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Superfields and Fermi-Bose symmetry

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The realization of supergauge transformations on fields defined over an 8-dimensional space whose points are labeled by x_μ and the anticommuting Majorana spinor θ_α is described. The covariant derivative is defined and applied to the problem of decomposing superfields into irreducible (chiral) parts and to the problem of constructing "supersymmetric" Lagrangians. Further, it is shown how to build internal symmetries (both global and local) into these Lagrangians. An example is discussed in which the internal (global) symmetry is spontaneously violated, giving rise to a supermultiplet of Goldstone particles (including fermions). When a local symmetry is broken the Higgs mechanism (for bosons and fermions) is shown to be operative. A possible solution to the problem of defining a conserved fermion number is indicated.

I. INTRODUCTION

The concept of a fundamental symmetry between fermions and bosons has begun recently to receive a good deal of attention. This symmetry was formulated first in the context of dual model theory.[1] In this 2-dimensional setting it takes the form of a *local* symmetry and plays a vital role in the elimination of ghosts. More recently, Wess and Zumino[2,3] took the decisive step of formulating a *global* Fermi-Bose symmetry in 4-dimensional spacetime.

An approach to the problem of implementing the global Fermi-Bose supersymmetry, which is somewhat different from that of Wess and Zumino, was proposed by ourselves.[4,5] This involved the consideration of a superfield $\Phi(x, \theta)$ defined on an 8-dimensional space which is the product of ordinary spacetime with a 4-dimensional space whose points are labeled by the anticommuting Majorana spinor θ_α.[6] The purpose of this article is to discuss in more detail the properties of such superfields and their products and to show how it is possible to set up supersymmetric Lagrangians which are compatible with local internal symmetries. The plan of the article is as follows.

In Sec. II a pseudogeometrical point of view is introduced and the action of the supersymmetry group on the space of x_μ and θ_α is defined. This leads in a natural way to the transformation rules for scalar, spinor, etc. superfields. The structure of such representations is discussed briefly. The important concept of covariant differentiation is introduced in Sec. III and its use in decomposing superfields into irreducible pieces is indicated. Detailed properties of the covariant derivative, including a number of identities, are set out in Appendix A. In Sec. IV the Lagrangian for a self-coupled scalar superfield first exhibited by Wess and Zumino[3] is given as an illustration. Section V is devoted to the problem of incorporating internal symmetries in a supersymmetric scheme. Those of the global kind are easily fitted in and so are mentioned only in passing. The main problem is to set up local (and particularly non-Abelian) internal symmetries which are compatible with supersymmetry. This is nontrivial. In particular, it is found that zero-mass "gauge" spinors must accompany the usual vector gauge fields.[7] Goldstone and Higgs mechanisms are discussed in Sec. VI and Goldstone fermions are, shown to arise when internal symmetry is spontaneously broken.

One of the disturbing features of the supersymmetry scheme is the prevalence of Majorana spinors. On the face of it, in combining fermions and bosons into a single multiplet it would seem to be inevitable that quantum numbers such as electric charge or baryon number must be shared between the fermions and the bosons in the multiplet, while neutral bosons would go together with neutral (i.e., Majorana) fermions. Fortunately, this difficulty is not inevitable. A counterexample is given at the end of Sec. V, where it is shown that if the system of gauge fields is allowed to interact with a massless scalar multiplet [belonging to the adjoint representation of an internal symmetry, e.g., SU(n)] then a new symmetry appears. The Majorana spinors from the gauge system can be combined with the spinors from the matter supermultiplets into *complex* Dirac spinors, with the Lagrangian exhibiting a fermion-number conservation.

With the incorporation of local internal symmetries into the framework of supersymmetric

Lagrangians and with fermion-number conservation implementable, the supersymmetric Lagrangian concept is now sufficiently developed to be considered for use in a realistic unified theory of weak, electromagnetic, and strong interactions.

II. SUPERFIELDS AND THEIR TRANSFORMATIONS

Superfields are defined over the 8-dimensional space whose points are represented by the pair (x_μ, θ_α), where x_μ denotes the usual (real) spacetime coordinate and θ_α is a Majorana spinor.[8] The variables θ_α differ radically from coordinates of the usual sort in that they anticommute,

$$\theta_\alpha \theta'_\beta + \theta'_\beta \theta_\alpha = 0 .$$

This has the important consequence that any *local* function $f(\theta)$ must be a polynomial. This can be seen from the fact that the monomials

$$\theta_{\alpha_1} \theta_{\alpha_2} \cdots \theta_{\alpha_n}$$

must be antisymmetric and therefore vanishing for $n > 4$. The local function $f(\theta)$ is fully specified by 16 elements: the coefficients in its expansion in powers of θ. Half of these elements are numbers of the ordinary sort while the rest are anticommuting quantities. One can think of the function $f(\theta)$ as a kind of 16-vector. Nevertheless, it is very useful to view such objects as functions defined over a 4-space, and this aspect will be emphasized in the following.

The action of the Poincaré group on the space of x and θ is given by

$$\begin{aligned} x_\mu &\to \Lambda_{\mu\nu} x_\nu + b_\mu , \\ \theta_\alpha &\to a_\alpha{}^\beta(\Lambda)\theta_\beta , \end{aligned} \tag{2.1}$$

where $a(\Lambda)$ denotes the Dirac spinor representation of the homogeneous Lorentz transformation Λ. In particular, space reflections are associated with the mapping

$$\theta_\alpha \to i(\gamma_0\theta)_\alpha . \tag{2.2}$$

The factor i is necessary here for compatibility with the Majorana constraint on θ.

The action of a supergauge transformation on the space of x and θ is defined by

$$\begin{aligned} x_\mu &\to x_\mu + \tfrac{1}{2} i\bar{\epsilon}\gamma_\mu\theta , \\ \theta_\alpha &\to \theta_\alpha + \epsilon_\alpha , \end{aligned} \tag{2.3}$$

where the parameter ϵ_α must, of course, be an anticommuting Majorana spinor. The group property of the mappings (2.1) and (2.3) is easily verified. However, it should perhaps be emphasized that our constructions are purely formal. For example, the spacetime translation $\frac{1}{2}i\bar{\epsilon}\gamma_\mu\theta$ is *not* a set of four ordinary real numbers such as x_μ is usually taken to be. These numbers are nilpotent, $(\bar{\epsilon}\gamma_\mu\theta)^5 = 0$. For consistency we should regard the coordinates x_μ also as belonging to some nontrivial algebra. Perhaps it may be possible to establish a rigorous geometry[9] on the space of x and θ.

The scalar **superfield** is naturally defined as one which transforms according to

$$\Phi'(x', \theta') = \Phi(x, \theta) . \tag{2.4}$$

Generalization to spinor and tensor superfields is equally natural. For example, the spinor would transform according to

$$\Psi'_\alpha(x', \theta') = a_\alpha{}^\beta(\Lambda)\Psi_\beta(x, \theta) .$$

As remarked above, any local function of θ must be a polynomial. To illustrate we give the expansion of the scalar superfield,

$$\begin{aligned} \Phi(x, \theta) = {} & A(x) + \bar{\theta}\psi(x) + \tfrac{1}{4}\bar{\theta}\theta F(x) + \tfrac{1}{4}\bar{\theta}\gamma_5\theta G(x) \\ & + \tfrac{1}{4}\bar{\theta} i\gamma_\nu\gamma_5\theta A_\nu(x) + \tfrac{1}{4}\bar{\theta}\theta\bar{\theta}\chi(x) \\ & + \tfrac{1}{32}(\bar{\theta}\theta)^2 D(x) , \end{aligned} \tag{2.5}$$

where the coefficients A, F, G, A_ν, D are ordinary Bose fields, and ψ and χ are Fermi fields. The behavior of these components under the action of the Poincaré group is clear: A, F, and D are scalars, G is a pseudoscalar, A_μ is an axial vector, ψ and χ are Dirac spinors. (The intrinsic parities are reversed in the case of a pseudoscalar superfield.) These components are all complex in general. However, it is possible to impose a reality condition on the superfield,

$$\Phi(x, \theta)^* = \Phi(x, \theta) ,$$

where the complex conjugation is understood to reverse the order of anticommuting factors. The real scalar superfield has Bose components which are real and Fermi components which are Majorana spinors.

The behavior of the component fields under the action of an infinitesimal supergauge transformation is easily deduced from (2.3) and (2.4),

$$\delta\Phi(x, \theta) = \bar{\epsilon}^\alpha \left(\frac{\partial\Phi}{\partial\bar{\theta}^\alpha} + \frac{i}{2}(\gamma_\mu\theta)_\alpha \frac{\partial\Phi}{\partial x_\mu} \right) , \tag{2.6}$$

by substituting the expansion (2.5). One finds

$$\begin{aligned} \delta A &= \bar{\epsilon}\psi , \\ \delta\psi &= \tfrac{1}{2}(F + \gamma_5 G + i\gamma_\mu\gamma_5 A_\mu - i\partial\!\!\!/ A)\epsilon , \\ \delta F &= \tfrac{1}{2}\bar{\epsilon}\chi - \tfrac{1}{2} i\bar{\epsilon}\partial\!\!\!/\psi , \\ \delta G &= \tfrac{1}{2}\bar{\epsilon}\gamma_5\chi - \tfrac{1}{2} i\bar{\epsilon}\gamma_5\partial\!\!\!/\psi , \\ \delta A_\nu &= \tfrac{1}{2}\bar{\epsilon} i\gamma_\nu\gamma_5\chi + \tfrac{1}{2} i\bar{\epsilon}\gamma_\mu i\gamma_\nu\gamma_5\partial_\mu\psi , \\ \delta\chi &= \tfrac{1}{2}(D - i\partial\!\!\!/ F - i\partial\!\!\!/\gamma_5 G - i\gamma_\nu\gamma_5 i\partial\!\!\!/ A_\nu)\epsilon , \\ \delta D &= -i\bar{\epsilon}\partial\!\!\!/\chi . \end{aligned} \tag{2.7}$$

In certain circumstances (which will be discussed in Sec. III) this representation turns out to be reducible.

A better insight into the structure of representations of the supersymmetry can be gained from an examination of the infinitesimal algebra. To obtain this algebra one represents the action of an infinitesimal transformation on $\Phi(x, \theta)$ by a commutator,

$$\frac{1}{i}\delta\Phi(x, \theta) = [\Phi(x, \theta), \bar{\epsilon}S]\,, \tag{2.8}$$

where the components of the generator S_α themselves comprise a Majorana spinor,

$$(\bar{\epsilon}S)^\dagger = \bar{\epsilon}S\,. \tag{2.9}$$

By considering the action of two infinitesimal transformations applied in succession and making use of the Jacobi identity one arrives at the consistency condition

$$[\bar{\epsilon}_1 S, \bar{\epsilon}_2 S] = \bar{\epsilon}_1\gamma_\mu\epsilon_2 P_\mu\,, \tag{2.10}$$

where P_μ is the generator of space translations. In fact it is clear from the rules (2.3) that the commutator of two supergauge transformations is a translation. At this point it is necessary to assume that the infinitesimal parameters ϵ anticommute with the generators S. From (2.10) one then extracts the anticommutation relation

$$\{S_\alpha, S_\beta\} = -(\gamma_\mu C)_{\alpha\beta}P_\mu \tag{2.11}$$

(where C is the charge-conjugation matrix). Since the matrices $\gamma_\mu C$ are symmetric[8] one sees that the left-hand side of (2.11) must be an anticommutator and hence the necessity of assuming that ϵ anticommutes with S.

