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Quantization of nonlinear waves*

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We suggest that certain nonlinear field theories possess a particle spectrum, richer than has been heretofore discussed. In addition to states associated with quantization of the free-field modes of oscillation—these are the conventional particles of the theory—there also appear heavy particles, which carry a new quantum number and are stable. An approximation scheme is developed in which the signal for these new particles is the existence of stable static solutions with finite energy to the classical equations of motion. We give a systematic expansion for the theory, with special emphasis on the translational motion.

I. INTRODUCTION

There exist many field theories for which the classical equations of motion have solutions independent of time but dependent on position in space, and with energy higher by a finite amount than the minimum energy of a constant field. In this paper we investigate the interpretation of such solutions in the corresponding quantum field theory, and show that it is consistent to relate them to a new particle, which is additional to the particles associated with quantized oscillations of the field. We consider models for which we find approximations in a weak-coupling limit. In this limit the new particles (which we call baryons) have mass large compared to the mass of the normal particles (which we call mesons).[1] Despite this, in the simplest models the baryons carry a new quantum number and are stable against decay into mesons.

Our models have a single scalar field in one space dimension. We show in Sec. II that in more than one dimension fields with spin are required to support the phenomena which we are discussing, but we believe that our interpretation also works in more realistic theories. In one-dimensional models, if the classical energy density of a constant field takes a minimum value of zero for two values ϕ_1, ϕ_2 of the field, there is also a field $\phi_c(x)$, satisfying the static field equations, with $\phi_c(x)\to\phi_1$ as $x\to-\infty$ and $\phi_c(x)\to\phi_2$ as $x\to+\infty$. The transition from ϕ_1 to ϕ_2 is localized in space and the total field energy is finite. In the quantum interpretation, the constant fields ϕ_1, ϕ_2 correspond to two vacuum states Ω_1, Ω_2; they are the vacuum expectation values of the quantum operator field $\Phi(x,t)$:

$$\langle\Omega_i|\Phi(x,t)|\Omega_i\rangle=\phi_i\,,\quad i=1,2\;. \tag{1.1}$$

If ϕ_1 and ϕ_2 are related by a symmetry, the states Ω_1, Ω_2 are degenerate even when quantum fluctuations are included. The quantized oscillations about ϕ_1, ϕ_2 correspond to multimeson states, and we can build a complete set of states on either Ω_1 or Ω_2.

We do not interpret $\phi_c(x)$ as determining a third "vacuum" with broken translational invariance. Instead, we claim that there exists a new sector, orthogonal to Ω_1 and Ω_2, in which the states are energy and momentum eigenstates. The lowest states contain one massive particle, the baryon, and the higher states consist of one baryon plus mesons. The function $\phi_c(x)$ appears in the quantum theory, not as the expectation value of $\Phi(x,t)$ in the baryon state, but as the Fourier transform of the field form factor of the baryon in a static limit. Because the baryon is very heavy, we can form a wave packet about any point x_0 with a position uncertainty much less than the meson Compton wavelength, which determines the rate of change of $\phi_c(x)$, but with a kinetic energy much less than a meson mass. In such a state the expectation value of $\Phi(x,t)$ is indeed $\phi_c(x-x_0)$.

In Sec. III we first set up the theory in terms of the effective action $\Gamma[\phi]$, the generating functional of single-particle-irreducible Green's functions. The equation

$$\frac{\delta\Gamma[\phi]}{\delta\phi(x,t)}=0 \tag{1.2}$$

is usually interpreted as determining the vacuum expectation value of $\Phi(x,t)$ and thus should only have constant solutions. In our models, the loop expansion for $\Gamma[\phi]$ is also an expansion in a coupling parameter λ. The weak-coupling limit is the tree approximation, which yields the classical equations for $\phi(x)$ and the classical value for the energy, and hence allows nonconstant solutions $\phi_c(x)$. We show that the one-loop corrections contain an unavoidable infrared divergence, so that $\phi_c(x)$ cannot be used as the leading term in a systematic approximation to the solution of (1.2).

The trouble appears when we consider time-dependent small oscillations of the field about $\phi_c(x)$. Because the theory is translation-invariant, $\phi_c(x+\delta x)$ also satisfies the equations of motion, and so $d\phi_c(x)/dx=\phi_c'(x)$ corresponds to a mode of oscillation with zero frequency (which we call the translation mode). This mode ruins the loop expansion.

We next make the *Ansatz* that a baryon sector as described above exists. We explore the consistency of this *Ansatz* by calculating the response of the system to a forcing term $-J(x)\Phi(x,t)$ in two ways: (i) by solving $\delta\Gamma[\phi]/\delta\phi(x,t)=J(x)$, using the loop expansion, which no longer diverges at the one-loop level, and (ii) by using the *Ansatz* and elementary quantum theory. The comparison shows that for a suitably chosen J the state of lowest energy is a localized baryon, plus meson excitations. The translation mode is exactly what is needed to give the baryon wave function the correct spread, provided that the baryon mass (i.e., the mass which occurs in the kinetic energy) is given by the classical energy corresponding to $\phi_c(x)$. The classical energy is also the leading term in the energy of a one-baryon state, so that this calculation verifies the Lorentz invariance of our interpretation.

In Sec. IV we set up a complete calculational scheme for the one-baryon sector, by using the equations of motion for all one-baryon multimeson matrix elements of the quantum field $\Phi(x,t)$. We construct a consistent coupling-constant expansion, in which to leading order the baryon appears as a static particle. However, when we examine sum rules arising from the commutation relations and the expressions for energy and momentum, we find that terms proportional to the baryon kinetic energy must be included; they exactly compensate for the contributions from the omitted translation mode.

Section V contains a discussion of some open questions. We propose a description of multibaryon states in our models, and attempt to explain in more detail how a quantum number making the single baryon stable arises. We also discuss the simplest example of a higher-order correction to the scheme proposed in Sec. IV and show that the necessary consistency condition is satisfied.[2]

II. CLASSICAL SOLUTIONS AND THE TRANSLATION MODE

We consider theories with a Lagrangian

$$\mathcal{L}=\frac{1}{2}\left(\frac{\partial\phi(x,t)}{\partial t}\right)^2-\frac{1}{2}\left(\frac{\partial\phi(x,t)}{\partial x}\right)^2-U(\phi)\ , \quad (2.1a)$$

where $U(\phi)$ depends on a parameter λ through

$$U(\phi;\lambda)=\frac{1}{\lambda}\ U(\lambda^{1/2}\phi;1)\ . \quad (2.1b)$$

$U(\phi)$ has two zero minimum values at ϕ_1, ϕ_2 and a symmetry which sends ϕ_1 into ϕ_2. The classical equation of motion is[3]

$$\frac{\partial^2\phi(x,t)}{\partial t^2}-\frac{\partial^2\phi(x,t)}{\partial x^2}+U'(\phi)=0 \quad (2.2)$$

and the classical energy of a static field $\phi(x)$ is

$$E_c[\phi]=E_T+E_V,\ \ E_T=\int dx\,\tfrac{1}{2}(\phi')^2,\ \ E_V=\int dx\,U(\phi)\ . \quad (2.3)$$

A static solution of (2.2) may be found by integrating once to give

$$\tfrac{1}{2}(\phi')^2=U(\phi)\ . \quad (2.4)$$

If $\phi_c(x)$ is a solution of (2.4) with $\phi_c(-\infty)=\phi_1$, $\phi_c(+\infty)=\phi_2$, the only solutions with finite energy are $\phi(x)=\phi_c(x-x_0)$ or $\phi(x)=\phi_c(x_0-x)$, for arbitrary x_0. Any other choice of integration constant in (2.4) leads to infinite energy.

