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Bhabha first-order wave equations. II. Mass and spin composition, Hamiltonians, and general Sakata-Taketani reductions*

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Beginning with the Bhabha first-order wave equation of maximum spin 1 [the Duffin-Kemmer-Petiau (DKP) equation], where Sakata and Taketani (ST) separated out the "particle components" from the built in "subsidiary components," we derive for the first time the Hamiltonian equation for the "subsidiary components," and show that its solution is an identity in terms of the particle-components solution. We then derive a set of general inverse and ST operators for arbitrary-spin Bhabha fields. With these generalized operators we can discuss and understand the mass and spin composition of a general Bhabha so(5) field, it being a particular sum of $[2\times(2S+1)]$ components for each particular mass and spin (S) state, as well as built in "subsidiary components" for integer spin. We then can use these general inverse and ST operators to (a) derive the general Bhabha Hamiltonian for arbitrary spin, (b) decouple the "particle components" from the "subsidiary components" in the Hamiltonian equations for integer spin (where, as was the case for DKP, we find that the Hamiltonian "subsidiary components" solution is an identity in terms of the particle-components solution), and (c) decouple the $\mathcal{S}+\frac{1}{2}$ ($\mathcal{S}$) different mass states for half-integer (integer) spin. We discuss the physical implications of this observation and other aspects of our results.

I. BACKGROUND AND INTRODUCTION

The Bhabha[1,2] first-order wave equations for particles of arbitrary spin are given by

$$(\partial\cdot\alpha+\chi)\psi=0\ , \tag{1.1}$$

where χ is either an integer or half-integer multiple of the mass, and the α_μ satisfy the algebra (with unity I added by hand for integer spin)

$$[[\alpha_\mu,\alpha_\nu],\alpha_\lambda]=\alpha_\mu\delta_{\nu\lambda}-\alpha_\nu\delta_{\mu\lambda},\quad \mu,\nu,\lambda=1,2,3,4\ . \tag{1.2}$$

[Our α_μ matrices will be self-adjoint, we will use the metric $\delta_{\mu\nu}$ relating four-vector quantities $x_\mu=(\vec{x}, ix_0)$, and $\partial\cdot\alpha\equiv\partial_\lambda\alpha_\lambda$.] The α_μ can be connected to the algebra so(5) by the identification[1-3]

$$\alpha_\mu=J_{\mu5}=-J_{5\mu},\quad J_{\mu\nu}=-i[\alpha_\mu,\alpha_\nu],\quad J_{55}=0\ , \tag{1.3a}$$

$$[J_{ab},J_{cd}]=i(\delta_{ac}J_{bd}+\delta_{bd}J_{ac}-\delta_{bc}J_{ad}-\delta_{ad}J_{bc})\ , \tag{1.3b}$$

$$J_{ab}=-J_{ba},\quad a,b,=1,2,3,4,5 \tag{1.3c}$$

meaning that the irreducible representations of the α_μ algebras have dimensions $d_5(\mathcal{S},S)$ labeled by two numbers, $\mathcal{S}$ and S both integer or half-integer, such that

$$\mathcal{S}\geq S\geq 0\ , \tag{1.4a}$$

$$d_5(\mathcal{S},S)=\tfrac{1}{6}(2\mathcal{S}+3)(2S+1)\times[(\mathcal{S}+1)(\mathcal{S}+2)-S(S+1)]\ . \tag{1.4b}$$

The above combined with the Cayley-Hamilton theorem implies that the α_μ satisfy the characteristic equation

$$\prod_{n=-\mathcal{S}}^{\mathcal{S}}(\alpha_\mu-nI)=0\ . \tag{1.5}$$

In Eq. (1.5), the unity operator I technically must be added by hand for integer-spin representations.[3] Having noted this, we will use I and 1 interchangeably.

In the special cases of $\mathcal{S}=\frac{1}{2}$ and 1 the above system reduces to the Dirac and Duffin-Kemmer-Petiau (DKP)[4,5] first-order wave equations. For the case $\mathcal{S}=\frac{1}{2}$ one has $d_5(\frac{1}{2},\frac{1}{2})=4$, $\alpha_\mu=\frac{1}{2}\gamma_\mu$, $\chi=\frac{1}{2}m$, and the characteristic equation is the well-known relation

$$\alpha_\mu{}^2-\tfrac{1}{4}\equiv\tfrac{1}{4}\gamma_\mu{}^2-\tfrac{1}{4}=0\ . \tag{1.6}$$

Combined with (1.2) this gives the Dirac algebra

$$\gamma_\mu\gamma_\nu+\gamma_\nu\gamma_\mu=2\delta_{\mu\nu}\ . \tag{1.7}$$

For the case $\mathcal{S}=1$ one has $d_5(1,1)=10$ and $d_5(1,0)=5$, i.e., the spin (S) 1 and 0 representations of the DKP equation, $\alpha_\mu=\beta_\mu$, $\chi=m$, and the characteristic equation is the well-known DKP relation

$$\alpha_\mu(\alpha_\mu{}^2-1)\equiv\beta_\mu(\beta_\mu{}^2-1)=0\ . \tag{1.8}$$

Equation (1.8) combined with (1.2) gives the DKP algebra

$$\beta_\mu\beta_\nu\beta_\lambda+\beta_\lambda\beta_\nu\beta_\mu=\beta_\lambda\delta_{\mu\nu}+\beta_\mu\delta_{\nu\lambda}\ . \tag{1.9}$$

Before going on, note that the Dirac equation has $[(2S+1)\times2$ (for particle-antiparticle)$]$ components. Further, the members of the algebra, γ_μ, have inverses (themselves), so that one can easily form the Hamiltonian equation, including the minimal electromagnetic substitution

$$\partial_\mu \to \partial_\mu^\pm = \partial_\mu \pm ieA_\mu \ , \tag{1.10}$$

as

$$-\partial_4\psi = H\psi = E\psi = [\gamma_4(\vec{\gamma}\cdot\vec{\partial}^- + m) + eA_0]\psi \ . \tag{1.11}$$

However, the DKP equation has more than $[2(2S+1)]$ components in both the $S=1$ and 0 representations. These remaining components are built in subsidiary conditions which, however, do not have to be put in externally as with other high-spin formalisms. Also, because the DKP characteristic equation (1.8) has some eigenvalues which are zero, the β_μ do not have inverses, so that the Hamiltonian equation must be obtained by using the fourth component of the "consequent equations" [Eq. (5.6) below with the minimal substitution (1.10)] to yield[5]

$$-\partial_4\psi = E\psi = H\psi \ , \tag{1.12}$$

$$H = \frac{1}{i}\vec{\partial}^-\cdot\left(\frac{\vec{\beta}\beta_4 - \beta_4\vec{\beta}}{i}\right) + m\beta_4 + eA_0 - \frac{ie}{2m}F_{\nu\rho}(\beta_\rho\beta_4\beta_\nu - \delta_{\rho 4}\beta_\nu) \ , \tag{1.13}$$

$$[\partial_\mu^-, \partial_\nu^-] = -ieF_{\mu\nu} \ . \tag{1.14}$$

Note that we also could have derived the Hamiltonian equation (1.13) by using the "first decoupling equation," Eq. (4.14) of paper I,

$$(\vec{\partial}^-\cdot\vec{\beta})\,\beta_4{}^2\psi + m(1-\beta_4{}^2)\psi = 0 \ . \tag{1.15}$$

In fact, in Sec. V we will use a combination of both methods to obtain the Hamiltonians for higher integer spins.

For $\mathcal{S} > 1$ the situation becomes quickly more complicated. First, from the characteristic equation, one sees that the algebra involves products of α_μ up to order $(2\mathcal{S}+1)$. (The explicit algebras for $\mathcal{S} = \frac{3}{2}$ and 2 were derived by Madhava Rao.[6]) Further, from (1.4a) each $\mathcal{S}$ algebra contains spin representations of $S = \mathcal{S}, \mathcal{S}-1, \mathcal{S}-2, \ldots$ ($\frac{1}{2}$ or 0 as $\mathcal{S}$ is a half-integer or an integer). Finally, by inserting (1.1) into $\partial_4{}^{2\mathcal{S}+1}$ times Eq. (1.5) taken in the rest frame, one can see that for $\mathcal{S} > 1$ the free Bhabha equation will no longer satisfy a single-mass-value Klein-Gordon (KG) equation, but rather will actually satisfy[1] the following equations.

$\mathcal{S}$ = integer:

$$0 = \chi[\Box - \chi^2][4\Box - \chi^2]\cdots \times[(\mathcal{S}-1)^2\Box - \chi^2][\mathcal{S}^2\Box - \chi^2]\psi \ ; \tag{1.16}$$

$\mathcal{S}$ = half-integer:

$$0 = [\tfrac{1}{4}\Box - \chi^2][\tfrac{9}{4}\Box - \chi^2]\cdots \times[(\mathcal{S}-1)^2\Box - \chi^2][\mathcal{S}^2\Box - \chi^2]\psi \ . \tag{1.17}$$

That is, for $\mathcal{S} > 1$ the Bhabha system has multiple-mass solutions. For example, when $\mathcal{S}$ is $\frac{3}{2}$ or 2 one has

$$\chi = \tfrac{3}{2}m, \tfrac{1}{2}m, \quad \mathcal{S} = \tfrac{3}{2} \tag{1.18}$$

$$\chi = 2m, m, \quad \mathcal{S} = 2 \ . \tag{1.19}$$

[The exponential part of ψ for a particular mass state is[2] $e^{ip(j)\cdot x}$, where $p(j)\cdot p(j) = -\chi^2/j^2$.]

Returning temporarily to the DKP system, Sakata and Taketani (ST)[7,8] observed that the Hamiltonian formulation could be decoupled into two separate equations by applying the operators $\mathcal{J}$ and $(1-\mathcal{J})$, where

$$\mathcal{J} \equiv (\beta_4{}^2) = (\mathcal{J})^2 \ , \tag{1.20a}$$

$$(1-\mathcal{J}) \equiv (1-\beta_4{}^2) = (1-\mathcal{J})^2 \ , \tag{1.20b}$$

$$\mathcal{J}(1-\mathcal{J}) = (1-\mathcal{J})\mathcal{J} = 0 \ , \tag{1.20c}$$

$$I = \mathcal{J} + (1-\mathcal{J}) \ , \tag{1.20d}$$

to the DKP Hamiltonian equation. That is, by writing

$$E \equiv E(\mathcal{J}) + E(1-\mathcal{J}) \ , \tag{1.21}$$

$$H \equiv \mathcal{J}H\mathcal{J} + \mathcal{J}H(1-\mathcal{J}) + (1-\mathcal{J})H\mathcal{J} + (1-\mathcal{J})H(1-\mathcal{J}) \ , \tag{1.22}$$

one can rewrite the Hamiltonian equation as

$$E[\mathcal{J}\psi] = [\mathcal{J}H\mathcal{J} + \mathcal{J}H(1-\mathcal{J})]\psi \ , \tag{1.23}$$

$$E[(1-\mathcal{J})\psi] = [(1-\mathcal{J})H\mathcal{J} + (1-\mathcal{J})H(1-\mathcal{J})]\psi \ . \tag{1.24}$$

Now by using the first decoupling Eq. (1.15) and a second decoupling equation [Eq. (2.13) below], it is in principle possible to decouple the two above equations by writing

$$\mathcal{J}H(1-\mathcal{J})\psi = \mathcal{J}\mathcal{O}_1\mathcal{J}[\mathcal{J}\psi] \ , \tag{1.25}$$

$$(1-\mathcal{J})H\mathcal{J}\psi = (1-\mathcal{J})\mathcal{O}_2(1-\mathcal{J})[(1-\mathcal{J})\psi] \ , \tag{1.26}$$

where $\mathcal{O}_1$ and $\mathcal{O}_2$ are to be determined. This is a Peirce decomposition.[9] The $[(2S+1)\times 2]$ "particle components" would be projected out by $\mathcal{J}$. The remaining "subsidiary components" would be projected out by $(1-\mathcal{J})$.

