Bhabha first-order wave equations. III. Poincaré generators*

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We construct the Poincaré generators for arbitrary-spin Bhabha fields. After showing that for high-spin fields Hermiticity does not mean that the operators are self-adjoint, but rather satisfy Eqs. (2.16) and (2.17) below, we observe that these generators are Hermitian in this generalized sense. We explicitly demonstrate that the generators algebraically satisfy the associated Lie algebra for arbitrary half-integer-spin representations, but only as an operator algebra on the fields themselves for integer-spin representations. Specifically, of the six independent commutation relations $[K_i, K_j] = -i\epsilon_{ijk}J_k$ is not satisfied algebraically, and of the three dependent commutation relations $[K_i, H] = iP_i$ is not satisfied algebraically. By looking at the Sakata-Taketani decomposition of the Duffin-Kemmer-Petiau case, we find that it is only the built-in subsidiary components, not the particle components, which need an operator equation on the fields to satisfy the above two commutation relations. We generalize this result and show that the particle components for arbitrary-integer-spin fields satisfy the commutation relations algebraically. Finally, we comment on the interacting field case and problems associated with high-spin interacting field theories.

I. INTRODUCTION

In paper I (Ref. 1) of this series we discussed the C, P, and T transformation properties of the Bhabha first-order wave equations for arbitrary spin[2-4]

$$(\partial \cdot \alpha + \chi)\psi = 0\,, \tag{1.1}$$

where the Bhabha algebra for representations up to maximum spin $\mathcal{S}$ is defined by the equations[2-5] (with unity I added by hand for integer spin)

$$[[\alpha_\mu, \alpha_\nu], \alpha_\lambda] = \alpha_\mu \delta_{\nu\lambda} - \alpha_\nu \delta_{\mu\lambda}\,, \tag{1.2}$$

$$\prod_{n=-\mathcal{S}}^{\mathcal{S}} (\alpha_\mu - nI) = 0\,. \tag{1.3}$$

These include the Dirac[6] and Duffin[7]-Kemmer[8]-Petiau[9] (DKP) equations as the special cases $\mathcal{S} = \frac{1}{2}$ and $\mathcal{S} = 1$. In paper II (Ref. 10) we discussed the mass and spin compositions, the Hamiltonians, and the general Sakata-Taketani reductions of these equations.

In this paper, we will give the Poincaré generators for the general Bhabha fields and prove that they satisfy the Poincaré commutation relations. Specifically, if we denote the generators of space translations, rotations, time translations, and velocity translations as $\vec{P}$, $\vec{J}$, H, and $\vec{K}$ respectively, then they must satisfy the following commutation relations among themselves:

$$[P_i, P_j] = 0\,, \tag{1.4a}$$

$$[P_i, H] = 0\,, \tag{1.4b}$$

$$[J_i, H] = 0\,, \tag{1.4c}$$

$$[J_i, J_j] = i\epsilon_{ijk}J_k\,, \tag{1.4d}$$

$$[J_i, P_j] = i\epsilon_{ijk}P_k\,, \tag{1.4e}$$

$$[J_i, K_j] = i\epsilon_{ijk}K_k\,, \tag{1.4f}$$

$$[K_i, P_j] = i\delta_{ij}H/c^2\,, \tag{1.4g}$$

$$[K_i, H] = iP_i\,, \tag{1.4h}$$

$$[K_i, K_j] = -i\epsilon_{ijk}J_k/c^2\,. \tag{1.4i}$$

[We will hereafter set $c=1$ in Eqs. (1.4g) and (1.4i). $c \neq 1$ is useful when discussing expansions in c^2 or nonrelativistic limits. Also, in the rest of this paper we will use $c = \hbar = 1$. For a discussion on the dimensions of Poincaré generators, see Ref. 11.]

Now, it is known that the nine Eqs. (1.4) are not independent. In fact, one can easily show by repeated use of the Jacobi identity that if one is given the five equations (1.4a), (1.4e), (1.4f), (1.4g), and (1.4i), then the three equations (1.4b), (1.4c), and (1.4h) follow directly.[12] Thus, only the above-named five commutation relations, as well as the sixth (1.4d), actually have to be proved to show that the Poincaré algebra is satisfied. Further, one can give physical insight into these six independent commutation relations. Equation (1.4a) is a manifestation that space-time is flat, Eqs. (1.4d), (1.4e), and (1.4f) show that space is isotropic, Eq. (1.4g) "projects H out of $\vec{K}$," and actually it is Eq. (1.4i) which finally represents special relativity.

The Bhabha generators which we will show satisfy the Poincaré commutation relations are

$$P_j = p_j = -i\partial_j\,, \tag{1.5}$$

$$J_k = -i\epsilon_{ijk}(x_i\partial_j + \alpha_i\,\alpha_j) \equiv L_k + S_k\,, \tag{1.6}$$

$$H=H(n+\tfrac{1}{2})=\alpha_4{}^{-1}(\vec{\partial}\cdot\vec{\alpha}+\chi)\,,$$
half-integer-spin representations (1.7a)

$$H=H(n)=Q(\vec{\partial}\cdot\vec{\alpha}+\chi)-\mathcal{I}_0(\vec{\partial}\cdot\vec{\alpha})\alpha_4[1+\chi^{-1}(\vec{\partial}\cdot\vec{\alpha})]\,,$$
integer-spin representations (1.7b)

and

$$K_j=x_jH-tp_j+t_{4j}\,, \tag{1.8}$$

$$t_{4j}\equiv[\alpha_4,\alpha_j]\,. \tag{1.9}$$

Equation (1.5) is the standard quantum-mechanical definition of three-momentum. Equation (1.6) is the total angular momentum, the two separate pieces being the orbital and spin angular momentum. For example, in the Dirac case, where $\alpha_\mu=\frac{1}{2}\gamma_\mu$, the spin piece of the angular momentum is the well known

$$S_k^D=[\gamma_i,\gamma_j]/(4i),\quad i,j,k \text{ cyclic}. \tag{1.10}$$

Equations (1.7) are the half-integer- and integer-spin Hamiltonians which we derived in paper II. The half-integer-spin Hamiltonian (1.7a) was derived in Eq. (II5.2). The derivation was simple since the half-integer-spin algebra matrices have inverses. We demonstrated in Eq. (II5.3) that this Hamiltonian reduces to the Dirac Hamiltonian in the case $\mathcal{S}=\frac{1}{2}$. The integer-spin Hamiltonian (1.7b) was derived in Eq. (II5.14), and we demonstrated in Eq. (II5.15) that it reduces to the DKP Hamiltonian in the case $\mathcal{S}=1$. The operators Q and $\mathcal{I}_0$ were derived and discussed in Sec. III of paper II. The reader is referred there for a detailed discussion of them, but we will also list their more important algebraic properties when they are needed, in Sec. V.

Equations (1.8) and (1.9) define the boost operators, which complete our definitions of the generators.

In Sec. II we will discuss the Hermiticity properties of Bhabha operators, and the Poincaré generators in particular. The way to properly define Hermiticity for higher-spin algebras other than the Dirac case has been subject to some confusion since Kemmer[8] first pointed out the problem for spin 0 and 1. Our discussion there will concentrate on the origin of the confusion and the resolution of the problem.

In Sec. III we will discuss the Poincaré commutation relations for the special cases $\mathcal{S}=\frac{1}{2}$ and 1, i.e., the Dirac and DKP cases. We will find that although the commutation relations are satisfied as an algebraic identity for the Dirac case (which, of course, is well known), this is no longer true for the DKP case. There the commutation relations are only valid as operator equations on the DKP field. In particular, the commutation relations (1.4h) and (1.4i) can only be satisfied by using the consequent equations (II5.5) and/or the free field Eq. (1.1).

In Sec. IV we will take a detailed look at the Sakata-Taketani (ST) version[13,14] of the DKP case. We will find that although the Poincaré commutation relations are now satisfied algebraically for the "ST particle components" generators the "ST subsidiary components" generators still only satisfy the Poincaré commutation relations as operator equations on the field. This illuminates the origin of the operator equations needed for both the particular DKP case and the general integer-spin Bhabha case discussed later.

In Sec. V we will go on to general half-integer- and integer-spin fields. The conclusions of Sec. III will remain in the general case. That is, the Poincaré generators (1.5)–(1.9) satisfy the Poincaré commutation relations (1.4) algebraically for half-integer-spin representations, but only as operator equations on the fields for integer-spin representations. One has to use the "consequent" and/or free field equations to satisfy the relations in the integer-spin case. However, as with DKP, we show that particle components of the integer-spin generators satisfy the commutation relations algebraically.

