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Model calculations of electroproduction and inclusive annihilation cross sections*

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We have studied "deep-inelastic electroproduction" and "inclusive e^+e^- annihilation" in massless ϕ^4 theory in the ladder approximation. The relevant Bethe-Salpeter-type equations can be completely solved and the physically important asymptotic limits are studied. The behavior of the moments of the structure functions is analyzed, and the "anomalous dimensions" which govern the asymptotic power behavior of the moments in the two cases are found. The anomalous dimensions are quite different in general, but are simply related in the weak-coupling limit.

I. INTRODUCTION

In this note we shall record the results of a ladder-model calculation of electroproduction and annihilationlike cross sections. Our motivation is to study the anomalous dimensions which govern scaling and, in particular, to ask whether their singularities bear any resemblance to the singularities as revealed by perturbation theory. At the same time we wish to see whether there is any simple relation between the anomalous dimensions which govern annihilation and those which govern electroproduction. Our purpose here is mainly illustrative: to show in a simple nontrivial example that the scaling properties of the inclusive annihilation cross section are perfectly analogous to those of the electroproduction cross section although governed by a different set of dimensions. Finally, we shall use the specific model cross sections we have obtained to test certain proposed[1] connections between properties of the anomalous dimensions and characteristic isolations of scaling in the asymptotic behavior of the cross sections themselves. In general, we propose to use this simple model as a laboratory to examine the limits of validity of schemes for converting renormalization-group information (i.e., properties of anomalous dimensions) into information about the directly measurable cross sections.

II. DEEP-INELASTIC SCATTERING

We begin by treating "electroproduction" but with all particles being treated as scalars. The process is, as usual, virtual photon (q) incident on "nucleon" (p) producing hadrons over which we sum. The kinematics are shown in Fig. 1, where we designate the absorptive part of the forward elastic amplitude by $A(q,p)$; the kernel in our model is a simple bubble. Quite generally, for any two-particle-irreducible kernel, I, we have the Amati-Bertocchi-Fubini-Stanghellini-Tonin (ABFST) equation[2]

$$A(q,p)=I(q,p)+\frac{2}{(2\pi)^4}\int d^4q'\,I(q,q')\times\Delta^2(q'^2)A(q,p), \qquad (2.1)$$

where $\Delta(q'^2)$ is the propagator along the ladder. Our normalization is such that if q and p were on the mass shell, the total cross section, $\sigma_{\rm tot}$, would be given by

$$\lambda^{1/2}((p+q)^2,p^2,q^2)\sigma_{\rm tot}=A(q,p), \qquad (2.2)$$

where $\lambda(a,b,c)=a^2+b^2+c^2-2ab-2ac-2bc$.

It is kinematically convenient to imagine that both q^2 *and* p^2 are spacelike. Of course this is no hardship as far as q^2 is concerned. The continuation to p^2 positive (on shell) presents no problem. We introduce conventional kinematic variables:

$$\begin{aligned}
s&=(p+q)^2,\\
u&=-q^2,\\
v&=-p^2,\\
u'&=-q'^2, \qquad (2.3)\\
\cosh\theta&=\frac{q\cdot p}{(uv)^{1/2}}=\frac{s+u+v}{2(uv)^{1/2}},\\
\cosh\theta'&=\frac{q'\cdot p}{(u'v)^{1/2}}=\frac{(q'+p)^2+u'+v}{2(u'v)^{1/2}},\\
\cosh\theta_0&=\frac{4\mu^2+u+v}{2(uv)^{1/2}}.
\end{aligned}$$

Here μ is the mass of the ϕ field. The kernel I in our model is easily found to be

$$I((q-q')^2, q^2, q'^2) = \frac{(2\pi)^4}{2} g^2 \int \frac{d^4q_1}{(2\pi)^3} \int \frac{d^4q_2}{(2\pi)^3} \delta(q_1{}^2-\mu^2)\delta(q_2{}^2-\mu^2)\delta(q-q'-q_1-q_2)$$
$$= \frac{g^2}{16\pi}\left(1-\frac{4\mu^2}{(q-q')^2}\right)^{1/2} . \tag{2.4}$$

It is convenient to effect a diagonalization of our integral equation for the absorptive part by making what is essentially a Laplace transform. This is a standard procedure[2] and we simply quote the results. We define

$$A_l(u,v) = \int_{4\mu^2}^{\infty} ds\, e^{-(l+1)\theta} A(s,u,v), \tag{2.5}$$

where θ is defined as above. The inversion formula, using $ds = 2(uv)^{1/2}\, d(\cosh\theta)$ is given by

$$2(uv)^{1/2}\sinh\theta A(s,u,v) = \int_{c-i\infty}^{c+i\infty} \frac{dl}{2\pi i} e^{(l+1)\theta} A_l(u,v), \tag{2.6}$$

and c must lie to the right of singularities of $A_l(u,v)$ in the l plane. With a correspondingly defined transform of I with respect to its subenergy $(q-q')^2$ and its "masses" q^2, q'^2 we have the transformed equation

$$A_l(u,v) = I_l(u,v) + \frac{1}{16\pi^3(l+1)} \int_0^{\infty} du' I_l(u,u')\, \Delta^2(u') A_l(u',v). \tag{2.7}$$

It is slightly more useful for purely technical reasons to deal with the amplitude $a_l(u,v)$, given by

$$a_l(u,v) = A_l(u,v)/(uv)^{(l+1)/2}, \tag{2.8}$$

and write the integral equation in terms of this and a similarly reduced kernel $i_l(u,u')$,

$$i_l(u,u') = I_l(u,u')/(uu')^{(l+1)/2}. \tag{2.9}$$

Our basic equation then becomes

$$a_l(u,v) = i_l(u,v) + \frac{1}{16\pi^3(l+1)} \int_0^{\infty} du' i_l(u,u') \Delta^2(u') u'^{(l+1)} a_l(u',v). \tag{2.10}$$