The rest of the infinitesimal algebra is deduced in the same way. In addition to the usual rules for the commutators among generators of Poincaré transformations one finds

$$\begin{aligned}&[S_\alpha, P_\mu] = 0\,,\\ &[S_\alpha, J_{\mu\nu}] = \tfrac{1}{2}(\sigma_{\mu\nu}S)_\alpha\,,\end{aligned} \tag{2.12}$$

indicating that S_α transforms like a Dirac spinor.

The rules (2.11) and (2.12) are fundamental. The construction of irreducible representations of this "algebra" and the development of rules for decomposing their products are the central problems of supersymmetry theory. We have chosen to regard this system as the infinitesimal algebra of a continuous group of point transformations in a space, some of whose coordinates are anticommuting c numbers. However, in the absence of a rigorous geometry in the space of x and θ this point of view is no more than a suggestive guide for one's intuition.

The construction of representations begins with the observation that supergauge transformations must leave invariant the manifold of states with fixed 4-momentum since S_α commutes with P_μ. On such a manifold the anticommutator (2.11) becomes a fixed set of numbers and we see that the operators S_α generate a *Clifford algebra*. Since this algebra has just 16 independent members, its one and only finite-dimensional irreducible representation is in terms of 4×4 matrices.[10] (There must exist infinite-dimensional representations as well, but we shall not consider them here.) The manifold of states with fixed 4-momentum is therefore reduced by the action of supergauge transformations into 4-dimensional invariant subspaces. However, these subspaces are not in general left invariant by the Wigner rotations. These transformations reduce the manifold into $(2\mathcal{J}+1)$-dimensional invariant subspaces. The subspaces which are invariant with respect to *both* supergauge and Wigner transformations are $4(2\mathcal{J}+1)$-dimensional.

The construction of unitary irreducible representations of the algebra (2.11) and (2.12) by Wigner's method has been treated elsewhere.[5] These representations are characterized by a mass, a spin ($\mathcal{J}$), and an intrinsic parity (η). Included in one of these representations are *four* irreducible representations of the Poincaré group. The $(\text{spin})^{\text{parity}}$ content is $(\mathcal{J}-\frac{1}{2})^\eta$, $\mathcal{J}^{i\eta}$, $\mathcal{J}^{-i\eta}$, $(\mathcal{J}+\frac{1}{2})^\eta$, where η takes one of the values $\pm i$ (for integer $\mathcal{J}$) or ± 1 (for half-integer $\mathcal{J}$). The rest mass is common. [The lightlike representations are 4-dimensional with helicity content $\pm\lambda$ and $\pm(\lambda+\frac{1}{2})$ with fixed λ.]

We sketch very briefly a method for constructing irreducible multiplets of fields (nonunitary representations). In a basis where γ_5 is diagonal the generators S_α become a pair of chiral spinors S_A and $S_{\dot{A}}$ $(A=1,2)$ which satisfy the algebra

$$\{S_A, S_B\} = 0, \quad \{S_{\dot{A}}, S_{\dot{B}}\} = 0, \quad \{S_A, S_{\dot{B}}\} = i\partial_{A\dot{B}}\,. \tag{2.13}$$

We shall treat S_A (or $S_{\dot{A}}$) as "raising operators" and $S_{\dot{A}}$ (or S_A) as "lowering operators." Let the lowest component $U(x)$ belong to some finite-dimensional representation $D(j_1, j_2)$ of the proper Lorentz group. Successive applications of S_A generate two new sets of components in the multiplet,

$$\begin{aligned}&S_A U(x) = M_A(x)\,,\\ &S_A S_B U(x) = \epsilon_{AB}V(x)\end{aligned} \tag{2.14}$$

(where ϵ_{AB} denotes the permutation symbol in two dimensions) *and no more* since the product of three undotted operators must vanish by (2.13). The complete table is then easily obtained:

$$S_A U(x) = M_A(x), \quad S_{\dot{A}} U(x) = 0,$$
$$S_A M_B(x) = \epsilon_{AB} V(x), \quad S_{\dot{A}} M_B(x) = -i\partial_{\dot{A}B} U(x), \quad (2.15)$$
$$S_A V(x) = 0, \quad S_{\dot{A}} V(x) = -i\partial_{\dot{A}}{}^B M_B(x).$$

A similar table [the parity transform of (2.15)] is obtained by treating $S_{\dot{A}}$ as the raising operator. Space reflections are incorporated by combining the two multiplets.[11]

Representations of the type (2.15) which have dimensionality $4(2j_1+1)(2j_2+1)$ are in general reducible. Thus, if $j_1 \geq j_2$ one can reduce the lowest component $U(x)$ to a tensor in $D(j_1-j_2, 0)$ by contraction with $2j_2$ powers of the operator $\partial/\partial x_\mu$. On this component one then constructs a representation of $4[2(j_1-j_2)+1]$ dimensions. This contraction is nothing more than the supersymmetry analog of, say, separating the longitudinal part $\partial_\mu V_\mu$ from a vector V_μ. In setting up a local action principle, one usually finds it necessary to employ fields which are reducible in this sense.

To conclude this section, it may be worth pointing out that the Majorana constraint is very potent in limiting the size of multiplets. If one were to treat S_α and $\bar{S}^\alpha$ as independent generators, for example, and replace (2.11) by the set

$$\{S_\alpha, S_\beta\} = 0, \quad \{\bar{S}^\alpha, \bar{S}^\beta\} = 0, \quad \{S_\alpha, \bar{S}^\beta\} = (\gamma_\mu)_\alpha{}^\beta P_\mu,$$

then the fundamental representation would have 16 rather than 4 dimensions. It would include vector as well as scalar and spinor components. The same thing happens when the generators are generalized so as to carry an internal quantum number such as isospin. This type of generalization was discussed in Ref. 5.

III. THE COVARIANT DERIVATIVE

Although it is not easy to define an integral over θ space, there is certainly no problem with differentiation. The ordinary derivative is defined by

$$f(\theta + \delta\theta) = f(\theta) + \delta\bar{\theta}^\alpha \frac{\partial f}{\partial \bar{\theta}^\alpha},$$

where the infinitesimal $\delta\bar{\theta}$ stands to the left of $\partial f/\partial\bar{\theta}$ (since these quantities may anticommute it is important to fix their order).

An important role is played in the following by the differential operator

$$D_\alpha = \frac{\partial}{\partial\bar{\theta}^\alpha} - \frac{i}{2}(\gamma_\mu\theta)_\alpha \frac{\partial}{\partial x_\mu}, \quad (3.1)$$

which we shall call the covariant derivative. This operator is covariant in the sense that it transforms as a Dirac spinor under Lorentz transformations and as an invariant with respect to the supergauge transformations (2.6).

The covariant derivative has the usual properties of a differential operator with two significant exceptions. Firstly, when applied to the product of two superfields its effects are distributed according to the rule

$$D_\alpha(\Phi_1\Phi_2) = (D_\alpha\Phi_1)\Phi_2 \pm \Phi_1(D_\alpha\Phi_2),$$

where the + (−) sign applies when Φ_1 is bosonic (fermionic), i.e., when $\delta\bar{\theta}$ commutes (anticommutes) with Φ_1. Secondly, the covariant derivatives neither commute nor anticommute. Their anticommutator is given by

$$\{D_\alpha, D_\beta\} = -(\gamma_\mu C)_{\alpha\beta}\, i\frac{\partial}{\partial x_\mu}. \quad (3.2)$$

The operator D_α is basically a Majorana spinor. For this reason it is useful to define another form

$$\bar{D}^\alpha = (C^{-1})^{\alpha\beta} D_\beta, \quad (3.3)$$

which is nothing more than a relabeling of components. The basic anticommutator can then be given in three equivalent forms:

$$\{D_\alpha, D_\beta\} = -(\not{p}C)_{\alpha\beta},$$
$$\{D_\alpha, \bar{D}^\beta\} = (\not{p})_\alpha{}^\beta, \quad (3.4)$$
$$\{\bar{D}^\alpha, \bar{D}^\beta\} = (C^{-1}\not{p})^{\alpha\beta}.$$

(It is often more convenient to work in momentum space.) The operators D_α clearly generate a Clifford algebra which is isomorphic to the superalgebra (2.11).

In view of the algebraic structure (3.4), only 16 independent operators can be made from products of the D_α. The most useful set is

$$1,\ D_\alpha,\ \bar{D}D,\ \bar{D}\gamma_5 D,\ \bar{D}i\gamma_\mu\gamma_5 D,\ \bar{D}DD_\alpha,\ (\bar{D}D)^2.$$

A product containing five or more factors must inevitably reduce. For example,

$$(\bar{D}D)^2 D_\alpha = -2\bar{D}D(\not{p}D)_\alpha.$$

Two other important identities are

$$\bar{D}\gamma_\mu D = 2p_\mu$$

and (3.5)

$$\bar{D}\sigma_{\mu\nu}D = 0.$$

A number of useful identities and multiplication rules are given in Appendix A.

It was mentioned before that the scalar superfield (2.5) is, in a sense, reducible. A reduction can be effected by imposing the condition

$$(\tfrac{1}{2}(1+i\gamma_5)D)_\alpha \Phi \equiv D_{L\alpha}\Phi = 0. \quad (3.6)$$

These linear differential equations are manifestly covariant (excluding space reflections). The general solution of (3.6) involves eight independent (real) components in contrast to the 16 of a general scalar superfield. This solution can be expressed in the form

$$\Phi_-(x,\theta)=\exp(\tfrac{1}{4}\bar\theta\partial\!\!\!/\gamma_5\theta)\times\left(A_-(x)+\bar\theta\psi_-(x)+\tfrac{1}{2}\bar\theta\,\frac{1-i\gamma_5}{2}\,\theta F_-(x)\right)\,, \quad (3.7)$$

where A_- and F_- are complex boson fields and ψ_- is a right-handed Dirac spinor, $i\gamma_5\psi_-=-\psi_-$. Likewise, the differential equations

$$D_{R\alpha}\Phi\equiv(\tfrac{1}{2}(1-i\gamma_5)D)_\alpha\Phi=0 \quad (3.6')$$

serve to define a left-handed superfield,

$$\Phi_+(x,\theta)=\exp(-\tfrac{1}{4}\bar\theta\partial\!\!\!/\gamma_5\theta)\times\left(A_+(x)+\bar\theta\psi_+(x)+\tfrac{1}{2}\bar\theta\,\frac{1+i\gamma_5}{2}\,\theta F_+(x)\right)\,. \quad (3.7')$$

It is possible (although not necessary) to identify Φ_- with the complex conjugate of Φ_+, i.e.,

$$A_-=A_+^*,\quad \psi_-=\psi_+^c,\quad F_-=F_+^*\,. \quad (3.8)$$

(Thus ψ_+ and ψ_- are identified as the left- and right-handed components, respectively, of a Majorana spinor.)

The components of the chiral superfields behave under an infinitesimal supergauge transformation according to

$$\begin{aligned}\delta A_\pm&=\bar\epsilon\psi_\pm\,,\\ \delta\psi_\pm&=\frac{1\pm i\gamma_5}{2}\,(F_\pm-i\partial\!\!\!/A_\pm)\epsilon\,,\\ \delta F_\pm&=-\bar\epsilon i\partial\!\!\!/\psi_\pm\,.\end{aligned} \quad (3.9)$$

These representations are irreducible.

