Representation constraint on asymptotic freedom in the high-symmetry limit*

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Asymptotic freedom of gauge coupling is studied in the high-symmetry limit where the rank of the symmetry group becomes large. It is shown that only particular irreducible representations for scalar and fermion multiplets are allowed in this high-symmetry limit. They are vector and tensor of the second rank (symmetric and antisymmetric) and adjoint representations. In particular, the spinor representations of the orthogonal group are ruled out.

Recently several theorists considered the large-N limit of field theories with O(N) or U(N) symmetry. This is of particular interest in the absence of exact solutions for a given theory. It contains much more of the nonlinear nature of the exact theory than the ordinary lowest-order perturbation expansion. Equivalently, one is able to sum a well-defined set containing an infinite number of Feynman diagrams which is the first term in a systematic expansion in $1/N$. In the O(N) model of N real scalar fields, considered by Schnitzer,[1] and Dolan and Jackiw,[2] and subsequently improved by Coleman, Jackiw, and Politzer,[3] the leading terms in the $1/N$ approximation for the effective potential and Green's functions are calculated. Gross and Neveu[4] analyzed O(N) models with four-Fermi interactions in two dimensions to investigate dynamical symmetry breaking in an asymptotically free theory. A gauge theory with color group U(N) has been considered by 't Hooft,[5] who showed that the topological structure of the terms in the perturbation series in $1/N$ can be identified with the expansion of dual resonance models in terms of planar and nonplanar diagrams.

On the other hand, asymptotic freedom has attracted a great deal of attention in particle physics.[6] In this theory the effective couplings as calculated from the renormalization-group equation[7] vanish asymptotically, and Bjorken scaling is attained with logarithmic corrections which are explicitly calculable from the perturbation expansion. The facts that only non-Abelian gauge theories can be asymptotically free[8] and that the presence of fermions and/or scalar mesons destabilizes the asymptotic freedom put severe limitations on the theory as to what representations we have to choose for the particle multiplets and how many multiplets we can accommodate in it.

The purpose of this paper is to study representation constraints on the allowed particle multiplets in the large-N limit of gauge theories retaining asymptotic freedom. We shall call this large-N limit the high-symmetry limit. It is the limit where the theory has the higher and higher internal symmetry, with the symmetries not necessarily confined to the orthogonal group SO(N) or the unitary group SU(N). The symplectic group Sp($2N$) may also be considered. The analysis of the representation constraints is greatly simplified in the high-symmetry limit, and so our study presents a useful limiting case for other examples.

Let us begin with the renormalization-group (RG) equation. Assuming that particle masses can be neglected at high energies, we have the asymptotic form of the RG equation for the one-particle-irreducible (1PI) Green's functions $\Gamma(p;g_1,\ldots,g_n;\mu)$:

$$\left[\mu\frac{\partial}{\partial\mu}+\sum_{i=1}^{n}\beta_i(g_1,\ldots,g_n)\frac{\partial}{\partial g_i}-\gamma_\Gamma(g_1,\ldots,g_n)\right]\times\Gamma(p;g_1,\ldots,g_n;\mu)=0\ . \quad (1)$$

Here μ is the subtraction mass for the renormalized Γ and γ_Γ is the anomalous dimension associated with Γ. The solution of Eq. (1) is well known. The Green's function in the deep-Euclidean region is determined by the same Green's function at some fixed nonexceptional momentum, but with the effective coupling constants $\bar{g}_i(t)$ in their asymptotic limit. These effective couplings satisfy a set of coupled, nonlinear, ordinary differential equations of first order,

$$\frac{d\bar{g}_i(t)}{dt}=\beta_i(\bar{g}_1,\bar{g}_2,\ldots,\bar{g}_n),\quad i=1,2,\ldots,n\ . \quad (2)$$