The relation between the transforms of $A(s,u,v)$ and the conventionally defined moments of structure functions can be illustrated by expressing $\cosh\theta$ in terms of the familiar variables ω and $-q^2 = u$. We have

$$\omega = 2p\cdot q/(-q^2) = 2p\cdot q/u, \quad \cosh\theta = \tfrac{1}{2}\omega\left(\frac{u}{v}\right)^{1/2}, \quad e^{\theta} = \omega\left(\frac{u}{v}\right)^{1/2} \frac{1+(1-4v/u\omega^2)^{1/2}}{2}, \tag{2.11}$$

so that

$$a_l(u,v) = \frac{1}{u^l} \int_{2(v/u)^{1/2}\cosh\theta_0}^{\infty} d\omega\, \omega^{-l-1} \left(\frac{2}{1+(1-4v/\omega^2 u)^{1/2}}\right)^{l+1} A(\omega,u,v), \tag{2.12}$$

which is the usual limit of large $(-q^2) = u$ becomes simply

$$a_l(u,v) = \frac{1}{u^l} \int_1^{\infty} d\omega\, \omega^{-l-1} A(\omega,u,v). \tag{2.13}$$

This is to be contrasted with the ordinary definition of the moments, call them $\tilde{a}_n$:

$$\tilde{a}_n \equiv \int_1^{\infty} d\omega\, \omega^{-n-2} A(\omega,u,v). \tag{2.14}$$

Thus

$$\tilde{a}_n(u,v) = u^{n+1} a_{n+1}(u,v). \tag{2.15}$$

For purposes of illustration we shall set the mass $\mu = 0$ as well as all other internal masses and replace the propagator $\Delta(u')$ by simply $(u')^{-1}$. In our bubble model we find for $i_l(u,u')$ the result

$$i_l(u,u') = \frac{g^2}{16\pi} \frac{1}{(u_< u_>)^{1/2}} \left(\frac{u_<}{u_>}\right)^{l/2} \left(\frac{1}{l} - \frac{u_<}{u_>}\frac{1}{l+2}\right), \tag{2.16}$$

where $u_<$ $(u_>)$ is the smaller (greater) of u, u'. With this kernel, the integral equation for $A_l(u,v)$ can be converted into a fourth-order differential equation with δ-function inhomogeneity which can be readily solved. It is more useful, however, to

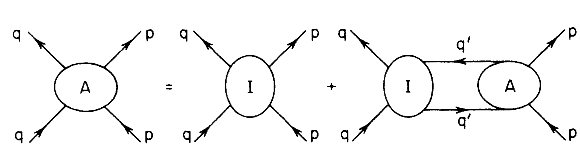

FIG. 1. Graphical expression of the integral equation for the electroproduction cross section.

solve the problem by a direct Mellin transform of the integral equation. We introduce

$$b_l(r,v)=\int_0^\infty du\,u^{-r-1}a_l(u,v)\,, \tag{2.17}$$

with the inversion

$$a_l(u,v)=\int_{c-i\infty}^{c+i\infty}\frac{dr}{2\pi i}\,u^r b_l(r,v)\,; \tag{2.18}$$

the contour is a line parallel to the imaginary axis lying in a strip in the complex r plane in which $b_l(r,v)$ is analytic. The transform of the kernel $j_l(r,v)$ is simply

$$j_l(r,v)=\frac{g^2}{16\pi}\frac{(l+1)v^{-l-r}}{r(r-1)(l+r)(l+r+1)}\,. \tag{2.19}$$

In deducing this transform we have required that $-\text{Re}\,l<\text{Re}\,r<0$, the tentative strip of analyticity for $b_l(r,v)$. With our approximation, $\Delta(u')=(u')^{-1}$, the Mellin transform of the integral equation reduces it to an algebraic one with the solution

$$b_l(r,v)=\frac{16\pi^3 f^2(l+1)v^{-l-r}}{r(r-1)(r+l)(r+l+1)-f^2}\,, \tag{2.20}$$

where $f^2=(g/16\pi^2)^2$.

It is elementary to solve for the roots $r_1,\ \ldots,r_4$ of the denominator. We find

$$\begin{aligned}
r_1&=-\frac{l}{2}+\left\{\frac{l^2}{4}+\frac{l+1}{2}-\left[\left(\frac{l+1}{2}\right)^2+f^2\right]^{1/2}\right\}^{1/2}\\
&\simeq-\frac{f^2}{l(l+1)}\,,\\
r_2&=-\frac{l}{2}-\left\{\frac{l^2}{4}+\frac{l+1}{2}-\left[\left(\frac{l+1}{2}\right)^2+f^2\right]^{1/2}\right\}^{1/2}\\
&\simeq-l+\frac{f^2}{l(l+1)}\,,\\
r_3&=-\frac{l}{2}+\left\{\frac{l^2}{4}+\frac{l+1}{2}+\left[\left(\frac{l+1}{2}\right)^2+f^2\right]^{1/2}\right\}^{1/2}\\
&\simeq 1+\frac{f^2}{(l+1)(l+2)}\,,\\
r_4&=-\frac{l}{2}-\left\{\frac{l^2}{4}+\frac{l+1}{2}+\left[\left(\frac{l+1}{2}\right)^2+f^2\right]^{1/2}\right\}^{1/2}\\
&\simeq-l-1-\frac{f^2}{(l+1)(l+2)}\,,
\end{aligned} \tag{2.21}$$

where we define the various roots to be real and positive when l is a very large, real, positive quantity. For orientation we have given the r_i for small f^2 or, what is equivalent, large l. To reconstruct $a_l(u,v)$ we have

$$a_l(u,v)=\frac{1}{v^l}\int_{c-i\infty}^{c+i\infty}\frac{dr}{2\pi i}\,\frac{16\pi^3 f^2(l+1)(u/v)^r}{r(r-1)(l+r)(l+r+1)-f^2}\,, \tag{2.22}$$

and we see that if $u/v>1$, we can close the contour to the left, and if $u/v<1$, to the right. Imagining l to be very large, real, and positive we see that the strip of analyticity is still roughly $-l<\text{Re}\,r<0$, and r_2, r_4 lie to the left and r_1, r_3 to the right of the strip. Writing $r_{1,2}=-l/2\pm\sigma_1$, $r_{3,4}=-l/2\pm\sigma_2$, we find for $a_l(u,v)$ the result

$$a_l(u,v)=\frac{16\pi^3 f^2(l+1)}{4\{[\frac{1}{2}(l+1)]^2+f^2\}^{1/2}}\,\frac{1}{(u_<u_>)^{l/2}}\times\left(\frac{(u_</u_>)^{\sigma_1}}{\sigma_1}-\frac{(u_</u_>)^{\sigma_2}}{\sigma_2}\right). \tag{2.23}$$