Equation (2.4) implies that $E_T=E_V$. This virial theorem may also be proved as follows. The static solutions of (2.2) are those fields $\phi(x)$ which make $E_c[\phi]$ stationary. If $\phi(x)$ is such a solution, and $\phi_a(x)=\phi(x/a)$, then $E_c[\phi_a]$ must be stationary at $a=1$. A change of integration variable shows that

$$E_c[\phi_a]=\frac{1}{a}\,E_T+aE_V\ , \quad (2.5a)$$

which is stationary only if $E_T=E_V$. Since $E_T>0$, $a=1$ is in fact a minimum of $E_c[\phi_a]$. Moreover, the same argument used in D space dimensions, with any number of fields $\phi^{(i)}(x)$ and

$$E_T=\sum_i\tfrac{1}{2}\int d^Dx|\vec{\nabla}\phi^{(i)}|^2,\quad E_V=\int d^Dx\,U(\phi^{(i)})\ ,$$

gives

$$E_c[\phi_a]=a^{D-2}E_T+a^DE_V \quad (2.5b)$$

and hence $E_V=[(2-D)/D]E_T$. This means that for $D>2$, $E_V<0$ and $E_c[\phi_a]$ is a maximum for variations in a at $a=1$, which implies that the field is not stable against oscillations in a.[4] Consequently in three dimensions, something more complicated than a set of spinless fields must be used if one wants to consider classical solutions which lead to a finite and minimized field energy—for example, fields with spin.[5]

To investigate stability in more detail, we look for solutions of the field equations (returning to one dimension) with $\phi(x,t)=\phi_c(x)+\psi(x)e^{i\omega t}$. The perturbation ψ must satisfy

$$\left[-\frac{d^2}{dx^2}+U''(\phi_c)\right]\psi=\omega^2\psi \tag{2.6}$$

and the stability condition is that all eigenvalues ω^2 of this Schrödinger-like equation should be non-negative. This is, of course, precisely the requirement that $\delta^2E_c(\phi)/\delta\phi(x)\delta\phi(y)|_{\phi=\phi_c}$ have a non-negative spectrum, which is necessarily true if $E_c[\phi]$ is to have a local minimum at $\phi=\phi_c$. The assumed symmetry of $U(\phi)$ implies that $U''(\phi_1)=U''(\phi_2)=\mu^2>0$, where μ is the lowest approximation to the meson mass in the sector built on either of the vacuum states with $\phi=\phi_1, \phi_2$. However, since $U(\phi)$ must have at least one maximum between the two minima at $\phi=\phi_1, \phi_2$, $U''(\phi)$ must certainly go negative between ϕ_1 and ϕ_2. In general (2.6) has a continuous spectrum of eigenvalues $\omega^2=\mu^2+k^2$, with $\psi\sim e^{ikx}$ at large $|x|$ and possibly some isolated bound states with $\omega^2<\mu^2$. But, by differentiating (2.2) we see that

$$\left[-\frac{d^2}{dx^2}+U''(\phi_c)\right]\phi_c'=0 \tag{2.7}$$

and so ϕ_c' is always an isolated solution of (2.6) with $\omega^2=0$. This is, of course, the "translation mode" described above; it must be present because of translation invariance. For stability, this must be the lowest mode of (2.6). [In more than one dimension, we find a translation mode in each direction ($\vec{\nabla}\phi_c$), so that $\omega^2=0$ is a degenerate eigenvalue and therefore not the lowest. In that case the solution ϕ_c is necessarily unstable. For example, for a spherically symmetric ϕ_c, the translation modes form a p state and there is an s state "breathing mode" with negative ω^2. This is expected from the argument above which showed that the energy is a maximum for breathing mode oscillations.]

As a typical example in one dimension, we take

$$U(\phi)=\frac{1}{2\lambda}(m^2-\lambda\phi^2)^2$$
$$=\frac{m^4}{2\lambda}-m^2\phi^2+\tfrac{1}{2}\lambda\phi^4 \tag{2.8}$$

(which we refer to as the ϕ^4 theory). The symmetry $\phi\to-\phi$ is broken in the vacuum states with $\phi_{1,2}=\mp m/\lambda^{1/2}$. The nonconstant solution is

$$\phi_c(x)=\frac{m}{\lambda^{1/2}}\tanh mx\,, \tag{2.9a}$$

with energy

$$E_c[\phi_c]=\frac{4}{3}\,\frac{m^3}{\lambda}\,. \tag{2.9b}$$

Equation (2.6) becomes

$$\left[-\frac{d^2}{dx^2}+4m^2-\frac{6m^2}{\cosh^2 mx}\right]\psi=\omega^2\psi\,. \tag{2.10}$$

This is the case $L=2$ of the class of Schrödinger equations

$$\left[-\frac{d^2}{dz^2}+L^2-\frac{L(L+1)}{\cosh^2 z}\right]\psi(z)=\omega^2\psi(z) \tag{2.11}$$

with very simple properties.[6] There is a continuous spectrum for $\omega^2=k^2+L^2$, $k^2>0$, with $\psi(z)\sim e^{ikz}$ multiplied by a Jacobi polynomial of degree L in $\tanh z$. (There is no reflection, only transmission.) In addition ω^2 takes the discrete values L^2-n^2, $n=L, L-1, \ldots, 1$.[7]

For (2.10) this means that $\omega^2=4m^2+k^2$, $\omega^2=3m^2$, and $\omega^2=0$ are the eigenvalues, the last being the translation mode. The (unnormalized) wave functions are

$$\psi_0(x)=\frac{1}{\cosh^2 mx}\,, \tag{2.12a}$$

$$\psi_1(x)=\frac{\sinh mx}{\cosh^2 mx}\,, \tag{2.12b}$$

$$\psi_k(x)=e^{ikx}\left(3\tanh^2 mx-3i\frac{k}{m}\tanh mx-1-\frac{k^2}{m^2}\right). \tag{2.12c}$$

We shall show that the level $\omega^2=3m^2$ should be associated with an excited state of the baryon; and the level $\omega^2=4m^2+k^2$, with the spectrum of baryon plus meson states. Indeed $\mu=2m$ is the meson mass calculated in the simplest approximation to the vacuum sector.

III. QUANTUM THEORY USING THE EFFECTIVE ACTION

We now investigate what happens if we try to use $\phi_c(x)$ as a zeroth-order approximation to the solution of (1.2). For a potential of the form (2.1b) the n-loop term in the effective action is of the form $\Gamma^{(n)}(\phi;\lambda)=\lambda^{n-1}\Gamma^{(n)}(\lambda^{1/2}\phi;1)$, which suggests that a systematic expansion with $\phi=\sum\lambda^{n-1/2}\phi^{(n)}$ and $\lambda^{-1/2}\phi^{(0)}=\phi_c$ should succeed. Since we are interested in this paper in time-independent solutions, it is convenient to use the entirely equivalent method of evaluating the minimum value of the expectation value of the energy, $\langle\Psi|H|\Psi\rangle=E[\phi,G]$, in states constrained by

$$\langle\Psi|\Phi(x,t)|\Psi\rangle=\phi(x)$$

and

$$\langle\Psi|\Phi(x,t)\Phi(y,t)|\Psi\rangle=\phi(x)\phi(y)+G(x,y)\,.$$

The equations

$$\delta E[\phi,G]/\delta\phi(x)=\delta E[\phi,G]/\delta G(x,y)=0$$

are then equivalent to $\delta\Gamma[\phi]/\delta\phi(x,t)=0$.[8] With ϕ of order $\lambda^{-1/2}$ and G of order λ^0, the loop expansion of $E[\phi,G]$ is again an expansion in powers of λ, of which the first two terms are[8]

$$E^{(0)}[\phi, G] = E_c[\phi]$$
$$= \int dx[\tfrac{1}{2}(\phi')^2 + U(\phi)] \ ,$$
$$E^{(1)}[\phi, G] = \tfrac{1}{8}\int dx\, G^{-1}(x, x) + \tfrac{1}{2}\int dx\, dy\, G(x, y) \times\left[-\frac{d^2}{dx^2} + U''(\phi)\right]\delta(x-y) \ . \tag{3.1}$$

Therefore ϕ and G satisfy

$$\phi'' - U'(\phi) - \tfrac{1}{2}G(x, x)U'''(\phi) = 0 \ , \tag{3.2}$$

$$\tfrac{1}{4}G^{-2}(x, y) = \left[-\frac{d^2}{dx^2} + U''(\phi)\right]\delta(x-y) \ . \tag{3.3}$$

An iterative solution of Eqs. (3.2) and (3.3) is attempted. First the last term in (3.2) is ignored. [It is $O(\lambda^{1/2})$ while the other terms are $O(\lambda^{-1/2})$.] With $\phi = \phi_c(x)$ solving (3.2) to leading order, the solution of (3.3) for $\phi = \phi_c(x)$ is obtained by finding a complete orthonomal set of functions ψ_n satisfying [compare with (2.6)]

$$\left[-\frac{d^2}{dx^2} + U''(\phi_c)\right]\psi_n = \omega_n{}^2\psi_n \ . \tag{3.4}$$

It follows that

$$G(x, y) = \sum_n \frac{1}{2\omega_n}\psi_n^*(x)\psi_n(y) \tag{3.5}$$

(of course G is the equal-time value of the Green's function for small deviations of ϕ from ϕ_c). The translation-mode solution of (3.4) has $\omega_0 = 0$. The corresponding normalized eigenfunction is

$$\psi_0(x) = N\phi_c'(x) \ , \tag{3.6a}$$

$$\frac{1}{N^2} = \int dx(\phi_c')^2 = E_c[\phi_c] \ , \tag{3.6b}$$

where the virial theorem (2.4) is used. This gives an isolated divergent term in (3.5) so that the leading approximation to G is infinite.