A general scalar superfield contains another (nonchiral) piece Φ_1 which is singled out by means of nonlinear differential conditions

$$\bar D\,\frac{1\pm i\gamma_5}{2}\,D\Phi_1=0\,. \quad (3.10)$$

The resolution

$$\Phi=\Phi_++\Phi_-+\Phi_1 \quad (3.11)$$

is effected by the projection operators

$$\begin{aligned}E_+&=-\frac{1}{\partial^2}\,\bar D\,\frac{1-i\gamma_5}{2}\,D\bar D\,\frac{1+i\gamma_5}{2}\,D\,,\\ E_-&=-\frac{1}{\partial^2}\,\bar D\,\frac{1+i\gamma_5}{2}\,D\bar D\,\frac{1-i\gamma_5}{2}\,D\,,\\ E_1&=1+\frac{1}{4\partial^2}\,(\bar DD)^2\,.\end{aligned} \quad (3.12)$$

In terms of the components (2.5), the resolution (3.11) takes the explicit form

$$\begin{aligned}A&=A_++A_-+A_1\,,\\ \psi&=\psi_++\psi_-+\psi_1\,,\\ F&=F_++F_-\,,\\ G&=iF_+-iF_-\,,\\ A_\mu&=i\partial_\mu A_+-i\partial_\mu A_-+A_{1\mu}\,,\\ \chi&=-i\partial\!\!\!/\psi_+-i\partial\!\!\!/\psi_-+i\partial\!\!\!/\psi_1\,,\\ D&=-\partial^2A_+-\partial^2A_-+\partial^2A_1\,,\end{aligned} \quad (3.13)$$

where $A_{1\mu}$ is transverse, $\partial_\mu A_{1\mu}=0$. This decomposition is useful in constructing local field theories if the resulting components are local fields. Such is the case only when the original components χ and D in (2.5) are themselves first and second derivatives, respectively, of local fields. If this is so we may say that the superfield is "locally reducible." Otherwise, it is not.

Because of their being defined by a linear differential condition, (3.6) or (3.6′), the chiral superfields are closed under multiplication. That is, we have

$$\Phi_{1+}\Phi_{2+}=\Phi_{3+}$$

and (3.14)

$$\Phi_{1-}\Phi_{2-}=\Phi_{3-}\,.$$

This remarkable property is very important in the construction of Lagrangians. The mixed product $\Phi_{1+}\Phi_{2-}$, on the other hand, is a general superfield, one which is not locally reducible. Details are given in Appendix B.

The spinor superfield $\Psi_\alpha=D_\alpha\Phi_+$ is left-handed with respect to the spinor index α, but is of the nonchiral (Φ_1) type in the complexion of its component fields. Conversely, the spinor $\Psi_\alpha=D_\alpha\Phi_1$ is a mixture of left- and right-handed chiral superfields (see Appendix B).

IV. A SIMPLE LAGRANGIAN

To illustrate the application of the superfield notation in a simple dynamical system we consider here the Lagrangian given by Wess and Zumino[3] for a scalar multiplet. This Lagrangian, $\mathcal{L}(\Phi_+,\Phi_-)$, must itself transform as a scalar superfield. In order that the equations of motion should be supercovariant, the action integral must be an invariant,

$$\begin{aligned}\delta\int dx\,\mathcal{L}&=\int dx\,\bar\epsilon\left(\frac{\partial}{\partial\bar\theta}+\frac{i}{2}\,(\gamma_\mu\theta)\,\frac{\partial}{\partial x_\mu}\right)\mathcal{L}\\ &=\bar\epsilon\,\frac{\partial}{\partial\bar\theta}\int dx\,\mathcal{L}+\text{surface term}\\ &=0\,.\end{aligned}$$

The action will be invariant (up to a variationally insignificant surface term) if it is independent of

θ. This means that every θ-dependent term in $\mathcal{L}(\Phi_+, \Phi_-)$ must have the form of a spacetime divergence. *Such a form can always be achieved by applying the covariant operator D_α a sufficient number of times: twice on a chiral or locally reducible superfield, four times on a general superfield.*

Consider the Lagrangian

$$\mathcal{L} = \tfrac{1}{8}(\bar{D}D)^2(\Phi_+\Phi_-) - \tfrac{1}{2}\bar{D}D[V(\Phi_+) + V(\Phi_-)]$$
$$= \tfrac{1}{2}\bar{D}D[\tfrac{1}{4}\Phi_+\bar{D}D\Phi_- + \tfrac{1}{4}\Phi_-\bar{D}D\Phi_+ - V(\Phi_+) - V(\Phi_-)] , \quad (4.1)$$

where $\Phi_- = \Phi_+^*$ and V is a smooth function (typically a polynomial[12]). The spacetime integral of this expression is certainly invariant. The variationally important part is obtained by setting $\theta = 0$. With the help of the formulas in Appendix B one finds, for this part,

$$\mathcal{L} = \partial_\mu A_+ \partial_\mu A_- + F_+F_- + \tfrac{1}{2}\bar{\psi}i\partial\!\!\!/\psi + [V'(A_+)F_+ + V'(A_-)F_-]$$
$$- \frac{1}{2}\left(V''(A_+)\bar{\psi}\,\frac{1+i\gamma_5}{2}\,\psi + V''(A_-)\bar{\psi}\,\frac{1-i\gamma_5}{2}\,\psi\right) , \quad (4.2)$$

where $V'(A) = dV/dA$, etc. The equations of motion are

$$\partial^2 A_\pm = V''(A_\mp)F_\mp - \tfrac{1}{2}V'''(A_\mp)\bar{\psi}\,\frac{1\mp i\gamma_5}{2}\,\psi ,$$
$$F_\pm = -V'(A_\mp) , \quad (4.3)$$
$$i\partial\!\!\!/\psi = V''(A_+)\,\frac{1+i\gamma_5}{2}\,\psi + V''(A_-)\,\frac{1-i\gamma_5}{2}\,\psi .$$

The Lagrangian (4.2) and equations of motion (4.3) could easily be expressed in terms of real scalar fields A, F and pseudoscalars B, G defined by

$$A_\pm = \frac{1}{\sqrt{2}}(A \pm iB) ,\quad F_\pm = \frac{1}{\sqrt{2}}(F \pm iG) ,$$

but there is no great advantage to be gained by this.

According to (4.3) the vacuum expectation values must, in the tree approximation, satisfy the equations

$$0 = V''(\langle A_\pm\rangle)V'(\langle A_\mp\rangle) , \quad (4.4)$$

i.e., one or other of the factors V', V'' must vanish in the vacuum. Consider the possibilities. If $\langle V'\rangle = 0$ then $\langle V''\rangle$ is identified as the (common) mass of the multiplet. On the other hand, if $\langle V''\rangle = 0$ then the fermion is massless. This is a Goldstone (fermion) solution. However, it is an unstable one since the boson field equations take the form

$$\partial^2 A_\pm = -\langle V'\rangle\langle V'''\rangle(A_\mp - \langle A_\mp\rangle)$$
$$+ \text{ interaction terms} ,$$

indicating that one boson component must be a tachyon. This situation may be altered when quantum corrections are included.[13]

V. LOCAL SYMMETRY

It is natural to ask whether supersymmetry is compatible with internal symmetries of the usual sort, both global and local. One finds that indeed it is: Global internal symmetries can be incorporated quite easily; local symmetries are more difficult.[7]

Consider first the global case. Suppose, for example, that the superfields Φ_+ and Φ_- transform as doublets of SU(2),

$$\Phi_+(x,\theta) \to \Omega\Phi_+(x,\theta) ,$$
$$\Phi_-(x,\theta) \to \Omega\Phi_-(x,\theta) , \quad (5.1)$$

where Ω is an SU(2) matrix (independent of x *and* θ). The Lagrangian

$$\mathcal{L}_0 = \tfrac{1}{8}(\bar{D}D)^2(\Phi_+^\dagger\Phi_+ + \Phi_-^\dagger\Phi_-)$$
$$- \tfrac{1}{2}M\bar{D}D(\Phi_-^\dagger\Phi_+ + \Phi_+^\dagger\Phi_-) \quad (5.2)$$

is both supersymmetric (up to a surface term) and SU(2)-invariant. Unfortunately, the simplest SU(2)-invariant interaction for this system,

$$\mathcal{L}_1 = g\bar{D}D[(\Phi_-^\dagger\Phi_+)^2 + (\Phi_+^\dagger\Phi_-)^2] ,$$

is not renormalizable. To have a renormalizable (i.e., trilinear) interaction it is necessary to bring in singlet and/or triplet superfields. Thus, if $\Phi'_\pm$ transforms according to

$$\Phi'_\pm \to \Omega\Phi'_\pm\Omega^{-1} \quad (5.3)$$

then a renormalizable and SU(2)-invariant interaction is given by

$$g\bar{D}D(\Phi_-^\dagger\Phi'_+\Phi_+ + \Phi_+^\dagger\Phi'_-\Phi_-) . \quad (5.4)$$

As a second example, suppose Φ_+ is a 3×3 matrix of superfields which belongs to the (real) (3, 3) representation of SU(2)×SU(2),

$$\Phi_+ \to R_1\Phi_+R_2^T , \quad (5.5)$$

where R_1 and R_2 are orthogonal matrices. Suppose, moreover, that $\Phi_- = \Phi_+^*$ so that

$$\Phi_-^T \to R_2\Phi_-^TR_1^T . \quad (5.6)$$

Then a renormalizable and invariant Lagrangian is given by

$$\mathcal{L} = \tfrac{1}{8}(\bar{D}D)^2\,\mathrm{Tr}(\Phi_-^T\Phi_+) - \tfrac{1}{4}\bar{D}D\,\mathrm{Tr}(\Phi_+^T\Phi_+ + \Phi_-^T\Phi_-)$$
$$+ g\bar{D}D(\det\Phi_+ + \det\Phi_-) . \quad (5.7)$$

This Lagrangian will be considered further in Sec. VI.

The problem of setting up a *local* symmetry which is compatible with the supersymmetry is more interesting. First of all, if the matrix Ω which appears in (5.1) is to depend on x, then it must also depend on θ. In fact, it must be a set of chiral-type superfields,

$$\Phi_\pm(x,\theta)\to\Omega_\pm(x,\theta)\Phi_\pm(x,\theta)\,, \tag{5.8}$$

since the transformed field must, if it is to have the same number of independent components as the original, be chiral. The consistency of (5.8) relies on the closure property (3.14) of chiral superfields under multiplication.

Representing the doublet $\Phi_+(x,\theta)$ and the matrix Ω_+ by the expansions

$$\Phi_+(x,\theta)=\exp(-\tfrac{1}{4}\bar\theta\partial\!\!\!/\gamma_5\theta)\times[A_+(x)+\bar\theta\psi_+(x)+\tfrac{1}{4}\bar\theta(1+i\gamma_5)\theta F_+(x)]\,,$$

$$\Omega_+(x,\theta)=\exp(-\tfrac{1}{4}\bar\theta\partial\!\!\!/\gamma_5\theta)\times[U_+(x)+\bar\theta V_+(x)+\tfrac{1}{4}\bar\theta(1+i\gamma_5)\theta W_+(x)]$$

the explicit form of the transformation rule (5.8) is

$$A_+(x)\to U_+(x)A_+(x)\,,$$

$$\psi_+(x)\to U_+(x)\psi_+(x)+V_+(x)A_+(x)\,,$$

$$F_+(x)\to U_+(x)F_+(x)-\bar V_+(x)\psi_+(x)+W_+(x)A_+(x)\,.$$

The matrices U_+, V_+, and W_+ are complex, and V_+ carries a Dirac spinor index.[14]

It is possible to regard Ω_- as completely independent of Ω_+ or one can impose the constraints

$$\det\Omega_\pm=1,\ \Omega_+^\dagger=(\Omega_-)^{-1}, \tag{5.9}$$

which we shall adopt in the following. This means that the supersymmetric mass term

$$\bar DD(\Phi_-^\dagger\Phi_+ +\Phi_+^\dagger\Phi_-) \tag{5.10}$$

is an invariant of the local symmetry.