Since $\sigma_1\to l/2$, $\sigma_2\to(l/2)+1$ as $f^2\to 0$, it is clear that $a_l(u,v)\to i_l(u,v)$, as it should. The quantity of interest in discussing the moments of the structure functions for large u is easily seen to be the large-u limit of

$$u^l a_l\sim u^{r_2+l}=u^{l/2-\sigma_1}.$$

The perturbation expansion of r_2+l is $f^2/l(l+1)$, but the superficial pole at $l=0$ is obviously absurd. The correct singularity of r_2+l is at $l=-1+(1+4f)^{1/2}$ and is a branch point, not a pole. This has recently been discussed by Lovelace and Gross.[3]

Depending upon what limit of the full amplitude $A(s,u,v)$ one is interested in studying, it may or may not be advantageous to have carried out the r integration explicitly. We can in fact go directly to the inversion formula

$$2(uv)^{1/2}\sinh\theta A(s,u,v)=\int\frac{dl}{2\pi i}\,e^{(l+1)\theta}(uv)^{(l+1)/2}a_l(u,v)\,, \tag{2.24}$$

to obtain

$$\frac{\sinh\theta A(s,u,v)}{8\pi^3 f^2}=\frac{\partial}{\partial\theta}\int\frac{dr}{2\pi i}\left(\frac{u}{v}\right)^r\int\frac{dl}{2\pi i}\,\frac{e^{(l+1)\theta}(u/v)^{l/2}}{r(r-1)(l+r)(l+r+1)-f^2}\,. \tag{2.25}$$

Now we carry out the integration over l completing the contour to the left. This is legitimate because $\theta\geq\theta_0$, where in our zero-mass limit

$$e^{\theta_0}=(u/v)^{1/2}\,.$$

With the trivial replacement $r-\frac{1}{2}=\sigma$ we have

$$\frac{\sinh\theta A(s,u,v)}{8\pi^3 f^2} = \frac{\partial}{\partial\theta}\int \frac{d\sigma}{2\pi i} \frac{e^{\sigma(\theta-\theta_0)}\sinh\{[\frac{1}{4}+f^2/(\sigma^2-\frac{1}{4})]^{1/2}(\theta+\theta_0)\}}{(\sigma^2-\frac{1}{4})[\frac{1}{4}+f^2/(\sigma^2-\frac{1}{4})]^{1/2}} . \tag{2.26}$$

Although we shall not make much use of it, it is possible to evaluate this integral in terms of elementary functions. To do so, expand the hyperbolic sine in a power series leading to a series in $[\frac{1}{4}+f^2/(\sigma^2-\frac{1}{4})]$ expand the latter into a series and then carry out the σ integration using

$$\oint \frac{d\sigma}{2\pi i} \frac{e^{\sigma x}}{(\sigma^2-\frac{1}{4})^{l+1}} = (\pi x)^{1/2}\frac{x^l}{l!} I_{l+1/2}(x/2), \tag{2.27}$$

where $I_\nu(z)$ is the usual Bessel function with the behavior

$$\begin{aligned} I_\nu(z) &\sim z^\nu/\Gamma(\nu+1), \quad z\to 0 \\ &\sim e^z/(2\pi z)^{1/2}, \quad z\to\infty . \end{aligned} \tag{2.28}$$

The double series can be summed into a single one, with the result

$$\frac{\sinh\theta A(s,u,v)}{8\pi^3 f^2} = \pi\frac{\partial}{\partial\theta}(xy)^{1/2} \times \sum_{l=0}^{\infty}\frac{(f^2xy)^l}{[l!]^2} I_{l+1/2}\left(\frac{x}{2}\right) I_{l+1/2}\left(\frac{y}{2}\right), \tag{2.29}$$

where $x=\theta-\theta_0$, $y=\theta+\theta_0$. The symmetry in x, y is not obvious from our original integral representation although it is expected on general grounds. The amplitude A is obviously symmetric in u and v; thus regarded as a function of $\theta_0=\frac{1}{2}\ln(u/v)$ it must be an even function of $\theta_0=(y-x)/2$. Since $\theta=(y+x)/2$, the whole amplitude is thus a symmetric function of x and y.

There are two other related observations we make. Writing

$$G(x,y)=\int\frac{d\sigma}{2\pi i}\frac{e^{\sigma x}\sinh\{[\frac{1}{4}+f^2/(\sigma^2-\frac{1}{4})]^{1/2}y\}}{(\sigma^2-\frac{1}{4})[\frac{1}{4}+f^2/(\sigma^2-\frac{1}{4})]^{1/2}} , \tag{2.30}$$

we note that $G(x,y)$ satisfies the differential equation

$$\left(\frac{\partial^2}{\partial x^2}-\frac{1}{4}\right)\left(\frac{\partial^2}{\partial y^2}-\frac{1}{4}\right)G(x,y)=f^2G(x,y); \tag{2.31}$$

and by taking a Laplace transform with respect to y of the integral representation of G and writing a formal inversion of it, we have

$$G(x,y)=\int\frac{d\sigma}{2\pi i}\int\frac{d\tau}{2\pi i}\frac{e^{\sigma x+\tau y}}{(\sigma^2-\frac{1}{4})(\tau^2-\frac{1}{4})-f^2} , \tag{2.32}$$

which makes the symmetry in x and y manifest.

We shall not stop to discuss these results further, but pass on to treat the problem of inclusive annihilation in a very similar manner. We shall find, in fact, almost identical formal results in spite of profound differences at intermediate steps.

III. INCLUSIVE ANNIHILATION

The process of interest is a virtual (timelike) photon (q) decaying into unobserved hadrons and a distinguished one (p). The kinematics of the process are shown in Fig. 2, where we show the general structure of the process. In general I is the absorptive part of a two-particle irreducible kernel; in our model calculation we shall take it to be a simple bubble as before. We now write $q^2=u$, $p^2=v$, where u, v are both positive. The general integral equation takes the form

$$A(q,p)=I(q,p) + \frac{2}{(2\pi)^4}\int d^4q' I(q,q')|\Delta(q')|^2 A(q',p). \tag{3.1}$$

It is necessary to write the absolute square of the propagators because, in contrast to the electroproduction case, they are complex.