In general the β_i's are unknown and very complicated, so that our knowledge is limited to perturbation expansions in practice. Asymptotically free theories are by definition those in which all the effective coupling constants vanish in the ultraviolet (UV) limit. Therefore, in these theories perturbation expansions are justified, and we are able to calculate the β_i's. It is still not easy to solve the differential equations (2). Since the β_i's are not linear in the effective couplings, formulating a general criterion for asymptotic freedom is further hampered by a lack of mathematical knowledge of the general stability criteria for equations of the type (2). Furthermore, the number of the effective couplings is not fixed by the symmetry of the theory alone, but is also dependent upon the representation content of the various multiplets of the theory. In an asymptotically free theory, however, the equations for gauge couplings in Eqs. (2) are decoupled from the remaining differential equations and we are able to study the *necessary condition* for asymptotic freedom on gauge couplings without recourse to detailed knowledge of the asymptotic behavior of the other effective coupling constants as long as these are driven to zero.

In this paper we confine ourselves to the discussion of the asymptotic freedom of gauge couplings of a non-Abelian gauge theory invariant under some compact, simple group. This is sufficient for general purposes, since any semisimple group is a direct product of simple groups. Let us consider a Yang-Mills theory described by the Lagrangian density

$$\mathcal{L} = -\tfrac{1}{4}(\partial_\nu A^a_\mu - \partial_\mu A^a_\nu + g C_{abc} A^b_\mu A^c_\nu)^2 + \cdots , \tag{3}$$

where C_{abc} is the structure constant of the gauge group.[9] Then the equation for the effective gauge coupling is found from the one-loop calculation to be

$$\frac{d\bar{g}(t)}{dt} = -\frac{b}{16\pi^2}\bar{g}^3(t) , \tag{4}$$

where

$$b = \tfrac{11}{3} C_2(G) - \sum_{\text{fermions}} \frac{4}{3}\,\frac{d(t)}{r}\,C_2(t) - \sum_{\text{scalars}} \frac{1}{6}\,\frac{d(\theta)}{r}\,C_2(\theta) . \tag{5}$$

Here r is the order of the group G, and $d(t)$ and $d(\theta)$ are the dimensions of the fermion and the scalar multiplets, respectively. Further, the C_2's are the eigenvalues of quadratic Casimir operators corresponding to the indicated irreducible representations. [In particular, $C_2(G)$ is the Casimir operator for the adjoint representation.] The necessary condition for $g=0$ to be ultraviolet stable is that the constant b be positive. Since this is true in the absence of any fermions and scalars, and their presence destabilizes the origin as the UV-stable fixed point, we have to calculate the eigenvalues of Casimir operators corresponding to an arbitrary representation in order to obtain a constraint on asymptotic freedom.

It is well known in the theory of Lie groups that there are only a finite number of algebraic structures[10]: Cartan's four families A_l, B_l, C_l, and D_l, and the five exceptional Lie algebras G_2, F_4, E_6, E_7, and E_8. The structure of any compact Lie group is specified by the so-called root vectors. Since the exceptional Lie algebras are finite in rank and order it is sufficient for us to consider Cartan's four families in the high-symmetry limit. It is also well established that an irreducible representation is completely characterized by its highest weight vector.[10] Furthermore, there are l fundamental weights $\vec{L}^{(i)}$ $(i=1,2,\ldots,l)$ for any family of rank l, and the highest weight $\vec{L}$ corresponding to an irreducible representation is a linear combination of the fundamental weights $\vec{L}^{(i)}$ with nonnegative integer coefficients,

$$\vec{L} = \sum_{i=1}^{l} \lambda_i \vec{L}^{(i)}, \quad \lambda_i \text{ nonnegative integers} . \tag{6}$$