It is no use trying to solve the equations for ϕ and G as a self-consistent pair rather than by systematic expansion in λ. This is most easily seen using the effective action $\Gamma[\phi]$. If $\phi(x, t) = \phi_c(x)$ satisfies $\delta\Gamma[\phi]/\delta\phi(x, t) = 0$, then so does $\phi_c(x+\delta x)$, and

$$\int dx'dt' \left.\frac{\delta^2\Gamma[\phi]}{\delta\phi(x, t)\delta\phi(x', t')}\right|_{\phi=\phi_c} \phi_c'(x') = 0 \ . \tag{3.7}$$

But $\delta^2\Gamma[\phi]/\delta\phi\,\delta\phi$ is the inverse of the one-meson propagator and according to (3.7) has an eigenvector at zero frequency with zero eigenvalue. The propagator therefore has an infinity exactly as in the approximation above, which leads to an infinite term in the energy. (The only way out would be by some summation of the entire series for $E[\phi, G]$. We assume this is not what happens.)[9]

This is analogous to the situation with a broken internal continuous symmetry, when this same zero in the inverse propagator is interpreted as indicating the presence of a zero-mass particle. Such an interpretation is neither necessary nor desirable here.[10]

An obvious way to obtain more information from the effective action formalism is to introduce an external source term $-\int dx\, J(x)\Phi(x, t)$ into the Hamiltonian. The difficulty in interpreting $\phi_c(x)$ arises from the translational degeneracy, and this is one way to remove that degeneracy. If we use the same formalism as above, Eq. (3.2) is modified by the addition of $J(x)$ on the left-hand side; (3.3) is unaltered. To develop a useful approximation scheme, we must decide how J is to behave as $\lambda \to 0$. It turns out to be convenient to keep J small but fixed as $\lambda \to 0$ and to approximate as follows:

$$\phi(x; J) = \phi_c(x - x_0) + \phi_1(x) + \phi_2(x) \ , \tag{3.8}$$

where ϕ_c is of order $\lambda^{-1/2}$ as before, ϕ_1 is of order J and ϕ_2 will turn out to be of order $\lambda^{1/4}J^{-1/2}$ (see below), which is much less than ϕ_1, provided $J \gg \lambda^{1/6}$. $\phi_c(x - x_0)$ is a solution of the order $\lambda^{-1/2}$ terms in (3.2), $\phi_c'' - U'(\phi_c) = 0$, and at this stage x_0 is undetermined. ϕ_1 obeys

$$\left[-\frac{d^2}{dx^2} + U''(\phi_c)\right]\phi_1(x) = J(x) \ . \tag{3.9}$$

Because the operator $-d^2/dx^2 + U''(\phi_c)$ has the isolated zero eigenvalue, (3.9) can only be solved if $J(x)$ is orthogonal to the translation-mode eigenfunction $\phi_c'(x - x_0)$, i.e., if $\int dx\, J(x)\phi_c'(x - x_0) = 0$. For interpretation, we write this in the form

$$\frac{d}{dx_0}V(x_0) = 0 \ ,$$
$$V(x_0) = -\int dx\, J(x)[\phi_c(x - x_0) - \phi_c(x - a_0)] \ . \tag{3.10}$$

Here a_0 is an arbitrarily chosen point, and the term with $\phi_c(x - a_0)$ is subtracted so that the integral for $V(x_0)$ converges even if $J(x) \to$ constant as $x \to \pm\infty$. In our typical ϕ^4 model, $\phi_c(x)$ tends to a negative value as $x \to -\infty$, and a positive value as $x \to +\infty$. Thus if we choose a $J(x)$ also tending to $\pm$ values as $x \to \pm\infty$, $V(x_0)$ rises like $|x_0|$ for large x_0 and has a unique minimum, which we henceforth take to be $x_0 = 0$. Thus, as is usual in degenerate perturbation theory, the first-order terms determine the choice of zeroth-order approximation. The solution of (3.9) is now

$$\phi_1(x) = \sum_{n\neq 0} \frac{1}{\omega_n^2}\, \psi_n^*(x) \int dy\, \psi_n(y) J(y) \; . \qquad (3.11)$$

[The possible addition of an arbitrary multiple of $\psi_0(x)$ can be absorbed in a first-order change in the choice of x_0 in the zeroth-order term.]

The term ϕ_2 arises from the term in G which becomes infinite when $J = 0$. Thus we take

$$\left[-\frac{d^2}{dx^2} + U''(\phi_c)\right]\phi_2 + \tfrac{1}{2} G_0(x, x) U'''(\phi_c) = 0 \; , \qquad (3.12a)$$

where

$$G_0(x, x) = \frac{1}{2\Delta\omega_0}\, [\psi_0(x)]^2 \qquad (3.12b)$$

and $\Delta\omega_0$ is the lowest-order perturbation in the zero eigenvalue of (3.4) when ϕ_c is replaced by $\phi_c + \phi_1$:

$$(\Delta\omega_0)^2 = \int dx [\psi_0(x)]^2 U'''(\phi_c)\phi_1(x) \qquad (3.13)$$

[$U'''(\phi_c)$ is of order $\lambda^{1/2}$, ψ_0, which is normalized, is of order λ^0, and ϕ_1 is of order J, hence $\Delta\omega_0$ is of order $\lambda^{1/4}J^{1/2}$]. Because of translation invariance, these equations have a remarkably simple solution. Differentiating (2.7) once more gives

$$\left[-\frac{d^2}{dx^2} + U''(\phi_c)\right]\phi_c'' + U'''(\phi_c)[\phi_c']^2 = 0 \; . \qquad (3.14)$$

Thus, comparing with (3.12) and using (3.6), we find

$$\phi_2(x) = \frac{1}{4\Delta\omega_0}\, \phi_c''(x)\, \frac{1}{E_c[\phi_c]} \; . \qquad (3.15a)$$

[Eq. (3.15a) shows that ϕ_2 is indeed of order $\lambda^{1/4}J^{-1/2}$.] Moreover, multiplying (3.14) by ϕ_1 and integrating gives, with the help of (3.6) and (3.9),

$$\begin{aligned}(\Delta\omega_0)^2 &= \frac{1}{E_c[\phi_c]} \int dx\, \phi_1(x) \left[\frac{d^2}{dx^2} - U''(\phi_c)\right]\phi_c''(x) \\ &= \frac{-1}{E_c[\phi_c]} \int dx\, J(x)\phi_c''(x) \\ &= \frac{1}{E_c[\phi_c]}\, V''(0) \; .\end{aligned}$$

The solution for ϕ_2 is

$$\phi_2(x) = \tfrac{1}{4}\{E_c[\phi_c] V''(0)\}^{-1/2}\phi_c''(x) \; . \qquad (3.15b)$$

The next terms in the expansion of $\phi(x;J)$ come from $U'''(\phi_c)\phi_1^2$ and the nondivergent term in $G(x, x)U'''(\phi_c)$.