Because we are dealing essentially with a decay process, the kinematics here are really dramatically different from the spacelike case. For example, the virtual photon q decays into unobserved hadrons of mass $(q-q')^2$ and the observed hadron p. Hence $q_0>q_0'>p_0$, and the integration over q' in our equation is over a finite timelike region. If we call $s_0=(q-q')^2$, $s'=(q'-p)^2$, $s=(q-p)^2$, $q'^2=u'$, and the threshold values of s_0 and s' $=m_T^2$, one easily deduces the further kinematic restrictions $v^{1/2}+m_T\le(u')^{1/2}\le u^{1/2}-m_T$ and $(s')^{1/2}+(s_0)^{1/2}\le s^{1/2}\le u^{1/2}-v^{1/2}$.

To proceed with the diagonalization in this case we introduce hyperbolic angles ξ, ξ_0, and ξ' defined by

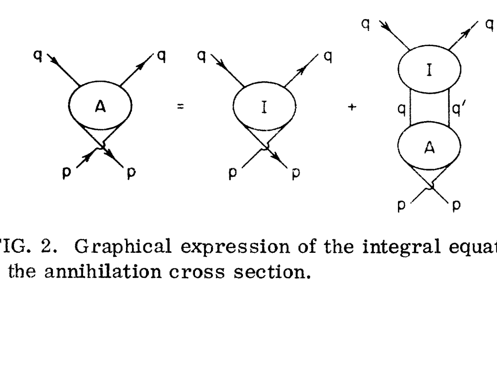

FIG. 2. Graphical expression of the integral equation for the annihilation cross section.

$$\cosh\xi = \frac{q\cdot p}{(uv)^{1/2}} = \frac{u+v-s}{2(uv)^{1/2}},$$

$$\cosh\xi_0 = \frac{q\cdot q'}{(uu')^{1/2}} = \frac{u+u'-s_0}{2(uu')^{1/2}},$$

$$\cosh\xi' = \frac{q'\cdot p}{(u'v)^{1/2}} = \frac{u'+v-s'}{2(u'v)^{1/2}}, \qquad (3.2)$$

$$\cosh\xi_m = \frac{u+v-m_T^2}{2(uv)^{1/2}},$$

where ξ_m evidently is the maximum value of ξ corresponding to the produced particle carrying off its greatest amount of energy. Introducing these variables into the integral equation we have the Jacobian

$$d^4p' = \frac{\pi u'\,du'}{\sinh\xi}\, d(\cosh\xi_0) d(\cosh\xi')\theta(X),$$

$$X = 1 + 2\cosh\xi\cosh\xi'\cosh\xi_0 - \cosh^2\xi - \cosh^2\xi' - \cosh^2\xi_0. \quad (3.3)$$

The step function $\theta(X)$ implies limits on the variables ξ' and ξ_0 aside from the limitations on u' noted above. With the understanding that the kernel I and the amplitude A have their own secret understanding of threshold θ functions that prevent their ξ's from getting to large, we can diagonalize the equation by defining a transform[4]

$$A_N(u,v) = \int d\xi \sinh\xi \sinh N\xi A(s,u,v) \qquad (3.4)$$

and recognizing that the step-function restriction in the variable change above is tantamount to the restriction $|\xi_0 - \xi'| \le \xi \le \xi_0 + \xi'$. We find then

$$A_N(u,v) = I_N(u,v) + \frac{1}{4\pi^3 N}\int_v^u du'\,u'\,|\Delta(u')|^2\, I_N(u,u') A_N(u',v), \qquad (3.5)$$

where I_N is defined precisely the same way as A_N as an integral over $I(s,u,v)$.

In order to show the relation between A_N and the conventionally defined moments of structure functions, we remark that

$$\cosh\xi = \frac{p\cdot q}{(uv)^{1/2}} = \tfrac{1}{2}\omega\left(\frac{u}{v}\right)^{1/2}, \qquad (3.6)$$

where $\omega \equiv 2p\cdot q/q^2$. Introducing ω as a variable in place of ξ, recalling that the upper limit on ξ is ξ_m, we find[5]

$$A_N(u,v) = \frac{1}{4}\left(\frac{u}{v}\right)^{1/2} \times \int_{2(v/u)^{1/2}}^{2(v/u)^{1/2}\cosh\xi_m} d\omega \left\{\omega^N\left(\frac{u}{v}\right)^{N/2}\left[\frac{1+(1-4v/u\omega^2)^{1/2}}{2}\right]^N - \omega^{-N}\left(\frac{u}{v}\right)^{-N/2}\left[\frac{1+(1-4v/u\omega^2)^{1/2}}{2}\right]^{-N}\right\} A(\omega,u,v). \qquad (3.7)$$

In the usually considered limit of large $q^2 = u$ we have

$$A_N(u,v) = \frac{1}{4}\left(\frac{u}{v}\right)^{(N+1)/2}\int_0^1 d\omega\,\omega^N A(\omega,u,v). \qquad (3.8)$$

This says that the quantity $a_N(u,v)$, defined by

$$a_N(u,v) = (v/u)^{(N+1)/2} A_N(u,v), \qquad (3.9)$$

is directly the usually defined Nth moment of the structure function.