For a study of asymptotic freedom we need to know the dimension and eigenvalue of the quadratic Casimir operator for an irreducible representation specified by its highest weight $\vec{L}$. These are found in the literature[11]:

$$d(\vec{L}) = \prod_{\vec{\alpha}_+} \frac{\vec{\alpha}\cdot(\vec{L}+\vec{R})}{\vec{\alpha}\cdot\vec{R}} ,$$
$$C_2(\vec{L}) = \vec{L}\cdot\vec{L} + 2\vec{R}\cdot\vec{L} , \tag{7}$$

where $2\vec{R} = \sum_{\vec{\alpha}_+}\vec{\alpha}$ is the sum of positive root vectors. Although representations characterized by the fundamental weights are not the most general representations, our analysis of these representations is very important in the discussion of any general representation. It is straightforward to calculate the dimensions and quadratic Casimir operators for these representations. The results are listed in Table I. From this table we observe that $C_2(G)$ is of $O(l)$ in the large-l limit. This is the high-symmetry limit, for N is of $O(l)$ for large N. $C_2(G)$ is the first term of b in Eq. (4), and thus plays a dominant role in determining whether a gauge coupling is asymptotically free. Since the order of the group is of $O(l^2)$, in this limit we obtain as a necessary condition that $d(\vec{L})C_2(\vec{L})$ should be at most of $O(l^3)$ if the gauge coupling is to be asymptotically free. For a general representation with the highest weight $\vec{L} = \sum_{i=1}^{l}\lambda_i\vec{L}^{(i)}$ we prove the following statements: If $\lambda_i \neq 0$ for some i, then we have

$$d\left(\sum_{k=1}^{l}\lambda_k\vec{L}^{(k)}\right) \geq d(\lambda_i\vec{L}^{(i)}) \geq d(\vec{L}^{(i)}) \tag{8}$$

and

$$C_2\left(\sum_{k=1}^{l}\lambda_k\vec{L}^{(k)}\right) \geq C_2(\lambda_i\vec{L}^{(i)}) \geq C_2(\vec{L}^{(i)}) . \tag{9}$$

Proof: For any of Cartan's four families A_l, B_l,

TABLE I. Dimensions and quadratic Casimir operators of the irreducible representations with the fundamental weights as their highest weight. For the definition of the fundamental weights, see Ref. 10.

Cartan family	A_l	B_l	C_l	D_l
Realizing group	$SU(l+1)$	$SO(2l+1)$	$Sp(2l)$	$SO(2l)$
Order	$l(l+2)$	$l(2l+1)$	$l(2l+1)$	$l(2l-1)$
Fundamental weights $\vec{L}^{(1)}$	$\frac{1}{l+1}(l,-1,-1,-1,\ldots,-1)$	$(\frac{1}{2},\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2})$	$(1,0,0,0,\ldots,0)$	$(\frac{1}{2},\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2},\frac{1}{2})$
$\vec{L}^{(2)}$	$\frac{1}{l+1}(l-1,l-1,-2,-2,\ldots,-2)$	$(1,0,0,\ldots,0)$	$(1,1,0,0,\ldots,0)$	$(\frac{1}{2},\frac{1}{2},\frac{1}{2},\ldots,\frac{1}{2},-\frac{1}{2})$
$\vec{L}^{(3)}$	$\frac{1}{l+1}(l-2,l-2,l-2,-3,\ldots,-3)$	$(1,1,0,\ldots,0)$	$(1,1,1,0,\ldots,0)$	$(1,0,0,\ldots,0)$
⋮	⋮	⋮	⋮	⋮
$\vec{L}^{(l)}$	$\frac{1}{l+1}(1,1,1,1,\ldots,-l)$	$(1,1,1,\ldots,1,0)$	$(1,1,1,1,\ldots,1)$	$(1,1,1,\ldots,1,0,0)$
$2\vec{R}=\sum\vec{\alpha}_+$	$(l,l-2,l-4,\ldots,-l+2,-l)$	$(2l-1,2l-3,\ldots,3,1)$	$(2l,2l-2,\ldots,4,2)$	$(2l-2,2l-4,\ldots,2,0)$
Dimension	$d(L^{(i)})=\frac{(l+1)!}{i!(l-i+1)!}$	$d(L^{(1)})=2^l$	$d(L^{(i)})=\frac{2l-2i+2}{2l-i+2}\,\frac{(2l+1)!}{i!(2l-i+1)!}$	$d(L^{(1)})=d(L^{(2)})=2^{l-1}$
		$d(L^{(i)})=\frac{(2l+1)!}{(i-1)!(2l-i+2)!}$ $(i\geq 2)$		$d(L^{(i)})=\frac{(2l)!}{(i-2)!(2l-i+2)!}$ $(i\geq 3)$
Eigenvalues of Casimir operators	$C_2(G)=2(l+1)$	$C_2(G)=2(2l-1)$	$C_2(G)=2(2l+2)$	$C_2(G)=2(2l-2)$
	$C_2(L^{(i)})=\frac{i(l-i+1)(l+2)}{l+1}$	$C_2(L^{(1)})=\frac{1}{2}l^2+\frac{1}{4}l$	$C_2(L^{(i)})=i(2l-i+2)$	$C_2(L^{(1)})=C_2(L^{(2)})=\frac{1}{2}l^2-\frac{1}{4}l$
		$C_2(L^{(i)})=(i-1)(2l-i+2)$ $(i\geq 2)$		$C_2(L^{(i)})=(i-2)(2l-i+2)$ $(i\geq 3)$