We now explain these results by recalculating $\phi(x;J)$ as the expectation value $\langle\Psi|\Phi(x, 0)|\Psi\rangle$ in a normalized state which minimizes

$$\langle\Psi|H_J|\Psi\rangle = \langle\Psi|H - \int dx\, J(x)[\Phi(x, 0) - \phi_c(x - a_0)]|\Psi\rangle \; .$$

Once more the c-number term involving an arbitrarily chosen a_0 is subtracted to make $\langle\Psi|H_J|\Psi\rangle$ finite. We make the following *Ansatz* about the relevant eigenstates of H. (i) There is a set of one-particle (baryon) states with momentum p and kinetic energy $p^2/2M$, where M is of order λ^{-1}. (ii) The matrix element $\langle p'|\Phi(x, 0)|p\rangle$ between one-baryon states (which we refer to as the form factor of the baryon) is given to order $\lambda^{1/2}$ by[11]

$$\langle p'|\Phi(x, 0)|p\rangle = \int dx_0 e^{i(p-p')x_0}\phi_c(x - x_0) \; . \qquad (3.16)$$

Certainly for a very heavy particle, the requirement of Lorentz invariance makes the form factor a function of $p - p'$ to leading order in $1/M$. The translational degeneracy of ϕ_c now becomes an unimportant phase ambiguity of the form factor. [The role that $\phi_c(x_0 - x)$ plays in the theory is discussed in Sec. V.] If we simply define a localized state

$$|x_0\rangle = \int \frac{dp}{(2\pi)}\, e^{-ipx_0}|p\rangle$$

(not an eigenstate of energy), then (3.16) is equivalent to

$$\langle x_1|\Phi(x, 0)|x_2\rangle = \delta(x_1 - x_2)\phi_c(x - x_1) \; . \qquad (3.17)$$

The meaning of (3.17) is obvious, but note that we do not need to construct explicitly the states $|x_i\rangle$. (iii) There are states of one baryon plus one meson corresponding to each solution ψ_n of (3.4) *except* ψ_0. The baryon-meson state $|p;n\rangle$ with total momentum p has energy ω_n above the rest energy of the baryon, neglecting the baryon kinetic energy which is of order λ. The matrix element $\langle p'|\Phi(x, 0)|p;n\rangle$ is assumed to depend on p and p' only through the difference:

$$\langle p'|\Phi(x, 0)|p;n\rangle = \int dx_0 e^{i(p-p')x_0}\, \frac{1}{(2\omega_n)^{1/2}}\, \psi_n(x - x_0) \; . \qquad (3.18)$$

(The labels n are, of course, nearly all continuous labels k, representing the asymptotic momentum of the meson. For theories with isolated ω_n, like the ϕ^4 theory, the corresponding $|p;n\rangle$ are interpreted as excited states of the baryon with excitation energies ω_n.)

If we take for $|\Psi\rangle$ a linear combination of one-baryon states $|\Psi\rangle = \int dx_0 \rho(x_0)|x_0\rangle$ and minimize $\langle\Psi|H_J|\Psi\rangle$ using *Ansätze* (i) and (ii), we find that ρ obeys the Schrödinger equation

$$\left[-\frac{1}{2M}\frac{d^2}{dx_0^2} + V(x_0)\right]\rho(x_0) = \epsilon\rho(x_0) \; , \qquad (3.19)$$

where $V(x_0)=-\int dx\,J(x)[\phi_c(x-x_0)-\phi_c(x-a_0)]$. Because M is very large, ρ is dominated by a Gaussian centered round the minimum of V, which we take to be at $x_0=0$, and with a spread

$$(\Delta x)^2=\int dx\,x^2|\rho(x)|^2$$
$$=\tfrac{1}{2}[MV''(0)]^{-1/2}$$

(harmonic-oscillator approximation). This leads to

$$\langle\Psi|\Phi(x,0)|\Psi\rangle=\int dx_0\phi_c(x-x_0)|\rho(x_0)|^2$$
$$=\phi_c(x)+\tfrac{1}{4}[MV''(0)]^{-1/2}\phi_c''(x)\ , \tag{3.20}$$

provided the spread Δx is small compared to the meson Compton wavelength. All this agrees exactly with (3.10) and (3.15b) if M is identified with $E_c[\phi_c]$. Consequently our calculation of the baryon's motion when constrained by a suitable external $J(x)$ leads to the conclusion that the mass in the kinetic energy is equal to the classical energy of a localized baryon. On the other hand, the leading-order term in the energy is given by $E^{(0)}[\phi_c,G]=E_c[\phi_c]$, which is again the classical energy. We have thus verified that the baryon energy is of the form $M+p^2/2M$, which is Lorentz-invariant to order λ.

This accounts for the terms ϕ_c and ϕ_2. φ_1, as given by (3.11), is identical to the first-order, in J, perturbation-theory response to the source J, with the baryon localized at $x_0=0$ and the various meson states excited, using our *Ansatz* (iii) for the meson matrix elements.

It should be emphasized that the method only gives information with J sufficiently strong to localize the baryon (in fact $J\gg\lambda^{1/6}$), and that it depends on the comparison of two calculations of the response. We therefore develop a more direct analysis of the problem in the next section.

IV. THE KERMAN-KLEIN METHOD

In this section we develop a procedure for calculating in a systematic fashion all the matrix elements of the quantum field Φ in the one-baryon sector. We have not proved that the expansion is indeed consistent when carried to arbitrary order, but in the first few terms that we have investigated, it gives sensible results. The method makes use of the quantum equations of motion; it was suggested to us by Kerman, who, with Klein, used a similar procedure in a many-body problem.[12] We generalize the *Ansatz* of Sec. III as follows: The states in the one-baryon sector are multimeson states labeled by the baryon momentum p and a set of meson momenta $\{k_1,\ldots,k_m\}$. The state $|p;\{k\}\rangle$ has momentum $p+\sum k_i$ and energy $E(p)+\sum\omega(k_i)$, with $E(p)=(p^2+M^2)^{1/2}$ and $\omega(k)=(k^2+\mu^2)^{1/2}$. [For simplicity we assume that the spectrum of (3.4) consists of the discrete point $\omega_0=0$ and a continuum.]

We have in mind two calculations, one for matrix elements of Φ between "in" states, the other between "out" states. We shall not exhibit the "in," "out" label explicitly; it will matter only in selecting boundary conditions for the differential equations which are encountered below. (Of course the single-baryon–no-meson state is both an "in" state and an "out" state, since the baryon is stable.) The two sets of matrix elements are related by the S matrix, which can therefore be calculated once both are known.

The matrix element $\langle p;\{k\}|\Phi|q;\{l\}\rangle$ is the sum of all possible terms of the form

$$(2\pi)\delta(k_1-l_1)(2\pi)\delta(k_2-l_2)\langle p;k_3\cdots|\Phi|q;l_3\cdots\rangle_c$$

in which any number of mesons are "disconnected" (but not the baryon).[13] The essential assumption is that the connected matrix elements between m and n mesons, denoted by the subscript c, have an expansion in powers of λ with leading order $\lambda^{(m+n-1)/2}$. Thus $\langle p|\Phi|q\rangle$ is of order $\lambda^{-1/2}$ and $\langle p|\Phi|q;k\rangle$ of order λ^0, while

$$\langle p;k|\Phi|q;l\rangle=(2\pi)\delta(k-l)\langle p|\Phi|q\rangle+\langle p;k|\Phi|q;l\rangle_c\ ,$$

where the connected part is of order $\lambda^{1/2}$. Finally we set to zero matrix elements of any product of Φ's between no-baryon and one-baryon states, since the baryon is stable (see Sec. V).