It is important to remark that for finite u, the quantity $A_N(u,v)$ is an *entire function* of N, since the defining integral extends over only a *finite* interval and the integrand is an entire function of N. This is to be contrasted with the function $A_l(u,v)$ introduced in the spacelike regime which has branch points in the l plane. The inversion formula expression $A(s,u,v)$ in terms of $A_N(u,v)$ is easily found to be (for $\xi < \xi_m$)

$$A(\xi,u,v) = \frac{i}{\pi}\int_{c-i\infty}^{c+i\infty} dN \frac{\sinh N\xi}{\sinh\xi} A_N(u,v), \qquad (3.10)$$

and the contour is any line parallel to the imaginary axis and might just as well be the imaginary axis. In the large-u limit, in terms of the auxiliary quantity $a_n(u,v)$ we have

$$A(\omega,u,v) = 4\int_{c-i\infty}^{c+i\infty}\frac{dN}{2\pi i}\,\omega^{-N-1} a_N(u,v). \qquad (3.11)$$

We now specialize the kernel I to the ϕ^4 bubble model considered in connection with "electroproduction." We again set all internal masses equal to zero and replace $\Delta(u')$ by $(u')^{-1}$. The maximum value of ξ becomes $\xi_m = \frac{1}{2}\ln(u/v)$. We have

$$I_N(u,u') = \frac{g^2}{16\pi}\int_0^{\xi_m'} d\xi \sinh\xi\sinh N\xi = \frac{g^2}{64\pi}\left(\frac{e^{(N+1)\xi_m'} - e^{-(N+1)\xi_m'}}{N+1} - \frac{e^{(N-1)\xi_m'} - e^{-(N-1)\xi_m'}}{N-1}\right), \qquad (3.12)$$

with $\xi_m' = \frac{1}{2}\ln(u/u')$. In terms of

$$a_N=\left(\frac{v}{u}\right)^{(N+1)/2}A_N$$

our integral equation becomes

$$a_N(u,v)=i_N(u,v)+\frac{1}{4\pi^3N}\int_v^u\frac{du'}{u'}\,i_N(u,u')a_N(u',v),\tag{3.13}$$

where

$$\begin{aligned}i_N(u,u')&=(u'/u)^{(N+1)/2}I_N(u,u')\\&=\frac{g^2}{64\pi}\left\{\frac{1}{N+1}\left[1-\left(\frac{u'}{u}\right)^{N+1}\right]\right.\\&\qquad\left.-\frac{1}{N-1}\left[\frac{u'}{u}-\left(\frac{u'}{u}\right)^{N}\right]\right\}.\end{aligned}\tag{3.14}$$

Evidently $a_N(u,v)$ is a function of the ratio u/v, and we introduce as new variables $w=u/v$, $w'=u'/v$. To solve the integral equation for $a_N(w)$ we use again a Mellin transform defined now as

$$b_N(r)=\int_1^\infty dw\,w^{-r-1}a_N(w),\tag{3.15}$$

with the inversion

$$a_N(w)=\int_{c-i\infty}^{c+i\infty}\frac{dr}{2\pi i}\,w^r b_N(r).\tag{3.16}$$

The quantity $b_N(r)$ is analytic in a half plane this time rather than a strip. We find

$$\int_1^\infty dw\,w^{-r-1}i_N(w)=\frac{g^2}{64\pi}\,\frac{N}{r(r+1)(N+r)(N+r+1)},\tag{3.17}$$

and the transform of the integral equation reduces it to an algebraic one for $b_N(r)$, with the solution

$$b_N(r)=\frac{4\pi^3f^2N}{r(r+1)(N+r)(N+r+1)-f^2},\tag{3.18}$$

with $f^2=(g/16\pi^2)^2$.

The inversion to recover $a_N(w)$ is easily done since we can recognize the roots of our quartic equation as those obtained in the spacelike case with $r\to-r$, $N\to-l-1$. We find

$$\frac{a_N(w)w^{(N+1)/2}}{4\pi^3f^2}=\frac{N}{(N^2/4+f^2)^{1/2}}\left(\frac{\sinh(\tilde\sigma_1\ln w)}{\tilde\sigma_1}-\frac{\sinh(\tilde\sigma_2\ln w)}{\tilde\sigma_2}\right),\tag{3.19}$$

where

$$\begin{aligned}\tilde\sigma_1&=\left[\frac{N^2+1}{4}+\left(\frac{N^2}{4}+f^2\right)^{1/2}\right]^{1/2},\\ \tilde\sigma_2&=\left[\frac{N^2+1}{4}-\left(\frac{N^2}{4}+f^2\right)^{1/2}\right]^{1/2}.\end{aligned}\tag{3.20}$$

One can be utterly cavalier about how the square roots are defined since there are, in fact, no branch points involved, as is easily seen. This may be inserted into the inversion formula to recover $A(\xi,u,v)$, but it is not particularly illuminating or useful. Rather, we formally represent $A_N(w)$ as an inverse Mellin transform (that is, as an integral over the transform variable called r) and use this representation to carry out the integration over N. With a trivial translation $r\to\sigma-\frac{1}{2}$ we find, recalling that $w=\exp(2\xi_m)$,

$$\frac{\sinh\xi A(\xi,\xi_m)}{8\pi^3f^2}=\frac{\partial}{\partial\xi}\int_{c-i\infty}^{c+i\infty}\frac{d\sigma}{2\pi i}\,\frac{e^{\sigma(\xi_m-\xi)}\sinh\{[\frac14+f^2/(\sigma^2-\frac14)]^{1/2}(\xi_m+\xi)\}}{(\sigma^2-\frac14)[\frac14+f^2/(\sigma^2-\frac14)]^{1/2}}.\tag{3.21}$$

We have used the fact that the above integral is actually an even function of ξ.

Our result for inclusive annihilation is essentially identical to that obtained in the electroproduction region, which perhaps comes as somewhat of a suprise because the singularity structure of the moments is so different. Because of the formal identity in the two cases we can discuss various mathematical limits simultaneously, although the physical interpretations are quite different. For example, in the spacelike region the limit $\omega\to\infty$, q^2 fixed corresponds to both $x=\theta-\theta_0$ and $y=\theta+\theta_0$ becoming large or, stated otherwise, $\frac12(y-x)=\theta_0=\frac12\ln(q^2/p^2)$ fixed, but $\frac12(y+x)=\theta=\ln\omega+\frac12\ln(q^2/p^2)$ growing. Another limit of interest is ω fixed, $-q^2\to\infty$, corresponding to x fixed, $y\to\infty$. The physically interesting limits in the annihilation (timelike) region are the following: (1) $\bar x=\xi_m-\xi=\ln(1/\omega)$ fixed and $q^2\to\infty$ or $\bar y=\xi_m+\xi\to\infty$ (this is mathematically identical with the fixed-ω large-$(-q^2)$ limit just discussed), and (2) fixed ξ, $\xi_m\to\infty$, which means both x and y become large with $\bar y-\bar x$ fixed and, of course, $\bar y+\bar x$ growing. This is the counterpart of the large-ω fixed-$(-q^2)$ limit of electroproduction. Here, however, it has the interpretation of fixed energy of the produced, distinguished hadron, in the center-of-mass system [where $q=(q_0,\vec 0)$] in the limit where the photon energy is very large. It is evident then that in our model similar features of the anomalous dimensions that determine what is usually called the Regge limit in electroproduction, namely, $\omega\to\infty$, q^2 fixed, simultaneously describe the behavior of the inclusive annihilation cross seccorresponding to a fixed energy of the produced hadron.

