p,\lambda;0)\,e^{-iP^-x^+}\ ,\\ d^\dagger(p,\lambda;x^+)&=e^{iP^-x^+}d^\dagger(p,\lambda;0)\,e^{-iP^-x^+}\ , \end{aligned} \tag{2.4}$$

so that these operators are in general no longer proportional to their values at $x^+=0$. They cannot in general be simply interpreted as creation and destruction operators for single quanta of definite masses[15]; nevertheless, they can be considered as creation and destruction operators in the present null-plane formulation. This has to do with one of the essential differences between null-plane and spacelike-plane kinematics, namely the translations which leave the plane invariant. In the case of the null plane $x^+=0$, these translations are generated by P^+, $\vec{P}_\perp$, while in the spacelike case the three translations are generated by P_1, P_2, P_3. The point is that P^+ has a positive spectrum, since $P^0>|P^3|$, while none of P_1, P_2, P_3 is positive. Referring back to the expansion of ψ given above, Eq. (2.1), we see that by definition the operator b lowers the component p^+ of the momentum by η, while the operator d^+ raises this component by η. Since the integration is restricted to positive values of η, and since the momentum component p^+ of any physical state is positive ($p^+=0$ only for the vacuum) we must conclude that

$$\begin{aligned} &b(p,\lambda;0)|0\rangle=0\ ,\\ &d(p,\lambda;0)|0\rangle=0\ , \end{aligned} \tag{2.5}$$

which enables us to interpret b and d as destruction operators. We thus *postulate* the covariant anticommutation relations

$$\begin{aligned}\{b(p,\lambda;0),b^\dagger(p',\lambda';0)\}&=\{d(p,\lambda;0),d^\dagger(p',\lambda';0)\}\\ &=\delta_{\lambda\lambda'}\,\eta\delta(\eta'-\eta)\,\delta^{(2)}(\vec{p}_\perp-\vec{p}\,'_\perp)\ ,\end{aligned} \tag{2.6}$$

with all other anticommutators vanishing. Owing to the relation Eq. (2.4) between these operators at arbitrary values of x^+ and their values at $x^+=0$, it follows that the anticommutation relations hold for all x^+:

$$\begin{aligned}\{b(p,\lambda;x^+),b^\dagger(p',\lambda';x^+)\}&\\ =\{d(p,\lambda;x^+),d^\dagger(p',\lambda';x^+)\}&\\ =\delta_{\lambda\lambda'}\,\eta\delta(\eta'-\eta)\,\delta^{(2)}(\vec{p}_\perp-\vec{p}\,'_\perp)\ .&\end{aligned} \tag{2.7}$$

In order to proceed further, we must specify the meaning of the label λ which appears above in the expression for the field $\psi(x)$ both in the spinors $u(p,\lambda)$ and in the operators $b(p,\lambda;0)$ and $d^+(p,\lambda;0)$. To do this, it is convenient to begin with a description of the transformation properties of single particle states under the Poincaré group and then continue to a discussion of the field operators ψ.

In general, the states of a massive particle of spin S can be labeled by the momentum $\eta,\vec{p}_\perp$ of the particle and some $(2S+1)$-valued spin index λ. For a state at rest, we define

$$J_3|m/\sqrt{2},\vec{0};\lambda\rangle=\lambda|m/\sqrt{2},\vec{0};\lambda\rangle\ , \tag{2.8}$$

so that λ is just the z component of spin for a rest state. For a moving state, however, this does not specify λ until we have defined how to boost the rest state to arbitrary momentum. Furthermore, by saying that the state has angular momentum S we have still only said something about the rest state, namely that it transforms according to the spin-S representation of SU(2):

$$U[\Lambda]\,|m/\sqrt{2},\vec{0};\lambda\rangle=\sum_{\lambda'}D^{(S)}[\Lambda]_{\lambda'\lambda}|m/\sqrt{2},\vec{0};\lambda'\rangle \tag{2.9}$$

for Λ a pure rotation. The states $|\eta,\vec{p}_\perp;\lambda\rangle$ of a particle with arbitrary momentum are defined by applying a particular Lorentz transformation to the rest states. We will discuss two different choices for this Lorentz boost.

In the conventional formalism, one defines[1]

$$|\eta,\vec{p}_\perp;\lambda;\text{constituent}\rangle=e^{-i\vec{\beta}\cdot\vec{K}}|m/\sqrt{2},\vec{0};\lambda\rangle\ , \tag{2.10}$$

where

$$\beta=\hat{p}\,\text{arcsinh}(|\vec{p}|/m)\ , \tag{2.11}$$

and we have added the notation "constituent" to the state vector to emphasize the dependence of the construction on the choice of the boosting operation. If we define

$$\mathcal{J}_i=e^{-i\vec{\beta}\cdot\vec{K}}J_i\,e^{i\vec{\beta}\cdot\vec{K}}\ , \tag{2.12}$$

then

$$\mathcal{J}_3|\eta,\vec{p}_\perp;\lambda;\text{constituent}\rangle=\lambda|\eta,\vec{p}_\perp;\lambda;\text{constituent}\rangle\ , \tag{2.13}$$

and it is easy to verify that the states transform simply under $\vec{\mathcal{J}}$ rotations and $\vec{K}$ boosts.