Since the number of particle components is 6 and 2 for the spin-1 and -0 representations, the subsidiary equations contain 4 and 3 components. Physical analogies to these components can be made. For spin 1 consider the massive-photon[10] Proca equation. There the six particle components are proportional to mixtures of the electric and vector potential fields $\vec{E}$ and $\vec{A}$, while the four subsidiary components are proportional to the magnetic field $\vec{B}$ and the electric potential V. (See Refs. 11 and 12 for the explicit DKP spin-1 coupled solution. Reference 12 is paper I of this series, which discusses C, P, and T for Bhabha fields.) For the spin-0 case the particle components[8, 13] are proportional to mixtures of a Klein-Gordon (KG) field and its time derivative, while the subsidiary components[8] are the space derivatives of the KG field.

Sakata and Taketani[7, 8] explicitly obtained the particle-components equation projected out by $\mathcal{G}$. However, neither they nor, to our knowledge, anyone else ever projected out the subsidiary-components equation. This is a more difficult procedure, and will be done in Sec. II. We will find, surprisingly, that even though this second equation is necessary to have the ST system be manifestly covariant, the solution to the subsidiary components will turn out to be an identity in terms of the particle components. We will comment on the physical implications of this.

In Sec. III we will derive some important algebraic relationships concerning general inverse and Sakata-Taketani operators. These relationships are necessary for obtaining the general Bhabha Hamiltonians and for obtaining the generalization of the Sakata-Taketani reduction in the general Bhabha case.

In Sec. IV we will describe the reduction of an arbitrary Bhabha equation into all of its mass and spin states. This will show that the general ST operators obtained in Sec. III are exactly mass-state projection operators in the limit $\vec{p}=0$. For $\vec{p}\neq 0$, the general Bhabha equation will mix up the mass states contained in a particular Bhabha field. Then one uses the general ST operators to decouple the mass states by a Peirce decomposition,[9] as was done by Sakata and Taketani[7, 8] for the DKP case.

In Sec. V we will proceed to obtain the general Bhabha Hamiltonians for arbitrary $\mathcal{S}$. Having these, we can describe the general Sakata-Taketani reduction for an arbitrary Bhabha field in Sec. VI. This will include the division into particle and subsidiary components for general integer spin, and the decoupling into specific mass states for both integer and half-integer spin.

We will conclude in Sec. VII with a short discussion of our results. This will include a comparision of the general ST Bhabha reduction with the particular ST reduction for the DKP case $\mathcal{S}=1$. We will also comment on ST reductions for the Harish-Chandra modification[14, 15] of the Bhabha equations, where the α_μ, instead of satisfying the so(5) characteristic equation (1.5), are forced to satisfy

$$\alpha_\mu{}^{2\mathcal{S}-1}(1-\alpha_\mu{}^2)=0 \ . \tag{1.27}$$

In paper III of this series,[16] we will explicitly give the Poincaré generators $\vec{P}$, $\vec{J}$, $\vec{K}$, and H for the general Bhabha case of arbitrary spin, and then explicitly verify that these generators satisfy the commutation relations of the Poincaré group. Interestingly, the commutation relations are satisfied algebraically for half-integer-spin Bhabha fields, but are only satisfied as an operator algebra on the Bhabha fields themselves for integer-spin representations.

We simply note here that the general Bhabha Hamiltonians and general ST operators developed in this paper are necessary to derive the results in paper III.

II. ST PARTICLE AND SUBSIDIARY REDUCTIONS OF THE DKP SYSTEM

A. Particle-components equation

As discussed in Eqs. (1.20)–(1.26), the key to obtaining the particle components of the ST reduction, i.e., Eq. (1.23), is to find the operator $\mathcal{O}_1$ of Eq. (1.25), where H is the DKP Hamiltonian of Eq. (1.13). For DKP, $\mathcal{O}_1$ is found by inserting the "first decoupling equation," Eq. (1.15), into the left-hand side of Eq. (1.25), so that[17]

$$\mathcal{G}H(1-\mathcal{G})\psi=-m^{-1}\mathcal{G}H\vec{\partial}^-\cdot\vec{\beta}\mathcal{G}[\mathcal{G}\psi] \ . \tag{2.1}$$

Inserting this into Eq. (1.23) gives us the result

$$E[\mathcal{G}\psi]=\mathcal{H}_P[\mathcal{G}\psi] \ , \tag{2.2}$$

$$\mathcal{H}_P=\mathcal{G}[\mathcal{H}-m^{-1}\mathcal{H}\vec{\beta}\cdot\vec{\partial}^-]\mathcal{G} \tag{2.3}$$

$$=m\beta_4+eA_0\mathcal{G}-m^{-1}\beta_4\beta_k\beta_i\partial_k^-\partial_i^- \tag{2.4}$$

$$=m\beta_4+eA_0\mathcal{G}-\beta_4\left(\frac{1+\eta}{2}\right)m^{-1}\vec{\partial}^-\cdot\vec{\partial}^- +\beta_4\eta m^{-1}(\vec{S}\cdot\vec{\partial}^-)^2-\beta_4\left(\frac{1+\eta}{2}\right)\frac{e}{m}(\vec{S}\cdot\vec{B}), \tag{2.5}$$

where, from (1.14), $\vec{B}$ is the magnetic field, and the spin $\vec{S}$ and η are given by

$$S_i\equiv-i\epsilon_{ijk}\beta_j\beta_k \ , \tag{2.6}$$

$$\eta\equiv\eta_1\eta_2\eta_3, \quad \eta_\lambda\equiv 2\beta_\lambda{}^2-1 \ . \tag{2.7}$$

In obtaining Eq. (2.5) from (2.4), the following relations are helpful:

$$\beta_i{}^2=\tfrac{1}{2}(1+\eta)-\eta S_i{}^2 \ , \tag{2.8}$$

$$(\beta_i\beta_j+\beta_j\beta_i)=\eta_k(S_iS_j+S_jS_i), \quad i,j,k, \text{ cyclic} \tag{2.9a}$$

$$=-\eta(S_iS_j+S_jS_i) \ . \tag{2.9b}$$

The final form of the ST particle components comes from observing that among themselves the surrounded operators

$$\mathcal{I}(\mathcal{I})\mathcal{I} \sim 1, \quad \mathcal{I}(-i\beta_4)\mathcal{I}\mathcal{I}(\eta)\mathcal{I} \sim \tau_y ,$$
$$\mathcal{I}(\eta)\mathcal{I} \sim \tau_x, \quad \mathcal{I}(\beta_4)\mathcal{I} \sim \tau_z \tag{2.10}$$

form a Pauli algebra and that this algebra commutes with the surrounded spin algebra. Thus, one has

$$\psi_P^{\mathrm{ST}} \equiv \mathcal{I}\psi , \tag{2.11}$$

$$E\psi_P^{\mathrm{ST}} = \mathcal{H}_P \psi_P^{\mathrm{ST}} , \tag{2.12}$$

$$\mathcal{H}_P = m\tau_z + eA_0 - (\tau_z + i\tau_y)(\vec{\partial}^- \cdot \vec{\partial}^- + e\vec{S}\cdot\vec{\beta})(2m)^{-1} + i\tau_y(\vec{S}\cdot\vec{\partial}^-)^2 m^{-1} . \tag{2.13}$$

B. Subsidiary-components equation

Obtaining the subsidiary-components equation is more complicated. From Eq. (1.24) one wants to calculate the operator $\mathcal{O}_2$ of Eq. (1.26). To do this, first multiply the DKP equation

$$(\partial^- \cdot \beta + m)\psi = 0 \tag{2.14}$$

by $(\partial_4^- \beta_4 - m)\beta_4^2$ and rearrange terms to yield

$$\beta_4^2\psi = \mathcal{I}\psi \equiv Y(1-\mathcal{I})\psi , \tag{2.15a}$$

$$Y = [(\partial_4^-)^2 - m^2]^{-1}(-\partial_4^-\beta_4 + m)(\vec{\partial}^- \cdot \vec{\beta}) , \tag{2.15b}$$

$$[(\partial_4^-)^2 - m^2]\psi \neq 0 , \tag{2.15c}$$

i.e., the "second decoupling equation." For reference, the first decoupling equation (1.15) can be put in the same form,

$$(1-\beta_4^2)\psi = (1-\mathcal{I})\psi \equiv X\mathcal{I}\psi , \tag{2.16a}$$

$$X = -m^{-1}(\vec{\partial}^- \cdot \vec{\beta}) . \tag{2.16b}$$

Putting (2.15) into (1.24) and (1.26) gives us the subsidiary-components Hamiltonian equation

$$\mathcal{H}_S(1-\mathcal{I})\psi = E(1-\mathcal{I})\psi , \tag{2.17a}$$

$$\mathcal{H}_S = (1-\mathcal{I})H[1+Y](1-\mathcal{I}) , \tag{2.17b}$$

where H is given by Eq. (1.13). The subsidiary-components Hamiltonian $\mathcal{H}_S$ cannot be considered a Hamiltonian in the ordinary sense since it envolves the time derivative (or E) explicitly. The solution to Eq. (2.17) must therefore be considered to be an identity in terms of the solution to the particle-components Hamiltonian equation (2.12). What has happened is that all of the physics has been transferred into the particle components. Be that as it may, one can obtain a fair amount of physical insight by studying this equation in detail.

First note that the operator Y of Eq. (2.15b), which couples the particle components to the subsidiary components, goes to infinity as $1/|\vec{p}^-|$ as $\vec{p}^-$ goes to zero. On the other hand, the operator X, which couples the subsidiary components to the particle components, goes to zero like $|\vec{p}^-|$ as $\vec{p}^-$ goes to zero. Therefore, the product XY is finite, a fact which is clearest in the free case when the product XY occurs in the Hamiltonian as

$$\mathcal{H}_S(A_\lambda = 0) = EXY(1-\mathcal{I}) = \left[\frac{-E(\vec{\partial}\cdot\vec{\beta})(\vec{\partial}\cdot\vec{\beta})}{\vec{p}\cdot\vec{p}}\right](1-\mathcal{I}) . \tag{2.18a}$$

In fact, one can also show in the free case that

$$XY(1-\mathcal{I})\psi = (1-\mathcal{I})\psi , \tag{2.18b}$$

which means that Eq. (2.18a) is a manifestation that the subsidiary-components solution is an identity in terms of the particle-components solution.

The decoupling equations show that in the limit $\vec{p}\to 0$, the subsidiary components are automatically decoupled from the particle components. Further, since the coupling is proportional to $|\vec{p}^{\pm 1}|$, the coupling of the particle states of mass m with the subsidiary components of mass ∞ (see Sec. III B) is singular as $\vec{p}\to 0$.