We write down the equation of motion for $\Phi(x,t)$, at first for the ϕ^4 theory

$$\frac{\partial^2\Phi(x,t)}{\partial x^2}-\frac{\partial^2\Phi(x,t)}{\partial t^2}+2m^2\Phi(x,t)=2\lambda\Phi^3(x,t)\ , \tag{4.1}$$

and take matrix elements. The left-hand side gives

$$\Big\{-\Big(p+\sum k_i-q-\sum l_i\Big)^2+\Big[E(p)+\sum\omega(k_i)-E(q)-\sum\omega(l_i)\Big]^2+2m^2\Big\}\times\langle p;\{k\}|\Phi|q;\{l\}\rangle\ .$$

To evaluate the matrix element of Φ^3, we insert complete sets of multimeson states. It is then easy to see that we obtain a set of equations for the connected matrix elements of the form

$$[-(\Delta p)^2+(\Delta E)^2+2m^2]\langle p;\{k\}|\Phi|q;\{l\}\rangle_c=2\lambda\langle p;\{k\}|\Phi^3|q;\{l\}\rangle_c\ , \tag{4.2a}$$

$$\langle p;\{k\}|\Phi^3|q;\{l\}\rangle_c = \sum\langle p;\{k_1\}|\Phi|r;\{l_1\}\rangle_c\langle r;\{k_2\}|\Phi|s;\{l_2\}\rangle_c \times\langle s;\{k_3\}|\Phi|q;\{l_3\}\rangle_c\ . \tag{4.2b}$$

The sum is over baryon momenta r, s and sets of meson momenta $\{k_1\}\cdots\{l_3\}$ consisting of (i) the external momenta $\{k\}$ divided among $\{k_1\}, \{k_2\}, \{k_3\}$ in any way and $\{l\}$ divided among $\{l_1\}, \{l_2\}, \{l_3\}$ in any way; (ii) internal momenta each occurring twice, once in a set $\{l\}$ and once in a set $\{k\}$ to its right, i.e., in $\{l_1\}$ and $\{k_2\}$, or $\{l_1\}$ and $\{k_3\}$, or $\{l_2\}$ and $\{k_3\}$. The crucial point is that the leading term on the right-hand side of (4.2a) is now the term with no internal meson momenta and is of order $\lambda^{(m+n-1)/2}$, consistent with the order of the left-hand side.

[Our formula can be written in operator form as follows. There are creation and annihilation operators, $a^\dagger(k)$ and $a(k)$, for mesons, and the field Φ taken between baryon states $|p\rangle$, $|q\rangle$ is the meson operator

$$\Phi_{pq} = \sum_{\{k\}\{l\}}\frac{1}{m!n!}\int dk\,dl\langle p;\{k\}|\Phi|q;\{l\}\rangle_c \times a^\dagger(k_1)\cdots a^\dagger(k_m)a(l_1)\cdots a(l_n)\ .$$

The disentangling of Φ^3 in (4.2b) is then just the use of Wick's theorem to normal order a product of a, $a^\dagger$.]

We now make the further assumption that the baryon mass M is of order λ^{-1} and expand $E(p) = M + p^2/2M + \cdots$, so that $E(p) - E(q)$ is of order λ and can be dropped on the left-hand side of (4.2a). Consistently with this, we assume that the matrix elements can be written in the form

$$\langle p;\{k\}|\Phi(x,0)|q;\{l\}\rangle_c = \int dx_0 \exp\left[i\left(q+\sum l_i - p - \sum k_i\right)x_0\right]f(\{k\},\{l\};x-x_0)\ . \tag{4.3}$$

We call these two assumptions the static approximation. They amount to using Galilean invariance for the baryon. The integrals over internal baryon momenta now simply make each x_0 the same point, and the leading-order equations become

$$\left[\frac{d^2}{dx^2} + \left(\sum\omega(k_i) - \sum\omega(l_i)\right)^2 + 2m^2\right]f(\{k\},\{l\};x) = 2\lambda\sum f(\{k_1\},\{l_1\};x)f(\{k_2\},\{l_2\};x)f(\{k_3\},\{l_3\};x)\ . \tag{4.4}$$

In particular, the no-meson matrix element obeys the classical static equation

$$f''(x) - U'(f) = 0\ , \tag{4.5}$$

with solution $f(x) = \phi_c(x)$, so that as before the classical solution appears as the leading term in the Fourier transform of the baryon form factor. For the no-meson-to-one-meson matrix element, the external momentum k can be put in any of the three sets $\{k_1\}, \{k_2\}, \{k_3\}$ and (4.4) becomes

$$\left[-\frac{d^2}{dx^2} + U''(\phi_c)\right]f(k;x) = \omega^2(k)f(k;x)\ , \tag{4.6}$$

which is (3.4) once more. Higher matrix elements obey equations of the form

$$\left[\frac{d^2}{dx^2} + (\Delta\omega)^2 - U''(\phi_c)\right]f(\{k\},\{l\};x) = \text{a sum of terms involving only fewer mesons than } f(\{k\},\{l\};x)\ . \tag{4.7}$$

We thus obtain a sensible hierarchy of equations for the leading order of all the connected matrix elements in the static approximation. It is clear that (4.5)–(4.7) are valid for an arbitrary potential, not just the one of the ϕ^4 theory.

It is crucial to our interpretation that the solution for $\omega = 0$ of (4.6) does not correspond to a state. In fact, (4.6) does not determine the normalization of $f(k;x)$ (unlike all the other equations). We must therefore examine some nonlinear expression in $f(k;x)$ to find the correct normalization. The field commutation relations are an obvious possibility. We look at the no-meson matrix element of the canonical commutation

$$\left\langle p\left|\left[\Phi(x,t), \frac{\partial\Phi(y,t)}{\partial t}\right]\right|q\right\rangle\Bigg|_{t=0} = i(2\pi)\delta(p-q)\,\delta(x-y)\ . \tag{4.8a}$$

Insertion of a complete set of intermediate states gives

$$\sum_{r,\{k\}}\langle p|\Phi(x,0)|r;\{k\}\rangle\langle r;\{k\}|\Phi(y,0)|q\rangle\left[E(r) - E(q) + \sum\omega(k_i)\right] - \langle p|\Phi(y,0)|r;\{k\}\rangle\langle r;\{k\}|\Phi(x,0)|q\rangle\left[E(p) - E(r) - \sum\omega(k_i)\right] = \delta(x-y)(2\pi)\delta(p-q)\ . \tag{4.8b}$$

The no-meson terms give a contribution of order λ^0, since each matrix element is of order $\lambda^{-1/2}$ while the energy differences are of order λ. The one-meson terms are of the same order, since each matrix element is of order λ^0 and so is $\omega(k)$. All other contributions are of higher order in λ. Thus we obtain two types of contribution to this sum rule.

(i) From the one-meson states

$$\sum_{r,k} \omega(k)[\langle p|\Phi(x,0)|r;k\rangle\langle r;k|\Phi(y,0)|q\rangle + (x\leftrightarrow y)] = \sum_k \omega(k)\left[\int dx_0 f^*(k;x-x_0)f(k;y-x_0)e^{i(q-p)x_0} + (x\leftrightarrow y)\right]. \tag{4.9a}$$

If we take for $f(k;x)$ the orthonormal solutions of (4.6) $\psi_k(x)$ divided by the usual boson factor of $[2\omega(k)]^{1/2}$, then

$$f(k;x) = \{1/[2\omega(k)]^{1/2}\}\psi_k(x)$$

and

$$\sum_k 2\omega(k)f^*(k;x)f(k;y) + N^2\phi_c'(x)\phi_c'(y) = \delta(x-y),$$

where the second term is the contribution from the $\omega=0$ state. [Any bound-state solutions of (4.6) are included in the sum, with f interpreted as the matrix element of Φ between the baryon and a baryon excited state.] The contribution of the one-meson terms to the sum rule is therefore

$$\delta(x-y)(2\pi)\delta(p-q) - N^2\int dx_0 e^{i(q-p)x_0}\phi_c'(x-x_0)\phi_c'(y-x_0). \tag{4.9b}$$

(ii) From the kinetic energy of the no-meson states

$$\begin{aligned}\sum_r \left(\langle p|\Phi(x,0)|r\rangle\langle r|\Phi(y,0)|q\rangle \frac{r^2-q^2}{2M} + \langle p|\Phi(y,0)|r\rangle\langle r|\Phi(x,0)|q\rangle\frac{r^2-p^2}{2M}\right)\\
&= \int\frac{dr}{(2\pi)}\int dx_0dy_0\phi_c(x-x_0)\phi_c(y-y_0)\left(\frac{r^2-q^2}{2M}e^{i(r-p)x_0+i(q-r)y_0} + \frac{r^2-p^2}{2M}e^{i(q-r)x_0+i(r-p)y_0}\right)\\
&= \int\frac{dr}{(2\pi)}\int dx_0dy_0\phi_c(x-x_0)\phi_c(y-y_0)e^{i(r-p)x_0+i(q-r)y_0}\frac{(r-p)(r-q)}{M}\\
&= \frac{1}{M}\phi_c'(x)\phi_c'(y)(2\pi)\delta(p-q)\end{aligned} \tag{4.10}$$

[the second equality follows from the first by changing r to $p+q-r$ in the second term]. Thus provided $M=1/N^2=\int dx(\phi_c')^2 = E_c[\phi_c]$, we find that the translation-mode state is not needed to saturate the sum rule to leading order in λ. This identifies M and shows that it is indeed of order λ^{-1}.