On the other hand, we might choose to define our states so that they transform simply under the Poincaré generators J_3, $\vec{E}_\perp$, and K_3 which leave the plane $x^+=0$ invariant. These states have arisen naturally recently in discussions of field theories in the infinite-momentum frame. They are defined by

$$|\eta,\vec{p}_\perp;\lambda;\text{current}\rangle=e^{-i\vec{v}_\perp\cdot\vec{E}_\perp}e^{-i\omega K_3}|m/\sqrt{2},\vec{0};\lambda\rangle\ , \tag{2.14}$$

where

$$\vec{v}_\perp=\vec{p}_\perp/\eta\ \text{and}\ e^{\omega}=\sqrt{2}\,\eta/m\ , \tag{2.15}$$

and we have added the notation "current" in analogy with the "constituent" notation above. Furthermore, if we define

$$\begin{aligned} j_3&=e^{-i\vec{v}_\perp\cdot\vec{E}_\perp}e^{-i\omega K_3}J_3\,e^{i\omega K_3}e^{i\vec{v}_\perp\cdot\vec{E}_\perp}\\ &=e^{-i\vec{v}_\perp\cdot\vec{E}_\perp}J_3\,e^{i\vec{v}_\perp\cdot\vec{E}_\perp}\\ &=J_3-\frac{1}{\eta}(\vec{E}_\perp\times\vec{P}_\perp)\ ,\end{aligned} \tag{2.16}$$

then

$$j_3 |\eta, \vec{p}_\perp; \lambda; \text{current}\rangle = \lambda |\eta, \vec{p}_\perp; \lambda; \text{current}\rangle , \quad (2.17)$$

and the states $|\eta, \vec{p}_\perp; \lambda; \text{current}\rangle$ can be shown to transform simply under j_3 rotations, $\vec{E}_\perp$ boosts, and K_3 boosts. These states are eigenstates of the ordinary helicity operator, but with respect to a reference frame moving with infinite velocity in the $-z$ direction.

It will be clear later why we have used the terms "constituent" and "current" to distinguish these alternatives to the construction of moving states.

Now we will apply these results to a discussion of field operators. If the field $\psi(x)$ transforms according to the $(S, 0) \oplus (0, S)$ representation of the Lorentz group, then

$$U[\Lambda]\psi_\alpha(x)U^{-1}[\Lambda] = \sum_\beta \mathfrak{D}^{(S)}_{\alpha\beta}[\Lambda^{-1}]\psi_\beta(\Lambda x) , \quad (2.18)$$

where

$$\mathfrak{D}^{(S)}[\Lambda] = \begin{bmatrix} D^{(S)}[\Lambda] & 0 \\ 0 & \bar{D}^{(S)}[\Lambda] \end{bmatrix} \quad (2.19)$$

and $D^{(S)}$, $\bar{D}^{(S)}$ are the usual $(2S+1)$-dimensional matrices representing the Lorentz transformation Λ in the $(S, 0)$ and $(0, S)$ representations, respectively. Weinberg[16] has shown that the spinors which appear in the Fourier expansion of $\psi(x)$ are simply proportional to these matrices. In particular, referring back to our expansion Eq. (2.1) for the four-component quark field operator we have

$$\begin{aligned} u_\alpha(p, \pm\tfrac{1}{2}) &= ND^{(1/2)}[\Lambda(p)]_{\alpha, \pm 1/2} \text{ for } \alpha = 1, 2 \\ &= N\bar{D}^{(1/2)}[\Lambda(p)]_{\alpha, \pm 1/2} \text{ for } \alpha = 3, 4 , \end{aligned} \quad (2.20)$$

where N is a normalization constant to be fixed below. But what is $\Lambda(p)$? For the conventional formalism which is that treated by Weinberg, $\Lambda(p)$ is just the K boost we used to define "constituent" basis states; on the other hand, the "current" basis states are defined in terms of $\vec{E}_\perp, K_3$ boosts, and it is this Lorentz transformation which must be used to define spinors in that context. In sum, we are led to define two sets of spinors (and associated creation and destruction operators) corresponding respectively to our two choices of boosting rest states; these will be written $u_\alpha(p, \lambda;$ constituent), $b(p, \lambda; x^+;$ constituent), and so on.

One further comment on the Lorentz transformation properties of spinors is appropriate at this juncture. There is another inherent kinematic aspect of the null-plane formalism which differs from that of the spacelike plane which we must use here. The stability group of a spacelike plane has the structure $SO_3 \times_s T(3)$, while that of a null plane is of the form $[E(2) \times D] \times_s T(3)$. The group T(3) is the translation group generated by P_1, P_2, P_3 in the spacelike case and by P^+, P_1, P_2 in the null-plane case. The group SO_3 is the rotation group generated by J_1, J_2, J_3, while the group $E(2) \times D$ is the group generated by E_1, E_2, K_3, J_3 and is isomorphic to the direct product of dilatations and the Euclidean group in two dimensions. The symbol $\times$ denotes a direct product, $\times_s$ a semidirect one. In terms of the covering group of the homogeneous Lorentz group, SL(2, C), the factor E(2) $\times$ D, which leaves the null plane $x^+ = 0$ invariant, consists of matrices of the form[17]

$$A = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \quad (2.21)$$

rather than of unitary matrices as in the spacelike case. This implies that the representation $D^{(S)}[\Lambda]$ is reducible if Λ is restricted to $E(2) \times D$. For the four-component quark spinors, the matrix $G = \frac{1}{2}(1 + \gamma_0\gamma_3)$ projects onto a two-dimensional invariant subspace spanned by spinors of the form

$$\psi_+(x) = G\psi(x) . \quad (2.22)$$

The projected spinor has two linearly independent components for each kind of quark. The spinor representation does not decompose, however; i.e., the subspace orthogonal to that spanned by ψ_+ is not invariant. In other words, if we define

$$\psi(x) = \psi_+(x) + \psi_-(x) , \quad (2.23)$$

then when $\psi_-(x)$ is Lorentz transformed by operators leaving the $x^+ = 0$ null plane invariant, it cannot be expressed solely in terms of ψ_- but must involve ψ_+ as well.