IV. ASYMPTOTIC BEHAVIOR OF ELECTROPRODUCTION AND INCLUSIVE ANNIHILATION

We conclude our formal discussion of the model by considering the interesting physical limits. The more complicated one is the large-ω fixed-q^2 limit in the spacelike region (S) and the mathematically equivalent large-q^2 fixed-energy limit of produced hadron in the timelike case (T). The two cases can be treated together.

We recall

$$\frac{\sinh\theta A(\theta,\theta_0)}{8\pi^3 f^2}=\frac{\partial}{\partial\theta}G(\theta,\theta_0)\,,\quad (S)$$
$$\frac{\sinh\xi A(\xi,\xi_m)}{8\pi^3 f^2}=-\frac{\partial}{\partial\xi}G(\xi_m,\xi)\,,\quad (T) \tag{4.1}$$

where

$$G(\alpha,\beta)=\int_{c-i\infty}^{c+i\infty}\frac{d\sigma}{2\pi i}\,\frac{e^{\sigma(\alpha-\beta)}\sinh\{[\frac{1}{4}+f^2/(\sigma^2-\frac{1}{4})]^{1/2}(\alpha+\beta)\}}{(\sigma^2-\frac{1}{4})[\frac{1}{4}+f^2/(\sigma^2-\frac{1}{4})]^{1/2}}\,. \tag{4.2}$$

What we require is the limit of $G(\alpha,\beta)$ for large α, fixed β. This can be treated by the method of steepest descents. In the desired limit, it is only the positive exponent in the hyperbolic sine that is important. We have then

$$G(\alpha,\beta)\cong\int_{c-i\infty}^{c+i\infty}d\sigma e^{\Phi(\sigma)}\,,\qquad \Phi(\sigma)=\alpha\left[\sigma+\left(\frac{1}{4}+\frac{f^2}{\sigma^2-\frac{1}{4}}\right)^{1/2}\right]-\beta\left[\sigma-\left(\frac{1}{4}+\frac{f^2}{\sigma^2-\frac{1}{4}}\right)^{1/2}\right]-\ln(\sigma^2-\tfrac{1}{4})-\tfrac{1}{2}\ln\left(\frac{1}{4}+\frac{f^2}{\sigma^2-\frac{1}{4}}\right). \tag{4.3}$$

It is easy to see that the coefficient of α has minima on the real σ axis at the points $\sigma=(\frac{1}{4}\pm f)^{1/2}$; since we are interested in large α we concentrate our attention on that saddle point that is near $\sigma_0=(\frac{1}{4}+f)^{1/2}$ and pass the contour through that point. The calculation is straightforward, standard, and tedious. We find, keeping only terms of leading and next-to-leading order in the small quantity α^{-1}, the result

$$G(\alpha,\beta)=\frac{\exp(2\sigma_0\alpha-2\beta^2/b_0\alpha)}{2f\sigma_0(2\pi b_0\alpha)^{1/2}}\left(1+\frac{b_1}{\alpha}\right), \tag{4.4}$$

where

$$b_0=\frac{2(1+1/2f)}{\sigma_0}\,,\qquad \sigma_0=(\tfrac{1}{4}+f)^{1/2}\,, \tag{4.5}$$

and b_1 is a constant involving f that is not needed for the moment. The fact that only even powers of β occur is a consequence of previously noted symmetries. The factor $\exp(-2\beta^2/b_0\alpha)$ should really properly be regarded as $(1-2\beta^2/b_0\alpha)$ to the order calculated.

It is now a simple matter to compute the amplitude A for the two cases:

$$\frac{\sinh\theta\, A(\theta,\theta_0)}{8\pi^3 f^2}=\frac{\exp(2\sigma_0\theta-2\theta_0{}^2/b_0\theta)}{f(2\pi b_0\theta)^{1/2}}\left(1+\frac{\bar{b}_1}{\theta}\right), \quad (S)$$

$$\frac{\sinh\xi A(\xi,\xi_m)}{8\pi^3 f^2}=2\,\frac{\exp(2\sigma_0\xi_m-2\xi^2/b_0\xi_m)}{f\sigma_0(2\pi b_0\xi_m)^{1/2}}\,\frac{\xi}{b_0\xi_m}\,, \quad (T) \tag{4.6}$$

where

$$\bar{b}_1=b_1-\frac{1}{4\sigma_0}=\frac{29f^2+21f+4}{32f^3(1+1/2f)^2\sigma_0}\,. \tag{4.7}$$

Befor discussing these results let us consider the other interesting limit, namely, fixed ω and large q^2 in both the (S) and (T) cases.

This large-q^2 fixed-ω limit is much simpler technically than the one just considered. In addition, as we have discussed in a previous paper, this limit depends essentially only on the behavior of the anomalous dimensions for large n. The limit of interest requires a study of the quantity $G(\alpha,\beta)$ for $\alpha-\beta=x$ fixed, $\alpha+\beta=y$ large. We note that for the spacelike case, $\alpha-\beta=x=\theta-\theta_0\cong\ln\omega$, $\alpha+\beta=y=\theta+\theta_0\cong\ln\omega+\ln(q^2/p^2)$; and for the timelike case, $\alpha-\beta=\xi_m-\xi\cong\ln(1/\omega)$, and $\alpha+\beta=\xi_m+\xi\cong\ln(q^2/p^2)-\ln(1/\omega)$.