We ought also to determine the rest energy of the baryon. The energy is given by the matrix element of the Hamiltonian density $\mathcal{H}$:

$$\begin{aligned}&E(p) = \langle p|\mathcal{H}(0)|p\rangle,\\
&\mathcal{H}(x) = \frac{1}{2}\Pi^2 + \frac{1}{2}\left(\frac{\partial\Phi}{\partial x}\right)^2 - m^2\Phi^2 + \frac{\lambda}{2}\Phi^4 + \frac{m^4}{2\lambda},\\
&\Pi = \frac{\partial\Phi}{\partial t}.\end{aligned} \tag{4.11}$$

The matrix elements of products of fields are evaluated by expanding in intermediate states. When the computation is carried to terms of order λ^{-1} and λ^0, the relevant intermediate states are the following: (i) The no-meson state gives an $O(\lambda^{-1})$ contribution $\int dx[\frac{1}{2}(\phi_c')^2 + (1/2\lambda)(m^2-\lambda\phi_c^2)^2]$. (The matrix element $\langle p|\Pi|q\rangle = i[E(p)-E(q)]\langle p|\Phi|q\rangle$ is order $\lambda^{1/2}$ and is dropped.) (ii) The $O(\lambda^0)$ terms come from one-meson states. Π^2 gives

$$\begin{aligned}\int\frac{dq}{(2\pi)}\sum_k\langle p|\Pi|q;k\rangle\langle q;k|\Pi|p\rangle\\
&= \int\frac{dq}{(2\pi)}\sum_k[\omega(k)]^2\langle p|\Phi|q;k\rangle\langle q;k|\Phi|p\rangle\\
&= \int dx\sum_k \tfrac{1}{2}\omega(k)\psi_k^*(x)\psi_k(x).\end{aligned}$$

The remaining terms are similarly evaluated and the total energy becomes

$$\begin{aligned}E(p) = \int dx[\tfrac{1}{2}(\phi_c')^2 + U(\phi_c)] + \tfrac{1}{8}\int dx\,\tilde{G}^{-1}(x,x)\\
+ \tfrac{1}{2}\int dx\,dy\,\tilde{G}(x,y)\left[-\frac{d^2}{dx^2} + U''(\phi_c)\right]\delta(x-y),\end{aligned} \tag{4.12a}$$

$$\tilde{G}^{\pm 1}(x,y)=\sum_k [2\omega(k)]^{\mp 1}\psi_k^*(x)\psi_k(y) \qquad (4.12b)$$

(contributions from baryon excited states are to be included in the sum). This formula is similar to the one obtained from the effective action (3.1), with two important differences. The quantities ϕ_c and $\tilde{G}$ are not variational parameters, but are fixed matrix elements of the quantum field. Also, only the infrared-finite $\tilde{G}$ occurs; the pole due to the translation mode is absent. When we make use of (4.6), the result is

$$E(p)=\int dx[\tfrac{1}{2}(\phi_c')^2+U(\phi_c)]+\tfrac{1}{4}\int dx\,\tilde{G}^{-1}(x,x) + O(\lambda)\,. \qquad (4.13)$$

[It is easy to see that any $O(\lambda^{1/2})$ correction to the baryon form factor does not contribute to this order.] The first term is the classical energy $E_c[\phi_c]=M$. The second term is the first quantum correction. It is essentially equal to $\frac{1}{2}\sum_k\omega(k)$, the zero-point energy of the fluctuations of the field. Of course the corresponding vacuum energy must be subtracted, and the result is an integral of the phase shift in $f(k;x)$, which is still logarithmically ultraviolet-divergent. The divergence is canceled by the renormalization of the meson mass.[14] Since this contribution is independent of p, we interpret it as the first correction to the mass of the baryon. The kinetic p-dependent term does not appear until $O(\lambda)$, which is to be expected since the kinetic energy is $p^2/2M$.

Another sum rule which establishes the consistency of our procedure arises from the matrix elements of the total momentum operator

$$P=-\int dx\,\Pi(x,0)\partial\Phi(x,0)/\partial x \quad .$$

Between no-meson states, the momentum density should give

$$p=-\left\langle p\left|\Pi\frac{\partial\Phi}{\partial x}\right|p\right\rangle \ . \qquad (4.14)$$

As in the energy calculation, we saturate with intermediate states. It is easy to show that the one-meson intermediate states give zero, while the no-meson intermediate states contribute

$$\begin{aligned}\int\frac{dq}{(2\pi)}\frac{(p^2-q^2)}{2M}F(p-q)(p-q)F(q-p) &= \int\frac{dq}{(2\pi)}\frac{q^2(2p-q)}{2M}F(q)F(-q)\\ &=\frac{p}{M}\int\frac{dq}{(2\pi)}q^2F(q)F(-q)\\ &=\frac{p}{M}\int dx(\phi_c')^2\\ &=p\frac{E_c[\phi_c]}{M}\,. \end{aligned}\qquad (4.15)$$

[$F(p)$ is the baryon form factor, i.e., the Fourier transform of ϕ_c, and symmetric integration eliminates the integral $\int dq\,q^3F(q)F(-q)$.] Again we see that for consistency we must have $E_c[\phi_c]=M$.

The decision whether we are computing "in" or "out" matrix elements determines the boundary conditions for (4.6) or (4.7). For "out" states, the boundary conditions on the continuum solutions of (4.6) are

$$\begin{aligned}\psi_k(x)&=[2\omega(k)]^{1/2}f_k(x)\\ &\underset{x\to\infty}{\sim} e^{ikx}+e^{i|k|x}A(k)\\ &\underset{x\to-\infty}{\sim} e^{ikx}+e^{-i|k|x}B(k)\,. \end{aligned}\qquad (4.16a)$$

The constants A and B satisfy the following relations, which follow by evaluating at $\pm\infty$ the Wronskian of $\psi_k(x)$ with $\psi_k^*(x)$ and $\psi_{-k}(x)$:

$$\begin{aligned}&|A(k)|^2+2\,\mathrm{Re}A(k)+|B(k)|^2=0, \quad k\geq 0\\ &|B(k)|^2+2\,\mathrm{Re}B(k)+|A(k)|^2=0, \quad k\leq 0\end{aligned}\qquad (4.16b)$$

$$A(k)=B(-k), \quad k\geq 0\ . \qquad (4.16c)$$

Therefore it is also true that

$$|B(k)|^2=|A(-k)|^2, \quad k\geq 0\ . \qquad (4.16d)$$

For "in" states the asymptotic behavior is obtained by complex-conjugating (4.16a) and replacing k by $-k$. To calculate the S matrix for baryon–one-meson scattering, we consider the expression $\langle p'|\Phi|p;k\,\mathrm{out}\rangle$, and insert a set of one-meson "in" states:

$$\langle p'|\Phi|p;k\,\mathrm{out}\rangle=\sum_{r,l}\langle p'|\Phi|r;l\,\mathrm{in}\rangle\langle r;l\,\mathrm{in}|p;k\,\mathrm{out}\rangle\ . \qquad (4.17)$$

The relation of the matrix elements to Eq. (4.6) makes it clear that the S matrix is the same as the S matrix for scattering of a meson in the static potential $U''(\phi_c)$. For the ϕ^4 theory, our discussion of Eqs. (2.10) and (2.12c) shows that there is no reflected wave, only a transmitted wave with a phase shift,

$$\tan\delta(k)=\frac{-3|k|/m}{2-k^2/m^2}\ , \qquad (4.18)$$

where the S matrix is $e^{2i\delta(k)}$.