Recall now the Fourier expansion of $\psi(x)$. Clearly, since the projection operator G is linear we may write an expansion for $\psi_+(x)$ where the only difference is that we must use projected spinors

$$u_+(p, \lambda) \equiv Gu(p, \lambda) . \quad (2.24)$$

Now we shall choose the normalization of the spinors so that[18]

$$\sum_\lambda u^\dagger_{+\alpha}(p, \lambda)\, u_{+\beta}(p, \lambda) = \sqrt{2}\,\eta G_{\alpha\beta} . \quad (2.25)$$

Having done this, we may return to the anticommutation relation Eqs. (2.7) and transform back to configuration space; we find[19]

$$\{\psi_+(x^+,x^-,\vec{x}_\perp),\psi_+^\dagger(x^+,y^-,\vec{y}_\perp)\}=(G/\sqrt{2})\,\delta(x^--y^-)\,\delta^{(2)}(\vec{x}_\perp-\vec{y}_\perp)\,,$$
$$\{\psi_+(x^+,x^-,\vec{x}_\perp),\psi_+(x^+,y^-,\vec{y}_\perp)\}=0\,,\qquad(2.26)$$
$$\{\psi_+^\dagger(x^+,x^-,\vec{x}_\perp),\psi_+^\dagger(x^+,y^-,\vec{y}_\perp)\}=0\,.$$

Note that these relations have no analog for the "bad" components, ψ_-. This is because, as we have noted above, the subspace spanned by ψ_- is not invariant under the stability group of the null plane. Therefore, anticommutation relations for ψ_- analogous to those for ψ_+ could not possibly be covariant.

III. CONSTRUCTION OF THE TRANSFORMATION[20]

For the quark field, transforming as spin $\frac{1}{2}$, the action of an arbitrary Lorentz transformation is usually written[21]

$$U[\Lambda]\,\psi_\alpha(x)\,U^{-1}[\Lambda]=\sum_\beta S^{-1}{}_{\alpha\beta}[\Lambda]\,\psi_\beta(\Lambda x)\,,\qquad(3.1)$$

where S is a 4×4 matrix which operates on the four-component column vector $\psi(x)$. For spatial rotations, S is unitary, but this does not hold true for Lorentz boosts; in general,

$$S^{-1}=\gamma_0 S^\dagger\gamma_0\,.\qquad(3.2)$$

The important attribute of the matrix $S[\Lambda]$ that we need now is that it may be used to express the spinors $u(p,\lambda)$ for arbitrary momentum in terms of spinors describing rest states ($\vec{p}=0$)

$$u_\beta(p,\lambda)=\sum_\alpha S_{\beta\alpha}u_\alpha(\text{rest},\lambda)\,.\qquad(3.3)$$

Actually, since we have defined two sets of spinors we must be more explicit; that is, we must say how the boost is to be effected. In other words, we write

$$u_\beta(p,\lambda;\text{constituent})=\sum_\alpha S_{\beta\alpha}[\Lambda_{\text{constituent}}]\times u_\alpha(\text{rest},\lambda)\,,\qquad(3.4)$$

$$u_\beta(p,\lambda;\text{current})=\sum_\alpha S_{\beta\alpha}[\Lambda_{\text{current}}]\,u_\alpha(\text{rest},\lambda)\,,\qquad(3.5)$$

where $\Lambda_{\text{constituent}}$ and Λ_{current} are the Lorentz boosts appropriate to the constituent and current bases described above.

Next, we project the "good" spinors and write

$$u_{+\beta}(p,\lambda)=\sum_\alpha S_{+\beta\alpha}u_\alpha(\text{rest},\lambda),\qquad(3.6)$$

where of course we have defined the matrix

$$S_+[\Lambda]=GS[\Lambda]\,.\qquad(3.7)$$

Although S_+ is not a unitary matrix, we may construct a unitary matrix from it by multiplication by a simple factor; to this end, we define

$$\mathcal{S}[\Lambda]=\left(\frac{2m}{\sqrt{2}\,\eta}\right)^{1/2}S_+[\Lambda]\,,\qquad(3.8)$$

and note that

$$\begin{aligned}\delta_{\lambda\lambda'}&=\frac{1}{\sqrt{2}\,\eta}\sum_\beta u_{+\beta}^\dagger(p,\lambda)\,u_{+\beta}(p,\lambda')=\frac{1}{\sqrt{2}\,\eta}\sum_{\alpha,\beta,\gamma}u_\alpha^\dagger(\text{rest},\lambda)\,S_{+\alpha\beta}^\dagger S_{+\beta\gamma}u_\gamma(\text{rest},\lambda')\\&=\frac{1}{2m}\sum_{\alpha,\beta,\gamma}u_\alpha^\dagger(\text{rest},\lambda)\,\mathcal{S}^\dagger_{\alpha\beta}\mathcal{S}_{\beta\gamma}u_\gamma(\text{rest},\lambda')\\&=\frac{1}{2m}\sum_\alpha u_\alpha^\dagger(\text{rest},\lambda)\,u_\alpha(\text{rest},\lambda')\,,\end{aligned}\qquad(3.9)$$

since $\sum_\alpha u_\alpha^\dagger(\text{rest},\lambda)\,u_\alpha(\text{rest},\lambda)=2m\delta_{\lambda\lambda'}$. Therefore, we conclude that $\mathcal{S}[\Lambda]$ is a unitary matrix; that is to say, $\mathcal{S}[\Lambda_{\text{constituent}}]$ and $\mathcal{S}[\Lambda_{\text{current}}]$ are unitary matrices when acting on constituent and current basis spinors, respectively. It is then natural to use the matrices $\mathcal{S}[\Lambda]$ in the Fourier expansions of $\psi_+(x)$ to define new functions $\varphi(x;\text{constituent})$ and $\varphi(x;\text{current})$:

$$\psi_{+\alpha}(x)=\hat{\mathcal{S}}_{\alpha\beta}(\text{constituent})\,\varphi_\beta(x;\text{constituent}),$$