We should emphasize here that there is nothing surprising in the Poincaré commutation relations only being satisfied algebraically for half-integer spin. In general it is sufficient to have only an operator algebra on a field. The fact that one needs an operator algebra on the fields for integer spin is just one more aspect of the more complicated and less fundamental nature of integer-spin fields and representations. Other aspects are the lack of α_μ inverses and the built-in consequent equations for integer-spin fields we have discussed at length in papers I and II.

In Sec. VI we will conclude with a discussion. This discussion will first go over our results and then mention their implications for the Foldy-Wouthuysen transformations on general Bhabha fields which we will present in paper V.[15] We will also comment on what changes are involved when the minimal electromagnetic substitution

$$\partial_\lambda \to \partial_\lambda^{\mp}=\partial_\lambda \mp ieA_\lambda \tag{1.11}$$

is included. This will bring up two fundamental problems associated with high-spin field theories: (1) Introducing minimal electromagnetic interactions often destroys the second-quantized field commutation relations of high-spin fields; (2) even in the c-number theory, introducing electromagnetic interactions (minimal or higher multipole, depending on the type of field) will yield noncausal solutions for at least some high-spin field theories. Causality will be discussed in paper IV.[15]

II. HERMITICITY

As we mentioned in paper I, the Hermiticity requirement for Dirac operators turns out to be that the operators themselves are self-adjoint. This is because the field adjoint operator, η_4 of Eq. (I3.47), for the special Dirac case is $\eta_4^D = 2\alpha_4^D = \gamma_4$, the inverse of the γ_4 density operator (the fourth component of the current operator). That is,

$$\begin{aligned}\langle \mathcal{O}\rangle &= \int \bar{\psi}\,\alpha_4 \mathcal{O}\psi d\tau \\ &= \int \psi^\dagger \eta_4\,\alpha_4\,\mathcal{O}\psi d\tau \\ &= \int \psi^\dagger (\tfrac{1}{2})\,\mathcal{O}\psi d\tau \\ &= \int (\psi^D)^\dagger\,\mathcal{O}\psi^D d\tau\,, \end{aligned} \tag{2.1}$$

where the factor $\frac{1}{2}$ is absorbed in going from the explicit Bhabha to the Dirac field normalization. Thus, for Dirac operators Hermiticity demands that

$$\mathcal{O}^\dagger = \mathcal{O}\,. \tag{2.2}$$

It was Kemmer[8] who pointed out that this is no longer true for higher-spin algebras since, as in the DKP case, one then has

$$\eta_4\alpha_4 \neq I\,,\quad \mathcal{S} > \tfrac{1}{2}\,. \tag{2.3}$$

Therefore, the general Hermiticity requirement we need is that in some sense

$$[\eta_4 \langle\langle \alpha_4 \mathcal{O}\rangle\rangle]^\dagger = [\eta_4 \langle\langle \alpha_4\,\mathcal{O}\rangle\rangle]\,. \tag{2.4}$$

For the DKP case Kemmer suggested that the solution would be to interpret $\langle\langle \alpha_4\mathcal{O}\rangle\rangle$ as being either $\alpha_4\mathcal{O}$ itself, or some symmetric combination, like $(\frac{1}{2})(\alpha_4\mathcal{O}+\mathcal{O}\,\alpha_4)$. He felt that it should be clear which solution to use in any particular case.

However, we will now proceed to show that there is a way to resolve this ambiguity and to give a general prescription. The prescription will be that $\langle\langle \alpha_4\mathcal{O}\rangle\rangle$ should be interpreted as

$$\langle\langle \alpha_4\mathcal{O}\rangle\rangle \equiv \alpha_4\mathcal{O}\,. \tag{2.5}$$

This will yield the results that Hermiticity for half-integer-spin representations requires that

$$[\eta_4\,\alpha_4\,\mathcal{O}]^\dagger = [\eta_4\,\alpha_4\,\mathcal{O}]\,,\quad \mathcal{S} = n + \tfrac{1}{2} \tag{2.6}$$

while for integer-spin representations Hermiticity will require the weaker condition

$$\psi^\dagger[\eta_4\,\alpha_4\,\mathcal{O}]^\dagger\,\psi = \psi^\dagger[\eta_4\,\alpha_4\,\mathcal{O}]\psi\,,\quad \mathcal{S} = n\,. \tag{2.7}$$

The weaker condition for integer spin is a manifestation of the built-in "subsidiary components," as we show below. (More commonly, this Hermiticity condition is called "pseudo-Hermiticity," and is related to the metric $\eta_4\,\alpha_4$. This aspect will be covered in more detail in paper V.[15])

Now let us look at the Poincaré generators. Clearly for both integer- and half-integer-spin representations we have

$$[\eta_4\,\alpha_4 p_j]^\dagger = [\eta_4\,\alpha_4 p_j]\,, \tag{2.8}$$

$$[\eta_4\,\alpha_4 J_k]^\dagger = [\eta_4\,\alpha_4 J_k]\,. \tag{2.9}$$

Equation (2.8) is trivial when one remembers that

$$x_j^\dagger = x_j\,,\quad p_j^\dagger = p_j\,,\quad \partial_j^\dagger = -\partial_j\,. \tag{2.10}$$

Equation (2.9) follows from Eq. (I3.47) and the equation [(II3.40)]

$$[[\alpha_i,\alpha_j], f(\alpha_4)] = 0\,. \tag{2.11}$$

It is the Hamiltonian which brings us to the problem Kemmer referred to. First note that for the half-integer-spin Hamiltonian of Eq. (1.7a), one again has that

$$[\eta_4\,\alpha_4 H(n+\tfrac{1}{2})]^\dagger = [\eta_4\,\alpha_4 H(n+\tfrac{1}{2})]\,. \tag{2.12}$$

However, for the integer-spin Hamiltonian (1.7b) one has

$$\begin{aligned}[\eta_4\,\alpha_4 H(n)]^\dagger - [\eta_4\,\alpha_4 H(n)] = (\vec{\partial}\cdot\vec{\alpha})(\mathcal{I}_0\eta_4) \\ + (\eta_4\,\mathcal{I}_0)(\vec{\partial}\cdot\vec{\alpha})\,. \end{aligned} \tag{2.13}$$

Despite Eq. (2.13), our prescription (2.5) can be maintained by observing that if one uses the wave equation (1.1), its adjoint Eq. (I2.45), and Eq. (II3.23),

$$\alpha_4\,\mathcal{I}_0 = 0\,, \tag{2.14}$$

then one has that the "expectation value" of Eq. (2.13) is zero. That is,

$$\psi^\dagger[\eta_4\,\alpha_4 H(n)]^\dagger\,\psi = \psi^\dagger[\eta_4\,\alpha_4 H(n)]\psi\,. \tag{2.15}$$

One can understand this by observing that if instead of using the entire integer-spin Hamiltonian $H(n)$ of Eq. (1.7b), one uses the "particle components" Hamiltonian of Eq. (II6.13),

$$\mathcal{H}_P = Q(\vec{\partial}\cdot\vec{\alpha}+\chi) - Q(\vec{\partial}\cdot\vec{\alpha})\,\mathcal{I}_0[1+\chi^{-1}(\vec{\partial}\cdot\vec{\alpha})]\,, \tag{2.16}$$

then one has

$$[\eta_4\,\alpha_4\mathcal{H}_P]^\dagger = [\eta_4\,\alpha_4\mathcal{H}_P]\,. \tag{2.17}$$

Thus, the fact that the complete integer-spin Hamiltonian has pieces left over when compared to its $(\eta_4\,\alpha_4)$ adjoint simply reflects the presence of the built-in "subsidiary components" Hamiltonian, which we showed in paper II to have had no physical content beyond that which results from considering the "particle components" Hamiltonian alone.

By taking the expectation value of the complete integer-spin Hamiltonian one removes these extra coupled pieces, and the physical result is that of the "proper" Hermitian operator in Eq. (2.17). Thus, the definition of Hermiticity given in Eqs. (2.6) and (2.7) is physically consistent and can be used throughout. We also note that the above results hold when minimal electromagnetic substitution is introduced, at least for all the half-integer-spin and DKP cases (see Sec. VI).