The main problem is to construct a gauge-invariant kinetic energy for the doublets $\Phi_\pm$. The expression (5.2) is certainly not satisfactory since $\Omega_+^\dagger\Omega_+\neq 1$. It is necessary to introduce some gauge fields. Following Wess and Zumino, who solved the problem of making supersymmetry compatible with a local U(1) symmetry,[7] we deal with the compatibility problem for the case of the local internal symmetry SU(2) [in fact, the SU(2) could be generalized to a local internal SU(n)].

The gauge field Ψ is a general (not chiral) pseudoscalar Hermitian matrix superfield which transforms under the local symmetry according to the rule

$$e^{g\Psi}\to\Omega_-e^{g\Psi}\Omega_+^{-1}\,. \tag{5.11}$$

With the help of this field, the kinetic term can be expressed in the invariant and supersymmetric form

$$\tfrac{1}{8}(\bar DD)^2(\Phi_+^\dagger e^{g\Psi}\Phi_+ +\Phi_-^\dagger e^{-g\Psi}\Phi_-)\,. \tag{5.12}$$

Although this is not in general a polynomial, there does exist a special gauge in which $\Psi^n=0$ for $n\geq 3$, where (5.12) defines a renormalizable interaction. We shall come back to this in the following.

Having introduced a gauge coupling into the system of matter fields $\Phi_\pm$, we find it necessary now to set up a gauge-invariant kinetic term for Ψ. By exploiting the chiral properties of $\Omega_\pm$,

$$((1\mp i\gamma_5)D)_\alpha\Omega_\pm(x,\theta)=0\,,$$

one can prove that the vector superfield V_μ defined by

$$V_\mu=-\frac{1}{g}\left(C^{-1}\gamma_\mu\,\frac{1+i\gamma_5}{2}\right)^{\alpha\beta}D_\alpha(e^{-g\Psi}D_\beta e^{g\Psi}) \tag{5.13}$$

transforms under the local symmetry like a Yang-Mills gauge field,

$$V_\mu\to\Omega_+V_\mu\Omega_+^{-1}+\frac{2i}{g}\,\Omega_+\partial_\mu\Omega_+^{-1}\,. \tag{5.14}$$

The Hermitian conjugate transforms according to

$$V_\mu^\dagger\to\Omega_-V_\mu^\dagger\Omega_-^{-1}+\frac{2i}{g}\,\Omega_-\partial_\mu\Omega_-^{-1}\,.$$

Notice that the combinations

$$\left(\frac{1-i\gamma_5}{2}\,D\right)_\alpha V_\mu$$

and

$$\left(\frac{1+i\gamma_5}{2}\,D\right)_\alpha V_\mu^\dagger$$

transform homogeneously, i.e., like field strengths. Further, one can show that these field strengths are chiral,

$$\left(\frac{1-i\gamma_5}{2}\,D\right)_\alpha\left(\frac{1-i\gamma_5}{2}D\right)_\alpha V_\mu=0 \tag{5.15}$$

(and likewise for $V_\mu^\dagger$). This means that the expression

$$-\frac{1}{32}\,\bar DD\,\mathrm{Tr}\left(\left(C^{-1}\,\frac{1-i\gamma_5}{2}\right)^{\alpha\beta}(D_\alpha V_\mu)(D_\beta V_\mu)+\left(C^{-1}\,\frac{1+i\gamma_5}{2}\right)^{\alpha\beta}(D_\alpha V_\mu^\dagger)(D_\beta V_\mu^\dagger)\right)$$

is an invariant of the local symmetry and is supersymmetric. It can serve as a Lagrangian for the gauge field. Making use of the identity (5.15) and discarding a variationally insignificant surface term, one can put this Lagrangian into the compact form

$$\tfrac{1}{128}(\bar DD)^2\,\mathrm{Tr}(V_\mu V_\mu+V_\mu^\dagger V_\mu^\dagger)\,. \tag{5.16}$$

Gathering together the terms (5.10), (5.12), and (5.16) we obtain the gauge-invariant and supersymmetric Lagrangian

$$\mathcal{L} = \tfrac{1}{8}(\bar{D}D)^2[\tfrac{1}{16}\,\mathrm{Tr}(V_\mu V_\mu + V_\mu^\dagger V_\mu^\dagger) + (\Phi_+^\dagger e^{g\Psi}\Phi_+ + \Phi_-^\dagger e^{-g\Psi}\Phi_-)] - \tfrac{1}{2}M\bar{D}D(\Phi_-^\dagger\Phi_+ + \Phi_+^\dagger\Phi_-)\,, \qquad (5.17)$$

where V_μ is defined by (5.13).

To write out the Lagrangian (5.17) explicitly in terms of component fields would be a complicated and unrewarding task. Fortunately, there exists a remarkable gauge[7] in which the Lagrangian assumes polynomial form. The infinitesimal form of the transformation law (5.11) is, to lowest order in Ψ,

$$\delta\Psi = \delta\Omega_-\Psi - \Psi\,\delta\Omega_+ + \frac{1}{g}\,\delta\Omega_- - \frac{1}{g}\,\delta\Omega_+\,. \qquad (5.18)$$

This indicates that half of the components in Ψ can be transformed away, leaving it in the special form

$$\Psi = \tfrac{1}{4}\,\bar{\theta}i\gamma_\nu\gamma_5\theta A_\nu + \frac{1}{2\sqrt{2}}\,\bar{\theta}\theta\,\bar{\theta}\gamma_5\lambda + \tfrac{1}{16}\,(\bar{\theta}\theta)^2 D_5\,, \qquad (5.19)$$

where A_ν is transverse.[15] In the SU(2) space the matrices A_ν, $\bar{\theta}\gamma_5\lambda$, and D_5 are Hermitian and traceless. In the special gauge, V_μ is given by

$$V_\mu^k = \bar{D}\gamma_\mu\,\frac{1+i\gamma_5}{2}\,D\Psi^k - \frac{ig}{2}\,\epsilon^{klm}\left(\bar{D}\Psi^l\gamma_\mu D\Psi^m + 2\Psi^l\,\bar{D}\gamma_\mu\,\frac{1+i\gamma_5}{2}\,D\Psi^m\right). \qquad (5.20)$$

After some tedious labor one finds

$$\tfrac{1}{128}\,(\bar{D}D)^2\,\mathrm{Tr}(V_\mu V_\mu + V_\mu^\dagger V_\mu^\dagger) = -\tfrac{1}{4}(\partial_\mu A_\nu^k - \partial_\nu A_\mu^k + g\epsilon^{klm}A_\mu^l A_\nu^m)^2 + \tfrac{1}{2}i\bar{\lambda}^k\gamma_\mu(\partial_\mu\lambda^k + g\epsilon^{klm}A_\mu^l\lambda^m) + \tfrac{1}{2}(D_5^k)^2\,, \qquad (5.21)$$

i.e., the Lagrangian for a Yang-Mills field A_μ^k in interaction with a triplet of Majorana spinors λ^k. The matter terms in the Lagrangian (5.17) reduce to

$$\begin{aligned}
&\tfrac{1}{8}(\bar{D}D)^2(\Phi_+^\dagger e^{g\Psi}\Phi_+ + \Phi_-^\dagger e^{-g\Psi}\Phi_-) - \tfrac{1}{2}M(\bar{D}D)(\Phi_-^\dagger\Phi_+ + \Phi_+^\dagger\Phi_-)\\
&\quad= (\partial_\mu A_+^\dagger + \tfrac{1}{2}igA_+^\dagger A_\mu)(\partial_\mu A_+ - \tfrac{1}{2}igA_\mu A_+) + (\partial_\mu A_-^\dagger + \tfrac{1}{2}igA_-^\dagger A_\mu)(\partial_\mu A_- - \tfrac{1}{2}igA_\mu A_-)\\
&\qquad + F_+^\dagger F_+ + F_-^\dagger F_- + i\bar{\psi}\gamma_\mu(\partial_\mu - \tfrac{1}{2}igA_\mu)\psi + M(A_-^\dagger F_+ + A_+^\dagger F_- + F_-^\dagger A_+ + F_+^\dagger A_- - \bar{\psi}\psi) + \tfrac{1}{2}g(A_+^\dagger D_5 A_+ - A_-^\dagger D_5 A_-)\\
&\qquad + \frac{ig}{\sqrt{2}}A_+^\dagger\bar{\lambda}\,\frac{1+i\gamma_5}{2}\,\psi + \frac{ig}{\sqrt{2}}A_-^\dagger\bar{\lambda}\,\frac{1-i\gamma_5}{2}\,\psi - \frac{ig}{\sqrt{2}}\,\bar{\psi}\,\frac{1+i\gamma_5}{2}\,\lambda A_- - \frac{ig}{\sqrt{2}}\,\bar{\psi}\,\frac{1-i\gamma_5}{2}\,\lambda A_+\,,
\end{aligned} \qquad (5.22)$$

in which $A_\mu = A_\mu^k\tau^k$, $\lambda = \lambda^k\tau^k$, $D_5 = D_5^k\tau^k$. One may, if this is desired, replace the chiral combinations $A_\pm$ and $F_\pm$ by definite parity combinations

$$A_\pm = \frac{1}{\sqrt{2}}\,(A \pm iB),\quad F_\pm = \frac{1}{\sqrt{2}}\,(F \pm iG)$$

(bearing in mind that A, B, F, and G are all isodoublets). For a triplet of matter fields

$$\Phi_\pm = \Phi_\pm^k\tau^k \qquad [\Phi_-^k = (\Phi_+^k)^*]$$

the gauge-invariant kinetic term is

$$\begin{aligned}
\tfrac{1}{16}\,(\bar{D}D)^2\,\mathrm{Tr}(\Phi_- e^{g\Psi}\Phi_+ e^{-g\Psi}) &= (\partial_\mu A_-^k + g\epsilon^{klm}A_\mu^l A_-^m)(\partial_\mu A_+^k + g\epsilon^{klm}A_\mu^l A_+^m) + F_-^k F_+^k + \tfrac{1}{2}\bar{\psi}^k i\gamma_\mu(\partial_\mu\psi^k + g\epsilon^{klm}A_\mu^l\psi^m)\\
&\quad - g\epsilon^{klm}\left(\sqrt{2}\,A_+^k\bar{\lambda}^l\,\frac{1-i\gamma_5}{2}\,\psi^m + \sqrt{2}\,A_-^k\bar{\lambda}^l\,\frac{1+i\gamma_5}{2}\,\psi^m - iA_+^k A_-^l D_5^m\right).
\end{aligned} \qquad (5.23)$$

This expression is particularly interesting because the sum of (5.23) and the gauge Lagrangian (5.21) possesses a new symmetry, viz.,

$$\begin{aligned}
\lambda^k &\to \lambda^k\cos\alpha - \psi^k\sin\alpha\,,\\
\psi^k &\to \lambda^k\sin\alpha + \psi^k\cos\alpha\,,
\end{aligned} \qquad (5.24)$$

with all boson components treated as scalars. In other words, if we introduce the complex Dirac field

$$\chi^k = \frac{1}{\sqrt{2}}\,(\lambda^k + i\psi^k) \qquad (5.25)$$

the fermionic part of the Lagrangian takes the form

$$\bar{\chi}^k i\gamma_\mu(\partial_\mu\chi^k + g\epsilon^{klm}A_\mu^l\chi^m) + ig\epsilon^{klm}\left(\sqrt{2}\,A_+^k\bar{\chi}^l\,\frac{1-i\gamma_5}{2}\,\chi^m + \sqrt{2}\,A_-^k\bar{\chi}^l\,\frac{1+i\gamma_5}{2}\,\chi^m\right), \qquad (5.26)$$

which is manifestly invariant with respect to the phase transformations

$$\chi^k \to e^{-i\alpha}\chi^k, \quad \bar{\chi}^k \to e^{i\alpha}\bar{\chi}^k \ . \tag{5.27}$$

The new symmetry could be associated with a conserved quantity (such as baryon or lepton number) which is carried only by the fermions.[16] To maintain this symmetry the matter multiplet must have no mass.