$$\varphi_\beta(x;\text{constituent})=[2(2\pi)^3]^{-1/2}\sum_\lambda\int\frac{d^2p_\perp d\eta}{\eta}\times[\eta^{1/2}b(p,\lambda;\text{constituent})\,w_\beta(\lambda)\,e^{-ipx}+\eta^{1/2}d^\dagger(p,\lambda;\text{constituent})\,w_\beta(-\lambda)\,e^{ipx}]\,,$$

$$w_\beta(\lambda)=2^{-1/4}m^{-1/2}u_\beta(\text{rest},\lambda)\qquad(3.10)$$

and analogous expressions for the current-quark basis. The operators $\hat{\mathcal{S}}$ are defined in terms of the Fourier expansion of the φ fields: One multiplies the integrand by the appropriate unitary matrix $\mathcal{S}[\Lambda]$ constructed above. We now will discuss the two cases separately.

For the current quark basis, we recall that the appropriate boost is one which involves $\vec{E}_\perp$ and K_3. These operators, however, are contained in the set of generators of the stability group of the null plane. This implies then that the matrix $S[\Lambda_{\text{current}}]$ transforms good fields only into good fields. In fact, one may see from the work of Kogut and Soper[18] that the operator $\hat{\mathcal{S}}$(current) is a trivial one and that

$$\psi_+(x) = \varphi_+(x;\text{current}), \tag{3.11}$$

where $\varphi_+(x)$ is the good component of $\varphi(x)$.

In the constituent quark basis, on the other hand, the appropriate boost involves the operators K_1, K_2, K_3 which are not contained in the null-plane stability group. This implies that the matrix $S[\Lambda_{\text{constituent}}]$ mixes good and bad fields, and *a fortiori* so does $\hat{\mathcal{S}}$(constituent). Nevertheless, $\hat{\mathcal{S}}$ is a unitary operator, so we may invert the equation relating φ and ψ_+,

$$\varphi(x;\text{current}) = \hat{\mathcal{S}}^\dagger \psi_+(x) , \tag{3.12}$$

and the action of $\hat{\mathcal{S}}^\dagger$ on ψ_+ may be computed by multiplying the integrand in the Fourier expansion of $\psi_+(x)$ by the matrix $\hat{\mathcal{S}}^\dagger[\Lambda_{\text{constituent}}]$. Furthermore, even though $\hat{\mathcal{S}}^\dagger$ mixed good and bad components of ψ, if we use the equations of motion to express the dependent field $\psi_-(x)$ in terms of the independent one $\psi_+(x)$ then it follows that $\varphi_+(x)$ may be expressed entirely in terms of $\psi_+(x)$:

$$\begin{aligned}&\varphi_+(x;\text{constituent}) = T\psi_+(x) ,\\ &T^\dagger = T^{-1} ,\end{aligned} \tag{3.13}$$

where T is obtained from $\hat{\mathcal{S}}^\dagger$ by expressing all operators in terms of good ones. We will compute T explicitly below, so that this rather abstract description of it may be made more concrete.

This is the natural place, however, to point out that it follows from Eq. (3.11) and Eq. (3.13) that T is a unitary transformation which relates the current and constitutent quark basis fields φ_+,

$$\varphi_+(x;\text{constituent}) = T\varphi_+(x;\text{current}), \tag{3.14}$$

and that the $\varphi_+(x)$ fields satisfy the usual anticommutation relations on the null plane.

In the absence of interaction, the matrix S is well known and may be found in standard texts on relativistic quantum mechanics

$$S = \frac{m + |p_0| - i\vec{\gamma}\cdot\vec{p}}{[2m(m+|p_0|)]^{1/2}} . \tag{3.15}$$

From S we form

$$\left(\frac{\sqrt{2}\,\eta}{2m}\right)^{1/2} S^{-1} = \left(\frac{p_0+p_3}{p_0}\right)^{1/2}\left(\frac{m}{2p_0}\right)^{1/2} \times \frac{m+|p_0|+i\vec{\gamma}\cdot\vec{p}}{[2m(m+|p_0|)]^{1/2}} , \tag{3.16}$$

which is the operator that acts on good spinors $u_+(p,\lambda)$ appearing in the expansion of $\psi_+(x)$.

In general, the equation of motion for $\psi(x)$ may be written

$$(\beta p_0 - i\vec{\gamma}\cdot\vec{p} - m)\psi = 0 . \tag{3.17}$$

Using the relation

$$\vec{\gamma} = -i\beta\vec{\alpha} \tag{3.18}$$

we have

$$(m + i\vec{\gamma}_\perp\cdot\vec{p}_\perp)\psi = (p_0 + \alpha_3 p_3)\beta\psi , \tag{3.19}$$

so that the nontrivial part of the matrix S^{-1} satisfies

$$\begin{aligned}(m+|p_0|+i\vec{\gamma}\cdot\vec{p})\psi &= (m+|p_0|+i\vec{\gamma}_\perp\cdot\vec{p}_\perp)\psi - \alpha_3\beta\psi\\ &= (m+|p_0|+i\vec{\gamma}_\perp\cdot\vec{p}_\perp)\psi .\end{aligned} \tag{3.20}$$

But for the good component ψ_+ we have $\alpha_3 = 1$; therefore,

$$(m+|p_0|+i\vec{\gamma}\cdot\vec{p})\psi_+ = \frac{p_0}{p_0+p_3}\,(m+|p_0+p_3|+i\vec{\gamma}_\perp\cdot\vec{p}_\perp)\psi_+ . \tag{3.21}$$