However, if one had chosen to try to determine $(1-\mathcal{I})H\mathcal{I}$ in Eq. (1.24) as being the operator $(1-\mathcal{I})\mathcal{O}_2(1-\mathcal{I})$ in Eq. (1.26) containing no time derivatives, and in a manner which did not use the singular operator, one would not succeed. The best one could do would be to determine, after a great deal of algebra, that

$$\mathcal{H}_S \to \hat{\mathcal{H}}_S = (1-\mathcal{I})(E+h)(1-\mathcal{I}) , \tag{2.19a}$$

$$0 \equiv (1-\mathcal{I})h(1-\mathcal{I}) . \tag{2.19b}$$

Thus, the physics would still be the same (a solution in terms of the particle-components solution, but not as transparent). The preferability of the Eq. (2.17) viewpoint is that the free-case Hamiltonian (2.18a) will turn out to be the formally correct Poincaré Hamiltonian generator needed to satisfy the Poincaré commutation relations for the subsidiary components [even though by Eq. (2.18a) this Hamiltonian is an identity in terms of the particle-components solution]. We will show this in paper III.[16]

Finally we mention that limited though the interpretation of this subsidiary-components Hamiltonian was for the DKP case, even this will not be possible for the subsidiary components of Bhabha integer-spin fields when $\mathcal{S} > 1$. There the decoupling equations will always involve at least two mass states in such a way that an interpretation such as Eq. (2.17b) will not be possible. For $\mathcal{S} > 1$, there will be no subsidiary components Hamiltonian in the sense of Eq. (2.17b), only in the sense of Eq. (2.19).

III. GENERAL INVERSE AND SAKATA-TAKETANI OPERATORS

A. Inverses to α_μ

As mentioned in Sec. I, the DKP representation of the general Bhabha algebra is such that the β_μ do not have inverses. This is bothersome because it means the Hamiltonian equation cannot be formed by directly multiplying the wave equation by $(\beta_4)^{-1}$. Rather, one must use the algebra to end up with a Hamiltonian equation (1.12) accompanied by a decoupling equation (1.15). The reason the β_μ do not have inverses is clearly evident from Eq. (1.5): The α_μ have eigenvalues $\mathcal{S}, \mathcal{S}-1, \ldots, -\mathcal{S}+1, -\mathcal{S}$, which for $\mathcal{S}$ an integer includes the eigenvalue 0. However, the eigenvalue 0 is not included for half-integer $\mathcal{S}$. Thus, for half-integer $\mathcal{S}$, like the Dirac case, the α_μ have inverses.

For the half-integer case we can calculate these inverses by writing (1.5) in the form

$$0=[\alpha_\mu{}^2-\mathcal{S}^2][\alpha_\mu{}^2-(\mathcal{S}-1)^2]\cdots[\alpha_\mu{}^2-(\tfrac{3}{2})^2][\alpha_\mu{}^2-(\tfrac{1}{2})^2] \tag{3.1}$$

$$=(\alpha_\mu)^{2\mathcal{S}+1}+\sum_{k=1}^{\mathcal{S}-1/2}(-1)^k(\alpha_\mu)^{2\mathcal{S}+1-2k}\Bigg[\sum_{n_1>n_2>\cdots>n_k\geq 1}^{\mathcal{S}+1/2}S^2(n_1)S^2(n_2)\cdots S^2(n_k)\Bigg]+(-1)^{\mathcal{S}+1/2}\prod_{n=1}^{\mathcal{S}+1/2}S^2(n)I\ , \tag{3.2}$$

$$S(n)\equiv(n-\tfrac{1}{2})\ . \tag{3.3}$$

The sums of products of the $S^2(n_k)$ in Eq. (3.2) can be understood as simply being the sum of all the products of the (S^2)'s in Eq. (3.1), taking k at a time, multiplying the factor $(\alpha_\mu)^{2\mathcal{S}+1-2k}$ in the expansion of (3.1); that is, they are the elementary symmetric functions. Since we want to find the inverse $(\alpha_\mu)^{-1}$ such that

$$(\alpha_\mu)^{-1}\alpha_\mu=\alpha_\mu(\alpha_\mu)^{-1}=I\ , \tag{3.4}$$

we now take the last term on the right of Eq. (3.2), put it on the left, divide out the numerical factor multiplying I, multiply the resulting equation by $(\alpha_\mu)^{-1}$, and find

$$(\alpha_\mu)^{-1}=\left[\frac{(-1)^{\mathcal{S}-1/2}\,2^{2\mathcal{S}+1}}{[(2\mathcal{S})!!]^2}\right]\alpha_\mu\left\{(\alpha_\mu)^{2\mathcal{S}-1}+\sum_{k=1}^{\mathcal{S}-1/2}(-1)^k(\alpha_\mu)^{2\mathcal{S}-1-2k}\Bigg[\sum_{n_1>n_2>\cdots n_k\geq 1}^{\mathcal{S}+1/2}S^2(n_1)S^2(n_2)\cdots S^2(n_k)\Bigg]\right\}\ , \tag{3.5}$$

where only the first term is to be used for the case $\mathcal{S}=\frac{1}{2}$. Specific examples are

$$\begin{aligned}
\mathcal{S}&=\tfrac{1}{2}:\quad \alpha_\mu{}^{-1}=4\alpha_\mu\ ,\\
\mathcal{S}&=\tfrac{3}{2}:\quad \alpha_\mu{}^{-1}=-\tfrac{16}{9}\alpha_\mu(\alpha_\mu{}^2-\tfrac{5}{2})\ ,\\
\mathcal{S}&=\tfrac{5}{2}:\quad \alpha_\mu{}^{-1}=\tfrac{64}{225}\alpha_\mu(\alpha_\mu{}^4-\tfrac{35}{4}\alpha_\mu{}^2+\tfrac{259}{16})\ ,\\
\mathcal{S}&=\tfrac{7}{2}:\quad \alpha_\mu{}^{-1}=\left[\frac{-2^8}{(7!!)^2}\right]\alpha_\mu(\alpha_\mu{}^6-21\alpha_\mu{}^4+\tfrac{987}{8}\alpha_\mu{}^2-\tfrac{3229}{16})\ .
\end{aligned} \tag{3.6}$$

B. Sakata-Taketani operators

From Eqs. (1.20)–(1.26) one sees that the ST decomposition uses combinations of $\beta_4{}^2$ as operators to project out the mass eigenvalues m and ∞. The operators themselves came from the characteristic equation multiplied by β_4. The generalization of ST, then, will be to project out the various mass states for both integer and half-integer spin, and also to project out the "infinite-mass subsidiary states" for integer spin. This can be done by starting from the α_4 characteristic equation (multiplied by α_4 for integer spin)

$$0=\prod_{j=(0\text{ or }1/2)}^{\mathcal{S}}(\alpha_4{}^2-j^2)\ . \tag{3.7}$$

(We are using α_4 to build up our ST operators. This is the most physically motivated method, although in general one could use an arbitrary direction in four-dimensional space-time to build up the operators, as was done for DKP in Ref. 18.)

The next step is to recall that since α_4 is a generator of so(5), it can always be rotated into J_z, and hence can be put in diagonal form, with diagonal blocks SI, where S runs from $-\mathcal{S}$ to $\mathcal{S}$. Dealing with $\alpha_4{}^2$, the blocks run from $[0$ or $(\frac{1}{2})^2]$ to $\mathcal{S}^2$ times I, and represent the mass states or the zero-eigenvalue subsidiary components. Our generalized ST operators, then, will want to pick out the various blocks and have them normalized to unity. Therefore, if one just takes (3.7) without the factor representing a particular mass state, the remaining operator will pick out just that mass state (since the remaining factors piece-by-piece project to zero the rest of the mass states). The only thing necessary is to normalize. When this is done, one has

$$\mathcal{P}_j(\mathcal{S}) = \frac{1}{N(\mathcal{S}, j)} \prod_{\substack{k=(0 \text{ or } 1/2) \\ k \neq j}}^{\mathcal{S}} (\alpha_4^{\,2} - k^2) \ , \qquad (3.8\text{a})$$

$$N(\mathcal{S}, j) = \prod_{\substack{i=(0 \text{ or } 1/2) \\ i \neq j}}^{\mathcal{S}} (j^2 - i^2) \ , \qquad (3.8\text{b})$$

yielding the desired projection operator properties

$$\mathcal{P}_j(\mathcal{S})\mathcal{P}_i(\mathcal{S}) = \mathcal{P}_i(\mathcal{S})\mathcal{P}_j(\mathcal{S}) = \mathcal{P}_i(\mathcal{S})\,\delta(i,j) \ , \qquad (3.9\text{a})$$

$$\sum_j \mathcal{P}_j(\mathcal{S}) = I \ , \qquad (3.9\text{b})$$

$$\mathcal{P}_j(\mathcal{S}) \equiv \mathcal{P}_j^+(\mathcal{S}) + \mathcal{P}_j^-(\mathcal{S}), \quad j \neq 0 \qquad (3.9\text{c})$$

with mass states corresponding to (we use the particle positive-sign convention)

$$\mathcal{P}_j^{\pm}(\mathcal{S}) \Rightarrow m = \pm \chi / j \ . \qquad (3.10)$$

[When the reader is checking (3.9) for integer spin he should keep in mind that in $\mathcal{P}_j^{\,2}(\mathcal{S})$ the factor $\alpha_4^{\,4}/j^4$ reduces to $\alpha_4^{\,2}/j^2$ when multiplied by the rest of the projection operator.]

Specific examples of (3.8)–(3.9) are given below.

$$\begin{aligned}
\mathcal{S} = 1\!: \ & \mathcal{P}_0 = (1 - \alpha_4^{\,2}) \ , \\
& \mathcal{P}_1 = \alpha_4^{\,2}; \\
\mathcal{S} = \tfrac{3}{2}\!: \ & \mathcal{P}_{1/2} = -\tfrac{1}{2}(\alpha_4^{\,2} - \tfrac{9}{4}) \ , \\
& \mathcal{P}_{3/2} = \tfrac{1}{2}(\alpha_4^{\,2} - \tfrac{1}{4}); \qquad (3.11) \\
\mathcal{S} = 2\!: \ & \mathcal{P}_0 = \tfrac{1}{4}(\alpha_4^{\,2} - 4)(\alpha_4^{\,2} - 1) \ , \\
& \mathcal{P}_1 = -\tfrac{1}{3}\alpha_4^{\,2}(\alpha_4^{\,2} - 4) \ , \\
& \mathcal{P}_2 = \tfrac{1}{12}\alpha_4^{\,2}(\alpha_4^{\,2} - 1) \ .
\end{aligned}$$

Note that the Dirac case ($\mathcal{S} = \frac{1}{2}$) is not included in the above. This is because there is only one operator for Dirac, and that is just unity projecting onto itself:

$$\mathcal{S} = \tfrac{1}{2}\!: \ \mathcal{P}_{1/2} = 4\alpha_4^{\,2} = I \ , \qquad (3.12)$$

i.e., for Dirac there are no extra mass states and no subsidiary components. This case has the most well-behaved algebra and solution, as is well known.