Finally, when one considers the boost generators,[16] one has analogous results. That is, for the half-integer-spin case one has

$$[\eta_4 \alpha_4 K_j(n+\tfrac{1}{2})]^\dagger = [\eta_4 \alpha_4 K_j(n+\tfrac{1}{2})], \quad \mathcal{S} = n+\tfrac{1}{2} \tag{2.18}$$

whereas for the integer-spin case one gets

$$[\eta_4 \alpha_4 K_j(n)]^\dagger - [\eta_4 \alpha_4 K_j(n)] = (\vec{\partial}\cdot\vec{\alpha})(\eta_4 \mathcal{G}_0) x_j + x_j(\eta_4 \mathcal{G}_0)(\vec{\partial}\cdot\vec{\alpha}), \tag{2.19}$$

$$\psi^\dagger[\eta_4 \alpha_4 K_j(n)]^\dagger \psi = \psi^\dagger[\eta_4 \alpha_4 K_j(n)]\psi, \quad \mathcal{S} = n \tag{2.20}$$

and, from Eq. (5.39) below, the particle components of $K_j(n)$ are $(\eta_4\alpha_4)$-self-adjoint.

III. DIRAC AND DKP SPECIAL CASES

A. Dirac case

The Dirac case is, of course, well known.[17] Just to quickly review, the generators, when written in terms of the common Dirac notation ($\gamma_\mu = 2\alpha^D_\mu$, $m = 2\chi^D$), are

$$P_j = p_j = -i\partial_j, \tag{3.1}$$

$$J_k = -i\epsilon_{ijk}(x_i\partial_j + \tfrac{1}{4}\gamma_i\gamma_j), \tag{3.2}$$

$$H = \gamma_4(\vec{\partial}\cdot\vec{\gamma} + m), \tag{3.3}$$

$$K_j = x_j H - t p_j + \tfrac{1}{4}[\gamma_4, \gamma_j]. \tag{3.4}$$

By then using the explicit Dirac algebra

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}, \tag{3.5}$$

it is a short exercise to verify that the generators (3.1)–(3.4) satisfy the Poincaré commutation relations (1.4) algebraically.

B. DKP case

Writing the Poincaré generators explicitly in terms of the DKP notation ($\beta_\mu = \alpha_\mu{}^{\mathrm{DKP}}$, $m = \chi^{\mathrm{DKP}}$) gives

$$P_j = p_j = -i\partial_j, \tag{3.6}$$

$$J_k = -i\epsilon_{ijk}(x_k\partial_j + \beta_i\beta_j), \tag{3.7}$$

$$H = \beta_4(\vec{\partial}\cdot\vec{\beta} + m) - (\vec{\partial}\cdot\vec{\beta})\beta_4, \tag{3.8}$$

$$K_j = x_j H - t p_j + [\beta_4, \beta_j]. \tag{3.9}$$

By then using the DKP algebra

$$\beta_\mu\beta_\nu\beta_\lambda + \beta_\lambda\beta_\nu\beta_\mu = \beta_\lambda\delta_{\mu\nu} + \beta_\mu\delta_{\nu\lambda}, \tag{3.10}$$

a fair amount of algebra will verify that the generators (3.6)–(3.9) algebraically satisfy all the commutation relations (1.4) *except* (1.4h) and (1.4i). The reason for this is as follows.

Because the β_μ have no inverses, to obtain the generator H (Sec. I of paper II) it was first necessary to derive from the free wave equation,

$$(\partial\cdot\beta + m)\psi^{\mathrm{DKP}} = 0, \tag{3.11}$$

a set of consequent equations,

$$\partial_\lambda\psi^{\mathrm{DKP}} = (\partial\cdot\beta)\beta_\lambda\psi^{\mathrm{DKP}}, \tag{3.12}$$

and then to use only one of them, namely $\lambda = 4$. Equations (3.11) and (3.12) are operator equations on the fields. But since it is only the commutator of $\vec{K}$ with itself which carries the entire content of special relativity [remember, Eq. (1.4h) can be derived from the other independent commutation relations], one might expect that this is the only independent commutation relation which cannot be satisfied algebraically. Further, one might also expect that one could use the wave equation and the remaining three pieces of the consequent equations,

$$\partial_i\psi^{\mathrm{DKP}} = (\partial_4\beta_4\beta_i + \vec{\partial}\cdot\vec{\beta}\beta_i)\psi^{\mathrm{DKP}}, \qquad i = 1, 2, 3 \tag{3.13}$$

to satisfy (1.4i) as an operator equation, and thus indicate where the rest of the content of Eq. (3.12) needed for covariance is used. As we will see, this is in fact the case, and it singles out a special useful role for the consequent equations both here and in Sec. IV. (We should add that in Sec. V B, when we discuss the general integer-spin case, it turns out to be algebraically simpler to use the free wave equation instead of the $\lambda = 1, 2, 3$ consequent equations. However, after noting that the consequent equations come from the free wave equation, one can satisfy oneself that one is effectively using those pieces of the free wave equation necessary to make the procedure covariant.)

Using only Eq. (3.10), the algebraic results of commuting $\vec{K}$ with H and with itself are

$$[K_i, H] = iP_i + G_i, \tag{3.14}$$

$$[K_i, K_j] = -i\epsilon_{ijk}J_k - x_i G_j + x_j G_i, \tag{3.15}$$

where

$$G_i = -(1-\beta_4{}^2)(\partial_i - \vec{\partial}\cdot\vec{\beta}\beta_i) - \beta_i(1-\beta_4{}^2)(\vec{\partial}\cdot\vec{\beta} + m) \tag{3.16a}$$

$$\equiv \{[K_i, H] - iP_i\}. \quad (3.16b)$$

This shows that Eqs. (1.4h) and (1.4i) are not satisfied algebraically. Although (3.14) can be derived from (3.15), it is much easier to obtain (3.14) first. Given Eq. (3.14), Eq. (3.15) is most easily obtained by explicitly writing K_j in the form of Eq. (1.8) and commuting it term by term with K_i.

Now consider the free equation (3.11) and the $\lambda = 1, 2, 3$ consequent equations (3.13). Multiplying each on the left by $(1-\beta_4{}^2)$ and using an algebraic result from (3.10),

$$(1-\beta_4{}^2)\beta_4 = 0, \quad (3.17)$$

one has

$$(1-\beta_4{}^2)(\partial\cdot\beta + m)\,\psi^{\mathrm{DKP}} = (1-\beta_4{}^2)(\vec{\partial}\cdot\vec{\beta} + m)\,\psi^{\mathrm{DKP}} = 0, \quad (3.18)$$

$$(1-\beta_4{}^2)(\partial_i - \vec{\partial}\cdot\vec{\beta}\,\beta_i)\,\psi^{\mathrm{DKP}} = 0, \quad i = 1, 2, 3 \quad (3.19)$$

so that

$$G_i\,\psi^{\mathrm{DKP}} = 0, \quad (3.20)$$

and we have the operator statements

$$[K_i, H]\,\psi^{\mathrm{DKP}} = iP_i\,\psi^{\mathrm{DKP}}, \quad (3.21)$$

$$[K_i, K_j]\,\psi^{\mathrm{DKP}} = -i\epsilon_{ijk}J_k\,\psi^{\mathrm{DKP}}. \quad (3.22)$$

As already mentioned, in Sec. V B we will find that the results of this subsection generalize to arbitrary integer-spin Bhabha fields. That is, all the commutation relations (1.4) are satisfied algebraically for arbitrary integer-spin Bhabha fields, *except* (1.4h) and (1.4i). These are only satisfied as operator equations for integer-spin Bhabha fields.