After some algebra we find

$$\begin{aligned}\sqrt{2}\,\varphi_+ &= \left(\frac{p_0+p_3}{p_0}\right)^{1/2}\left(\frac{m}{p_0}\right)^{1/2}\frac{m+|p_0|+i\vec{\gamma}\cdot\vec{p}}{[2m(m+|p_0|)]^{1/2}}\,\psi_+\\ &= \frac{m+|p_0+p_3|+i\vec{\gamma}_\perp\cdot\vec{p}_\perp}{[(m+|p_0+p_3|)^2+p_\perp{}^2]^{1/2}}\,\psi_+\\ &= \exp\left(i\arctan\frac{\vec{\gamma}_\perp\cdot\vec{p}_\perp}{m+|p_0+p_3|}\right)\psi_+ ,\end{aligned} \tag{3.22}$$

which displays $\varphi_+(x)$ in terms of $\psi_+(x)$. It is important to realize that the matrix $S[\Lambda_{\text{constituent}}]$ is the appropriate one for the canonical Wigner spin basis, which we are calling the "constituent" basis, while it is not appropriate for the infinite momentum helicity basis, which we are calling the "current" basis. The explicit form given in Eq. (3.15) is correct for the case of no interaction, while in the general case $S[\Lambda_{\text{constituent}}]$ will depend on interaction since it is defined in terms of the K boosts; these boosts are interaction dependent in null-plane dynamics, in contrast with the $E_\perp$ boosts which are interaction-free in this formalism.

IV. THE GENERATORS OF $SU(6)_{W,\,\mathrm{strong}}$

The generators F_i^α of $SU(6)_{W,\,\mathrm{currents}}$ can be defined in terms of bilinear products of ψ_+, in the free-quark model, and we will assume the same form for them in general. If further (good) fields are found to be necessarily included in F_i^α then the discussion can be suitably extended. Therefore, we will define

$$F_i^\alpha=\frac{1}{\sqrt{2}}\int d^4x\,\delta(x^+)\,\psi_+^\dagger(x)\,\Gamma^\alpha\tfrac{1}{2}\lambda_i\,\psi_+(x)\ , \qquad (4.1)$$

where λ_i is an SU(3) matrix and $\Gamma^\alpha=(2,\beta\sigma^1,\beta\sigma^2,\beta\sigma^3)$. The SU(6) generators are defined analogously

$$W_i^\alpha=\frac{1}{\sqrt{2}}\int d^4x\,\delta(x^+)\,\varphi_+^\dagger(x;\text{constituent})\times\Gamma^\alpha\tfrac{1}{2}\lambda_i\,\varphi_+(x;\text{constituent})\ . \qquad (4.2)$$

As Melosh has noted, the structure of the F_i^α is rather complicated under rotations; they are not scalars and components of vectors but rather contain components of all angular momenta. Thus states which transform irreducibly under $SU(6)_{W,\,\mathrm{currents}}$ cannot have definite spin. We claim that the charges W_i^α defined above do not suffer from this drawback, and so the resulting $SU(6)_{W,\,\mathrm{strong}}$ may be used to classify hadron states. This will be shown in this section.

First we note that the W_i^α as defined above do indeed generate an SU(6) algebra, since φ_+ and ψ_+ are unitarily related by the operator T, and so the structure of the algebra generated by the F_i^α is preserved.

Next we make use of the Fourier expansion of $\varphi_+(x)$ as derived in the preceding section, Eq. (3.10), to obtain the Fock space form of W_i^α

$$W_i^\alpha=\sum_{\lambda\lambda'}\int\frac{d^2p\,d\eta}{2\eta}\,[b^\dagger(p,\lambda;\text{constituent})\,\tfrac{1}{2}\lambda_i\,b(p,\lambda';\text{constituent})\,w_+^\dagger(\lambda)\,\Gamma^\alpha w_+(\lambda') + d^\dagger(p,\lambda;\ \text{constituent})\,\tfrac{1}{2}\lambda_i\,d(p,\lambda';\text{constituent})\,w_+^\dagger(-\lambda)\,\Gamma^\alpha w_+(-\lambda')]\ . \qquad (4.3)$$

Although this expression for W_i^α has essentially the same appearance as that of Melosh, we have shown a bit more, even disregarding the fact that our work is not restricted to the free-quark model. There are two points relevant here: One involves the transformation properties of W_i^α under rotations while the other concerns the behavior of W_i^α under Lorentz boosts.

Both the F_i^α and the W_i^α are lightlike charges, and though they all preserve η, lightlike charges do change p^3 when they connect states of different mass. It is necessary, therefore, to be careful about what one means by spin: It is the angular momentum of a state in its rest frame. The appropriate operators for states defined with respect to the constituent quark basis were defined earlier as

$$\vec{\mathcal{J}}=e^{-i\vec{B}\cdot\vec{K}}\,\vec{J}\,e^{i\vec{B}\cdot\vec{K}}\ . \qquad (4.4)$$

Acting on states with $\vec{p}_\perp=0$ these reduce to

$$\mathcal{J}_1=\frac{J_1P^0+K_2P^3}{M}\ ,$$
$$\mathcal{J}_2=\frac{J_2P^0-K_1P^3}{M}\ , \qquad (4.5)$$
$$\mathcal{J}_3=J_3\ ,$$

which are exactly those used by Melosh. The $\mathcal{J}_i$ in general form an SU(2) algebra and have matrix elements equal to those of J_i in the rest frame of a state,

$$\vec{\mathcal{J}}\,|p,\lambda;\text{constituents}\rangle=\vec{\mathcal{J}}\,e^{-i\vec{B}\cdot\vec{K}}|\text{rest},\lambda;\text{constituents}\rangle = e^{-i\vec{B}\cdot\vec{K}}\,\vec{J}\,|\text{rest},\lambda;\text{constituents}\rangle \qquad (4.6)$$

So, acting on a state with spin j, for example,

$$(\mathcal{J}_x+i\mathcal{J}_y)\,|p,\lambda;\ \text{constituents}\rangle=e^{-i\vec{B}\cdot\vec{K}}[(j-\lambda)(j+\lambda+1)]^{1/2}|\text{rest},\lambda+1;\text{constituents}\rangle = [(j-\lambda)(j+\lambda+1)]^{1/2}\,|p,\lambda+1;\text{constituents}\rangle\ , \qquad (4.7)$$

with the usual conventions for $\vec{J}$.