Before we close this section, remark that in the one-loop approximation, contributions to the Callan-Symanzik function $\beta(g)$ for the gauge supermultiplet equals $-(g^3/16\pi^2)(3C_2(G))$, while the contribution from the matter-supermultiplet equals $+(g^3/16\pi^2)C_2(G)$, where $C_2(G)$ is the value of the quadratic Casimir operator for the adjoint representation of the internal-symmetry group. If we introduce *three* matter supermultiplets (as we do in the next section), $\beta(g)=0$ in the one-loop approximation, so that the charge renormalization of g is finite.

VI. GOLDSTONE AND HIGGS PHENOMENA

It is of prime importance to demonstrate the feasibility of spontaneous-symmetry-breaking mechanisms in supersymmetric systems. It has already been pointed out that the spontaneous breakdown of supersymmetry cannot occur (at least in the tree approximation) in the case of a self-interacting scalar multiplet. Our aim now is to show that *internal* symmetries, on the other hand, can be spontaneously violated even when they are embedded in a supersymmetric scheme. This means that it will be possible to set up renormalizable supersymmetric Lagrangian models in which the gauge particles (vector and spinor) are massive.

A suitable system on which to test for spontaneous breaking is the SU(2)×SU(2)-invariant Lagrangian (5.7). To begin with we shall treat this as a global symmetry and obtain a stable Goldstone solution which carries a residual SU(2) symmetry. (This is only one of a number of possible solutions.) Then we shall go on to consider the symmetry $SU(2)_{\rm local}\times SU(2)_{\rm global}$ and show that the Higgs mechanism is operative.

The scalar multiplet $\Phi_{\pm}^{ia}$ belongs to the real representation (3, 3) of SU(2)×SU(2) and it satisfies the reality condition $\Phi_{-}^{ia}=(\Phi_{+}^{ia})^*$. In terms of component fields the Lagrangian (5.7) takes the form

$$\begin{aligned}\mathcal{L} = {}& \partial_\mu A_-^a\cdot\partial_\mu A_+^a + F_-^a\cdot F_+^a + \tfrac{1}{2}\bar{\psi}^a\cdot i\not{\partial}\psi^a + M(A_+^a\cdot F_+^a + A_-^a\cdot F_-^a - \tfrac{1}{2}\bar{\psi}^a\cdot\psi^a)\\ &+ g_1\epsilon^{abc}\left(A_+^a\cdot A_+^b\times F_+^c - A_+^a\cdot\bar{\psi}^b\times\frac{1+i\gamma_5}{2}\psi^c + A_-^a\cdot A_-^b\times F_-^c - A_-^a\cdot\bar{\psi}^b\times\frac{1-i\gamma_5}{2}\psi^c\right)\ ,\end{aligned} \tag{6.1}$$

where all the fields are 9-folds ($F_-^a\cdot F_+^a = F_-^{ia}F_+^{ia}$, etc.) and the fermion components ψ^{ia} are Majorana spinors. The equations of motion are

$$\begin{aligned}&-\partial^2 A_\pm^a + MF_\mp^a + g_1\epsilon^{abc}\left(2A_\mp^b\times F_\mp^c - \bar{\psi}^b\times\frac{1\mp i\gamma_5}{2}\psi^c\right)=0\,,\\ &F_\mp^a + MA_\pm^a + g_1\epsilon^{abc}A_\pm^b\times A_\pm^c = 0\,,\\ &(i\not{\partial}-M)\psi^a - 2g_1\epsilon^{abc}\left(A_+^b\times\frac{1+i\gamma_5}{2}\psi^c + A_-^b\times\frac{1-i\gamma_5}{2}\psi^c\right)=0\,.\end{aligned} \tag{6.2}$$

From these equations it follows that the vacuum expectation values, in the tree approximation, must satisfy the algebraic equations

$$\begin{aligned}&M\langle A_\pm^a\rangle + g_1\epsilon^{abc}\langle A_\pm^b\rangle\times\langle A_\pm^c\rangle = -\langle F_\mp^a\rangle\ ,\\ &M\langle F_\mp^a\rangle + 2g_1\epsilon^{abc}\langle A_\mp^b\rangle\times\langle F_\mp^c\rangle = 0\ .\end{aligned} \tag{6.3}$$

(We are of course requiring that the vacuum be Poincaré-invariant so that $\langle\partial^2 A_\pm\rangle = 0$ and $\langle\psi\rangle = 0$.) Equations (6.3) are very much simplified if we choose the matrix $\langle A_+^{ia}\rangle$ to be diagonal. [No loss of generality is implied since any one of the matrices involved here can be diagonalized by means of an SU(2)×SU(2) transformation.] The equations themselves then imply that the other matrices $\langle A_-^{ia}\rangle$, $\langle F_\pm^{ia}\rangle$ must be diagonal as well. Representing $\langle A_+\rangle$ by

$$\langle A_+^{ia}\rangle = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3) \tag{6.4}$$

one finds

$$\begin{aligned}&\langle A_-^{ia}\rangle = \mathrm{diag}(\lambda_1^*, \lambda_2^*, \lambda_3^*)\,,\\ &\langle F_-^{ia}\rangle = -\mathrm{diag}(M\lambda_1 + 2g_1\lambda_2\lambda_3, M\lambda_2 + 2g_1\lambda_3\lambda_1, M\lambda_3 + 2g_1\lambda_1\lambda_2)\,,\end{aligned} \tag{6.5}$$

and Eqs. (6.3) reduce to the form

$$M^2\lambda_1 + 2Mg_1(\lambda_2\lambda_3 + \lambda_2\lambda_3^* + \lambda_2^*\lambda_3) + 4g_1^2\lambda_1(\lambda_2\lambda_2^* + \lambda_3\lambda_3^*) = 0\,, \tag{6.6}$$

and cyclic permutations thereof.

We shall not pursue the general solution of (6.6) but instead make the restrictive assumption that $\lambda_1=\lambda_2=\lambda_3$. In effect, we are selecting solutions in which a global SU(2) symmetry is preserved. With this restriction Eqs. (6.6) reduce to

$$\lambda(M+2g_1\lambda)(M+4g_1\lambda^*)=0 \tag{6.7}$$

and there are three distinct (parity-conserving) solutions.

The solution $\lambda=0$ corresponds to the case where no symmetry is broken. The solution $\lambda=-M/4g_1$ gives

$$\langle A_\pm^{ia}\rangle=-\frac{M}{4g_1}\,\delta^{ia}\,,\quad \langle F_\pm^{ia}\rangle=-\frac{M^2}{8g_1}\,\delta^{ia}\,,$$

and corresponds to the case where the supersymmetry is broken as well as SU(2)×SU(2). This solution can be shown to be unstable: Some of the bosons turn out to be tachyons.

The solution $\lambda=-M/2g_1$ gives $\langle F_\pm\rangle=0$ and so preserves the supersymmetry.[13] Indeed, by testing the propagation character of weak perturbations about this solution one finds that it is stable: The superfield Φ^{ia} breaks into three pieces [belonging to the representations $I=0, 1, 2$ of the unbroken SU(2)] with the respective mass values

$$M_0=M,\ M_1=0,\ M_2=2M. \tag{6.8}$$

The isovector piece is a Goldstone superfield. *It is perhaps worth emphasizing that the Goldstone multiplet includes a massless Majorana fermion along with the scalar and pseudoscalar bosons.* Supersymmetry is not broken, though the internal SU(2)×SU(2) is. It is the breaking of this internal symmetry which is responsible for the occurrence of the Goldstone fermions (together with Goldstone bosons).

Now consider what happens when the Lagrangian symmetry is generalized to $SU(2)_{\text{local}}\times SU(2)_{\text{global}}$. In the special gauge discussed in Sec. V the Lagrangian assumes the renormalizable form

$$\begin{aligned}\mathcal{L}=&-\tfrac{1}{4}(\partial_\mu A_\nu-\partial_\nu A_\mu+gA_\mu\times A_\nu)^2+\tfrac{1}{2}\bar\lambda\cdot i\gamma_\mu(\partial_\mu\lambda+gA_\mu\times\lambda)+\tfrac{1}{2}D_5^2\\&+(\partial_\mu A_-^a+gA_\mu\times A_-^a)\cdot(\partial_\mu A_+^a+gA_\mu\times A_+^a)+\tfrac{1}{2}\bar\psi^a\cdot i\gamma_\mu(\partial_\mu\psi^a+gA_\mu\times\psi^a)\\&-\sqrt{2}\,g\left(A_+^a\times\bar\lambda\cdot\frac{1-i\gamma_5}{2}\,\psi^a+A_-^a\times\bar\lambda\cdot\frac{1+i\gamma_5}{2}\,\psi^a\right)+igA_+^a\times A_-^a\cdot D_5+F_-^a\cdot F_+^a+M(A_+^a\cdot F_+^a+A_-^a\cdot F_-^a-\tfrac{1}{2}\bar\psi^a\cdot\psi^a)\\&+g_1\epsilon^{abc}\left(A_+^a\cdot A_+^b\times F_+^c-A_+^a\cdot\bar\psi^b\times\frac{1+i\gamma_5}{2}\,\psi^c+A_-^a\cdot A_-^b\times F_-^c-A_-^a\cdot\bar\psi^b\times\frac{1-i\gamma_5}{2}\,\psi^c\right),\end{aligned}\tag{6.9}$$

where the gauge condition $\partial_\mu A_\mu=0$ is understood. Our purpose here is only to discover the excitation spectrum implicit in (6.9) when the symmetry SU(2)×SU(2) is broken down to SU(2). We shall therefore take advantage of the manifest $SU(2)_{\text{local}}$ symmetry in (6.9) to change over from the renormalizable Landau gauge to the unitary gauge in which the spectrum is simplified. That is, we shall adopt the gauge condition

$$A^{[ia]}\equiv\tfrac{1}{2}(A^{ia}-A^{ai})=0\,, \tag{6.10}$$

where A^{ia} denotes the real (scalar) part of $A_\pm^{ia}$. The scalar isovector part of the superfield is, in effect, "gauged away." The pseudoscalar and spinor parts must of course remain.