IV. ST REDUCTION OF THE DKP CASE

A. Particle and subsidiary components of DKP generators

Any operator $\mathcal{O}$ can be written in the following form:

$$\mathcal{O} = \mathcal{I}\mathcal{O}\mathcal{I} + \mathcal{I}\mathcal{O}(1-\mathcal{I}) + (1-\mathcal{I})\mathcal{O}\mathcal{I} + (1-\mathcal{I})\mathcal{O}(1-\mathcal{I}), \quad (4.1)$$

where $\mathcal{I}$ is again any operator. If $\mathcal{I}$ is idempotent, then

$$\mathcal{I}^2 = \mathcal{I}, \quad (1-\mathcal{I})^2 = (1-\mathcal{I}), \quad (4.2)$$

$$\mathcal{I}(1-\mathcal{I}) = 0. \quad (4.3)$$

Furthermore, if there exist decoupling equations such that

$$(1-\mathcal{I})\psi = X\mathcal{I}\psi, \quad (4.4)$$

$$\mathcal{I}\psi = Y(1-\mathcal{I})\psi, \quad (4.5)$$

then the operator $\mathcal{O}$ can be decoupled into two disjoint pieces:

$$\mathcal{I}\mathcal{O}\psi = [\mathcal{I}\mathcal{O}(1+X)\mathcal{I}]\,\mathcal{I}\psi, \quad (4.6a)$$

$$(1-\mathcal{I})\mathcal{O}\psi = [(1-\mathcal{I})\mathcal{O}(1+Y)(1-\mathcal{I})](1-\mathcal{I})\psi. \quad (4.6b)$$

This is the Peirce decomposition discussed in paper II.

From the above we might only say that any operator can be decomposed into two disjoint pieces associated with the square brackets in the operator field Eqs. (4.6). However, even though Eqs. (4.6a) and (4.6b) were derived by and are operator equations on the fields, it is a consistent statement to consider the decoupled operators to be defined by the square brackets alone which can then operate on the associated decoupled fields $\mathcal{I}\psi$ and $(1-\mathcal{I})\psi$, respectively.

Specifically, and DKP operator can be decomposed into two disjoint pieces. The idempotent $\mathcal{I}$ is easily found from the characteristic equation (1.3),

$$\mathcal{I} = \beta_4{}^2 = (\beta_4{}^2)^2, \quad (4.7)$$

and from Eqs. (II2.16) and (II2.15) the decoupling operators X and Y are

$$X = -i(\vec{p}\cdot\vec{\beta})m^{-1}, \quad (4.8)$$

$$Y = \frac{i(m+\beta_4 E_p)}{p^2}(\vec{p}\cdot\vec{\beta}), \quad p^2 \neq 0 \quad (4.9a)$$

$$E_p = [p^2 + m^2]^{1/2}. \quad (4.9b)$$

We have been calling the fields and operators associated with $\beta_4{}^2$ the "particle components" and the fields and operators associated with $(1-\beta_4{}^2)$ the "subsidiary components." More specifically still, if $\mathcal{O}$ is any of the DKP Poincaré generators [Eqs. (3.6)–(3.9)], then the particle components of $\mathcal{O}$ are

$$\mathcal{O}^{(P)} = \beta_4{}^2\mathcal{O}[1 - i(\vec{p}\cdot\vec{\beta})m^{-1}]\beta_4{}^2, \quad (4.10)$$

or (changing to the now more convenient notation $H^{(P)} = \mathcal{H}_P$)

$$P_i^{(P)} = p_i\,\beta_4{}^2, \quad (4.11)$$

$$J_i^{(P)} = J_i\,\beta_4{}^2, \quad (4.12)$$

$$H^{(P)} = m\beta_4 + \beta_4(\vec{p}\cdot\vec{\beta})(\vec{p}\cdot\vec{\beta})m^{-1}, \quad (4.13)$$

$$K_i^{(P)} = x_i H^{(P)} - tP_i^{(P)} - i\beta_4\beta_i(\vec{p}\cdot\vec{\beta})m^{-1}. \quad (4.14)$$

Similarly, the subsidiary components of a DKP Poincaré generator $\mathcal{O}$ are

$$\mathcal{O}^{(S)} = (1-\beta_4{}^2)\mathcal{O}\left[1 + \frac{i(m+\beta_4 E_p)}{p^2}(\vec{p}\cdot\vec{\beta})\right](1-\beta_4{}^2), \quad (4.15)$$

or

$$P_i^{(S)} = p_i(1-\beta_4^2), \tag{4.16}$$

$$J_i^{(S)} = J_i(1-\beta_4^2), \tag{4.17}$$

$$H^{(S)} = E_p \frac{(\vec{p}\cdot\vec{\beta})(\vec{p}\cdot\vec{\beta})}{p^2}(1-\beta_4^2), \tag{4.18}$$

$$K_i^{(S)} = x_i H^{(S)} - t P_i^{(S)} - i\beta_i\beta_4 \frac{(m+\beta_4 E_p)}{p^2}(\vec{p}\cdot\vec{\beta})(1-\beta_4^2). \tag{4.19}$$

The peculiar properties associated with the operator $H^{(S)}$ have been discussed elsewhere (Sec. II B of paper II). What is of interest here is first that both the particle and subsidiary components of the DKP Poincaré generators can be defined by the formal procedure above. Second, being so defined, each individual set formally satisfies the Lie algebra of the Poincaré group in a way that is understandable.

B. The Lie algebra

Having the ST particle-components Poincaré generators (4.11)–(4.14) and the subsidiary-components generators (4.16)–(4.19), one could just take them and explicitly verify that they satisfy the Lie algebra commutation relations (1.4), obtaining the conclusions we will come to below. However, the simplest and most illuminating method is to take the DKP commutation relations themselves and decompose them into particle and subsidiary components. This is done in the Appendix.

The results of this analysis are that the particle-components generators now satisfy all the Poincaré commutation relations *algebraically*, whereas the subsidiary components remain the ones which only satisfy the commutation relations (1.4h) and (1.4i) as operators on the fields. Specifically, from Eqs. (A33), (A36), and (A37) in the Appendix, one has

$$[K_i^{(S)}, H^{(S)}]\psi_S = iP_i^{(S)}\psi_S, \tag{4.20}$$

$$[K_i^{(S)}, K_j^{(S)}]\psi_S = -i\epsilon_{ijk}J_k^{(S)}\psi_S, \tag{4.21}$$

$$\psi_S = (1-\beta_4^2)\psi. \tag{4.22}$$

Further, it is also shown in the Appendix that the consequent equations reduce to an identity for the particle components, while they remain nontrivial for the subsidiary components. In fact, it is the subsidiary components of the consequent equations which allows the commutation relations (1.4h) and (1.4i) to be satisfied as operators on the fields in the subsidiary case.

In conclusion, these results illuminate what we will discover in Sec. V about arbitrary-spin Bhabha fields and adds to our understanding of the difference between integer- and half-integer-spin Bhabha fields. Also, from the viewpoint of paper II,[10] we might already have suspected that the reason for the difference between the DKP and Dirac cases [that is, the DKP commutation relations (1.4h) and (1.4i) can only be a satisfied as operator equations on the fields] is that the β_λ, having zero eigenvalues, do not have inverses and therefore have the associated "infinite mass" subsidiary components. We have shown this to be true. The ST decomposition decouples the subsidiary components from the particle components, and when this is done we find that the particle components behave just like the half-integer-spin field (which contains only particle components) in that all the commutation relations are satisfied algebraically. In this section the need for the operator equations in integer-spin Bhabha fields has been explicitly associated with the subsidiary components.

V. POINCARÉ COMMUTATION RELATIONS FOR GENERAL BHABHA FIELDS

A. Half-integer-spin Bhabha fields

For the general half-integer-spin Bhabha fields, the commutation relations (1.4) are all satisfied algebraically, as was the case for the specific spin-$\frac{1}{2}$ Dirac case. Explicitly demonstrating this involves some extremely complicated algebra. However, there are some useful tricks which greatly aid one in this demonstration, and we list them below.

First recalling the fundamental algebraic double commutation relation statement (1.2),

$$[[\alpha_\mu, \alpha_\nu], \alpha_\lambda] = \alpha_\mu\delta_{\lambda\nu} - \alpha_\nu\delta_{\mu\lambda}, \tag{1.2}$$

we note the following:

(i) One can use the double commutation relation (1.2) with particular values of μ, ν, and λ to either reduce products of three α's to a single α, or else to change the orders of particular products of α's. This latter operation often is helpful of itself or in conjunction with trick (ii), which follows.