We require

$$G(x,y)=\int_{c-i\infty}^{c+i\infty}\frac{d\sigma}{2\pi i}\,\frac{e^{\sigma x}\sinh\{[\frac{1}{4}+f^2/(\sigma^2-\frac{1}{4})]^{1/2}y\}}{(\sigma^2-\frac{1}{4})[\frac{1}{4}+f^2/(\sigma^2-\frac{1}{4})]^{1/2}}\,, \tag{4.8}$$

for large y, fixed x. This can be discussed in a variety of ways. For example, we have seen that G can be evaluated exactly in terms of a sum over a product of Bessel functions. Starting from this we can use the asymptotic form of $I_{l+1/2}(y/2)$ and sum the resultant series to obtain

$$G(x,y)=e^{y/2}\int_{c-i\infty}^{c+i\infty}\frac{d\sigma}{2\pi i}\frac{\exp\left(\sigma x+\frac{f^2y}{\sigma^2-\frac{1}{4}}\right)}{\sigma^2-\frac{1}{4}}$$

$$=e^{y/2}\frac{d}{d(f^2y)}\int_{c-i\infty}^{c+i\infty}\frac{d\sigma}{2\pi i}\exp\left(\sigma x+\frac{f^2y}{\sigma^2-\frac{1}{4}}\right). \tag{4.9}$$

Calling $f^2y=z$ we must evaluate, for large z,

$$g(x,z)=\int_{c-i\infty}^{c+i\infty}\frac{d\sigma}{2\pi i}\exp\left(\sigma x+\frac{z}{\sigma^2-\frac{1}{4}}\right); \tag{4.10}$$

there is obviously a term $\delta(x)$ in this integral which we ignore since we shall ultimately take a derivative with respect to z. Clearly the quantity

$$\Phi(\sigma)=\sigma x+\frac{z}{\sigma^2-\frac{1}{4}} \tag{4.11}$$

has a minimum on the real axis for some large σ. The actual location is at a point

$$\sigma\cong\bar\sigma+1/6\bar\sigma, \tag{4.12}$$

where

$$\bar\sigma=(2z/x)^{1/3}. \tag{4.13}$$

If z is sufficiently large that $\bar\sigma^2\gg\frac{1}{4}$ wa may approximate ϕ by

$$\Phi(\sigma)\cong\sigma x+z/\sigma^2. \tag{4.14}$$

The error made thereby is controllably small and we will quote it below. Keeping only leading terms, we have from the standard steepest-descent procedure,

$$\Phi(\bar\sigma)=3(\tfrac{1}{4}x^2z)^{1/3} \tag{4.15}$$

and

$$g(x,z)=\frac{e^{\Phi(\bar\sigma)}}{[2\pi\Phi''(\bar\sigma)]^{1/2}}, \tag{4.16}$$

where

$$\Phi''(\bar\sigma)=6z/\bar\sigma^4=3x^{4/3}/(2z)^{1/3}. \tag{4.17}$$

We have then quite explicitly

$$g(x,z)=\frac{\exp[3(\frac{1}{4}x^2z)^{1/3}]}{[6\pi x^{4/3}/(2z)^{1/3}]^{1/2}}\,[1+O(1/z^{1/3})]. \tag{4.18}$$

We have indicated here the next correction terms in the steepest-descent procedure. It is easy to show that the errors associated with our approximation of $\Phi(\sigma)$ are also of order $z^{-1/3}$; these are both easily computed, but are not very interesting. We have finally

$$G(x,y)\cong e^{y/2}\frac{\exp3(\frac{1}{4}f^2x^2y)^{1/3}}{(12\pi f^2y)^{1/2}}. \tag{4.19}$$

It is straightforward to show that the large-y limit of the exact $G(x,y)$ from which we started leads, by saddle-point integration, to the same result again to within terms of order $y^{-1/3}$. This is an illustration of the general feature, discussed in I, that the large-q^2 fixed-ω limit depends only on the behavior of the anomalous dimensions for large n. We have tested this principle with a number of examples, and there is little doubt that, barring pathologies, it is a theorem. We mention one example:

$$\int_{c-i\infty}^{c+i\infty}\frac{d\sigma}{2\pi i}\exp\{\sigma x+y[\sigma-(\sigma^2-a^2)^{1/2}]\}$$

$$=\delta(x)+ay\frac{I_1(a(x^2+2yx)^{1/2})}{(x^2+2yx)^{1/2}}. \tag{4.20}$$

If we replace the integrand by its large σ value, we expect that we will get the correct large-y limit, and we find

$$\int_{c-i\infty}^{c+i\infty}\frac{d\sigma}{2\pi i}\exp(\sigma x+ya^2/2\sigma)$$

$$=\delta(x)+\left(\frac{a^2y}{2x}\right)^{1/2}I_1((2a^2yx)^{1/2}), \tag{4.21}$$

which is, in fact, the same as the large-y limit of the exact answer. It is, of course, no accident that the large-σ limit of the anomalous dimensions corresponds to the perturbation theoretic expansion: Large "angular momentum" is tantamount to weak coupling.

We now write explicitly the large-q^2 fixed-ω limits of the two cases (S) and (T).

$$\frac{A(\theta,\theta_0)}{16\pi^3f^2}=e^{-(\theta-\theta_0)/2}\left[\frac{f^2}{12\pi z(\theta,\theta_0)}\right]^{1/2}e^{3z(\theta,\theta_0)}, \quad (S)$$

$$\frac{A(\xi,\xi_m)}{16\pi^3f^2}=e^{(\xi_m-\xi)/2}\left[\frac{f^2}{12\pi z(\xi_m,\xi)}\right]^{1/2}e^{3z(\xi_m,\xi)}, \quad (T) \tag{4.22}$$

where

$$z(\alpha,\beta)=[\tfrac{1}{4}f^2(\alpha-\beta)^2(\alpha+\beta)]^{1/3}. \tag{4.23}$$

Now we note that for large q^2, $\theta-\theta_0\cong\ln\omega$ and $\xi_m-\xi\cong\ln(1/\omega)$. Hence to the extent that $\theta+\theta_0\approx2\theta_0$ $[=\ln(q^2/p^2)]$ and $\xi_m+\xi\approx2\xi_m$ $[=\ln(q^2/p^2)]$, we see that the above expressions satisfy the Gribov-Lipatov[6] relation

$$\bar F(\omega,\ln(q^2/p^2))=(1/\omega)F(1/\omega,\ln(q^2/p^2)), \tag{4.24}$$

where $\bar F$ is the annihilation structure function, and F is the deep-inelastic (spacelike) one. Quite generally we note that if [writing $\lambda=\ln(q^2/p^2)$]

$$F(\omega,\lambda)=\int\frac{d\sigma}{2\pi i}\,e^{(\sigma+1)\ln\omega}g(\sigma,\lambda),$$
$$\tilde F(\omega,\lambda)=\int\frac{d\sigma}{2\pi i}\,e^{(\sigma+1)\ln(1/\omega)}\tilde g(\sigma,\lambda), \tag{4.25}$$

and $\tilde g(\sigma,\lambda)=g(\sigma-1,\lambda)$. Then indeed

$$\tilde F(\omega,\lambda)=\frac{1}{\omega}F(1/\omega,\lambda). \tag{4.26}$$

This relation does not hold for our model exactly, as we will see later on more explicitly, but only in this extreme large-$\ln(q^2/p^2)$ limit.