Into (6.9) substitute

$$\sqrt{2}\,A_\pm^{ia}=-\frac{M}{\sqrt{2}\,g_1}\,\delta^{ia}+A^{(ia)}\pm iB^{ia} \tag{6.11}$$

and collect the bilinear terms

$$\begin{aligned}\mathcal{L}^{(2)}=&-\tfrac{1}{4}(\partial_\mu A_\nu^j-\partial_\nu A_\mu^j)^2+\tfrac{1}{2}\bar\lambda^j i\partial\!\!\!/\lambda^j+\tfrac{1}{2}(D_5^j)^2+\frac{1}{2}\left(\partial_\mu A^{(ia)}-\frac{Mg}{\sqrt{2}\,g_1}\,\epsilon^{ija}A_\mu^j\right)^2+\tfrac{1}{2}(\partial_\mu B^{ia})^2+\tfrac{1}{2}\bar\psi^{ia}i\partial\!\!\!/\psi^{ia}\\&-\tfrac{1}{2}(F^{ia})^2+\tfrac{1}{2}(G^{ia})^2+M(2A^{(ia)}F^{(ia)}-A^{(ii)}F^{(jj)}-2B^{(ia)}G^{(ia)}+B^{(ii)}G^{(jj)})\\&-\tfrac{1}{2}M(2\bar\psi^{(ia)}\psi^{(ia)}-\bar\psi^{(ii)}\psi^{(jj)})-\frac{Mg}{\sqrt{2}\,g_1}\,\epsilon^{ija}\bar\lambda^j\psi^{ia}\,,\end{aligned}\tag{6.12}$$

where $\psi^{(ia)}=(\tfrac{1}{2})(\psi^{ia}+\psi^{ai})$, etc. This free Lagrangian can be separated into three independent pieces if the fields are decomposed into their $I=0, 1, 2$ components by writing, for example,

$$\psi^{ia}=\psi_2^{(ia)}+\psi_1^{[ia]}+\tfrac{1}{3}\,\delta^{ia}\psi_0\,, \tag{6.13}$$

where $\psi_2^{(ii)}=0$. Suppressing I-spin indices, the three pieces are given by

$$\begin{aligned}\mathcal{L}_0=&\tfrac{1}{2}(\partial_\mu A_0)^2+\tfrac{1}{2}(\partial_\mu B_0)^2+\tfrac{1}{2}\bar\psi_0 i\partial\!\!\!/\psi_0+\tfrac{1}{2}(F_0^2+G_0^2)\\&-M(A_0F_0-B_0G_0-\tfrac{1}{2}\bar\psi_0\psi_0)\,,\end{aligned}\tag{6.14}$$

$$\begin{aligned}\mathcal{L}_1=&-\tfrac{1}{4}(\partial_\mu A_\nu-\partial_\nu A_\mu)^2+\frac{1}{2}\left(\frac{Mg}{g_1}\right)^2A_\mu^2+\tfrac{1}{2}D_5^2\\&+\tfrac{1}{2}(\partial_\mu B_1)^2+\tfrac{1}{2}(F_1^2+G_1^2)\\&+\tfrac{1}{2}\bar\psi_1 i\partial\!\!\!/\psi_1+\tfrac{1}{2}\bar\lambda i\partial\!\!\!/\lambda-\frac{Mg}{g_1}\,\bar\lambda\psi_1\,,\end{aligned}\tag{6.15}$$

$$\begin{aligned}\mathcal{L}_2=&\tfrac{1}{2}(\partial_\mu A_2)^2+\tfrac{1}{2}(\partial_\mu B_2)^2+\tfrac{1}{2}\bar\psi_2 i\partial\!\!\!/\psi_2+\tfrac{1}{2}(F_2^2+G_2^2)\\&+2M(A_2F_2-B_2G_2-\tfrac{1}{2}\bar\psi_2\psi_2)\,.\end{aligned}\tag{6.16}$$

The Lagrangians (6.14) and (6.16), respectively, describe the propagation of an $I=0$ multiplet with mass M and an $I=2$ multiplet with mass $2M$. The Lagrangian (6.15) describes an $I=1$ system containing a massless pseudoscalar, a vector with mass Mg/g_1, and a pair of Majorana spinors,

$$\frac{1}{\sqrt{2}}(\psi_1+\lambda) \text{ and } \frac{\gamma_5}{\sqrt{2}}(\psi_1-\lambda),$$

with mass Mg/g_1 and opposite parities.

This shows that the Higgs mechanism is operating here in the usual way. The fact that a pseudoscalar Goldstone particle remains in the system merely indicates the need for a larger local symmetry: one which involves axial-vector gauge particles as well as vectors.[17] It is certainly possible to construct such a scheme by the methods described in Sec. V.

APPENDIX A: THE ALGEBRA OF COVARIANT DERIVATIVES

The differential operators D_α defined by

$$D_\alpha = \frac{\partial}{\partial\bar\theta^\alpha} - \tfrac{1}{2}(\not p\theta)_\alpha \tag{A1}$$

[and $\bar D^\alpha$ defined by $\bar D^\alpha=(C^{-1})^{\alpha\beta}D_\beta$] are easily seen to generate a Clifford algebra which is isomorphic to the supersymmetry algebra, viz.,

$$\{D_\alpha, D_\beta\} = -(\not pC)_{\alpha\beta}\ . \tag{A2}$$

The purpose of this Appendix is to list some of its properties.

First of all, the algebra contains 16 independent basis elements,

$$1,\ D_\alpha,\ \bar DD,\ \bar D\gamma_5 D,\ \bar D i\gamma_\mu\gamma_5 D,\ \bar DDD_\alpha,\ (\bar DD)^2\ . \tag{A3}$$

Any product of D's can be reduced to a linear combination of these. A complete multiplication table for these basis elements would be too bulky to reproduce here. Instead, we shall list only the more difficult products from which, with the help of (A2), any other product can be deduced without too much effort.

To begin with, the product of two D's is given by

$$D_\alpha D_\beta = -\tfrac{1}{2}(\not pC)_{\alpha\beta} + \tfrac{1}{4}C_{\alpha\beta}\bar DD - \tfrac{1}{4}(\gamma_5 C)_{\alpha\beta}\bar D\gamma_5 D - \tfrac{1}{4}(i\gamma_\nu\gamma_5 C)_{\alpha\beta}\bar D i\gamma_\nu\gamma_5 D\ . \tag{A4}$$

Multiplication of this formula by $-(C^{-1}\gamma_\mu)^{\alpha\beta}$ and by $-(C^{-1}\sigma_{\mu\nu})^{\alpha\beta}$ yields the identities

$$\bar D\gamma_\mu D = 2p_\mu$$

and (A5)

$$\bar D\sigma_{\mu\nu}D = 0\ .$$

Products of three D's are comprised in the formulas

$$\begin{aligned} &D_\alpha\bar DD = \bar DDD_\alpha + (2\not pD)_\alpha\ ,\\ &D_\alpha\bar D\gamma_5 D = -\bar DD(\gamma_5 D)_\alpha + (2\not p\gamma_5 D)_\alpha\ ,\\ &D_\alpha\bar D i\gamma_\mu\gamma_5 D = -\bar DD(i\gamma_\mu\gamma_5 D)_\alpha + 2ip_\mu(\gamma_5 D)_\alpha\ ,\\ &\bar D\gamma_5 DD_\alpha = -\bar DD(\gamma_5 D)_\alpha\ ,\\ &\bar D i\gamma_\mu\gamma_5 DD_\alpha = -\bar DD(i\gamma_\mu\gamma_5 D)_\alpha - (2p_\lambda\sigma_{\lambda\mu}\gamma_5 D)_\alpha\ . \end{aligned} \tag{A6}$$

Products of four D's can sometimes be reduced by multiplying a single D into one of the formulas (A6). For example,

$$\begin{aligned}(\bar D\gamma_5 D)^2 &= \bar D\gamma_5 DD_\alpha(-C^{-1}\gamma_5 D)^\alpha\\ &= -\bar DD(\gamma_5 D)_\alpha(-C^{-1}\gamma_5 D)^\alpha\\ &= -\bar DD(-\bar D\gamma_5 C)_\alpha(-C^{-1}\gamma_5 D)^\alpha\\ &= (\bar DD)^2\ .\end{aligned}$$

Any combination of four D's can be reduced with the help of (A4) and the multiplication table

	$\bar DD$	$\bar D\gamma_5 D$	$\bar D i\gamma_\nu\gamma_5 D$
$\bar DD$	$(\bar DD)^2$	$-2ip_\rho\bar D i\gamma_\rho\gamma_5 D$	$2ip_\nu\bar D\gamma_5 D$
$\bar D\gamma_5 D$	$2ip_\rho\bar D i\gamma_\rho\gamma_5 D$	$(\bar DD)^2$	$-2ip_\nu\bar DD$
$\bar D i\gamma_\mu\gamma_5 D$	$-2ip_\mu\bar D\gamma_5 D$	$2ip_\mu\bar DD$	$\eta_{\mu\nu}(\bar DD)^2 - 2ip_\lambda\epsilon_{\lambda\mu\nu\rho}\bar D i\gamma_\rho\gamma_5 D - 4(\eta_{\mu\nu}p^2 - p_\mu p_\nu)$

(A7)

where left (right) factors are listed in the rows (columns).

Products of five or more D's can be reduced with the help of the above results together with the fundamental formula

$$(\bar DD)^2 D_\alpha = -\bar DD(2\not pD)_\alpha\ . \tag{A8}$$

Of some importance are the identities

$$((1\pm i\gamma_5)D)_\alpha\bar D(1\pm i\gamma_5)D = 0\ , \tag{A9}$$

which become more transparent in the notation of 2-component spinors. In a basis where γ_5 is diagonal, the anticommutators (A2) take the form

$$\{D_A, D_B\}=0\ ,\ \{D_{\dot A}, D_{\dot B}\}=0,\ \{D_A, D_{\dot B}\}=p_{A\dot B}\ .$$

Since they anticommute, the product of three operators D_A must vanish, e.g.,

$$0 = D_A D_B D_C = \epsilon_{BC}D_A\bar D\,\frac{1+i\gamma_5}{2}\,D,\ \text{etc.}$$

This is one of the identities (A9).

Frequently useful are the following formulas which give explicitly the action of some operators on a general superfield Φ whose components are given in (2.5):

$$-\bar{D}\,\frac{1\pm i\gamma_5}{2}\,D\Phi = F\mp iG+\bar{\theta}\,\frac{1\mp i\gamma_5}{2}\,(\chi-i\not\partial\psi)$$
$$+\frac{1}{4}\,\bar{\theta}\,\frac{1\mp i\gamma_5}{2}\,\theta(D-\partial^2A\pm 2i\partial_\mu A_\mu)$$
$$+\tfrac{1}{4}\bar{\theta}i\gamma_\nu\gamma_5\theta(\mp i\partial_\nu)(F\mp iG)$$
$$+\tfrac{1}{4}\bar{\theta}\theta\bar{\theta}\,\frac{1\pm i\gamma_5}{2}\,(-i\not\partial)(\chi-i\not\partial\psi)$$
$$+\tfrac{1}{32}(\bar{\theta}\theta)^2(-\partial^2)(F\mp iG)\,, \qquad \text{(A10)}$$

$$\bar{D}i\gamma_\mu\gamma_5 D\Phi = 2A_\mu+\bar{\theta}(i\gamma_\mu\gamma_5\chi-\not\partial\gamma_\mu\gamma_5\psi)$$
$$+\tfrac{1}{4}\bar{\theta}\theta(2\partial_\mu G)+\tfrac{1}{4}\bar{\theta}\gamma_5\theta(-2\partial_\mu F)$$
$$+\tfrac{1}{4}\bar{\theta}i\gamma_\nu\gamma_5\theta[\eta_{\mu\nu}D-2\epsilon_{\mu\nu\lambda\rho}\partial_\lambda A_\rho$$
$$+(\eta_{\mu\nu}\partial^2-2\partial_\mu\partial_\nu)A]$$
$$+\tfrac{1}{4}\bar{\theta}\theta\bar{\theta}[\gamma_\mu\not\partial\gamma_5\chi-(\eta_{\mu\nu}\partial^2-2\partial_\mu\partial_\nu)i\gamma_\nu\gamma_5\psi]$$
$$+\tfrac{1}{32}(\bar{\theta}\theta)^2(\eta_{\mu\nu}\partial^2-2\partial_\mu\partial_\nu)A_\nu\,, \qquad \text{(A11)}$$