(ii) Especially in commutation relations involving $\vec{J}$'s and $\vec{K}$'s, complicated sums of products of three and four α's will often occur. By using the algebraic identities below, these sums of products can be reduced to combinations of the double commutation relation (1.2), and therefore to single sums of α's:

$$0 = [[\alpha_\nu, \alpha_\lambda], \alpha_\mu] + [[\alpha_\lambda, \alpha_\mu], \alpha_\nu] + [[\alpha_\mu, \alpha_\nu], \alpha_\lambda], \tag{5.1}$$

$$[[\alpha_\mu, \alpha_\nu], \alpha_\rho\alpha_\sigma] = [[\alpha_\mu, \alpha_\nu], \alpha_\rho]\alpha_\sigma + \alpha_\rho[[\alpha_\mu, \alpha_\nu], \alpha_\sigma], \quad (5.2)$$

$$[t_{\mu\nu}, t_{\rho\sigma}] \equiv [[\alpha_\mu, \alpha_\nu], [\alpha_\rho, \alpha_\sigma]] = [[\alpha_\mu, \alpha_\nu], \alpha_\rho]\alpha_\sigma + \alpha_\rho[[\alpha_\mu, \alpha_\nu], \alpha_\sigma] - [[\alpha_\mu, \alpha_\nu], \alpha_\sigma]\alpha_\rho - \alpha_\sigma[[\alpha_\mu, \alpha_\nu], \alpha_\rho]. \quad (5.3)$$

(iii) In commutation relations involving ϵ_{ijk} either explicitly or because they involve J_k, which contains ϵ_{ijk} in its definition (1.6) in the form

$$J_k = \epsilon_{ijk} A_i B_j, \quad (5.4)$$

the above tricks will often be easier to use if one just dispenses with the ϵ's, and writes the operators in an explicit (i,j) index form, like

$$J_k = A_i B_j - B_j A_i, \quad i,j,k \text{ cyclic}. \quad (5.5)$$

(iv-a) In commutation relations involving H and/or $\vec{K}$, where the operators contain $(\alpha_\mu)^{-1}$, a useful device is to multiply terms on the left by $I = (\alpha_4^{-1})\alpha_4$ and/or on the right by $I = \alpha_4(\alpha_4)^{-1}$. Then one can use tricks (i) and (ii) to change the positions of the α_4's in the I operators so that they cancel the $(\alpha_4)^{-1}$ matrices in the H and $\vec{K}$ operators.

B. Integer-spin Bhabha fields

Demonstrating the commutation relations (1.4) for the integer-spin Bhabha fields is even more complicated than for the half-integer-spin fields, simply because the integer-spin Hamiltonian (1.7b) is much more complicated than the half-integer-spin Hamiltonian (1.7a). To perform the demonstration, one first proceeds with the algebra as in the half-integer-spin case. However although one can still use tricks (i), (ii), and (iii), one cannot use (iv-a). The reason for not using (iv-a) is, of course, that $(\alpha_4)^{-1}$ does not exist for integer-spin fields. In its place one has the operators Q and $\mathcal{I}_0$, which were defined in paper II[10] and which take the place of $(\alpha_4)^{-1}$ in defining the mass content of the fields.

We first quickly recall from Eq. (II3.8) the definitions of the operators $\mathcal{I}_j(\mathcal{S})$:

$$\mathcal{I}_j(\mathcal{S}) = \frac{1}{N(\mathcal{S},j)} \prod_{\substack{k=0 \\ k\neq j}}^{\mathcal{S}} (\alpha_4^2 - k^2), \quad (5.6)$$

$$N(\mathcal{S},j) = \prod_{\substack{i=0 \\ i\neq j}}^{\mathcal{S}} (j^2 - i^2). \quad (5.7)$$

In the rest system the $\mathcal{I}_j(\mathcal{S})$ project out the mass states χ/j for $j \neq 0$, and project out the subsidiary "infinite mass" components for $j=0$. In Eqs. (3.21) and (3.22) of paper II we also defined the operators $\mathcal{Q}_j(\mathcal{S})$ and $Q(\mathcal{S})$ by

$$\mathcal{Q}_j(\mathcal{S})\alpha_4 = \alpha_4\mathcal{Q}_j(\mathcal{S}) \equiv \mathcal{I}_j(\mathcal{S}), \quad 0 \neq j = 1, 2, \ldots, \mathcal{S} \quad (5.8a)$$

$$\mathcal{I}_0(\mathcal{S})\,\mathcal{Q}_j(\mathcal{S}) = 0, \quad (5.8b)$$

$$Q(\mathcal{S}) = \sum_{j=1}^{\mathcal{S}} \mathcal{Q}_j(\mathcal{S}). \quad (5.9)$$

Deleting the label $\mathcal{S}$ for a particular Bhabha algebra, these operators, specifically Q and $\mathcal{I}_0$, are the quantities in the integer-spin Hamiltonian (1.7b). We therefore list a series of equations involving Q and $\mathcal{I}_0$ which were derived in Sec. III D of paper II, and which constitute a new trick for integer-spin calculations in place of (iv-a).

(iv-b) For integer spin, in place of (iv-a) the following equations are useful for explicitly verifying the Poincaré commutation relations:

$$\alpha_4\mathcal{I}_0 = \mathcal{I}_0\alpha_4 = 0, \quad (5.10)$$

$$\alpha_4 Q = I - \mathcal{I}_0, \quad (5.11)$$

$$\mathcal{I}_0 Q = 0, \quad (5.12)$$

$$0 = \mathcal{I}_0\alpha_\lambda(I - \alpha_4^2), \quad (5.13)$$

$$0 = \mathcal{I}_0\alpha_\lambda\mathcal{I}_0, \quad (5.14)$$

$$0 = \mathcal{I}_0\alpha_\lambda(\alpha_4 - Q), \quad (5.15)$$

$$0 = [[\alpha_i, \alpha_j], \mathcal{I}_0], \quad (i,j) \neq 4 \quad (5.16)$$

$$0 = [[\alpha_i, \alpha_j], Q], \quad (i,j) \neq 4. \quad (5.17)$$

Equations (5.16) and (5.17) follow immediately from the definitions of Q and $\mathcal{I}_0$ coupled with the useful special case of (1.2),

$$0 = [[\alpha_i, \alpha_j], \alpha_4], \quad (i,j) \neq 4. \quad (5.18)$$

Having the tricks (i), (ii), (iii), and (iv-b), one can explicitly verify that all the commutation relations (1.4) are satisfied algebraically *except* (as for DKP) the relations (1.4h) and (1.4i).

Starting with (1.4h), one finds, after algebraic manipulation, that

$$[K_j, H] = -[H, x_j]H + [t_{4j}, H] \quad (5.19)$$

can be put in the form

$$[K_j,H] - iP_j = [Q\alpha_j\,\alpha_4 + \alpha_4\alpha_j\,Q - Q\alpha_jQ - 2(1-\mathcal{J}_0)\alpha_j](\vec{\partial}\cdot\vec{\alpha}+\chi)$$
$$+\chi^{-1}\mathcal{J}_0[(\vec{\partial}\cdot\vec{\alpha})\alpha_j - (\vec{\partial}\cdot\vec{\alpha})\alpha_4\alpha_j\alpha_4 + \alpha_j(\vec{\partial}\cdot\vec{\alpha})](\vec{\partial}\cdot\vec{\alpha}+\chi)$$
$$+[(1-\mathcal{J}_0)\alpha_j - \chi^{-1}\mathcal{J}_0\,\alpha_j(\vec{\partial}\cdot\vec{\alpha}) - \chi^{-1}\partial_j]\mathcal{J}_0(\vec{\partial}\cdot\vec{\alpha}+\chi) \tag{5.20a}$$
$$\equiv G_j\,. \tag{5.20b}$$

Since the DKP case is a special case of (5.20), we know that we must use at least the free wave equation to obtain our final result. For DKP it was also useful to use the $\lambda = j = 1, 2, 3$ consequent equations, but these, one must remember, come from the free equation and the DKP algebra. However, for the general Bhabha integer-spin case, since the consequent equations (II5.5) are rather complicated, it turns out to be easiest to work with the free equation alone.