C. Commutation relations of the $\mathcal{P}_j$ with the α_μ

Because the $\mathcal{P}_j$ are composed of products of $\alpha_4^{\,2}$, one trivially has that (always working in a given representation $\mathcal{S}$)

$$[\mathcal{P}_j, \alpha_4] = 0 \ . \qquad (3.13)$$

The problem is to find the commutation relations $[\mathcal{P}_j, \alpha_k]$, $k = 1, 2, 3$. To find them, one begins by using the trick[12, 19] we used to find the C, P, and T transformation matrices for general Bhabha equations.[12] Going to the representation where α_4 is diagonal we can write (no sum involved in the i's or j's here or later—they represent components)

$$(\alpha_4)_{ij} = d_i \delta_{ij}, \quad (\alpha_k)_{ij} = c_{ij} \ . \qquad (3.14)$$

Putting (3.14) into the basic double commutation relation (1.2) with $\nu = k$ and $\mu = \lambda = 4$, one has

$$c_{ij} = (d_i - d_j)^2 c_{ij} \ . \qquad (3.15)$$

From (3.15) one sees that

$$c_{ij} = 0, \text{ unless } (d_i - d_j)^2 = 1 \ \text{ or } \ d_i \pm 1 = d_j \ . \qquad (3.16)$$

Equation (3.16) tells us that α_k has nonzero matrix elements only in the off-diagonal blocks where α_4 has its diagonal matrix elements changing from some value S to a value $S \pm 1$. But starting with integer-spin representations, these are the same positions where $\alpha_4^{\,2}$ changes values, and hence the same places where the nonzero matrix elements of the projection operators $\mathcal{P}_j$ change over into the nonzero matrix elements of the projection operators $\mathcal{P}_{j\pm 1}$. Thus, we can write the sum $(I + \alpha_k)$ for integer $\mathcal{S}$ as

$$I + \alpha_k = \begin{bmatrix}
\mathcal{P}_{\mathcal{S}}^+ & c_{\mathcal{S}} & & & & & & & \\
c_{\mathcal{S}}^\dagger & \mathcal{P}_{\mathcal{S}-1}^+ & c_{\mathcal{S}-1} & & & & & & \\
 & c_{\mathcal{S}-1}^\dagger & \mathcal{P}_{\mathcal{S}-2}^+ & c_{\mathcal{S}-2} & & & & & \\
 & & c_{\mathcal{S}-2}^\dagger & \ddots & & & & & \\
 & & & & c_2 & & & & \\
 & & & c_2^\dagger & \mathcal{P}_1^+ & c_1 & & & \\
 & & & & c_1^\dagger & \mathcal{P}_0 & b_1 & & \\
 & & & & & b_1^\dagger & \mathcal{P}_1^- & b_2 & \\
 & & & & & & b_2^\dagger & & \ddots
\end{bmatrix} \qquad (3.17)$$

Note that $\mathcal{P}_j \alpha_k$ only has the matrix elements of α_k contained in the rows where $\mathcal{P}_j$ is nonzero and $\alpha_k \mathcal{P}_j$ only has the matrix elements of α_k in the columns where $\mathcal{P}_j$ is nonzero. Thus, $\mathcal{P}_{\mathcal{S}} \alpha_k$ would only have the matrix elements $c_{\mathcal{S}}$. These matrix elements would also be contained in $\alpha_k \mathcal{P}_{\mathcal{S}-1}$, but in addition there would be the term $c_{\mathcal{S}-1}^\dagger$. One could cancel that with $\mathcal{I}_{\mathcal{S}-2}\alpha_k$, but then the extra term $c_{\mathcal{S}-2}$ would be left over. By iterating down the diagonal, it is clear that one in general obtains a complicated set of commutation relations involving all the $\mathcal{I}_j(\mathcal{S})$ and any α_k. Specifically,

$$\alpha_k\left[\sum_{j=0} \mathcal{I}_{2j}(\mathcal{S})\right] = \left[\sum_{j=0} \mathcal{I}_{2j+1}(\mathcal{S})\right]\alpha_k\,,$$

$$\mathcal{S} \text{ an integer.} \qquad (3.18)$$

Equation (3.18) is simple only for the DKP case $\mathcal{S}=1$, as then there are only two $\mathcal{I}_j$. For $\mathcal{S}=2$ there are three $\mathcal{I}_j$, and matters rapidly become more complicated for $\mathcal{S}>2$. We also note that since our argument hinged only on making a particular α_μ diagonal, we could get the same type of commutation relations if we made our projection operators out of an in general different α_μ than α_4, as was done for DKP in Ref. 18.

For half-integer spin $\mathcal{S}$, there are no commutation relations as Eq. (3.18). This is because the central blocks corresponding to Eq. (3.17) have two spin-$\frac{1}{2}$ blocks, so that they look like

$$I+\alpha_k = \begin{bmatrix} \cdot & & & & & & \\ & \cdot & & & & & \\ & & \cdot & & & & \\ & & & \mathcal{I}^{+}_{1/2} & a & & \\ & & & a^{\dagger} & \mathcal{I}^{-}_{1/2} & & \\ & & & & & \cdot & \\ & & & & & & \cdot \end{bmatrix}. \qquad (3.19)$$

This additional piece mixes all of the matrix elements together so that one can only derive the identity

$$I\alpha_k = \left(\sum_{j=1/2}^{\mathcal{S}} \mathcal{I}_j\right)\alpha_k$$

$$= \alpha_k\left(\sum_{j=1/2}^{\mathcal{S}} \mathcal{I}_j\right)$$

$$= \alpha_k I\,. \qquad (3.20)$$

D. Other useful operator relationships

In this subsection we will discuss other operator relationships that will be useful both in this paper and in paper III.[16] Note that the results of this subsection also hold with the order of the operators reversed because all the operators are self-adjoint. [See, for example, Eqs. (3.21) and (3.23) below.]

We start by defining the operators $\mathcal{Q}_j(\mathcal{S})$:

$$\mathcal{Q}_j(\mathcal{S})\alpha_4 = \alpha_4\mathcal{Q}_j(\mathcal{S}) \equiv \mathcal{I}_j(\mathcal{S}), \quad \begin{cases} j,\,\mathcal{S} \text{ integers} \\ 0 \neq j = 1, 2, \ldots, \mathcal{S} \end{cases} \qquad (3.21\text{a})$$

$$\mathcal{I}_0(\mathcal{S})\mathcal{Q}_j(\mathcal{S}) = 0\,. \qquad (3.21\text{b})$$

We also define

$$Q(\mathcal{S}) = \sum_{j=1}^{\mathcal{S}} \mathcal{Q}_j(\mathcal{S})\,. \qquad (3.22)$$

Then, since Eqs. (1.5) and (3.8) for $j=0$ give us that

$$\mathcal{I}_0(\mathcal{S})\alpha_4 = \alpha_4\mathcal{I}_0(\mathcal{S}) = 0\,, \qquad (3.23)$$

we have from Eqs. (3.9b) and (3.21)–(3.23) that

$$\alpha_4 Q(\mathcal{S}) = I - \mathcal{I}_0(\mathcal{S})\,. \qquad (3.24)$$

Further, by the same type of reasoning that leads to Eq. (3.9a), one has

$$\mathcal{Q}_j(\mathcal{S})\mathcal{Q}_k(\mathcal{S}) = 0, \quad j \neq k \qquad (3.25)$$

and also

$$\mathcal{I}_0(\mathcal{S})\mathcal{Q}_j(\mathcal{S}) = 0\,. \qquad (3.26)$$

Equations (3.22) and (3.26) immediately give

$$\mathcal{I}_0(\mathcal{S})Q(\mathcal{S}) = 0\,. \qquad (3.27)$$

We can also show that

$$0 = \mathcal{I}_0(\mathcal{S})\alpha_\lambda(I-\alpha_4^{\,2})\,. \qquad (3.28)$$

For $\lambda=4$, Eq. (3.28) follows from Eq. (3.23). For $\lambda=k=1,2,3$, refer back to Eq. (3.17). $\mathcal{I}_0\alpha_k$ will only keep the elements $c_1^\dagger$ on the upper-left-hand side of the $\mathcal{I}_0$ submatrix, and the corresponding element b_1 on the lower right-hand side of the matrix. But since $(1-\alpha_4^{\,2})$ is zero in the two $\mathcal{I}_1$ boxes, the multiplication of the matrix with only $c_1^\dagger$ and b_1 by $(1-\alpha_4^{\,2})$ on the right is zero. Further, since $\mathcal{I}_0$ has a factor $(I-\alpha_4^{\,2})$ in it, Eq. (3.28) also implies that

$$0 = \mathcal{I}_0(\mathcal{S})\alpha_\lambda\mathcal{I}_0(\mathcal{S})\,. \qquad (3.29\text{a})$$

In fact, for every integer j, $\mathcal{I}_j(\mathcal{S})$ has the factor $(I-\alpha_4^{\,2})$ in it, except for $\mathcal{I}_1(\mathcal{S})$. Therefore, we can also write

$$0 = \mathcal{I}_0(\mathcal{S})\alpha_\lambda\mathcal{I}_j(\mathcal{S}), \quad j \neq 1\,. \qquad (3.29\text{b})$$

A similar result is

$$0 = \mathcal{I}_0(\mathcal{S})\alpha_\lambda[\alpha_4 - Q(\mathcal{S})]\,. \qquad (3.30)$$

By the same method one can demonstrate for $\mathcal{S}$ an integer and $k=1,2,3$ that

$$0 = \mathcal{I}_{\mathcal{S}}(\mathcal{S})\alpha_k[(\mathcal{S}-1)^2 - \alpha_4^{\,2}]\,, \qquad (3.31)$$

$$0 = \mathcal{I}_{\mathcal{S}}(\mathcal{S})\alpha_k\mathcal{I}_j(\mathcal{S}), \quad j \neq \mathcal{S}-1 \qquad (3.32)$$

$$0 = \mathcal{I}_j(\mathcal{S})\alpha_k[(j+1)^2 - \alpha_4^{\,2}][(j-1)^2 - \alpha_4^{\,2}]\,, \qquad (3.33)$$

$$0 = \mathcal{I}_j(\mathcal{S})\alpha_k\mathcal{I}_i(\mathcal{S}), \quad i \neq j \pm 1\,. \qquad (3.34)$$

For half-integer $\mathcal{S}$, equations similar to (3.31)–(3.34) have the added complication of the two central $\mathcal{G}_{1/2}(\mathcal{S})$ blocks shown in Eq. (3.19). When this is taken into consideration, similar methods as above allow one to demonstrate for half-integer $\mathcal{S}$ that

$$0=\mathcal{G}_{\mathcal{S}}(\mathcal{S})\alpha_k[(\mathcal{S}-1)^2-\alpha_4{}^2], \quad \mathcal{S}\neq\tfrac{1}{2} \tag{3.35}$$

$$0=\mathcal{G}_{\mathcal{S}}(\mathcal{S})\alpha_k\mathcal{G}_j(\mathcal{S}), \quad j\neq\mathcal{S}-1, \quad \mathcal{S}\neq\tfrac{1}{2} \tag{3.36}$$

$$0=\mathcal{G}_j(\mathcal{S})\alpha_k[(j+1)^2-\alpha_4{}^2][(j-1)^2-\alpha_4{}^2], \quad j\neq\tfrac{1}{2} \tag{3.37}$$

$$0=\mathcal{G}_j(\mathcal{S})\alpha_k\mathcal{G}_i(\mathcal{S}), \quad \begin{cases} i\neq j\pm 1 \\ i,j\neq\tfrac{1}{2}\ . \end{cases} \tag{3.38}$$

Finally we note a result which is useful for commuting spin operators with operator functionals of α_4. From (1.2) we have that

$$[[\alpha_i\,,\alpha_j],\alpha_4]=0, \quad i,j\neq 4\ . \tag{3.39}$$

But this means that

$$[[\alpha_i\,,\alpha_j],f(\alpha_4)]=0, \quad i,j\neq 4 \tag{3.40}$$

or, in particular,

$$[[\alpha_i\,,\alpha_j],\mathcal{G}_0(\mathcal{S})]=0, \quad i,j\neq 4 \tag{3.41}$$

$$[[\alpha_i\,,\alpha_j],\mathcal{G}_k(\mathcal{S})]=0, \quad i,j\neq 4 \tag{3.42}$$

$$[[\alpha_i\,,\alpha_j],Q(\mathcal{S})]=0, \quad i,j\neq 4\ . \tag{3.43}$$

These identities will be useful in deriving the results of later sections, and also in paper III.[16]

IV. MASS AND SPIN SPECTRUM OF BHABHA'S EQUATIONS

As stated in Sec. I the Bhabha algebra corresponds to the general so(5) algebra, with a particular explicit algebra being labeled by the number $\mathcal{S}$, and the various irreducible representations of the particular algebra having dimensions $d_5(\mathcal{S},S)$ given by

$$d_5(\mathcal{S},S)=\tfrac{1}{6}(2\mathcal{S}+3)(2S+1) \times[(\mathcal{S}+1)(\mathcal{S}+2)-S(S+1)]\ , \tag{4.1}$$

$$\mathcal{S}\geqslant S\geqslant 0, \quad \text{both integers or half-integers.} \tag{4.2}$$

To understand the mass and spin content, one first recalls the results of Eqs. (1.16) and (1.17), concerning the multiple-mass Klein-Gordon equation, that the Bhabha fields satisfy the following.