$$\tfrac{1}{2}(\bar{D}D)^2\Phi = D-\partial^2A-\bar{\theta}(\partial^2\psi+i\not\partial\chi)-\tfrac{1}{4}\bar{\theta}\theta(2\partial^2F)$$
$$-\tfrac{1}{4}\bar{\theta}\gamma_5\theta(2\partial^2G)-\tfrac{1}{4}\bar{\theta}i\gamma_\nu\gamma_5\theta(2\partial_\nu\partial_\mu A_\mu)$$
$$+\tfrac{1}{4}\bar{\theta}\theta\bar{\theta}(i\not\partial)(\partial^2\psi+i\not\partial\chi)$$
$$-\tfrac{1}{32}(\bar{\theta}\theta)^2\partial^2(D-\partial^2A)\,, \qquad \text{(A12)}$$

$$D_\alpha\Phi = \psi_\alpha+\tfrac{1}{2}[-i(\gamma_\mu\theta)_\alpha\partial_\mu A+\theta_\alpha F+(\gamma_5\theta)_\alpha G+(i\gamma_\nu\gamma_5\theta)_\alpha A_\nu]$$
$$+\tfrac{1}{4}\bar{\theta}\theta[\tfrac{1}{2}i(\not\partial\psi)_\alpha+\tfrac{1}{2}\chi_\alpha]$$
$$+\tfrac{1}{4}\bar{\theta}\gamma_5\theta[-\tfrac{1}{2}i(\not\partial\gamma_5\psi)_\alpha+\tfrac{1}{2}(\gamma_5\chi)_\alpha]$$
$$+\tfrac{1}{4}\bar{\theta}i\gamma_\nu\gamma_5[-\tfrac{1}{2}i\not\partial(i\gamma_\nu\gamma_5\psi)_\alpha+\tfrac{1}{2}(i\gamma_\nu\gamma_5\chi)_\alpha]$$
$$+\tfrac{1}{4}\bar{\theta}\theta\,\tfrac{1}{2}[-i(\gamma_\mu\theta)_\alpha\partial_\mu F+(i\gamma_\mu\gamma_5\theta)_\alpha\partial_\mu G$$
$$-(\gamma_\mu\gamma_\nu\gamma_5\theta)_\alpha\partial_\mu A_\nu+\theta_\alpha D]$$
$$+\tfrac{1}{32}(\bar{\theta}\theta)^2(i\not\partial\chi)_\alpha\,. \qquad \text{(A13)}$$

APPENDIX B: PRODUCTS

The product of two expansions of the form (2.5) can be rearranged into

$$\Phi(1,\theta)\Phi(2,\theta) = A(1)A(2)+\bar{\theta}[A(1)\psi(2)+\psi(1)A(2)]+\tfrac{1}{4}\bar{\theta}\theta[A(1)F(2)-\bar{\psi}^c(1)\psi(2)+F(1)A(2)]$$
$$+\tfrac{1}{4}\bar{\theta}\gamma_5\theta[A(1)G(2)+\bar{\psi}^c(1)\gamma_5\psi(2)+G(1)A(2)]$$
$$+\tfrac{1}{4}\bar{\theta}i\gamma_\nu\gamma_5\theta[A(1)A_\nu(2)+\bar{\psi}^c(1)i\gamma_\nu\gamma_5\psi(2)+A_\nu(1)A(2)]$$
$$+\tfrac{1}{4}\bar{\theta}\theta\bar{\theta}[A(1)\chi(2)+\psi(1)F(2)-\gamma_5\psi(1)G(2)+i\gamma_\nu\gamma_5\psi(1)A_\nu(2)$$
$$+\chi(1)A(2)+F(1)\psi(2)-G(1)\gamma_5\psi(2)-A_\nu(1)i\gamma_\nu\gamma_5\psi(2)]$$
$$+\tfrac{1}{32}(\bar{\theta}\theta)^2[A(1)D(2)+2F(1)F(2)+2G(1)G(2)+2A_\nu(1)A_\nu(2)$$
$$+D(1)A(2)-2\bar{\psi}^c(1)\chi(2)-2\bar{\chi}^c(1)\psi(2)]\,, \qquad \text{(B1)}$$

where ψ^c denotes the charge conjugate of ψ, i.e.,

$$(\bar{\psi}^c)^\alpha = (C^{-1})^{\alpha\beta}\psi_\beta\,.$$

The formula (B1) comprises the multiplication table for the anticommuting Majorana spinor θ and the various monomials made from it.

A particular case of (B1) is the product of left- and right-type chiral superfields,

$$\Phi_+(1,\theta)\Phi_-(2,\theta) = A_+(1)A_-(2)+\bar{\theta}[A_+(1)\psi_-(2)+\psi_+(1)A_-(2)]+\tfrac{1}{4}\bar{\theta}\theta[A_+(1)F_-(2)+F_+(1)A_-(2)]$$
$$+\tfrac{1}{4}\bar{\theta}\gamma_5\theta[-iA_+(1)F_-(2)+iF_+(1)A_-(2)]$$
$$+\tfrac{1}{4}\bar{\theta}i\gamma_\nu\gamma_5\theta[i\partial_\nu A_+(1)A_-(2)-iA_+(1)\partial_\nu A_-(2)-\bar{\psi}_+^c(1)\gamma_\nu\psi_-(2)]$$
$$+\tfrac{1}{4}\bar{\theta}\theta\bar{\theta}[-i\not\partial\psi_+(1)A_-(2)+i\gamma_\nu\psi_+(1)\partial_\nu A_-(2)-iA_+(1)\not\partial\psi_-(2)$$
$$+i\partial_\nu A_+(1)\gamma_\nu\psi_-(2)+2\psi_+(1)F_-(2)+2F_+(1)\psi_-(2)]$$
$$+\tfrac{1}{32}(\bar{\theta}\theta)^2[-\partial^2A_+(1)A_-(2)+2\partial_\nu A_+(1)\partial_\nu A_-(2)-A_+(1)\partial^2A_-(2)$$
$$+4F_+(1)F_-(2)+2\bar{\psi}_+^c(i\overleftarrow{\not\partial}-i\overrightarrow{\not\partial})\psi_-(2)]\,. \qquad \text{(B2)}$$

The multiplication of two left-type fields yields again a left-type field with

$$A_+(3) = A_+(1)A_+(2)\,,$$
$$\psi_+(3) = A_+(1)\psi_+(2)+\psi_+(1)A_+(2)\,, \qquad \text{(B3)}$$
$$F_+(3) = A_+(1)F_+(2)-\bar{\psi}_+^c(1)\psi_+(2)+F_+(1)A_+(2)\,.$$

Repeated application of this rule gives the components of the superfield $(\Phi_+(x,\theta))^n$, viz.,

$$A_+ = A_+(x)^n\,,$$
$$\psi_+ = nA_+(x)^{n-1}\psi_+(x)\,, \tag{B4}$$
$$F_+ = nA_+(x)^{n-1}F_+(x) - \frac{n(n-1)}{2}A_+(x)^{n-2}\overline{\psi}{}^c_+(x)\psi_+(x)\,.$$

The vector multiplet Φ_1 contained in (3.8) can be expressed in chiral form as a *spinor* superfield,

$$\Psi_{\alpha\pm} = \left(\frac{1\pm i\gamma_5}{2}\,\partial\!\!\!/ D\right)_\alpha \Phi_1$$
$$= \exp(\mp\tfrac{1}{4}\overline{\theta}\partial\!\!\!/\gamma_5\theta)\left\{\left(\frac{1\pm i\gamma_5}{2}\lambda\right)_\alpha + \overline{\theta}^\beta\left[\frac{i}{2}\left(\frac{1\pm i\gamma_5}{2}\right)_{\beta\alpha}D \mp i\left(\frac{1\pm i\gamma_5}{2}\sigma_{\mu\nu}C\right)_{\beta\alpha}A_{\mu\nu}\right] + \tfrac{1}{4}\overline{\theta}(1\pm i\gamma_5)\theta\,\frac{1\pm i\gamma_5}{2}\,i\partial\!\!\!/\lambda\right\}\,, \tag{B5}$$

where

$$D = \partial^2 A_1\,,$$
$$\lambda = \partial\!\!\!/\psi_1\,, \tag{B6}$$
$$A_{\mu\nu} = \partial_\mu A_{1\nu} - \partial_\nu A_{1\mu}\,.$$

Conversely,

$$\Phi_1 = \frac{1}{2i\partial^2}\,\overline{D}^\alpha(\Psi_{\alpha+} + \Psi_{\alpha-})\,, \tag{B7}$$

which may be a nonlocal superfield.

APPENDIX C: SUPERFIELD EQUATIONS

In the text we have been concerned mainly with the construction of supersymmetric Lagrangians. For this task the superfield concept has provided a compact and suggestive notational framework. One may ask whether the concept can be usefully pursued and, in particular, whether it would be advantageous to set up an apparatus of Feynman rules with superfield propagators, superfield vertices, etc. Little has been done in this direction. The purpose of this Appendix is merely to sketch a preliminary idea of how such a development would proceed.

Corresponding to the Lagrangian (4.1) the superfield equations of motion are

$$\overline{D}D\Phi_+ = 2V'(\Phi_-)\,,$$
$$\overline{D}D\Phi_- = 2V'(\Phi_+)\,. \tag{C1}$$

These equations can be solved perturbatively. The first step is to linearize them and work out the propagator. Make the replacement

$$V(\Phi_\pm) \to (M/2)\Phi_\pm{}^2 + J_\pm\Phi_\pm\,, \tag{C2}$$

where $J_\pm(x,\theta)$ denote external-source distributions of the usual chiral types

$$J_\pm(x,\theta) = \exp(\mp\tfrac{1}{4}\overline{\theta}\partial\!\!\!/\gamma_5\theta)\left(J_{F_\pm} - \overline{\theta}\,\frac{1\pm i\gamma_5}{2}J_\psi + \tfrac{1}{4}\overline{\theta}(1\pm i\gamma_5)\theta J_{A_\pm}\right)\,, \tag{C3}$$

where the components are labeled such that $J_{A_\pm}$ acts as the source of $A_\pm$, etc. The linear inhomogeneous equations of motion

$$\overline{D}D\Phi_+ - 2M\Phi_- = 2J_-\,,$$
$$\overline{D}D\Phi_- - 2M\Phi_+ = 2J_+ \tag{C4}$$

are easily solved. Making use of the identity

$$(\overline{D}D)^2\Phi_\pm = -4\partial^2\Phi_\pm\,, \tag{C5}$$

which can be deduced from the formula (A10), one finds

$$\Phi_\pm = -\frac{1}{\partial^2 + M^2}\,(MJ_\pm + \tfrac{1}{2}\overline{D}DJ_\mp)\,. \tag{C6}$$