Writing the free equation (1.1) in the form

$$(\vec{\partial}\cdot\vec{\alpha}+\chi)\psi = -\partial_4\,\alpha_4\psi \tag{5.21}$$

and putting it into the right-hand side of Eq. (5.20) operating on the field, one finds with the aid of (i)–(iv) that the three square-bracket terms in (5.20), which as a result of Eq. (5.21) are now all multiplied on the right by α_4, are then all individually zero. Therefore, we have

$$[K_j,H]\,\psi(\mathcal{S}=n) = iP_j\psi(\mathcal{S}=n)\,. \tag{5.22}$$

Finally, note that with our generators the result (3.15) coupled with (3.16b) is in general true. That is, given that H is not a functional of x_μ, explicitly putting K_i and K_j in the form of Eq. (1.8) into the left-hand side of Eq. (1.4i) yields

$$[K_i,K_j] = -i\epsilon_{ijk}J_k + x_i\{iP_j - [K_j,H]\}$$
$$-x_j\{iP_i - [K_i,H]\}$$
$$= -i\epsilon_{ijk}J_k + x_iG_j - x_jG_i\,. \tag{5.23}$$

Thus, putting (5.20) and (5.22) into (5.23) gives

$$[K_i,K_j]\,\psi(\mathcal{S}=n) = -i\epsilon_{ijk}J_k\,\psi(\mathcal{S}=n)\,. \tag{5.24}$$

C. Particle components of integer-spin Bhabha fields

From the results of Sec. IV, one might suspect that the particle components of the Poincaré generators for arbitrary integer-spin Bhabha fields no longer need the operator equations on the fields, but rather satisfy the Poincaré commutation relations algebraically. Indeed, this is the case. One could show this by direct algebraic manipulation, but an easier way is to use the method of the Appendix which directly generalizes from the DKP case to arbitrary integer-spin Bhabha fields.

By recalling Eq. (II6.12), one first has that the particle components $\mathcal{G}^{(P)}$ of an integer-spin operator $\mathcal{G}$ are given by

$$\mathcal{G}^{(P)} = \mathcal{J}_P\mathcal{G}(1+X_P)\mathcal{J}_P\,, \tag{5.25}$$

$$\mathcal{J}_P = 1-\mathcal{J}_0\,, \tag{5.26}$$

$$X_P = -\mathcal{J}_0(\vec{\partial}\cdot\vec{\alpha})\chi^{-1}\,. \tag{5.27}$$

Therefore, the particle components of the arbitrary integer-spin generators are

$$P_j^{(P)} = P_j(1-\mathcal{J}_0)\,, \tag{5.28}$$

$$J_j^{(P)} = J_j(1-\mathcal{J}_0)\,, \tag{5.29}$$

$$H^{(P)} = [Q(\vec{\partial}\cdot\vec{\alpha}+\chi) - Q(\vec{\partial}\cdot\vec{\alpha})\mathcal{J}_0\chi^{-1}(\vec{\partial}\cdot\vec{\alpha})](1-\mathcal{J}_0)$$
$$= Q(\vec{\partial}\cdot\vec{\alpha}+\chi) - Q(\vec{\partial}\cdot\vec{\alpha})\mathcal{J}_0[1+\chi^{-1}(\vec{\partial}\cdot\vec{\alpha})]\,, \tag{5.30}$$

$$K_j^{(P)} = x_jH^{(P)} - tP_j^{(P)} + (1-\mathcal{J}_0)[\alpha_4,\,\alpha_i](1-\mathcal{J}_0)$$
$$-\chi^{-1}\alpha_4\,\alpha_j\mathcal{J}_0(\vec{\partial}\cdot\vec{\alpha})(1-\mathcal{J}_0)\,. \tag{5.31}$$

Now one can directly follow from the beginning of the Appendix, with $\xi\equiv P$ and Eqs. (5.26) and (5.27) substituted for the first lines of Eqs. (A7) and (A8). One proceeds to Eq. (A17), showing that the first five independent commutation relations [in fact, all the first seven Eqs. (1.4a)–(1.4g)] are satisfied algebraically for the particle components.

Going on to the generalization of Eq. (A18), where now G_i is given by Eq. (5.20), one can show that

$$G_i^{(P)} = 0\,. \tag{5.32}$$

Actually, this demonstration is algebraically similar to showing that $G_i\psi = 0$ after Eq. (5.21). There the trick of having α_4 multiply G_i on the right was used. Here, multiplying G_i on the right by $(1-\mathcal{J}_0) = Q\alpha_4$ is useful.

Continuing, the equations which are generalizations of (A20) and (A21) are

$$[\mathcal{J}_P, K_i] = x_i\{\alpha_4(\vec{\partial}\cdot\vec{\alpha})\mathcal{J}_0 + \mathcal{J}_0(\vec{\partial}\cdot\vec{\alpha})\alpha_4[1+(\vec{\partial}\cdot\vec{\alpha})\chi^{-1}]\}$$
$$+\alpha_4\,\alpha_i\mathcal{J}_0 + \mathcal{J}_0\,\alpha_i\,\alpha_4\,, \tag{5.33}$$

$$[X_P\mathcal{J}_P, K_i] = \mathcal{J}_0\{-(\vec{\partial}\cdot\vec{\alpha})\alpha_4x_i[1+(\vec{\partial}\cdot\vec{\alpha})\chi^{-1}]$$
$$+\alpha_i\,\alpha_4(\vec{\partial}\cdot\vec{\alpha})\chi^{-1}\}$$
$$+\alpha_4(\vec{\partial}\cdot\vec{\alpha})\mathcal{J}_0x_i(\vec{\partial}\cdot\vec{\alpha})\chi^{-1}\,. \tag{5.34}$$

One can then show that

$$[(1+X_P)\mathcal{G}_P, K_i] = \alpha_4(\vec{\partial}\cdot\vec{\alpha})\mathcal{G}_0 x_i + \alpha_4(\vec{\partial}\cdot\vec{\alpha})\mathcal{G}_0 x_i(\vec{\partial}\cdot\vec{\alpha})\chi^{-1}, \quad (5.35)$$

which in the special DKP case trivially reduces to the sum of Eqs. (A20) and (A21). Combining all the above, algebraic manipulation using Eqs. (5.10)–(5.18) allows one to show that

$$\{K_j, [(1+X_P)\mathcal{G}_P, K_i]\}^{(P)} = 0, \quad (5.36)$$

meaning we have demonstrated that

$$[K_i^{(P)}, K_j^{(P)}] = -i\epsilon_{ijk}J_k^{(P)}. \quad (5.37)$$

Because of its dependence on the other commutation relations, this also means that

$$[K_i^{(P)}, H^{(P)}] = iP_i^{(P)}. \quad (5.38)$$

This completes our proof that the integer-spin particle-components generators of Eqs. (5.28)–(5.31) satisfy the Poincaré commutation relations algebraically and further elucidates the role of the subsidiary components of integer-spin Bhabha fields.

In this last respect, one can verify that the particle-components generators (5.28)–(5.31) now all satisfy the Hermiticity condition (2.6) instead of the weaker condition (2.15). That is,

$$[\eta_4\,\alpha_4\,\mathcal{G}^{(P)}]^\dagger = [\eta_4\,\alpha_4\,\mathcal{G}^{(P)}]. \quad (5.39)$$

VI. DISCUSSION

In this paper we have shown that the Poincaré generators for the general Bhabha field defined in Eqs. (1.5)–(1.9) do indeed satisfy the commutation relations (1.4). For half-integer-spin representations these commutation relations were satisfied algebraically, whereas for integer-spin representations they [specifically (1.4h) and (1.4i)] were only satisfied as an operator algebra on the fields themselves. One can understand this last result. Remembering that (1.4h) is dependent on (1.4i), and (1.4i) contains, as we noted earlier, the content of special relativity, one appears to be saying that the operator algebra on the fields is necessary in the integer-spin representations because these fields contain built-in subsidiary components. In fact, from our detailed discussions of the ST decomposition of the DKP case and the general integer-spin particle components one can see this is true. There we found that the particle-components generators satisfied the commutation relations algebraically, but the subsidiary components only satisfied (1.4h) and (1.4i) as operators on the fields.

We wish to point out that although our work in this series has concentrated on representation-independent analyses, any particular Bhabha field can be studied by using known explicit formulas for arbitrary representations of the so(5) matrices.[18-21] Also, we remind the reader that there exists a vast literature on the Poincaré structure of high-spin field equations, of which we just cite a few examples.[22-26] (Also, see below.)

A future step is to show that there exist "unitary transformations" which diagonalize the generators we have derived, and to show that these diagonalized generators have their particle and antiparticle components "separated." That is, we will soon, in paper V,[15] perform Foldy-Wouthuysen transformations[27] on the generators. This process is more complicated than in the Dirac case, primarily because of the nature of the field inner-product space, as was indicated in our discussion of Hermiticity in Sec. II of this paper, but also because of the presence of multiple-mass solutions.