C_l, and D_l it is easy to show that

$$\vec{\alpha}_+ \cdot \vec{R} > 0 \text{ and } \vec{\alpha}_+ \cdot \vec{L}^{(k)} \geq 0 .$$

Therefore,

$$d\Big(\sum \lambda_k \vec{L}^{(k)}\Big) = \prod_{\vec{\alpha}_+} \left(1 + \frac{\vec{\alpha}\cdot\vec{L}}{\vec{\alpha}\cdot\vec{R}}\right)$$
$$= \prod_{\vec{\alpha}_+} \left(1 + \sum_k \lambda_k \frac{\vec{\alpha}\cdot\vec{L}^{(k)}}{\vec{\alpha}\cdot\vec{R}}\right)$$
$$\geq \prod_{\vec{\alpha}_+} \left(1 + \lambda_i \frac{\vec{\alpha}\cdot\vec{L}^{(i)}}{\vec{\alpha}\cdot\vec{R}}\right) = d(\lambda_i \vec{L}^{(i)})$$
$$\geq \prod_{\vec{\alpha}_+} \left(1 + \frac{\vec{\alpha}\cdot\vec{L}^{(i)}}{\vec{\alpha}\cdot\vec{R}}\right) = d(\vec{L}^{(i)}) .$$

The last inequality comes from the fact that $\lambda_i \neq 0$ and thus $\lambda_i \geq 1$.

It can also be readily shown that for any family

$$\vec{L}^{(j)} \cdot \vec{L}^{(k)} \geq 0 \text{ and } \vec{R} \cdot \vec{L}^{(k)} \geq 0 .$$

Therefore,

$$C_2\Big(\sum \lambda_k \vec{L}^{(k)}\Big) = \sum_{j,k} \lambda_j \lambda_k \vec{L}^{(j)} \cdot \vec{L}^{(k)} + 2 \sum_k \lambda_k \vec{R} \cdot \vec{L}^{(k)}$$
$$\geq \lambda_i^2 \vec{L}^{(i)} \cdot \vec{L}^{(i)} + 2\lambda_i \vec{R} \cdot \vec{L}^{(i)} = C_2(\lambda_i \vec{L}^{(i)})$$
$$\geq \vec{L}^{(i)} \cdot \vec{L}^{(i)} + 2\vec{R} \cdot \vec{L}^{(i)} = C_2(\vec{L}^{(i)}).$$

Q.E.D.