$\mathcal{S}=$ *integer:*

$$0=\chi[\Box-\chi^2][4\Box-\chi^2]\cdots \times[(\mathcal{S}-1)^2\Box-\chi^2][\mathcal{S}^2\Box-\chi^2]\psi\ ; \tag{4.3}$$

$\mathcal{S}=$ *half-integer:*

$$0=[\tfrac{1}{4}\Box-\chi^2][\tfrac{9}{4}\Box-\chi^2]\cdots \times[(\mathcal{S}-1)^2\Box-\chi^2][\mathcal{S}^2\Box-\chi^2]\psi\ . \tag{4.4}$$

Equations (4.3) and (4.4) yield the mass, and also spin, eigenvalues that are contained in a particular Bhabha field. The question to be answered is: Under what circumstances do particular eigenvalues hold? To solve this question one proceeds to decompose the so(5) algebra into its subalgebras. One decomposes an individual so(5) representation into a sum of so(4) representations [or equivalently o(4) or Lorentz-algebra representations], and then into sums of so(3) representations. This decomposition is unique and complete.

The easiest way to do this decomposition is to use the Gel'fand pattern which uniquely labels the decomposition of the algebra so($2r$) into so($2r-1$) $\cdots$ into so(3) into so(2). As clearly explained by Louck and Galbraith,[20] the pattern is

$$\begin{bmatrix} l_{2r,1} & & l_{2r,2} & & \cdots & & l_{2r,r-1} & & l_{2r,r} \\ & l_{2r-1,1} & & l_{2r-1,2} & & \cdots & & l_{2r-1,r-1} & \\ & & \vdots & & & & \vdots & & \\ & & & l_{5,1} & & l_{5,2} & & & \\ & & & l_{4,1} & & l_{4,2} & & & \\ & & & & l_{3,1} & & & & \\ & & & & l_{2,1} & & & & \end{bmatrix}, \tag{4.5}$$

where the $l_{j,k}$ are the integers (or half-integers) which label the so(j) representations. We are concerned here with only the bottom piece of the pattern. The two numbers $l_{5,1}$ and $l_{5,2}$ are what we have been calling $\mathcal{S}$ and S.

Given a particular $l_{5,1}$ and $l_{5,2}$, this particular so(5) irreducible representation (irrep) decomposes into the unique sum of so(4) representations labeled by all those irreps with $l_{4,1}$ and $l_{4,2}$ satisfying

$$l_{5,1} \geq l_{4,1} \geq l_{5,2} \geq |l_{4,2}| \geq 0 \ , \qquad (4.6)$$

$$l_{4,1} \geq |l_{4,2}| \geq 0 \ ,$$

both integers or half-integers as $l_{5,1}$. (4.7)

The dimensions of the so(4) representations are

$$d_4(l_{4,1}, l_{4,2}) = (l_{4,1}+1)^2 - (l_{4,2})^2 \ . \qquad (4.8)$$

Note that for $l_{4,2} \neq 0$, there is a second representation with the same dimension, labeled by $l_{4,1}$ and $-l_{4,2}$. This is the so(4) doubling,[21] and if instead of so(4) we consider o(4) or the Lorentz algebra, this doubling is taken care of and the representations have dimensions

$$d_4^L(l_{4,1}, l_{4,2}^L) = [2 - \delta(l_{4,2}^L, 0)][(l_{4,1}+1)^2 - (l_{4,2}^L)^2] \ , \qquad (4.9)$$

$$l_{5,1} \geq l_{4,1} \geq l_{5,2} \geq l_{4,2}^L \geq 0 \ . \qquad (4.10)$$

The o(4) irreps are single-valued only for $l_{4,2}^L = 0$, and otherwise are the sum of the two so(4) representations

$$(l_{4,1}, l_{4,2}^L) = (l_{4,1}, l_{4,2}) \oplus (l_{4,1}, -l_{4,2}) \ , \qquad (4.11)$$

$$l_{4,2}^L = |l_{4,2}| \neq 0 \ . \qquad (4.12)$$

On decomposing so(4) into so(3) one has the rule[20]

$$l_{4,1} \geq l_{3,1} \geq |l_{4,2}| \geq 0 \ , \qquad (4.13)$$

with the dimension given by

$$d_3(l_{3,1}) = (2l_{3,1}+1) \ . \qquad (4.14)$$

One can already see that the so(3) irreps correspond to spin states. Further, since from Eq. (4.11) the double-valued d_4^L representations correspond to a sum of two so(4) representations, the decomposition into so(3) representations will be doubled, so that one will have $(2) \times (2l_{3,1}+1)$ states giving 2 for particle-antiparticle times $(2l_{3,1}+1)$ for spin $l_{3,1}$.

In Table I this decomposition has been done explicitly for the half-integer-spin representations up to maximum spin $\mathcal{S} = \frac{7}{2}$. The first columns list the possible so(5) representations and their dimensions. Next come the o(4) representations and their dimensions that a particular so(5) representation decomposes into, and the number of so(3) representations of dimension $(2l_{3,1}+1)$ that an individual o(4) representation decomposes into. Finally, the mass of a particular o(4) representation [including all the particular so(3) irreps that it contains] is given in the last column. We now describe how the mass values are obtained.

The basic idea for finding the mass states was first given by Bhabha,[2] and in Table I we have explicitly extended his idea to higher spin. The key is the realization by Bhabha[2] that if one puts α_4 into block diagonal form, then each eigenvalue block of α_4 contains a distinct number of complete so(3) spin representations. Further, if a spin irrep is in an α_4 block with eigenvalue S, it must also be in all the blocks $+S, S-1, \ldots, -S+1, -S$. This allows you to find the mass states.

As an example, consider the so(5) representation $(\frac{5}{2}, \frac{3}{2})$. There are 8 spin-$(\frac{3}{2})$ irreps, so these can be placed in each of the α_4 eigenvalue blocks with an extra set in the $+\frac{1}{2}$ and $-\frac{1}{2}$ blocks [see Eq. (4.15)]. [The reader might think that they could also all be put in the two $+\frac{1}{2}$ and $-\frac{1}{2}$ blocks, but because each row of so(3) states in Table I must have the same mass, the distribution turns out to be unique.]

Blocks contain the so(3) spin irreps:

$$\alpha_4(\tfrac{5}{2}, \tfrac{3}{2}) = \begin{vmatrix} \frac{5}{2}\mathcal{G}_{5/2}^- & & & & & 0 \\ & \frac{3}{2}\mathcal{G}_{3/2}^+ & & & & \\ & & \frac{1}{2}\mathcal{G}_{1/2}^+ & & & \\ & & & -\frac{1}{2}\mathcal{G}_{1/2}^- & & \\ 0 & & & & -\frac{3}{2}\mathcal{G}_{3/2}^- & \\ & & & & & -\frac{5}{2}\mathcal{G}_{5/2}^- \end{vmatrix} \begin{array}{l} (\frac{3}{2}) \\ (\frac{3}{2}) \oplus (\frac{1}{2}) \oplus (\frac{5}{2}) \\ (\frac{3}{2}) \oplus (\frac{3}{2}) \oplus (\frac{1}{2}) \oplus (\frac{5}{2}) \\ (\frac{3}{2}) \oplus (\frac{3}{2}) \oplus (\frac{1}{2}) \oplus (\frac{5}{2}) \\ (\frac{3}{2}) \oplus (\frac{1}{2}) \oplus (\frac{5}{2}) \\ (\frac{3}{2}) \end{array} . \qquad (4.15)$$

The four so(3) spin-$\frac{1}{2}$ irreps and the four so(3) spin-$\frac{5}{2}$ irreps are put in the α_4 blocks from $+\frac{3}{2}$ to $-\frac{3}{2}$.

Now we can see the mass eigenvalues. Remember, the mass eigenvalue is given by χ/j, where j is the eigenvalue of the α_4 block. (For definiteness we are keeping the sign of the mass positive. The negative eigenvalues correspond to the anti-

TABLE I. The mass and spin decomposition of half-integer Bhabha fields up to maximum spin $\frac{7}{2}$. The columns show (1) the so(5) representation, (2) the dimension of the so(5) representation, (3) the o(4) irreps that a particular so(5) irrep decomposes into, (4) the dimensions of the o(4) irreps, (5) the number and spin content of the so(3) irreps that a particular o(4) decomposes into, and finally (6) the mass of the o(4) irrep (and its component so(3) irreps) in a particular row. See the main text for a more detailed discussion.

				Number of so(3) spin representations with dimension $(2l_{3,1}+1)$, for $l_{3,1}=$				
$(\mathcal{S},S)$	d_5	$(l_{4,1},l^L_{4,2})$	d^L_4	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$	Mass
$(\frac{1}{2},\frac{1}{2})$	4	$(\frac{1}{2},\frac{1}{2})$	4	2				2χ
$(\frac{3}{2},\frac{3}{2})$	20	$(\frac{3}{2},\frac{3}{2})$	8		2			$2\chi/3$
		$(\frac{3}{2},\frac{1}{2})$	12	2	2			2χ
$(\frac{3}{2},\frac{1}{2})$	16	$(\frac{3}{2},\frac{1}{2})$	12	2	2			2χ
		$(\frac{1}{2},\frac{1}{2})$	4	2				$2\chi/3$
$(\frac{5}{2},\frac{5}{2})$	56	$(\frac{5}{2},\frac{5}{2})$	12			2		$2\chi/5$
		$(\frac{5}{2},\frac{3}{2})$	20		2	2		$2\chi/3$
		$(\frac{5}{2},\frac{1}{2})$	24	2	2	2		2χ
$(\frac{5}{2},\frac{3}{2})$	64	$(\frac{5}{2},\frac{3}{2})$	20		2	2		2χ
		$(\frac{5}{2},\frac{1}{2})$	24	2	2	2		$2\chi/3$
		$(\frac{3}{2},\frac{3}{2})$	8		2			$2\chi/5$
		$(\frac{3}{2},\frac{1}{2})$	12	2	2			2χ
$(\frac{5}{2},\frac{1}{2})$	40	$(\frac{5}{2},\frac{1}{2})$	24	2	2	2		2χ
		$(\frac{3}{2},\frac{1}{2})$	12	2	2			$2\chi/3$
		$(\frac{1}{2},\frac{1}{2})$	4	2				$2\chi/5$
$(\frac{7}{2},\frac{7}{2})$	120	$(\frac{7}{2},\frac{7}{2})$	16				2	$2\chi/7$
		$(\frac{7}{2},\frac{5}{2})$	28			2	2	$2\chi/5$
		$(\frac{7}{2},\frac{3}{2})$	36		2	2	2	$2\chi/3$
		$(\frac{7}{2},\frac{1}{2})$	40	2	2	2	2	2χ

				Number of so(3) spin representations with dimension $(2l_{3,1}+1)$, for $l_{3,1}=$				
$(\mathcal{S},S)$	d_5	$(l_{4,1},l^L_{4,2})$	d^L_4	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$	Mass
$(\frac{7}{2},\frac{5}{2})$	160	$(\frac{7}{2},\frac{5}{2})$	28			2	2	$2\chi/5$
		$(\frac{7}{2},\frac{3}{2})$	36		2	2	2	$2\chi/3$
		$(\frac{7}{2},\frac{1}{2})$	40	2	2	2	2	2χ
		$(\frac{5}{2},\frac{5}{2})$	12			2		$2\chi/7$
		$(\frac{5}{2},\frac{3}{2})$	20		2	2		$2\chi/3$
		$(\frac{5}{2},\frac{1}{2})$	24	2	2	2		2χ
$(\frac{7}{2},\frac{3}{2})$	140	$(\frac{7}{2},\frac{3}{2})$	36		2	2	2	2χ
		$(\frac{7}{2},\frac{1}{2})$	40	2	2	2	2	2χ
		$(\frac{5}{2},\frac{3}{2})$	20		2	2		$2\chi/3$
		$(\frac{5}{2},\frac{1}{2})$	24	2	2	2		$2\chi/3$
		$(\frac{3}{2},\frac{3}{2})$	8		2			$2\chi/7$
		$(\frac{3}{2},\frac{1}{2})$	12	2	2			$2\chi/5$
$(\frac{7}{2},\frac{1}{2})$	80	$(\frac{7}{2},\frac{1}{2})$	40	2	2	2	2	2χ
		$(\frac{5}{2},\frac{1}{2})$	24	2	2	2		$2\chi/3$
		$(\frac{3}{2},\frac{1}{2})$	14	2	2			$2\chi/5$
		$(\frac{1}{2},\frac{1}{2})$	4	2				$2\chi/7$

particle masses.)