However, first the question arises as to what happens to the results of this paper when minimal electromagnetic substitution is introduced into the problem.[3] Then the best way to proceed is via a Lagrangian second-quantized technique. (Remember, however, that then we would be talking about the entire system of coupled fields, as the interacting fields can transfer energy and momentum, such as in the Compton effect.) Having obtained the coupled field generators, in principle the most straightforward way to proceed would be to directly verify that they satisfy the Poincaré commutation relations. However, simply by first observing how complicated our free case demonstration was, and then by observing how much more complicated the Bhabha generators become in the presence of electromagnetic interactions (an example is Kemmer's form[8] of the meson pieces of the DKP coupled generators P_μ and M_{ik}), one can quickly satisfy oneself that although this would be an impressive exercise, it is probably not the way to proceed.

The clearest and most pragmatic way to proceed probably is to follow the discussion of, say, Jauch and Rohrlich[28] for the Dirac case. In fact, combined with our results here, the Jauch and Rohrlich argument may carry over directly. The interaction Lagrangian $\bar{\psi}\alpha_\lambda A_\lambda\psi$ contains no derivatives, so that, as with Dirac, the momentum operator for the coupled fields has the same form as for the free fields. Therefore, since one knows that the Poincaré commutation relations are true for the free fields, if one can also show that the commutation relations for the second-quantized Bhabha fields ψ and $\bar{\psi}$ are preserved for arbitrary spin in the interacting case, this will imply that the Poincaré commutation relations (1.4) remain true in the second-quantized interacting case.

Showing that the field commutation relations are preserved for high-spin Bhabha fields is important in its own right. Johnson and Sudarshan[29] (JS) long ago showed that they are not necessarily preserved. In particular, they found that the standard equal-time (anti-) commutation relations and relativistic covariance are not compatible for certain second-quantized spin-$\frac{3}{2}$ fields with an external minimally coupled electromagnetic field. They found this property for both the Rarita-Schwinger[30] spin-$\frac{3}{2}$ field and a mixed spin ($\frac{1}{2}$ and $\frac{3}{2}$) field invented by Bhabha[31] and discussed by K. K. Gupta.[32] However, as has been emphasized,[32-34] this mixed field of Bhabha *is not* a particular case of the general Bhabha fields we have been discussing. It is another field. For example, the spin-$\frac{3}{2}$ Lorentz-group content is not that which we discussed in detail in paper II,[10] but rather that of the Rarita-Schwinger (RS) field.[30]

Thus, as Wightman[35] has emphasized, it is of interest to discover which field theories preserve the field commutation relations and which do not. One spin-$\frac{3}{2}$ field which fails to do this is that of S. N. Gupta.[36] S. N. Gupta and Repko[37] have also proposed that the difficulties in the RS formalism can be removed by using a nonstandard definition of the canonical field variables. However, Mainland and Sudarshan[38] have disputed the significance of this result by constructing RS Poincaré generators and using them to transform the interacting field variables covariantly. Their claim is that this shows the normal Rarita-Schwinger problems are not due to using incorrect quantization (i.e., using the wrong canonical field variables). But in any event, it clearly is of interest[33,34] to study whether general Bhabha fields have the JS problem.

We should also mention a recent and related discovery concerning problems with high-spin interacting field theories, even in the c-number theory. Velo and Zwanziger[39] discovered that noncausal solutions result when minimal electromagnetic substitution is introduced into the Rarita-Schwinger spin-$\frac{3}{2}$ theory and the tensor spin-2 theory. They also found that the spin-1 Proca second-order field equation has noncausal solutions when an electric quadrapole interaction is introduced, but that minimal electromagnetic coupling caused no problems in the solution for spin-0 or spin-1 second-order fields equations.

Thus the question arises as to what happens when interactions are introduced into general Bhabha fields.[33,34] Wightman[40] has partially answered this question by showing that the DKP spin-0 field has noncausal solutions with dipole interactions, and that the DKP spin-1 field has noncausal solutions with an electric quadrapole interaction (just as for Proca). However, it remains an incompletely answered question as to whether causality breaks down when minimal electromagnetic substitution is introduced into an arbitrary-spin Bhabha field.

We will discuss both the JS and the c-number problems in paper IV.[15]

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APPENDIX: ST GENERATORS AND COMMUTATION RELATIONS

Recall from Sec. I that there are only six independent commutation relations among the generators of the Poincaré group.[11] Further, from Sec. III B the *algebraic* results for these six commutation relations in the DKP case are

$$[P_i, P_j] = 0 , \tag{A1}$$

$$[J_i, J_j] = i\epsilon_{ijk} J_k , \tag{A2}$$

$$[J_i, P_j] = i\epsilon_{ijk} P_k , \tag{A3}$$

$$[J_i, K_j] = i\epsilon_{ijk} K_k , \tag{A4}$$

$$[K_i, P_j] = i\delta_{ij} H , \tag{A5}$$

$$[K_i, K_j] = -i\epsilon_{ijk} J_k - x_i G_j + x_j G_i , \tag{A6}$$

where the DKP Poincaré generators are given by Eqs. (3.6)–(3.9) and G_i is given by Eq. (3.16). Next, note that both the particle and subsidiary components of an operator $\mathcal{O}$ have the same form:

$$\mathcal{O}^{(\xi)} = \mathcal{G}_\xi \mathcal{O}(1 + X_\xi)\mathcal{G}_\xi , \tag{A7}$$

where $\xi = P, S$,

$$\mathcal{G}_\xi = \begin{cases} \beta_4^{\,2} , & \xi = P \\ (1 - \beta_4^{\,2}), & \xi = S \end{cases} \tag{A8}$$

$$X_\xi = \begin{cases} -i(\vec{p}\cdot\vec{\beta})m^{-1} , & \xi = P \\ +i\,\dfrac{(m + \beta_4 E_p)}{p^2}(\vec{p}\cdot\vec{\beta}), & \xi = S . \end{cases} \tag{A9}$$

Thus, if $\mathcal{O}_1$ and $\mathcal{O}_2$ are any two DKP Poincaré generators, then the ξ components of the commutator of $\mathcal{O}_1$ with $\mathcal{O}_2$ can be written as

$$[\mathcal{O}_1, \mathcal{O}_2]^{(\xi)} = [\mathcal{O}_1^{(\xi)}, \mathcal{O}_2^{(\xi)}] - \{\mathcal{O}_1, [\mathcal{g}_\xi, \mathcal{O}_2]\}^{(\xi)} - \{\mathcal{O}_1, [X_\xi \mathcal{g}_\xi, \mathcal{O}_2]\}^{(\xi)}. \tag{A10}$$

Also, by interchanging 1 and 2 and reversing the order of the commutators in 1 and 2,

$$[\mathcal{O}_1, \mathcal{O}_2]^{(\xi)} = [\mathcal{O}_1^{(\xi)}, \mathcal{O}_2^{(\xi)}] + \{\mathcal{O}_2, [\mathcal{g}_\xi, \mathcal{O}_1]\}^{(\xi)} + \{\mathcal{O}_2, [X_\xi \mathcal{g}_\xi, \mathcal{O}_1]\}^{(\xi)}. \tag{A11}$$

What is interesting about Eqs. (A10) and (A11) is that if $\mathcal{O}_1$ or $\mathcal{O}_2$ are such that either one satisfies the conditions

$$[\mathcal{g}_\xi, \mathcal{O}_r] = [X_\xi \mathcal{g}_\xi, \mathcal{O}_r] = 0, \quad r = 1 \text{ or } 2 \tag{A12a}$$

then

$$[\mathcal{O}_1, \mathcal{O}_2]^{(\xi)} = [\mathcal{O}_1^{(\xi)}, \mathcal{O}_2^{(\xi)}], \tag{A12b}$$

and the DKP commutator between $\mathcal{O}_1$ and $\mathcal{O}_2$ trivially splits into two disjoint pieces, $\xi = P$ and $\xi = S$.