This equation defines the bare propagator. In a formal sense the propagator is given by equations like

$$\langle T\Phi_+(x_1,\theta_1)\Phi_-(x_2,\theta_2)\rangle = \frac{\hbar}{i}\,\frac{\delta\Phi_+(x_1,\theta_1)}{\delta J_-(x_2,\theta_2)}\,,$$

but the meaning of the functional derivative here needs to be clarified. To this end one can exploit the supposed invariance of the vacuum with respect to translations and supersymmetry transformations to write

$$\langle T\Phi_+(x_1,\theta_1)\Phi_-(x_2,\theta_2)\rangle = \langle T\Phi_+(x_1 + a + \tfrac{1}{2}i\overline{\epsilon}\gamma\theta_1,\,\theta_1+\epsilon)\times\Phi_-(x_2 + a + \tfrac{1}{2}i\overline{\epsilon}\gamma\theta_2,\,\theta_2+\epsilon)\rangle$$
$$= \langle T\Phi_+(x_1 - x_2 + \tfrac{1}{2}i\overline{\theta}_1\gamma\theta_2,\,\theta_1-\theta_2)\times\Phi_-(0,0)\rangle \tag{C7}$$

(on choosing $\epsilon = -\theta_2$, $a = -x_2$). Now $\Phi_-(0,0) = A_-(0)$ and one needs to evaluate only

$$\frac{i}{\hbar}\langle T\Phi_+(x,\theta)A_-(0)\rangle = \frac{\delta\Phi_+(x,\theta)}{\delta J_{A_-}(0)}$$
$$= -\frac{1}{\partial^2 + M^2}\,\tfrac{1}{2}\overline{D}D\,\frac{\delta J_-(x,\theta)}{\delta J_{A_-}(0)}$$
$$= -\frac{1}{\partial^2 + M^2}\,\tfrac{1}{2}\overline{D}D\times[\exp(\tfrac{1}{4}\overline{\theta}\partial\!\!\!/\gamma_5\theta)\tfrac{1}{4}\overline{\theta}(1-i\gamma_5)\theta\delta(x)]\,, \tag{C8}$$

where we have used (C3) and (C6). This formula can be simplified with the help of (A10), which implies the relation

$$-\tfrac{1}{2}\bar{D}D\Phi_- = \exp(-\tfrac{1}{4}\bar{\theta}\partial\!\!\!/\gamma_5\theta)[F_- + \bar{\theta}(-i\partial\!\!\!/\psi_-) + \tfrac{1}{4}\bar{\theta}(1+i\gamma_5)\theta(-\partial^2 A_-)]$$

and, therefore,

$$\frac{i}{\hbar}\langle T\Phi_+(x,\theta)A_-(0)\rangle = \frac{1}{\partial^2+M^2}\exp(-\tfrac{1}{4}\bar{\theta}\partial\!\!\!/\gamma_5\theta)\delta(x)\,. \qquad \text{(C9)}$$

The propagator (C7) now takes the form

$$\frac{i}{\hbar}\langle T\Phi_+(x_1,\theta_1)\Phi_-(x_2,\theta_2)\rangle = \exp\left([\tfrac{1}{2}\bar{\theta}_1\gamma_\mu\theta_2 + \tfrac{1}{4}(\bar{\theta}_1-\bar{\theta}_2)i\gamma_\mu\gamma_5(\theta_1-\theta_2)]i\frac{\partial}{\partial x_\mu}\right)\frac{1}{\partial^2+M^2}\delta(x_1-x_2)\,, \qquad \text{(C10)}$$

which is a polynomial in θ_1 and θ_2. (The component propagators can be obtained by comparing coefficients.) In similar fashion one obtains

$$\frac{i}{\hbar}\langle T\Phi_+(x_1,\theta_1)\Phi_+(x_2,\theta_2)\rangle = \exp\left([\tfrac{1}{4}\bar{\theta}_1 i\gamma_\mu\gamma_5\theta_1 - \tfrac{1}{4}\bar{\theta}_2 i\gamma_\mu\gamma_5\theta_2]i\frac{\partial}{\partial x_\mu}\right)(-\tfrac{1}{4}M)\bar{\theta}_{12}(1+i\gamma_5)\theta_{12}\frac{1}{\partial^2+M^2}\delta(x_1-x_2)\,. \qquad \text{(C11)}$$

With the bare propagators in hand one can contemplate the problem of computing scattering amplitudes. For example, one could put the equations (C1) into integral form

$$\Phi_\pm = -\frac{1}{\partial^2+M^2}[MV'_{\text{int}}(\Phi_\pm) + \tfrac{1}{2}\bar{D}DV'_{\text{int}}(\Phi_\mp)] \qquad \text{(C12)}$$

and go on to obtain perturbative developments of the superfields. This kind of approach may, perhaps, lead to useful insights.

A different sort of Lagrangian which can be used to characterize at least the free-field behavior is given by

$$\mathcal{L} = (\bar{D}D)^2[\tfrac{1}{2}\Phi(\bar{D}D-2M)\Phi - 2J\Phi]\,, \qquad \text{(C13)}$$

where Φ is a general superfield. The equations of motion are

$$(\bar{D}D-2M)\Phi = 2J\,. \qquad \text{(C14)}$$

In fact, these equations are equivalent to (C4). Substitute the resolution (3.11) for Φ and J. Then, since $\bar{D}D\Phi_1 = 0$, Eq. (C14) takes the form

$$\begin{aligned} &\bar{D}D\Phi_- - 2M\Phi_+ = 2J_+\,,\\ &\bar{D}D\Phi_+ - 2M\Phi_- = 2J_-\,, \qquad \text{(C15)}\\ &-2M\Phi_1 = 2J_1\,, \end{aligned}$$

and one sees that the nonchiral part, Φ_1, does not propagate. The equation (C14) can be solved directly with the help of (A6) to give

$$\Phi = -\frac{1}{\partial^2+M^2}\left[\tfrac{1}{2}\bar{D}D + \frac{1}{4M}(\bar{D}D)^2\right]J - \frac{1}{M}J\,, \qquad \text{(C16)}$$

and from this the form of the 2-point function can be obtained.

The methods can be applied to other kinds of superfield. For example, the spinor superfields, $\Psi_{\alpha\pm}$, may satisfy the equations

$$(i\partial\!\!\!/-2M)\Psi_\pm + \tfrac{1}{2}\bar{D}D\Psi_\mp = 2J_\pm\,, \qquad \text{(C17)}$$

which are solved by

$$\Psi_\pm = \frac{1}{i\partial\!\!\!/-M}\left(J_\pm + \frac{1}{2M}\bar{D}DJ_\mp\right) - \frac{1}{2M}J_\pm\,. \qquad \text{(C18)}$$

These equations describe the propagation of a supermultiplet of particles of mass M and $(\text{spin})^{\text{parity}}$ content 0^+, $(\tfrac{1}{2})^i$, $(\tfrac{1}{2})^{-i}$, and 1^+. It remains to be seen whether or not there exist any renormalizable interactions for this superfield.

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[1]P. Ramond, Phys. Rev. D 3, 2415 (1971); A. Neveu and J. H. Schwarz, Nucl. Phys. B31, 86 (1971); J.-L. Gervais and B. Sakita, *ibid.* B34, 632 (1971); Y. Iwasaki and K. Kikkawa, Phys. Rev. D 8, 440 (1973).

[2]J. Wess and B. Zumino, Nucl. Phys. B70, 39 (1974).

[3]J. Wess and B. Zumino, Phys. Lett. 49B, 52 (1974).

[4]Abdus Salam and J. Strathdee, Nucl. Phys. B76, 477 (1974).

[5]Abdus Salam and J. Strathdee, Nucl. Phys. B80, 499 (1974).

[6]Superfields defined over a space of 2-component spinors have been considered by C. Fronsdal, ICTP Report No. IC/74/21 (unpublished), and by S. Ferrara, J. Wess, and B. Zumino, Phys. Lett. 51B, 239 (1974). These latter authors independently defined the covariant derivative and the chiral superfields.

[7]The problem was solved for an Abelian local symmetry by J. Wess and B. Zumino, Nucl. Phys. B78, 1 (1974), who invented the special gauge used in the text. The non-Abelian case was solved by Abdus Salam and J. Strathdee, Phys. Lett. 51B, 353 (1974), and, independently, by S. Ferrara and B. Zumino, Nucl. Phys. B79, 413 (1974).

[8]Our notational conventions are as follows. The Dirac matrices satisfy $\frac{1}{2}\{\gamma_\mu,\gamma_\nu\} = \eta_{\mu\nu} = \text{diag}(+---)$. Ad-

joint spinors are defined by $\bar{\psi}=\psi^\dagger\gamma_0$. The matrices γ_0, $\gamma_0\gamma_\mu$, $\gamma_0\sigma_{\mu\nu}=\frac{1}{2}i\,\gamma_0[\gamma_\mu,\gamma_\nu]$, $\gamma_0 i\,\gamma_\mu\gamma_5$, and $\gamma_0\gamma_5=\gamma_1\gamma_2\gamma_3$ are Hermitian. The charge conjugate of ψ is defined by $\psi^c=C\bar{\psi}^T$, where $C^T=-C$ and $C^{-1}\gamma_\mu C=-\gamma_\mu^T$. By a Majorana spinor we mean $\psi^c=\psi$. It is useful to remember that the matrices $\gamma_\mu C$ and $\sigma_{\mu\nu}C$ are symmetric, while C, $\gamma_5 C$, and $i\gamma_\mu\gamma_5 C$ are antisymmetric. In particular, it follows that $\bar{\psi}\chi=\bar{\chi}\psi$, $\bar{\psi}\gamma_\mu\chi=-\bar{\chi}\gamma_\mu\psi$, $\bar{\psi}\sigma_{\mu\nu}\chi=-\bar{\chi}\sigma_{\mu\nu}\psi$, $\bar{\psi}i\,\gamma_\mu\gamma_5\chi=\bar{\chi}i\,\gamma_\mu\gamma_5\psi$, and $\bar{\psi}\gamma_5\chi=\bar{\chi}\gamma_5\psi$ if ψ and χ are anticommuting Majorana spinors.

[9]It may be remarked that the rank-4 "line element" $ds^2=(dx_\mu-\frac{1}{2}i\,\bar{\theta}\gamma_\mu d\theta)^2$ is an invariant.

[10]See, for example, J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison-Wesley, Cambridge, Mass., 1955), Appendix A.2, p. 425.

[11]In certain cases it is possible to avoid this doubling and so define an intrinsic parity. The "vector" part, Φ_1, of the scalar superfield illustrates this [see Eqs. (3.10)–(3.13)].

[12]For renormalizability, V must be a polynomial of order 3. This Lagrangian, given by Wess and Zumino (Ref. 3), has been analyzed in detail by Iliopoulos and Zumino for renormalizability [J. Iliopoulos and B. Zumino, Nucl. Phys. B76, 310 (1974)]. See also Hung-Sheng Tsao, Phys. Lett. 53B, 381 (1974).

[13]Abdus Salam and J. Strathdee, Phys. Lett. 49B, 465 (1974), Footnote 2. A heuristic argument has been given by Iliopoulos and Zumino, suggesting that supersymmetry cannot be broken spontaneously (Ref. 12, Appendix 1).

[14]The compact notation becomes somewhat ambiguous in this formula. In particular, $\bar{V}_+\psi_+\equiv(C^{-1})^{\alpha\beta}V_{+\beta}\psi_{+\alpha}$.

[15]In order to avoid the Faddeev-Popov complications, it might be better to choose the noncovariant "axial" gauge $n_\mu A_\mu=0$ rather than $\partial_\mu A_\mu=0$.

[16]The same proposal was made independently by Ferrara and Zumino and by ourselves (Ref. 7). Another and quite different proposal is contained in R. Delbourgo, Abdus Salam, and J. Strathdee, Phys. Lett. 51B, 475 (1974).

[17]See Delbourgo, Salam, and Strathdee, Ref. 16.