Looking at Eq. (4.15), we see that only a single so(3) spin-$\frac{3}{2}$ irrep is in the $\frac{5}{2}$ eigenvalue block. And, indeed, in Table I there is a row in the decomposition of the $(\mathcal{S},S)=(\frac{5}{2},\frac{3}{2})$ so(5) representation that has only one so(3) irrep, for spin $\frac{3}{2}$. Therefore, this state has the lowest mass eigenvalue, $m=2\chi/5$. Carrying on, one sees that the $\frac{3}{2}$ eigenvalue block of α_4 has $\frac{1}{2}$, $\frac{3}{2}$, and $\frac{5}{2}$ so(3) spin states [the complete $(\frac{5}{2},\frac{1}{2})$ o(4) irrep]. Therefore the mass is $2\chi/3$. Finally, there are two mass states with $m=2\chi$: the $(\frac{5}{2},\frac{3}{2})$ o(4) irrep with so(3) spin states $\frac{3}{2}$ and $\frac{5}{2}$, and the $(\frac{3}{2},\frac{1}{2})$ o(4) irrep with so(3) spin states of $\frac{1}{2}$ and $\frac{3}{2}$.

The same procedure can be done for all of the so(5) irreps, and Table I lists the results for all the half-integer irreps up to $\mathcal{S}=\frac{7}{2}$. The mass-spin content in Table I can easily be generalized to $\mathcal{S}>\frac{7}{2}$ from the form of the triangle and cut-off triangle patterns in the table.

The physically interesting things to note are that for a given so(5) irrep labeled by $(\mathcal{S},S)$, (1) $\mathcal{S}$ is the maximum spin contained in the representation, (2) the masses go from $\chi/\mathcal{S}$ up to 2χ for half-integer-spin irreps (to $\chi/0=\infty$ for the integer-spin irreps we shall come to next), and (3) the lowest mass state $\chi/\mathcal{S}$ will contain a single spin state S. (Bhabha[2] noticed this last point by looking at the nonrelativistic limits of his equations.)

Therefore, we now see that our operators $\mathcal{G}_j(\mathcal{S})$ are projection operators for the so(5) algebra labeled by $\mathcal{S}$ onto the mass states with $m=\chi/j$. Further, if the so(5) algebra is broken down into the particular so(5) irreps $(\mathcal{S},S)$, then the $\mathcal{G}_j(\mathcal{S})$, with α_4 given in the $(\mathcal{S}, S)$ irrep, picks out the mass state χ/j only in the $(\mathcal{S}, S)$ irrep. Thus, $\mathcal{G}_j(\mathcal{S})$ effectively becomes labeled $\mathcal{G}_j(\mathcal{S}, S)$. Finally, by coupling these mass projection operators with a set of spin projection operators, one can pick out any particular mass and spin state one wants from the general set of Bhabha equations.

Going over to the more complicated integer-spin representations, we look at Table II. The layout is similar to Table I, but there is a problem with the 0-eigenvalue block of α_4. This block has a mass eigenvalue $m = \chi/0 = \infty$, and corresponds to the subsidiary components that are eliminated by the Sakata-Taketani process. Because this α_4 block has eigenvalue zero, not ± some integer or half-integer, it contains only single so(3) blocks.

Looking at Table II in detail, the (0, 0) so(5) irrep is, of course, the trivial, identically zero irrep. (1, 1) is the DKP spin-1 irrep, with the indicated $m = \chi$ spin-1 state and the four subsidiary components. [However, recall the physical content of the two sets of components and how they are obtained, as described in the paragraph below Eq. (1.26).]

The (1, 0) so(5) irrep has two particle components and three subsidiary components, but they are obtained by the ST process of combining states [as also indicated below Eq. (1.26)]. More specifically, because the so(5) irrep (1, 0) has no doubling, obtaining the two particle components is not the simple particle-antiparticle statement of having doubled spin-0 states. What happens is that the single spin-0 piece of the o(4) (0, 0) irrep combines with the spin-0 piece of the o(4) (1, 0) 4-vector to form the particle components, and the spin-1 piece of the o(4) (1, 0) 4-vector becomes the subsidiary components. Thus, any time we obtain the combined o(4) irreps (1, 0) and (0, 0), we list the mass as a mixture of χ and ∞.

The rest of Table II is obtained as was done for Table I, except that the single so(3) irrep states will be infinite-mass subsidiary components. Also, when we have an so(5) irrep of the type $(\mathcal{S}, 0)$, again the lowest mass state will have spin zero, but the mass will be χ, having come from o(4) (1, 0) and (0, 0) irreps as in the DKP spin-0 case. The rest of the irrep will be called infinite-mass subsidiary components, even though some doubling could be done. We do this because the $(\mathcal{S}, 0)$ irreps can never have all the mass states, as is possible for every $(\mathcal{S}, S \neq 0)$ irrep.

Finally, we can now see that our general ST operators $\mathcal{G}_j(\mathcal{S})$ or $\mathcal{G}_j(\mathcal{S}, S)$ reduce to mass-state projection operators onto the mass states χ/j for the so(5) algebra $\mathcal{S}$ or the so(5) irreps $(\mathcal{S}, S)$, respectively, *when* $\vec{p} = 0$. This is simply because when $\vec{p} = 0$, the Bhabha equation reduces to

TABLE II. The mass and spin decomposition of integer Bhabha fields up to maximum spin 3. The columns are the same as in Table I. See the main text for a more detailed discussion, especially concerning the "infinite-mass" subsidiary components.

				Number of so(3) spin representations with dimension $(2l_{3,1}+1)$, for $l_{3,1}$ =				
$(\mathcal{S}, S)$	d_5	$(l_{4,1}, l^L_{4,2})$	d^L_4	0	1	2	3	Mass
(0, 0)	1	(0, 0)	1	1				∞
(1, 1)	10	(1, 1)	6		2			χ
		(1, 0)	4	1	1			∞
(1, 0)	5	(1, 0)	4	1	1			{χ
		(0, 0)	1	1				{∞
(2, 2)	35	(2, 2)	10			2		$\chi/2$
		(2, 1)	16		2	2		χ
		(2, 0)	9	1	1	1		∞
(2, 1)	35	(2, 1)	16		2	2		χ
		(2, 0)	9	1	1	1		∞
		(1, 1)	6		2			$\chi/2$
		(1, 0)	4	1	1			∞
(2, 0)	14	(2, 0)	9	1	1	1		∞
		(1, 0)	4	1	1			{χ
		(0, 0)	1	1				{∞
(3, 3)	84	(3, 3)	14				2	$\chi/3$
		(3, 2)	24			2	2	$\chi/2$
		(3, 1)	30		2	2	2	χ
		(3, 0)	16	1	1	1	1	∞
(3, 2)	105	(3, 2)	24			2	2	χ
		(3, 1)	30		2	2	2	$\chi/2$
		(3, 0)	16	1	1	1	1	∞
		(2, 2)	10			2		$\chi/3$
		(2, 1)	16		2	2		χ
		(2, 0)	9	1	1	1		∞
(3, 1)	81	(3, 1)	30		2	2	2	χ
		(3, 0)	16	1	1	1	1	∞
		(2, 1)	16		2	2		$\chi/2$
		(2, 0)	9	1	1	1		∞
		(1, 1)	6		2			$\chi/3$
		(1, 0)	4	1	1			∞
(3, 0)	30	(3, 0)	16	1	1	1	1	∞
		(2, 0)	9	1	1	1		∞
		(1, 0)	4	1	1			{χ
		(0, 0)	1	1				{∞

$$(\partial_4 \alpha_4 + \chi)\psi = 0 , \tag{4.16}$$

and the solutions are the uncoupled mass states.

However, when $\vec{p} \neq 0$, then the full Bhabha Eq. (1.1) is to be used, and one can see from Eq. (3.17) that the α_k will couple (i.e., mix up) the different mass states. In this case one must use the generalized ST operators in a generalized Sakata-Taketani reduction to decouple the mass states. This is what will be discussed in Sec. VI.

V. THE HAMILTONIANS

The Hamiltonian for a wave equation is that operator (H) which satisfies the eigenvalue equation

$$-\partial_4\psi = E\psi = H\psi \ , \tag{5.1}$$

where Eq. (5.1) is to be obtained from the fundamental wave equation (1.1). It will turn out to be trivial to obtain the Hamiltonian for half-integer-spin Bhabha wave equations such as the special Dirac case. However, the procedure is complicated for integer-spin Bhabha equations, as is already known from our previous remarks about the special DKP case.

A. Half-integer-spin equations

As already mentioned in Sec. III A, the α_μ have inverses given by Eq. (3.5) for half-integer-spin representations. In particular, the matrices α_4 have inverses. Thus, by simply multiplying Eq. (1.1) by α_4^{-1} one immediately has the Hamiltonian equation (5.1), with H given by

$$H = \alpha_4^{-1}(\vec{\partial}\cdot\vec{\alpha} + \chi) \ . \tag{5.2}$$

Note that for $\mathcal{S} = \frac{1}{2}$, one has $\vec{\alpha} = \frac{1}{2}\vec{\gamma}$, $\chi = \frac{1}{2}m$, and $\alpha_4^{-1} = 4\alpha_4 = 2\gamma_4$ yielding the Dirac equation

$$H\psi^D = \gamma_4(\vec{\partial}\cdot\vec{\gamma} + m)\psi^D \ . \tag{5.3}$$

B. Integer-spin equations

The complication for the integer-spin equations comes about because, as explained previously, there are no inverse matrices for these representations. Thus, one cannot simply multiply Eq. (1.1) by α_4^{-1} to obtain H. However, it is possible to define a Hamiltonian using a generalization of the method used by Kemmer[5] for the DKP equation.

First we must obtain the Bhabha "consequent equations," which are the generalizations to arbitrary $\mathcal{S}$ = integer of the Eqs. (2.14) in paper I (Ref. 12) for the DKP $\mathcal{S} = 1$ case. Remember, the consequent equations are built into the system from the beginning and are not external constraint equations that have to be imposed from the outside as, for example, in the Rarita-Schwinger case.