Note that $\vec{\mathcal{J}}$ as defined above commutes with $\vec{K}$ boosts but does not commute with $\vec{E}_\perp, K_3$ boosts which are the appropriate ones for current quark basis states. Therefore, our angular momentum operators $\vec{\mathcal{J}}$ are not the same as those used by Osborn,[10] since his are constructed so as to commute with $\vec{E}_\perp$ and K_3. For states with $\vec{p}_\perp=0$, however, the angular momentum operators of Melosh, of Osborn, and of the present work become identical. Nevertheless, it should not be surprising that our conclusions will differ somewhat from those of Osborn. In particular, his chief result that no constituent current quark transformation fully satisfies the angular constraints if it is generated by single quark operators, even for the case of free quarks, will not obtain in the present work.

The expression for W_i^α shows that it has $|\Delta \mathcal{J}| \leq 1$, since all it does is change one (constituent) quark from one type to another (e.g., change its isospin projection) and possibly rotate its spin. This is just the behavior that one expects for $SU(6)_{W,\,\mathrm{strong}}$ generators in naive quark models.

We now turn to an examination of the behavior of the W_i^α under boosts. From Eq. (4.3), we see that W_i^α is built out of terms bilinear in constituent quark creation and destruction operators. Let us write $b^\dagger(p,\lambda;n)$ for a creation operator, for the moment, where the extra index n serves to label the SU(3) properties of the quark that is to be created. Furthermore, write $U[p,n]$ to denote the standard K boost which takes a quark of type n at rest to momentum p. Then

$$\begin{aligned} U[p,n]\,b^\dagger(q,\lambda;n)\,b(q,\lambda';n') &= b^\dagger(p+q,\lambda;n)\,b(q,\lambda';n') \\ &= b^\dagger(p+q,\lambda;n)\,b(p+q,\lambda';n')\,U[p,n'] \;. \end{aligned} \tag{4.8}$$

But W_i^α is a covariant integral over such terms. Therefore, matrix elements of the form

$$\langle \eta, \vec{p}_\perp; \lambda; \text{constituent} | W_i^\alpha | \eta, \vec{p}_\perp; \lambda'; \text{constituent}\rangle \tag{4.9}$$

are independent of η and $\vec{p}_\perp$. In other words, the W_i^α may be used to classify states in a momentum-independent manner. We emphasize that this does not mean that the W_i^α commute with K_3, since the states are defined in the constituent quark basis to be boosted by $\exp(-i\vec{\beta}\cdot\vec{K})$. Only in the special case that $\vec{p}_\perp = 0$ does K_3 commute with the W_i^α:

$$\begin{aligned} |\eta', p'_\perp; \lambda\rangle &= e^{-i\vec{\beta}'\cdot\vec{K}} e^{-i\vec{\beta}\cdot\vec{K}} |\eta, \vec{p}_\perp; \lambda\rangle \\ &\neq e^{-i\beta K_3} e^{i\beta K_3} |\eta, \vec{p}_\perp; \lambda\rangle \text{ unless } \vec{p}_\perp = \vec{p}'_\perp = 0. \end{aligned} \tag{4.10}$$

It is worthwhile understanding why commutation with K_3 has been demanded by other workers and how the present approach relates to theirs insofar as this question of boost invariance is concerned.

As Melosh points out quite correctly,[22] one striking feature of $SU(6)_{W,\mathrm{strong}}$ is the fact that the $SU(6)_{W,\mathrm{strong}}$ classification of a particle appears to be independent of its momentum in the z direction. The validity of the Johnson-Treiman relations[23] over a wide range of particle momenta is evidence for this momentum independence of transformation properties.

The generators of $SU(6)_{W,\,\mathrm{currents}}$, F_i^α, are invariant under finite boosts along the z direction

$$[F_i^\alpha, K_3] = 0\,, \tag{4.11}$$

but they are not invariant under transverse $\vec{K}_\perp$ boosts. This has as a consequence that a state with a given $\mathcal{J}_3$ value and transverse momentum $\vec{p}_\perp \neq 0$ does not have the same $SU(6)_{W,\,\mathrm{currents}}$ classification as one with the same $\mathcal{J}_3$ value and different $\vec{p}_\perp$. On the other hand, since

$$[F_i^\alpha, \vec{E}_\perp] = 0\,, \tag{4.12}$$

states with transverse momenta $\vec{p}_\perp$ generated by $\vec{E}_\perp$ can be classified in the same $SU(6)_{W,\,\mathrm{currents}}$ representation independent of $\vec{p}_\perp$. These states so defined are of course the ones we have called $|\eta, p; \lambda; \text{current}\rangle$. These current quark basis states can be related to a certain mixture of constituent quark basis states (which are eigenstates of $\mathcal{J}_3$). However, the quark content need not be the same if there is interaction since quark number will most likely not be conserved;[24] in the free-quark model the quark content must be the same, so the current constituent quark transformation is just a change of spin basis, as Eichten *et al.*[25] have shown. We will return to this point in the next section.