V. FURTHER ASPECTS OF THE THEORY

One question we have not discussed in the previous sections is that of multibaryon states. There certainly exist time-dependent solutions of the classical field equations which as $t\to-\infty$ consist of widely separated baryons moving with uniform velocities. In the type of theory with two constant minimum-energy fields ϕ_1 and ϕ_2, related by a symmetry, such a solution would have regions where $\phi(x, t) = \phi_1$ alternating with regions where $\phi(x, t) = \phi_2$; the transition regions would have

$$\phi(x, t) = \phi_c\left(\frac{x - x_0 - ut}{(1-u^2)^{1/2}}\right)$$

or

$$\phi(x, t) = \phi_c\left(\frac{x_0 + ut - x}{(1-u^2)^{1/2}}\right).$$

Eventually the moving transition regions meet and an interaction takes place. A great deal is known about such solutions of some classical theories,[1] in particular that for some theories the "solitons" emerge unchanged from the interaction; but whether there is a deep relation between the soliton property and the existence of a quantum field theory is obscure.

To obtain a provisional picture of the nature of the quantized multibaryon states, it is perhaps helpful to look at a Schrödinger picture of the quantum field theory, in which a state is described by a functional Ψ of a c-number field $\phi(x)$. The quantum field $\Phi(x, 0)$ is represented by multiplication by $\phi(x)$ and the canonical momentum $\Pi(x, 0)$ by $(1/i)\delta/\delta\phi(x)$; $\Psi[\phi]$ satisfies the Schrödinger equation[15]

$$i\frac{\partial\Psi}{\partial t} = H\Psi, \tag{5.1a}$$

where

$$H = -\tfrac{1}{2}\int dx\, \frac{\delta^2}{\delta\phi(x)\delta\phi(x)} + E_c[\phi]. \tag{5.1b}$$

Thus the classical field energy $E_c[\phi]$ plays the role of a potential energy in the Schrödinger equation. $E_c[\phi]$ is infinite unless $\phi(x) \to \phi_1$ or ϕ_2 as $x \to \pm\infty$, and so the configuration space is divided into four regions separated by infinite potential barriers. The solutions of Eq. (5.1) may be divided into four sectors, in each of which $\Psi[\phi]$ is nonzero in only one of the regions. The whole space of states is divided into four orthogonal subspaces, listed below, with no transitions between subspaces.

Sector I. $\Psi[\phi] = 0$ unless $\phi(x) \to \phi_1$ as $x \to \pm\infty$. The lowest energy eigenstates are multimeson states built on a vacuum Ω_1. However, the classical multibaryon scattering states with an even number of transition regions alternately from ϕ_1 to ϕ_2 and from ϕ_2 to ϕ_1 must also correspond to quantum states in this sector.

Sector II. This is related to sector I by the symmetry which takes $\phi_1 \leftrightarrow \phi_2$ and consists of states built on a vacuum Ω_2.

Sector III. $\Psi[\phi] = 0$ unless $\phi(x) \to \phi_1$ as $x \to -\infty$ and $\phi(x) \to \phi_2$ as $x \to +\infty$. The lowest energy eigenstates are the baryon-multimeson states described in this paper. But there must also be scattering states corresponding to the classical solutions with an odd number of transition regions alternately from ϕ_1 to ϕ_2 and ϕ_2 to ϕ_1.

Sector IV. This is related to sector III by the symmetry $\phi_1 \leftrightarrow \phi_2$.

To make a sensible theory of a real (one-dimensional) world, we should retain only two sectors, one chosen arbitrarily from sectors I and II, and one chosen arbitrarily from sectors III and IV. The discarded sectors are identical doubles of those retained. Since no transitions take place between sector I (even number of baryons) and sector III (odd number of baryons), the baryons carry an additive quantum number +1, which is conserved modulo 2. Thus the one-baryon state is stable, as we have assumed. It should be noted that whether a baryon localized at x_0 is associated with a field expectation value $\phi_c(x - x_0)$ or $\phi_c(x_0 - x)$ depends on how many baryons are to its left; however, this does not obviously contradict local causality since insertion of an extra baryon far to the left does not affect the physical behavior far to its right, it merely changes the description in effect from sector I to II or III to IV.

If this is a correct description of the quantized theory, there is no baryon-antibaryon conjugation. The symmetry $\phi_1 \leftrightarrow \phi_2$ is broken, since it sends a retained sector into a discarded sector. However, within sector I $\Phi(x, t) \to \Phi(-x, t)$ is a parity transformation; within sector III this must be combined with the $\phi_1 \leftrightarrow \phi_2$ transformation. Thus for the ϕ^4 theory, the field Φ behaves as a scalar in sector I, but as a pseudoscalar in sector III.

This structure is specific to the particular type of model [with a $\phi_1 \leftrightarrow \phi_2$ symmetry], but should be capable of generalization. For example, still in one space dimension, in the theory with $U(\phi) = 1 - \cos\phi$ we would retain one sector for each integer N, with $\phi(\infty) - \phi(-\infty) = 2N\pi$; there are baryons and antibaryons with a conserved additive quantum number ± 1; the symmetry $\Phi(x, t) \to -\Phi(x, t)$ survives as a baryon-antibaryon conjugation.

We have no systematic method of verifying these conjectures or of calculating scattering of baryons. We should draw attention to the rather puzzling analytic properties of the matrix elements we have calculated. For the ϕ^4 theory, we found

$$\begin{aligned}\langle p'|\Phi|p\rangle &= \lambda^{-1/2}\int dx\, e^{i(p-p')x}\tanh mx \\ &= \frac{i\pi}{m}\lambda^{-1/2}\frac{1}{\sinh(\pi/2m)(p-p')}.\end{aligned} \tag{5.2a}$$

(The singularity at $p-p'=0$ is a principal part and is due to the large-x behavior of $\tanh x$.) We may write this in a Lorentz-covariant form as

$$\langle p'|\Phi|p\rangle = \frac{i\epsilon_{\mu\nu}p^{\mu}p'^{\nu}}{[2E(p')2E(p)]^{1/2}}\,G(t)\,, \tag{5.2b}$$

$$t=(p-p')_{\mu}(p-p')^{\mu}\,,$$
$$G(t)=\frac{2\pi\lambda^{-1/2}}{m}\;\frac{1}{\sqrt{t}\,\sin(\pi/2m)\sqrt{t}}\;. \tag{5.2c}$$

The pseudoscalar form is expected from the argument about parity above. There are poles at $t=(2nm)^2$; whether these can be associated with meson thresholds in the crossed two-baryon to vacuum matrix element is not clear. Indeed any attempt to interpret Eq. (5.2b) in the crossed channel immediately runs into the difficulty that it is antisymmetric under interchange of p and p'. Is this a peculiarity of the two-dimensional theory or a hint that the baryons are fermions? (The classical picture of the multibaryon states does not obviously force a particular statistics on the quantized theory.)

The stability of the baryons may also be understood in terms of a conservation law. All models in one dimension possess a conserved current $J^{\mu}=\epsilon^{\mu\nu}\partial_{\nu}\Phi(x,t)$. The charge is $\Phi(\infty,t)-\Phi(-\infty,t)$. In sector I (or II), matrix elements of this charge are zero, since the matrix elements of the field tend to the same value as $x\to\pm\infty$, while in sectors III (or IV) the matrix element is nonzero, since the field matrix elements have differing asymptotic values. It is the conservation of this charge that renders the baryons stable.[16]

In the one-baryon sector which we have examined, the question of higher-order calculations remains open. The corrections in λ come from two distinct places. First there are contributions of the multimeson states to the right-hand side of (4.4) which are of higher order in λ than the left-hand side. Secondly, the use of correct relativistic kinematics for the baryon will produce corrections in λ to the static approximation which we have employed.