We note that $C_2(\vec{L}^{(i)})$ is at least of $O(l)$ (see Table I) and therefore $C_2(\vec{L})$ is also at least of $O(l)$ [see Eq. (9)]. We observe that dimension $d(\vec{L})$ must be at most of $O(l^2)$, which means some of the λ_i's should be zero because of Eq. (8). As a result the allowed representations are

$$\begin{aligned} \vec{L} &= \lambda_1 \vec{L}^{(1)} + \lambda_2 \vec{L}^{(2)} + \lambda_{l-1} \vec{L}^{(l-1)} + \lambda_l \vec{L}^{(l)} && \text{for } A_l \\ &= \lambda_2 \vec{L}^{(2)} + \lambda_3 \vec{L}^{(3)} && \text{for } B_l \\ &= \lambda_1 \vec{L}^{(1)} + \lambda_2 \vec{L}^{(2)} && \text{for } C_l \\ &= \lambda_3 \vec{L}^{(3)} + \lambda_4 \vec{L}^{(4)} && \text{for } D_l \end{aligned} \tag{10}$$

Here the λ's are not arbitrary nonnegative integers. They are still to be determined by the constraint that $d(\vec{L})$ is at most of $O(l^2)$. Please note that the spinor representations of the orthogonal group (B_l or D_l) are ruled out. Their dimensions grow exponentially and thus destroy the asymptotic freedom of the gauge coupling in the large-l limit.

The dimension of the representation $\vec{L}$ in Eq. (10), $d(\vec{L})$, can be calculated straightforwardly. For the A_l family,

$$d(\vec{L}) = \frac{(\lambda_1+1)(l+\lambda_1+\lambda_2+\lambda_3-1)(l+\lambda_1+\lambda_2+\lambda_3+\lambda_4)(l+\lambda_2+\lambda_3-2)(l+\lambda_2+\lambda_3+\lambda_4-1)(\lambda_4+1)}{l!\,(l-1)!\,(l-2)!\,(l-3)!}$$
$$\times \frac{(l+\lambda_1+\lambda_2-2)!\,(l+\lambda_2-3)!\,(l+\lambda_3-3)!\,(l+\lambda_3+\lambda_4-2)!}{(\lambda_1+\lambda_2+1)!\,\lambda_2!\,\lambda_3!\,(\lambda_3+\lambda_4+1)!} \tag{11}$$

In the large-l limit this expression behaves as $O(l^{\lambda_1+2\lambda_2+2\lambda_3+\lambda_4})$. Therefore, we have the condition

$$\lambda_1 + 2\lambda_2 + 2\lambda_3 + \lambda_4 \leq 2 . \tag{12}$$

Since the λ's are nonnegative integers there are only a finite number of solutions. They are

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (1,0,0,0),\ (2,0,0,0),\ (0,1,0,0),\ (1,0,0,1),\ (0,0,1,0),\ (0,0,0,1),\ (0,0,0,2) . \tag{13}$$

The last three solutions are conjugate representations corresponding to the first three solutions, and thus there are four representations which allow asymptotic freedom in the high-symmetry limit. For the other families their weight vectors $\vec{L}$ have the same form $\vec{L} = (\lambda_1+\lambda_2, \lambda_2, 0, \ldots, 0)$ and the dimension for this representation is found to be

$$d(\vec{L}) = N(l;\lambda_1,\lambda_2)\, \frac{2l+2a+\lambda_1+2\lambda_2-3}{2l+2a-3}\, \frac{\Gamma(2l+2a+\lambda_1+\lambda_2-3)\Gamma(l+2a-1)\Gamma(2l+2a+\lambda_2-4)}{\Gamma(l+2a+\lambda_1+\lambda_2-1)\Gamma(2l+2a-3)\Gamma(l+2a+\lambda_2-2)}$$
$$\times \frac{\Gamma(l+2\lambda_1-2)\Gamma(l+\lambda_1+\lambda_2)\Gamma(l+\lambda_2-1)}{\Gamma(2l+2\lambda_1-4)\Gamma(\lambda_1+\lambda_2+2)\Gamma(l)\Gamma(l-1)}\, \frac{\lambda_1+1}{\Gamma(\lambda_2+1)} . \tag{14}$$

where

$$\begin{aligned} N(l;\lambda_1,\lambda_2) &= \frac{l+a+\lambda_1+\lambda_2-1}{l+a-1}\, \frac{l+a+\lambda_2-2}{l+a-2} && \text{for } B_l \text{ and } C_l \\ &= 1 && \text{for } D_l , \end{aligned}$$