The powerful result that the F_i^α classify states independently of momentum has led to the search for W_i^α generators of $SU(6)_{W,\mathrm{strong}}$ which satisfy similar boost invariance properties to those of the F_i^α and to the assumption that states of arbitrary momentum are to be defined in terms of $\vec{E}_\perp, K_3$ boosts; apparently, the motivation for this last (sometimes implicit) assumption is the fact that the W_i^α are also expressible as null-plane integrals (albeit over nonlocal operators even in the free-quark model). In any event, the W_i^α have traditionally[2] been required to commute with K_3. In the free-quark model, Melosh constructed such W_i^α, although with considerable difficulty in satisfying the K_3-boost invariance (somewhat awkwardly); however, the W_i^α was found to be noninvariant under E boosts, so that the $SU(6)_{W,\mathrm{strong}}$ classification of a state was dependent upon its transverse momentum in a quite complicated manner. This, then led to trouble in the treatment of matrix elements of the magnetic moment operator since states with transverse momentum must enter into consideration here. We will discuss this further in the next section.

In the present context, where conventional K boosts are used for constituent quark basis states, we have shown that the W_i^α may classify states independently of momentum. We note in passing that there exist proofs (deAlwis and Stern[3]) and suggestions (Bell[20]) in the literature that this in inherently impossible; such conclusions usually depend on either the assumption of E boosts for hadronic states (as criticized above) and/or the assumption that the W_i^α are integrals of local operators (which does not even hold in the free-quark model).

V. APPLICATION TO CURRENT MATRIX ELEMENTS

Now that we have constructed the generators of $SU(6)_{W,\text{strong}}$ and have given a prescription for computing the field operators $\varphi_+(x;\text{constituent})$ in terms of the field operators $\varphi_+(x;\text{current})$ we proceed to the consideration of hadronic matrix elements of currents and their various moments. The goal is to determine the algebraic structure of such matrix elements, assuming that hadronic states transform simply under $SU(6)_{W,\text{strong}}$; this will then lead to the predictions of relations between various matrix elements.

The essential problem here is the determination of the transformation properties of currents and their moments under the hadronic classification symmetry group $SU(6)_{W,\text{strong}}$ [or, in more detail, the subgroup $SU(3)\times SU(3)$]. Before attacking this problem, however, we must add an element to our classification of states. Since $\mathcal{J}_3$ commutes with the W_i^α, we define a quark "orbital angular momentum" component

$$L_3=\mathcal{J}_3-W_0^3 \tag{5.1}$$

to obtain a $SU(6)_W\times O(2)_{\text{strong}}$ classification. The low-lying hadronic states are classified as usual as $\underline{56}$, $L_3=0$ (for the baryons) and $\underline{35}$, $L_3=0$ (for the mesons).

Recall now that Eq. (3.23) gives an expression for $\varphi_+(x;\text{constituent})$ in terms of $\psi_+(x)=\varphi_+(x;\text{current})$. In principle, this relation can be inverted (since the operator transforming ψ_+ into φ_+ is unitary) to give us $\psi_+(x)$ in terms of φ_+ and whatever other good fields occur in the theory. In practice this may be quite difficult; however, we may write in general

$$\psi_+(x)=\sqrt{2}\exp\left(-i\arctan\frac{\vec{\gamma}_\perp\cdot\vec{p}_\perp}{m+|p_0+p_3|}\right)\varphi_+(x;\ \text{constituent})+\text{interaction-dependent terms}\,. \tag{5.2}$$

The field $\psi_+(x)$ may be written as the sum of two terms; the first one is independent of interaction and has exactly the form derived by Melosh for the free-quark model, while the second one depends on the interaction and vanishes when the interaction vanishes.

For the free-quark model, then, we recover the striking results previously obtained; namely, that operators bilinear in ψ_+ are also bilinear in φ_+. It then follows that such operators can transform according to $SU(6)_{W,\text{strong}}$ as $(1,8)\oplus(8,1)$ and $(3,\bar{3})\oplus(\bar{3},3)$ and nothing else.[26]

For the interacting quark model, this simple result will not obtain in general, since the interaction dependent terms will contribute pieces to φ_+ which will not always be linear in φ_+. The nature of the transformation properties of ψ_+ under $SU(6)_W\times O(2)_{\text{strong}}$ will depend on the corresponding properties of these terms and they will vary with the choice of interaction. It is an interesting question whether the choice of an $SU(6)_{W,\text{currents}}$ invariant interaction will guarantee that ψ_+ transforms as a $\underline{6}$, $L_3=0$; in fact, the whole problem of how the transformation properties of ψ_+ depend on interaction is worth investigating fully.

We would like to make a general observation on the present theoretical framework and how it contrasts with the approach generally found in the literature insofar as the determination of the algebraic structure of the matrix elements of currents and their moments is concerned. This, after all, is at the heart of any phenomenological application of the theory.

We remark that the low-lying baryons and mesons may be assumed to transform simply under $SU(6)_W\times O(2)_{\text{strong}}$ in a momentum-independent fashion, but that one must remember to use the appropriate K boosts to relate states of different momenta.

To illustrate this, we consider the total magnetic moment for a spin-$\frac{1}{2}$ particle A,

$$\frac{\mu_T}{2M}=i\left\langle A,\text{rest};\lambda=-\tfrac{1}{2}\right|\int d^4x\,\delta(x^+)x\mathfrak{F}^+_{\text{em}}(x)\left|A,\text{rest};\lambda=+\tfrac{1}{2}\right\rangle, \tag{5.3}$$

where $\mathfrak{F}^+_{\text{em}}$ is the good component of the electromagnetic current density. This identification can be verified by expanding out the matrix element in the conventional invariants. The evaluation must be performed in terms of symmetric wave packets, rather than plane-wave states. The algebraic structure of the operator, in the free-quark model, is easily determined; the results have been given by Melosh and include the famous ratio

$$\mu_T(\text{proton})/\mu_T(\text{neutron})=-\tfrac{3}{2}\,. \tag{5.4}$$

In the conventional framework as developed by Melosh, this gives rise to a puzzle. Since the discussion requires the use of wave packets whose momentum spread is small but essential, there is an implicit assumption that moving particles transform in the same way as stationary particles. But the W_i^α do not commute with $\vec{E}_\perp$ and K_3, so using these boosts to obtain moving particles the $SU(6)_W \times O(2)_{strong}$ properties appear to be momentum dependent. In fact, de Alwis and Stern find that no definite value for μ_T(proton)/μ_T(neutron) is obtained unless arbitrary additional assumptions are made.