To obtain the consequent equations, one first multiplies the wave function by the fundamental algebraic commutation relations (1.2), and then multiplies this by $\partial_\mu\partial_\lambda$, yielding

$$[2(\partial\cdot\alpha)\alpha_\nu(\partial\cdot\alpha) - (\partial\cdot\alpha)(\partial\cdot\alpha)\alpha_\nu - \alpha_\nu(\partial\cdot\alpha)(\partial\cdot\alpha)]\psi = [(\partial\cdot\alpha)\partial_\nu - \Box\alpha_\nu]\psi \ . \tag{5.4}$$

By then using the free wave equation (1.1) to eliminate the factors $(\partial\cdot\alpha)$ in the first and third terms on the left-hand side and the first term on the right-hand side, one obtains the consequent equations

$$\partial_\nu\psi = [2(\partial\cdot\alpha) + \chi^{-1}(\partial\cdot\alpha)(\partial\cdot\alpha) + \chi(\chi^2 - \Box)]\alpha_\nu\psi \ . \tag{5.5}$$

For the case $\mathcal{S} = 1$, when $\chi = m$, $\Box\psi = m^2\psi$, and one has the α_ν obeying the DKP algebra Eq. (1.9), Eq. (5.5) reduces to the DKP consequent equations

$$\partial_\nu\psi^{\mathrm{DKP}} = (\partial\cdot\beta)\beta_\nu\psi^{\mathrm{DKP}} \ . \tag{5.6}$$

Now we are ready to obtain the Hamiltonian. The trick is to realize one can write the Hamiltonian equation as

$$\begin{aligned}(-\partial_4)\psi &= H\psi \\ &= (-\partial_4)(I - \mathcal{I}_0)\psi + (-\partial_4)\mathcal{I}_0\psi \ .\end{aligned} \tag{5.7}$$

The second term on the right-hand side of (5.7) comes from the consequent Eq. (5.5) when $\nu = 4$. In that case, by using the free Eq. (1.1) a part of the second term in (5.5) can be rewritten as

$$\chi^{-1}(\partial\cdot\alpha)(\partial_4\alpha_4)\alpha_4\psi = \chi^{-1}(\partial\cdot\alpha)\alpha_4(-\vec{\partial}\cdot\vec{\alpha} - \chi)\psi \ . \tag{5.8}$$

Then, rearranging terms gives

$$\begin{aligned}\partial_4\psi = \{&(\partial\cdot\alpha)\alpha_4 + (\partial\cdot\alpha)[(\vec{\partial}\cdot\vec{\alpha})\alpha_4 - \alpha_4(\vec{\partial}\cdot\vec{\alpha})]\chi^{-1} \\ &+ \chi^{-1}(\chi^2 - \Box)\alpha_4\}\psi \ .\end{aligned} \tag{5.9}$$

Now multiplying (5.9) by $\mathcal{I}_0$ eliminates many of the pieces (including the troublesome term with the $\Box$) because from Eq. (3.23) one has $\mathcal{I}_0\alpha_4 = 0$. Finally, using the commutation relations (1.2) to rewrite the term in the square brackets,

$$\partial_4\mathcal{I}_0\psi = \mathcal{I}_0(\vec{\partial}\cdot\vec{\alpha})\alpha_4[1 + (\vec{\partial}\cdot\vec{\alpha})\chi^{-1}]\psi \ . \tag{5.10}$$

To get the first term on the right-hand side of (5.7), start by taking the free Eq. (1.1) with the $(\partial_4\alpha_4)$ piece on one side, and multiply it by the sum of all the $\mathcal{Q}_j$ of Eq. (3.21):

$$(-\partial_4)\left(\sum_{j=1}^{\mathcal{S}}\mathcal{Q}_j\alpha_4\right)\psi = \left(\sum_{j=1}^{\mathcal{S}}\mathcal{Q}_j\right)(\vec{\partial}\cdot\vec{\alpha} + \chi)\psi \ . \tag{5.11}$$

Then by using Eqs. (3.19)–(3.22) one has

$$(-\partial_4)(I - \mathcal{I}_0)\psi = Q(\vec{\partial}\cdot\vec{\alpha} + \chi) \ . \tag{5.12}$$

Thus, by inserting Eqs. (5.10) and (5.12) into Eq. (5.7), one has the Bhabha Hamiltonian equations for arbitrary integer spin,

$$(-\partial_4)\psi = E\psi = H\psi \ , \tag{5.13}$$

$$H = Q(\vec{\partial}\cdot\vec{\alpha} + \chi) - \mathcal{I}_0(\vec{\partial}\cdot\vec{\alpha})\alpha_4[1 + \chi^{-1}(\vec{\partial}\cdot\vec{\alpha})] \ . \tag{5.14}$$

Again, for the special case $\mathcal{S} = 1$, Eq. (5.14) reduces to the DKP Hamiltonian. This is because then $Q = \beta_4$, $\mathcal{I}_0 = (1 - \beta_4^2)$, and using the DKP algebra Eq. (1.9) to rewrite the last term in Eq. (5.14) yields

$$H(\mathcal{S}=1)=H^{\rm DKP} = \beta_4(\vec{\partial}\cdot\vec{\beta}+m)-(\vec{\partial}\cdot\vec{\beta})\,\beta_4\ . \qquad (5.15)$$

VI. SAKATA-TAKETANI REDUCTION FOR THE GENERAL BHABHA EQUATION

A. Method for a general ST (Peirce) reduction

From the material between Eqs. (1.20) and (1.26) of Sec. I, the generalization of the ST reduction in the DKP case to all Bhabha systems is now clear. One uses the projection operators derived in Sec. III to perform a Peirce decomposition into the different mass states as was done for the DKP case. In particular, the Hamiltonian equation

$$E\psi=H\psi \qquad (6.1)$$

can be written as

$$E\Big(\sum_{j=M}^{\mathcal{S}}\mathcal{G}_j\Big)\psi=\Big(\sum_{j=M}^{\mathcal{S}}\mathcal{G}_j\Big)H\Big(\sum_{j=M}^{\mathcal{S}}\mathcal{G}_j\Big)\psi\ , \qquad (6.2a)$$

$$M=\begin{cases}0, & \mathcal{S}\text{ an integer}\\ \tfrac{1}{2}, & \mathcal{S}\text{ a half-integer.}\end{cases} \qquad (6.2b)$$

Equation (6.2) can be written as $\mathcal{S}+1$ ($\mathcal{S}+\frac{1}{2}$) separate equations for $\mathcal{S}$ an integer ($\mathcal{S}$ a half-integer), of the form

$$E(\mathcal{G}_j)\psi=\Big[\mathcal{G}_jH\mathcal{G}_j+\mathcal{G}_jH\Big(\sum_{k=M;\,k\neq j}^{\mathcal{S}}\mathcal{G}_k\Big)\Big]\psi,\quad M\le j\le\mathcal{S}\ . \qquad (6.3)$$

The reduction into $\mathcal{S}+1$ ($\mathcal{S}+\frac{1}{2}$) uncoupled mass-state equations is done by finding the $\mathcal{O}_j$ such that the last term on the right-hand side of Eq. (6.3) can be written in the form

$$\Big[\mathcal{G}_jH\Big(\sum_{k=M;\,k\neq j}^{\mathcal{S}}\mathcal{G}_k\Big)\Big]\psi=[\mathcal{G}_j\mathcal{O}_j\mathcal{G}_j]\,\psi\ . \qquad (6.4)$$

Inserting Eq. (6.4) into Eq. (6.3) completes the ST reduction of the general Bhabha equation into its component mass states.

If one is only interested in decoupling one mass state from the others, then the process becomes simpler in principle, as only two operators of the form (6.4) have to be found. In particular, if one is only interested in decoupling the jth mass state, then one can use the identity

$$I=\sum_{j=M}^{\mathcal{S}}\mathcal{G}_j(\mathcal{S}) \qquad (6.5)$$

to write the two equations to be solved as

$$E(\mathcal{G}_j)\,\psi=[\mathcal{G}_jH\mathcal{G}_j+\mathcal{G}_jH(I-\mathcal{G}_j)]\,\psi\ , \qquad (6.6)$$

$$E(I-\mathcal{G}_j)\,\psi=[(I-\mathcal{G}_j)H(I-\mathcal{G}_j)+(I-\mathcal{G}_j)H\mathcal{G}_j]\,\psi\ . \qquad (6.7)$$

This means that the two operators needed to decouple the jth mass state are the $\mathcal{O}_j$ and $\hat{\mathcal{O}}_j$ satisfying

$$\mathcal{G}_jH(I-\mathcal{G}_j)\,\psi=\mathcal{G}_j\,\mathcal{O}_j\mathcal{G}_j\,\psi\ , \qquad (6.8)$$

$$(I-\mathcal{G}_j)H\mathcal{G}_j\,\psi=(I-\mathcal{G}_j)\,\hat{\mathcal{O}}_j\,(I-\mathcal{G}_j)\,\psi\ . \qquad (6.9)$$

As a matter of fact, the use of the single-mass-state reduction is the most useful, especially for the integer-spin case. There, one can use this method to decouple the "infinite-mass subsidiary components" from the "particle components," which contain $2\times(2S+1)$ for each mass and spin state, as explained in Sec. IV. This is what Sakata and Taketani did for the DKP case, and which we now do in Secs. VI B and VI C for the general integer-spin Bhabha case. Then we will proceed to discuss the decoupling of specific mass states for both integer spin and half-integer spin.

B. Reduction of the general integer-spin case "particle components"

The particle-components equation is (6.7) with $\mathcal{G}_j=\mathcal{G}_0$. [In the DKP case the operator $(1-\mathcal{G}_0)\equiv\mathcal{G}=\beta_4^{\,2}$.] To obtain the operator $\hat{\mathcal{O}}_0$ of Eq. (6.9) we need to derive the "first decoupling equation" [the generalization of Eq. (1.15) in the DKP case]. This is obtained by multiplying the free Bhabha equation (1.1) by $\mathcal{G}_0$ and using Eqs. (3.23) and (3.29) to yield.

$$\begin{aligned}\mathcal{G}_0\psi&=-\chi^{-1}\mathcal{G}_0(\vec{\partial}\cdot\vec{\alpha})\,\psi\\ &=-\chi^{-1}\mathcal{G}_0(\vec{\partial}\cdot\vec{\alpha})\,(I-\mathcal{G}_0)\,\psi\\ &\equiv X(I-\mathcal{G}_0)\,\psi\ .\end{aligned} \qquad (6.10)$$

Inserting this "first decoupling equation" (6.10) into the second term on the right-hand side of Eq. (6.7) with $j=0$ yields the "particle-components equation" for $\mathcal{S}$ an integer,

$$E(I-\mathcal{G}_0)\,\psi=\mathcal{H}_P(I-\mathcal{G}_0)\,\psi\ , \qquad (6.11)$$

$$\mathcal{H}_P=(I-\mathcal{G}_0)\,[H-H\mathcal{G}_0(\vec{\partial}\cdot\vec{\alpha})\,\chi^{-1}]\,(I-\mathcal{G}_0)\ . \qquad (6.12)$$

Using Eq. (5.14) for H, and the operator relationships in Sec. III D, $\mathcal{H}_P$ can be explicitly written as

$$\begin{aligned}\mathcal{H}_P&=[Q(\vec{\partial}\cdot\vec{\alpha}+\chi)-Q(\vec{\partial}\cdot\vec{\alpha})\mathcal{G}_0\chi^{-1}]\,(I-\mathcal{G}_0)\\ &=Q(\vec{\partial}\cdot\vec{\alpha}+\chi)-Q(\vec{\partial}\cdot\vec{\alpha})\,\mathcal{G}_0[1+\chi^{-1}(\vec{\partial}\cdot\vec{\alpha})]\ .\end{aligned} \qquad (6.13)$$

Note that the particle-components Hamiltonian is symbolically the same as the complete integer-spin Hamiltonian Eq. (5.14), only $\mathcal{G}_0$ becomes Q and α_4 becomes $\mathcal{G}_0$, respectively, in the second part of the right-hand side. Also, for the DKP case $Q=\beta_4$ and $\mathcal{I}_0 = I-\beta_4{}^2$. Then Eq. (6.13) reduces to the Sakata-Taketani particle-components free Hamiltonian, Eq. (2.4), with $A_\lambda = 0$.