and $a=\frac{1}{2}$, 1, and 0 for B_l, C_l, and D_l, respectively. For these three families $d(\vec{L})$ is of $O(l^{\lambda_1+2\lambda_2})$, and the necessary condition for asymptotic freedom becomes

$$\lambda_1+2\lambda_2 \leqslant 2 . \qquad (15)$$

The solution for this condition is

$$(\lambda_1, \lambda_2)=(1, 0),\ (2, 0),\ (0, 1) .$$

We have tabulated these solutions in Table II with their dimensions and the eigenvalues of the Casimir operators. This is briefly mentioned by Cheng, Eichten, and Li.[12] But our argument is general in two respects. Firstly, we include the symplectic groups also for completeness of our discussion. Secondly, all possible irreducible representations are considered in this work while Cheng, Eichten, and Li comment on only the kth-rank symmetric tensor representations with the highest weight $k\vec{L}^{(1)}$ for SU(N) and $k\vec{L}^{(2)}$ or $k\vec{L}^{(3)}$ [$=(k, 0, 0, \ldots, 0)$ in either case] for SO(N).

In conclusion, we have shown that asymptotic freedom for gauge couplings in the high-symmetry limit allows only a finite number of irreducible representations for the fermion and the scalar multiplets. They are vector, tensor (symmetric and antisymmetric tensor of the second rank), and adjoint representations (and their conjugate representations in the case of unitary symmetry). In particular, the spinor representations of orthogonal groups are ruled out. These are only necessary conditions for asymptotic freedom. There is still another constraint relating to the number of multiplets of the allowed representations in the theory. These conditions can readily be formulated from Eq. (5) with the aid of Table II.

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TABLE II. Allowed representations for scalar and fermion fields if the gauge coupling is to be asymptotically free. Dimensions and Casimir operators are also listed for these representations. Adjoint representations are marked with asterisks.

Family	Allowed representations ($\vec{L}$)	$d(\vec{L})$	$C_2(\vec{L})$	$\frac{d(\vec{L})}{r} C_2(\vec{L})$
A_l	$\lambda_1=1,\ \lambda_2=\lambda_3=\lambda_4=0$	$l+1$	$\frac{l(l+2)}{l+1}$	1
	$\lambda_1=2,\ \lambda_2=\lambda_3=\lambda_4=0$	$\frac{1}{2}(l+1)(l+2)$	$\frac{2l(l+3)}{l+1}$	$l+3$
	$\lambda_2=1,\ \lambda_1=\lambda_3=\lambda_4=0$	$\frac{1}{2}l(l+1)$	$\frac{2(l-1)(l+2)}{l+1}$	$l-1$
	$\lambda_1=\lambda_4=1,\ \lambda_2=\lambda_3=0$*	$l(l+2)$	$2(l+1)$	$2(l+1)$
B_l	$\lambda_2=1,\ \lambda_3=0$	$2l+1$	$2l$	2
	$\lambda_2=2,\ \lambda_3=0$	$l(2l+3)$	$4l+2$	$2(2l+3)$
	$\lambda_2=0,\ \lambda_3=1$*	$l(2l+1)$	$2(2l-1)$	$2(2l-1)$
C_l	$\lambda_1=1,\ \lambda_2=0$	$2l$	$2l+1$	2
	$\lambda_1=2,\ \lambda_2=0$*	$l(2l+1)$	$4l+4$	$4(l+1)$
	$\lambda_1=0,\ \lambda_2=1$	$(l-1)(2l+1)$	$4l$	$4(l-1)$
D_l	$\lambda_3=1,\ \lambda_4=0$	$2l$	$2l-1$	2
	$\lambda_3=2,\ \lambda_4=0$	$(l+1)(2l-1)$	$4l$	$4(l+1)$
	$\lambda_3=0,\ \lambda_4=1$*	$l(2l-1)$	$2(2l-2)$	$4(l-1)$

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