On the other hand, in the present work we have momentum-independent transformation properties so there is no objection to the use of wave packets.

A related point is worth mentioning here. Recall that because of his requirement that $[W_i^\alpha, K_3] = 0$, Melosh was forced to modify his transformation so that the $|p_0+p_3|$ term should not violate this condition; he did this by introducing a factor of $M/|P_0+P_3|$, where M, P_μ are the mass and four-momentum of the state acted on (not of individual quarks). In the rest frame $M/|P_0+P_3| = 1$, whereas the ratio $|p_0+p_3|/|P_0+P_3|$ is invariant[27] under boosts along z.

As he remarks, this factor leads to a puzzle in the following way. The free-quark model gives the algebraic structure of the first moments of the electromagnetic current, as we noted above; this has been applied successfully to electromagnetic decays of higher resonances and to photoproduction. This is puzzling, since matrix elements such as

$$\left\langle A \middle| \int d^4x\, \delta(x^+) x \mathfrak{F}^+_{em}(x) \middle| B \right\rangle,$$

where A and B have different masses, give rise to multilinear products of quark fields with coefficients proportional to $(M_A - M_B)/|P_0+P_3|$, so that use of the free-quark model where transitions between states with very different masses are involved seems unwarranted.

On the other hand, in the present work there is no need for the $M/|P_0+P_3|$ factor since the W_i^α are boost invariant (in our sense) without any additional modification, and the free-quark model relation between current- and constituent-quark basis fields is the same as that of Melosh if no such factor is introduced. Therefore, we feel this puzzle is resolved. There still remains, however, the question as to why the free-quark model works so well. On this we have no comment.

VI. SUMMARY

We have investigated the possible relation between the algebra of $SU(6)_{W,\text{currents}}$, whose generators F_i^α are integrals of the local currents which describe the electromagnetic and weak interactions of hadrons, and the algebra of $SU(6)_{W,\text{strong}}$, whose generators W_i^α are supposed to classify hadrons into approximately degenerate multiplets. The generators F_i^α are bilinear forms in field operators $\varphi_+(x;\text{current})$, while the W_i^α are bilinear forms in field operators $\varphi_+(x;\text{constituent})$. These fields separately satisfy canonical anticommutation relations on the null plane and are unitarily equivalent. We constructed a unitary transformation relating them in such a way that the W_i^α classify states in a momentum independent manner when the states for arbitrary momentum are obtained from rest states by means of K boosts, while the F_i^α classify states in a momentum independent manner when the states for arbitrary momentum are obtained from rest states by means of E boosts. We then found that the W_i^α have simple angular momentum properties, so that their action on hadronic states is just that expected in the naive quark model. Furthermore, in phenomenological application of the formalism to matrix elements of integrated currents and their moments we found no inconsistency in the use of wave packets. The results of this paper are not restricted, in general, to a specific choice of interaction, and in that regard we are optimistic that further work along these lines may answer some of the questions we have dodged: Why does the free-quark structure abstracted from the model work so well for matrix elements of F_i^3 and moments of the electromagnetic current, but not for the bilocal current operators? Why do the physical hadrons lie in approximately degenerate SU(6) multiplets?

Finally, we note that we have provided a precise formulation of the notion of an approximate symmetry.

Even though neither of the groups $SU(6)_{W,\text{currents}}$ and $SU(6)_{W,\text{strong}}$ are symmetry groups for the physical states (except for the special case of the mass-degenerate free-quark model) we have a well-defined prescription for constructing them. The generators of $SU(6)_{W,\text{current}}$ are obtained by null-plane integration of local currents, which may in turn be defined in terms of gauge invariance of the second kind for the Lagrangian. On the other hand, these operators, F_i^α, may then be modified by means of the procedures described in this paper so as to provide a classification symmetry group, called $SU(6)_{W,\text{strong}}$, for physical states. The generators of this group, W_i^α, are not constrained to commute with the Hamiltonian; in fact, after the construction of the W_i^α is performed one may *compute* the commutators $[W_i^\alpha, P^-]$ in order to find out how the Hamiltonian transforms. Thus a *definite* $SU(6)_{W,\text{strong}}$ symmetry breaking follows.

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[1]We use the following conventions for the Poincaré group: 1. The metric tensor $g_{\mu\nu}$ has $g_{00}=1$, $g_{11}=g_{22}=g_{33}=-1$. 2. The Poincaré algebra is

$$[P_\mu, P_\nu]=0,\ [M_{\mu\nu}, P_\rho]=i\,(g_{\nu\rho}P_\mu - g_{\mu\rho}P_\nu),$$

$$[M_{\mu\nu}, M_{\rho\sigma}]=i\,(g_{\mu\sigma}M_{\nu\rho}+g_{\nu\rho}M_{\mu\sigma}-g_{\mu\rho}M_{\nu\sigma}-g_{\nu\sigma}M_{\mu\rho}).$$

3. The generators of rotations and of boosts are, respectively, $M_{ij}=\epsilon_{ijk}J_k$ and $M_{i0}=K_i$. 4. Some operators which occur often in the null-plane formalism are

$$\eta=(P^0+P^3)/\sqrt{2},\qquad P^-=(P^0-P^3)/\sqrt{2},$$

$$\vec{P}_\perp=(P_1,P_2),\quad E_1=K_1+J_2,\ E_2=K_2-J_1,$$

$$x^\pm=(x^0\pm x^3)/\sqrt{2},\qquad \vec{x}_\perp=(x_1,x_2).$$

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