There are no particular surprises in the large-q^2 fixed-ω limit we have just discussed. The cross sections grow as q^2 increases, faster than any power of $\ln q^2$. Note, incidentally, that we cannot trust these formulas for $x\to 0$ or ω near unity (i.e., threshold) in either the (S) or (T) case; our steepest-descent method is simply inadequate in this case. In fact, we must require that $(x^2y)^{1/3}$ be large compared to unity, or $|\ln\omega|\gtrsim(\ln q^2)^{-1/2}$.

What we have found in the other interesting physical limits, i.e., large ω, fixed q^2 in the spacelike case (what is sometimes called the Regge limit) and the large-q^2 fixed-energy limit of produced particle in the timelike case are rather more striking. We see that

$$A(\theta,\theta_0)\to\frac{\exp\{[2(\frac14+f)^{1/2}-1]\theta\}}{\theta^{1/2}},\quad (S) \tag{4.27}$$

as $\theta\cong\ln\omega+\frac12\ln(q^2/p^2)\to\infty$, $\theta_0=\frac12\ln(q^2/p^2)$ fixed, which means

$$A\to\omega^P/(\ln\omega)^{1/2},\quad (S) \tag{4.28}$$

where P is the positive power $2(\frac14+f)^{1/2}-1$. This is no cause for alarm, because we are not putting in any s-channel unitarity constraints. The possible violation of the Froissart bound (which would be the case if $P>1$) cannot be taken seriously. We note that we have to do with a branch point in the "l plane," not a pole, reminiscent of the work of Cheng and Wu.[7] In the timelike region we have

$$A(\xi,\xi_m)\to\frac{\exp[2(\frac14+f)^{1/2}\xi_m]}{\xi_m^{3/2}}\,\frac{\xi}{\sinh\xi},\quad (T) \tag{4.29}$$

where we recall that in the center-of-mass system, $q=(q,0,0,0)$. $\cosh\xi=E/m$, with E (m) the energy (mass) of the distinguished hadron and $2\xi_m=\ln(q^2/p^2)$. Thus we have

$$A(\xi,\xi_m)\to\frac{(q^2/p^2)^{(1/2+f)^{1/2}}}{[\ln(q^2/p^2)]^{3/2}}\times\frac{\ln[E/m+(E^2/m^2-1)^{1/2}]}{(E^2/m^2-1)^{1/2}}. \tag{4.30}$$

For fixed E the cross section increases with q^2 (which is the general trend of the current SPEAR data). Again we emphasize that the results we have just recorded are strongly dependent on the details of the theory for their precise form, but that the behavior reflecting a singularity in what is essentially the anomalous dimension at some small σ (in this case near $\sigma=\frac12$) is quite general.

Finally, we record for completeness the precise connection between the conventionally defined anomalous dimensions and the quantities entering our model. For the spacelike case, as $-q^2\to\infty$,

$$\tilde a_n\equiv\int_1^\infty d\omega\,\omega^{-n-2}A(\omega,q^2,p^2)=C_n(p^2)(q^2)^{-\gamma_n}, \tag{4.31}$$

and γ_n is the anomalous dimension associated with the nth moment. Our model then tells us that

$$-\gamma_n=\frac{n+1}{2}-\left\{\frac{(n+2)^2+1}{4}-\left[\left(\frac{n+2}{2}\right)^2+f^2\right]\right\}^{1/2}\cong\frac{f^2}{(n+1)(n+2)}, \tag{4.32}$$

for small f^2 or large n. Similarly, in the annihilation case, we write

$$\tilde a_N\equiv\int_0^1 d\omega\,\omega^N A(\omega,q^2,p^2)=C_N(p^2)(q^2)^{-\tilde\gamma_N}, \tag{4.33}$$

where

$$-\tilde\gamma_N=-\frac{N+1}{2}+\left[\frac{N^2+1}{4}+\left(\frac{N^2+1}{4}+f^2\right)\right]^{1/2}\cong\frac{f^2}{N(N+1)}. \tag{4.34}$$

We see that in the small-f^2 limit γ_n and $\tilde\gamma_n$ are related by simple translation by one unit (this is the origin of the Gribov-Lipatov relation as we have noted) and possess simple poles in N. Neither feature survives in the exact expression: The pole singularities are converted to simple branch-point singularities, and there ceases to be any simple relation between γ_n and $\tilde\gamma_N$.

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[1]Curtis G. Callan, Jr. and M. L. Goldberger, preceding paper, Phys. Rev. D 11, 1542 (1975).

[2]For a relatively recent review of such equations see M. L. Goldberger, in Developments in High Energy Physics, *Proceedings of the International School of Physics "Enrico Fermi," Course 54*, edited by R. Gatto (Academic, New York, 1973).

[3]C. Lovelace and D. Gross, private communication.

[4]A. Mueller, Phys. Rev. D 9, 963 (1974).

[5]We are being somewhat casual about notation here and in some of the following equations: We use the same symbol A when we regard this quantity as a function of (s, u, v) or (ω, u, v) or (ξ, u, v). This should cause no problem.

[6]V. N. Gribov and L. N. Lipatov, Yad. Fiz. 15, 1218 (1972) [Sov. J. Nucl. Phys. 15, 675 (1972)].

[7]For a rather complete set of references to the work of Cheng and Wu, see H. Cheng, J. Walker, and T. T. Wu, Phys. Rev. D 9, 749 (1974).