Now look at the first five independent commutation relations given in Eqs. (A1)–(A5). Note that each contains either $\vec{P}$ or $\vec{J}$ in the commutators. But both $\vec{P}$ and $\vec{J}$ satisfy Eq. (A12a). (For $\vec{P}$ this statement is obvious. For $\vec{J}$, if it is not immediately clear, note that the X_ξ contain the scalar products $\vec{p}\cdot\vec{p}$ and $\vec{p}\cdot\vec{\beta}$, so that either directly or using the commutation relations, the result holds.) Therefore, regardless of the ultimate form of the generators $P_i^{(\xi)}$, $J_i^{(\xi)}$, $H^{(\xi)}$, and $K_i^{(\xi)}$, one has by inspection that both the particle and subsidiary components of the DKP Poincaré generators satisfy the first five independent Poincaré commutation relations algebraically. That is,

$$[P_i^{(\xi)}, P_j^{(\xi)}] = 0, \tag{A13}$$

$$[J_i^{(\xi)}, J_j^{(\xi)}] = i\epsilon_{ijk} J_k^{(\xi)}, \tag{A14}$$

$$[J_i^{(\xi)}, P_j^{(\xi)}] = i\epsilon_{ijk} P_k^{(\xi)}, \tag{A15}$$

$$[J_i^{(\xi)}, K_j^{(\xi)}] = i\epsilon_{ijk} K_k^{(\xi)}, \tag{A16}$$

$$[K_i^{(\xi)}, P_j^{(\xi)}] = i\delta_{ij} H^{(\xi)}, \tag{A17}$$

where $\xi = P$ or S.

The last independent commutation relation, Eq. (A6), will not be as easy to work since $\vec{K}$ does not satisfy Eq. (A12a).

Consider Eq. (A6) for $\xi = P$. The particle components of this equation are

$$[K_i, K_j]^{(P)} = -i\epsilon_{ijk} J_k^{(P)} - x_i G_j^{(P)} + x_j G_i^{(P)}. \tag{A18}$$

But from Eqs. (3.16) and (4.10) it is easy to show that

$$G_i^{(P)} = 0. \tag{A19}$$

It is also easy to show that

$$[\mathcal{g}_P, K_i] = i\{(\vec{p}\cdot\vec{\beta}), \beta_4\} x_i, \tag{A20}$$

$$[X_P \mathcal{g}_P, K_i] = -i(\vec{p}\cdot\vec{\beta})\beta_4 x_i - \beta_4(\vec{p}\cdot\vec{\beta}) x_i (\vec{p}\cdot\vec{\beta}) m^{-1}, \tag{A21}$$

and thus it is straightforward to prove that

$$\{K_j, [\mathcal{g}_P, K_i]\}^{(P)} + \{K_j, [X_P \mathcal{g}_P, K_i]\}^{(P)} = 0. \tag{A22}$$

Consequently, by Eq. (A11), the last independent commutation relation for $\xi = P$ becomes

$$[K_i^{(P)}, K_j^{(P)}] = -i\epsilon_{ijk} J_k^{(P)}. \tag{A23}$$

This implies that the particle-components generators defined in Eqs. (4.11)–(4.14) satisfy the entire Lie algebra of the Poincaré group *algebraically*.

Now consider Eq. (A6) for $\xi = S$. The subsidiary components of this equation are

$$[K_i, K_j]^{(S)} = -i\epsilon_{ijk} J_k^{(S)} - x_i G_j^{(S)} + x_j G_i^{(S)}. \tag{A24}$$

From Eqs. (3.16) and (4.15) one finds that

$$G_i^{(S)} = i[(\vec{p}\cdot\vec{\beta})\beta_i - p_i](1 - \beta_4^2). \tag{A25}$$

Also, with

$$[\mathcal{g}_S, K_i] = -i\{(\vec{p}\cdot\vec{\beta}), \beta_4\} x_i, \tag{A26}$$

$$[X_S \mathcal{g}_S, K_i] = \frac{(m + \beta_4 E_p)}{p^2}(\vec{p}\cdot\vec{\beta})(\vec{p}\cdot\vec{\beta})\beta_4 x_i - (\vec{p}\cdot\vec{\beta})\beta_4 x_i \frac{(m + \beta_4 E_p)}{p^2}(\vec{p}\cdot\vec{\beta}) - im\beta_4 x_i \frac{(m + \beta_4 E_p)}{p^2}(\vec{p}\cdot\vec{\beta}), \tag{A27}$$

one can eventually show that

$$\{K_j, [\mathcal{g}_S, K_i]\}^{(S)} + \{K_j, [X_S \mathcal{g}_S, K_i]\}^{(S)} = i(\vec{p}\cdot\vec{\beta}) x_j \frac{(\vec{p}\cdot\vec{\beta})}{p^2}[(\vec{p}\cdot\vec{\beta})\beta_i - p_i](1 - \beta_4^2) - (i \leftrightarrow j). \tag{A28}$$

Equations (A11), (A24), (A25), and (A28) then yield

$$[K_i^{(S)}, K_j^{(S)}] = -i\epsilon_{ijk} J_k^{(S)} + \left[(\vec{p}\cdot\vec{\beta}) x_i \frac{(\vec{p}\cdot\vec{\beta})}{p^2} - x_i\right] G_j^{(S)} - \left[(\vec{p}\cdot\vec{\beta}) x_j \frac{(\vec{p}\cdot\vec{\beta})}{p^2} - x_j\right] G_i^{(S)}. \tag{A29}$$

Thus, the subsidiary-components generators defined in Eqs. (4.16)–(4.19) do not satisfy the Lie algebra of the Poincaré group algebraically but, as we will see momentarily, only as an operator equation.

Before continuing, note that in this case the commutator of $K_i^{(S)}$ and $H^{(S)}$ can be derived from Eq. (A29) in exactly the same way[11] that it is derived in the absence of the additional terms depending upon $G_i^{(S)}$: Form the commutator of Eq. (A29) with $P_k^{(S)}$ and use the Jacobi identity with Eqs. (A15) and (A17) to yield

$$\left\{[K_i^{(S)},H^{(S)}] - iP_i^{(S)} + \left[\frac{(\vec{p}\cdot\vec{\beta})(\vec{p}\cdot\vec{\beta})}{p^2} - 1\right] G_i^{(S)}\right\}\delta_{jk} = \left\{[K_j^{(S)},H^{(S)}] - iP_j^{(S)} + \left[\frac{(\vec{p}\cdot\vec{\beta})(\vec{p}\cdot\vec{\beta})}{p^2} - 1\right] G_j^{(S)}\right\}\delta_{ik}. \quad \text{(A30)}$$

Since i, j, and k are arbitrary,

$$[K_i^{(S)},H^{(S)}] = iP_i^{(S)} - \left[\frac{(\vec{p}\cdot\vec{\beta})(\vec{p}\cdot\vec{\beta})}{p^2} - 1\right] G_i^{(S)}. \quad \text{(A31)}$$

Up to this point we have seen that the subsidiary-components independent commutation relation (A29) which corresponds to (1.4i) and the dependent commutation relation (A31) which corresponds to (1.4h) are not satisfied algebraically. However, we will now show that they are satisfied as operator equations on the fields by finding the subsidiary components of the consequent equations (3.12). For $\lambda = 4$, Eq. (3.12) reduces to

$$-\partial_4^{(S)}\psi_S = E_p(\vec{p}\cdot\vec{\beta})(\vec{p}\cdot\vec{\beta})p^{-2}(1-\beta_4{}^2)\psi_S$$
$$= H^{(S)}\psi_S, \quad \text{(A32)}$$

$$\psi_S \equiv (1-\beta_4{}^2)\psi, \quad \text{(A33)}$$

while for $\lambda = i = 1, 2, 3$, it yields

$$P_i^{(S)}\psi_S = (\vec{p}\cdot\vec{\beta})\beta_i(1-\beta_4{}^2)\psi_S, \quad \text{(A34)}$$

or

$$G_i^{(S)}\psi_S = 0. \quad \text{(A35)}$$

Putting Eq. (A35) into Eqs. (A31) and (A29) immediately yields the predicted results

$$[K_i^{(S)},H^{(S)}]\psi_S = iP_i^{(S)}\psi_S, \quad \text{(A36)}$$

$$[K_i^{(S)},K_j^{(S)}]\psi_S = -i\epsilon_{ijk}J_k^{(S)}\psi_S. \quad \text{(A37)}$$

The particle components of the consequent equations (3.12) are also of interest. They reduce to the identity

$$\partial_\lambda^{(P)}\psi_P = \partial_\lambda^{(P)}\psi_P, \quad \lambda = 1, 2, 3, 4 \quad \text{(A38)}$$

$$\psi_P \equiv \beta_4{}^2\psi, \quad \text{(A39)}$$

where $-\partial_4^{(P)}$ has been replaced by $H^{(P)}$ of Eq. (4.13) in order to simplify the results for $\lambda = 1, 2, 3$. Thus, the particle components are not burdened by a consequent equation, so that one could have expected that the particle-components generators would satisfy the Poincaré commutation relations algebraically.

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