As an example we derive a formula for the baryon form factor which includes first-order corrections to (4.5). In this particular case, the kinematic corrections only occur at second order. Returning to (4.4), we find that for the no-meson matrix element the order $\lambda^{1/2}$ terms on the right-hand side involve only one-meson terms. The equation satisfied by f therefore becomes

$$f''-U'(f)-\tfrac{1}{2}\tilde{G}(x,x)U'''(f)=0\;. \tag{5.3}$$

This is similar to (3.2), with the crucial difference that the propagator $\tilde{G}(x,x)$, given by (4.12b), is infrared finite. Equation (5.3) may be solved iteratively. Setting $f=\phi_c(x)+\delta\phi(x)$ and constructing $\tilde{G}(x,x)$ from the solutions of (4.6), we find[17]

$$\left[-\frac{d^2}{dx^2}+U''(\phi_c)\right]\delta\phi=\tfrac{1}{2}\tilde{G}(x,x)U'''(\phi_c)\;. \tag{5.4}$$

For consistency, the right-hand side must be orthogonal to the translation mode:

$$\int dx\,\tilde{G}(x,x)U'''(\phi_c)\phi_c'(x)=0\;. \tag{5.5}$$

To see that this vanishes we proceed as follows. The integral in (5.5) can be expressed as

$$\sum_k \frac{1}{2\omega(k)}\int dx\,\psi_k^*(x)\psi_k(x)U'''(\phi_c)\phi_c'(x)\;. \tag{5.6a}$$

But by differentiating (4.6) with respect to x, we see that the quantity $\psi_k(x)U'''(\phi_c)\phi_c'(x)$ can be replaced by $\psi_k'''(x)-[U''(\phi_c)-\omega^2(k)]\psi_k'(x)$. Thus using (4.6) once more to eliminate $\psi_k^*(x)U''(\phi_c)$, (5.6a) may be rewritten as

$$\sum_k \frac{1}{2\omega(k)}\int dx[\psi_k^*(x)\psi_k'''(x)-\psi_k^{*\prime\prime}(x)\psi_k'(x)]$$
$$=\sum_k \frac{1}{2\omega(k)}\left[\psi_k^*(x)\psi_k''(x)-\psi_k^{*\prime}(x)\psi_k'(x)\right]_{x=-\infty}^{x=\infty}\;. \tag{5.6b}$$

For discrete states this clearly vanishes; for continuum states (5.6b) can be evaluated from (4.16a). We find with the help of (4.16b)

$$2\int_0^{\infty} dk\,\frac{k^2}{\omega(k)}|B(k)|^2-2\int_{-\infty}^{0} dk\,\frac{k^2}{\omega(k)}|A(k)|^2\;. \tag{5.6c}$$

Changing k to $-k$ in the second integral and using (4.16d) shows that this quantity indeed vanishes. [Equation (5.4) does not determine contributions to $\delta\phi$ proportional to ϕ_c'. This is as it should be, since such terms can be compensated by adjusting the phase of the form factor.]

Consistency conditions, like (5.5), may be expected to arise at each new order of the calculation. It would be most interesting to formulate them in some general way, so that the consistency of our theory is evident to all orders.

VI. CONCLUSION

We have paid rather close attention to the quantum interpretation of classical solutions to field theory and to the translation mode. Though it was clear that such an interpretation had to be in terms of new particles, it was not so obvious how to carry it out in a way that would extend beyond the crudest approximation. Once the properties of the quantum theory have been established, one can proceed with more confidence to physically interesting calculations on field theories in three dimensions. Also one must develop approximation techniques suitable to larger coupling constants. It may be, as has been long speculated,[18] that the baryons occurring in nature will be found to coincide with the mathematical baryons which we have discussed.[19]

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[1]There exists a vast literature about solutions of nonlinear field equations. We are concerned with quantized solitary waves; much of the classical work concerns itself with "solitons," which are solitary waves that emerge from collisions unchanged. We do not use the name soliton since we do not know whether our baryons have, or possibly must have, the soliton property. For a review of solitons, see G. Whitham, *Linear and Non-linear Waves* (Wiley, New York, 1974); A. Scott, F. Chu, and D. McLaughlin, Proc. IEEE 61, 1443 (1973).

[2]R. Dashen, B. Hasslacher, and A. Neveu [Phys. Rev. D 10, 4114 (1974); 10, 4130 (1974); 10, 4138 (1974)] have also examined the quantum-mechanical interpretation of classical solutions. Their method is entirely different from ours, though some of their conclusions are similar.

[3]The notation $U'(\phi)$, $U''(\phi)$, etc. will always denote derivatives of $U(\phi)$ with respect to ϕ, while ϕ', ϕ'', etc. will mean derivatives of $\varphi(x)$ with respect to x.

[4]This stability theorem is generally known to workers in this field.

[5]H. B. Nielsen and P. Olesen, Nucl. Phys. B61, 45 (1973); G. 't Hooft, Nucl. Phys. B79, 276 (1974); L. D. Faddeev, Max-Planck Institut report (unpublished); A. M. Polyakov, Landau Institute report (unpublished); R. Dashen *et al.*, Ref. 2; T. Eguchi and H. Sugawara, Phys. Rev. D 10, 4257 (1974).

[6]See for example P. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953), p. 1650.

[7]Additional field-theoretic models which possess static, stable solutions in the classical approximation can be constructed in the following way. Let $f(x)$ be any integrable function without zeros. Define $\phi_c(x)=\int_{-\infty}^{x} dy\, f(y)+\phi_1$ and express $V(\phi_c)=\frac{1}{2}f^2$ in terms of ϕ_c. It follows that f is a zero-energy eigenstate of $-d^2/dx^2+V''(\phi_c)$; since f has no zeros, it is the lowest state. The class of field potentials that lead to (2.11) correspond to setting $f(x)=1/\cosh^L x$. For $L=1$ the sine-Gordon theory is obtained. The sine-Gordon equation in classical mechanics has been recently analyzed by L. Faddeev and L. Takhtajan, JINR report (unpublished), while S. Coleman [Phys. Rev. D (to be published)] has discussed the quantum theory.

[8]J. M. Cornwall, R. Jackiw, and E. Tomboulis, Phys. Rev. D 10, 2428 (1974).

[9]In other words, we are assuming that the exact equation $\delta\Gamma[\phi]/\delta\varphi(x,t)=0$ only has solutions with constant ϕ, and the lowest-order nontranslationally invariant solution $\phi_c(x)$ is an artifact of an approximation. It was emphasized to us by F. Low that the situation here is analogous to the polaron problem or the Hartree-Fock approximation to a nucleus. There too a lowest-order calculation violates translation invariance, which is an exact symmetry. Nevertheless, the approximation is physically relevant.

[10]There is an important difference. When an internal continuous symmetry is spontaneously broken, the singularity in the propagator is not isolated, as it is here.

[11]The idea that $\phi_c(x)$ should be related to a baryon form factor was also developed in conversations with F. Low.

[12]A. Kerman and A. Klein, Phys. Rev. 132, 1326 (1963).

[13]When the operator Φ occurs without an argument, it means that the argument is the origin: $\Phi=\Phi(0,0)$.

[14]Renormalization is discussed by S. Coleman, R. Jackiw, and H. D. Politzer, Phys. Rev. D 10, 2491 (1974). The explicit evaluation of (4.13) in the ϕ^4 theory is given by Dashen *et al.*, Ref. 2.

[15]Such a Schrödinger picture has been discussed most recently by J. Kuti (unpublished). An account of Kuti's work is given in Ref. 8.

[16]The interpretation which we are advocating is also supported by Coleman's investigation of the quantum sine-Gordon equation (Ref. 7). He finds that model to be equivalent to the massive Thirring model, with $\epsilon^{\mu\nu}\partial_\nu\Phi$ proportional to the fermion-number current. Since our methods apply equally well to the sine-Gordon theory (see Ref. 7), it is plausible to identify our baryons with the fermion field.

[17]$\tilde{G}(x,x)$ is ultraviolet-infinite; a mass renormalization must be performed in Eq. (5.4), analogous to that discussed in Ref. 14.

[18]An early reference is T. Skyrme, Proc. R. Soc. A262,

237 (1961); a recent one is W. Bardeen, M. Chanowitz, S. Drell, M. Weinstein, and T.-M. Yan, Phys. Rev. D 11, 1094 (1975).

[19]The classical statistical mechanics of the ϕ^4 theory, as a model for a ferroelectric, is discussed by J. A. Krumhansl and J. R. Schrieffer, Univ. of Pennsylvania report (unpublished). The "baryons" are their domain walls.