C. Reduction of the general integer-spin case "subsidiary components"

To obtain the "subsidiary components" equation, we first derive the "second decoupling equation" by multiplying the free Bhabha equation (1.1) by $(I-\mathcal{I}_0)$. Again using Eqs. (3.23) and (3.29), this yields

$$(I-\mathcal{I}_0)\psi = -\chi^{-1}[\partial_4\alpha_4 + (I-\mathcal{I}_0)(\vec{\partial}\cdot\vec{\alpha})(I-\mathcal{I}_0) + (\vec{\partial}\cdot\vec{\alpha})\mathcal{I}_0]\psi \ . \quad (6.14)$$

Now, if we multiply Eq. (6.14) by $(\partial_4 Q-\chi)$, use the algebraic results of Sec. III D, and rearrange, we have

$$(1-\mathcal{I}_0)\psi + (Z-Y)(1-\mathcal{I}_0)\psi = Y\mathcal{I}_0\psi \ , \quad (6.15a)$$

$$Y = (\partial_4{}^2-\chi^2)^{-1}(-\partial_4 Q+\chi)(\vec{\partial}\cdot\vec{\alpha}) \ , \quad (6.15b)$$

$$Z = (\partial_4{}^2-\chi^2)^{-1}(-\partial_4)\chi(\alpha_4 - Q) \ . \quad (6.15c)$$

Equation (6.15) is analogous to the "second decoupling equation" (2.15), except that there remains the extra $(Z-Y)(1-\mathcal{I}_0)$ term, which is algebraically zero in the DKP case. This term prevents the clean reduction of the subsidiary components that was possible for DKP. To understand more clearly why the $(Z-Y)$ term does not allow this decomposition, let us just try to decouple the subsidiary components from the $\mathcal{I}_1$, $\chi = m$, state.

Following the same procedure as was used to obtain (6.15), except that instead of multiplying by $(\partial_4 Q-\chi)(1-\mathcal{I}_0)$ one uses $(\partial_4 Q-\chi)\mathcal{I}_1$, one obtains

$$\mathcal{I}_1\psi + Z\mathcal{I}_1\psi = Y(\mathcal{I}_0+\mathcal{I}_2)\psi \ , \quad (6.16a)$$

$$Y = (\partial_4{}^2-\chi^2)^{-1}(-\partial_4\alpha_4+\chi)\mathcal{I}_1(\vec{\partial}\cdot\vec{\alpha}) \ , \quad (6.16b)$$

$$Z = (\partial_4{}^2-\chi^2)^{-1}(-\partial_4)\chi(\alpha_4-Q) \ . \quad (6.16c)$$

Now, the extra term is partially eliminated, since

$$Z\mathcal{I}_1 = 0 \ . \quad (6.17)$$

However, the term involving $Y\mathcal{I}_2$ is not zero. Thus, once the algebra involves $\mathcal{I}_j$ with $j>1$ this simple decoupling cannot be done, and the subsidiary-components Hamiltonian cannot simply have the form Eq. (2.17), as was the case in DKP.

However, one can still use Eq. (6.14) to place the subsidiary-components Hamiltonian equation into an identity. A fair amount of algebra allows one to obtain

$$\mathcal{H}_S \to \hat{\mathcal{H}}_S = (1-\mathcal{I}_0)[E+h](1-\mathcal{I}_0) \ , \quad (6.18a)$$

$$h = H[1-\chi^{-1}(\vec{\partial}\cdot\vec{\alpha})] - \chi^{-2}(\vec{\partial}\cdot\vec{\alpha})\,\alpha_4(\vec{\partial}\cdot\vec{\alpha})(\vec{\partial}\cdot\vec{\alpha}) \ , \quad (6.18b)$$

with H again given by Eq. (5.14). Then, as for DKP, one can show algebraically that

$$0 = (1-\mathcal{I}_0)h(1-\mathcal{I}_0) \ , \quad (6.19)$$

meaning $\hat{\mathcal{H}}_S$ is an identity in terms of the particle-components solution.

D. Single-mass-state ST reductions for integer spin

The particle-components equations (6.11) to (6.13) can be further reduced to obtain the individual mass-state equations. One starts with the analogs to Eqs. (6.6) and (6.7), noting that the subsidiary components have already been decoupled. The two equations are

$$E\mathcal{I}_j\psi = [\mathcal{I}_j\mathcal{H}_P\mathcal{I}_j + \mathcal{I}_j\mathcal{H}_P(1-\mathcal{I}_j-\mathcal{I}_0)]\psi \ , \quad (6.20)$$

$$E(1-\mathcal{I}_0-\mathcal{I}_j)\psi = [(1-\mathcal{I}_0-\mathcal{I}_j)\mathcal{H}_P(1-\mathcal{I}_0-\mathcal{I}_j) + (1-\mathcal{I}_0-\mathcal{I}_j)\mathcal{H}_P\mathcal{I}_j]\psi \ . \quad (6.21)$$

Explicitly putting Eq. (6.13) for $\mathcal{H}_P$ into Eq. (6.20), and using the results of Sec. III to eliminate most of the terms, one is left with

$$E\mathcal{I}_j\psi = \{\chi(\mathcal{Q}_j\mathcal{I}_j) + (\mathcal{Q}_j\mathcal{I}_j)(\vec{\partial}\cdot\vec{\alpha})[\mathcal{I}_{j+1}(1-\delta(j,\mathcal{S})) + \mathcal{I}_{j-1}(1-\delta(j,1))]\}\psi \ . \quad (6.22)$$

Equation (6.22) clearly shows that in the limit $\vec{p}=0$ the $\mathcal{I}_j$ are mass projection operators. In this case only the χ term remains on the right, and because of the definition of $\mathcal{Q}_j$ in Eq. (3.21), we see that

$$\mathcal{Q}_j\mathcal{I}_j^{\pm} = (\pm 1/j)\mathcal{I}_j^{\pm} \ , \quad (6.23)$$

the sign depending on which of the two $\mathcal{I}_j$ blocks one is in. Thus, the mass is χ/j.

For $\vec{p}\neq 0$, a χ/j mass state is coupled to $\chi/(j\pm 1)$ mass states, with special cases for $j=(1 \text{ or } \mathcal{S})$. Thus, for $\mathcal{S}$ mass states there will be an $\mathcal{S}\times\mathcal{S}$ matrix equation to solve to decouple the mass states. To see how this is done, we first consider the special case $\mathcal{S}=2$. Then we have

$$E\mathcal{I}_1\psi = [\chi(\mathcal{Q}_1\mathcal{I}_1) + (\mathcal{Q}_1\mathcal{I}_1)(\vec{\partial}\cdot\vec{\alpha})\mathcal{I}_2]\psi \ , \quad (6.24)$$

$$E\mathcal{I}_2\psi = [\chi(\mathcal{Q}_2\mathcal{I}_2) + (\mathcal{Q}_2\mathcal{I}_2)(\vec{\partial}\cdot\vec{\alpha})\mathcal{I}_1]\psi \ . \quad (6.25)$$

Equations (6.24) and (6.25) are easily solved to give the decoupled mass-state equations

$$E(\mathcal{J}_1\psi)=\mathcal{J}_1[\chi(\mathcal{Q}_1\mathcal{J}_1)+(\mathcal{Q}_1\mathcal{J}_1)(\vec{\partial}\cdot\vec{\alpha})(E-\chi\mathcal{Q}_2)^{-1}(\mathcal{Q}_2\mathcal{J}_2)(\vec{\partial}\cdot\vec{\alpha})\mathcal{J}_1](\mathcal{J}_1\psi)\ , \tag{6.26}$$

$$E(\mathcal{J}_2\psi)=\mathcal{J}_2[\chi(\mathcal{Q}_2\mathcal{J}_2)+(\mathcal{Q}_3\mathcal{J}_2)(\vec{\partial}\cdot\vec{\alpha})(E-\chi\mathcal{Q}_1)^{-1}(\mathcal{Q}_1\mathcal{J}_1)(\vec{\partial}\cdot\vec{\alpha})\mathcal{J}_2](\mathcal{J}_2\psi)\ . \tag{6.27}$$

Thus, what we see is a system of two equations which remind one of the ST system of equations for the DKP case. The second term on the right-hand side has two factors of $(\vec{\partial}\cdot\vec{\alpha})$, separated by an energy denominator. The generalization to $\mathcal{S}>2$ is clear. One will end up with terms having up to $\mathcal{S}$ factors of $(\vec{\partial}\cdot\vec{\alpha})$, separated by energy denominators. They will come because in a chain sequence all the mass states will be coupled, and uncoupling them involves eliminating the mass states one at a time.

E. Single-mass-state ST reductions for half-integer spin

Starting with Eqs. (6.6) and (6.7), we want to calculate $\mathcal{J}_jH\mathcal{J}_j$ and $\mathcal{J}_jH(I-\mathcal{J}_j)$ for some half-integer j. Taking H given by Eq. (5.2) and using Eq. (3.38), we have

$$\begin{aligned}\mathcal{J}_jH\mathcal{J}_j&=\mathcal{J}_j[\alpha_4^{-1}(\chi+\vec{\partial}\cdot\vec{\alpha})]\mathcal{J}_j\\&=\chi(\alpha_4^{-1}\mathcal{J}_j)+\alpha_4^{-1}\mathcal{J}_j(\vec{\partial}\cdot\vec{\alpha})\mathcal{J}_j\delta(j,\tfrac{1}{2})\ .\end{aligned} \tag{6.28a}$$

Similarly,

$$\begin{aligned}\mathcal{J}_jH(I-\mathcal{J}_j)=(\alpha_4^{-1}\mathcal{J}_j)(\vec{\partial}\cdot\vec{\alpha})[&\mathcal{J}_{j+1}(1-\delta(j,\mathcal{S}))\\&+\mathcal{J}_{j-1}(1-\delta(j,\tfrac{1}{2}))]\ ,\end{aligned} \tag{6.28b}$$

so that

$$\begin{aligned}E\mathcal{J}_j\psi=\{&\chi(\alpha_4^{-1}\mathcal{J}_j)\\&+(\alpha_4^{-1}\mathcal{J}_j)(\vec{\partial}\cdot\vec{\alpha})[\mathcal{J}_{j+1}(1-\delta(j,\mathcal{S}))+\mathcal{J}_j\delta(j,\tfrac{1}{2})\\&+\mathcal{J}_{j-1}(1-\delta(j,\tfrac{1}{2}))]\}\psi\ .\end{aligned} \tag{6.29}$$

Looking at (6.29) when $\vec{p}=0$, we see that the $\mathcal{J}_j$ again become mass-state projection operators. In that case, only the first term survives, and

$$\alpha_4^{-1}\mathcal{J}_j^{\pm}=\pm(1/j)\mathcal{J}_j^{\pm}\ , \tag{6.30}$$

the sign depending on which $\mathcal{J}_j$ box one is in, so that the mass is χ/j.

When $\vec{p}\neq 0$, we see that a particular $\mathcal{J}_j$ in general couples to the $\mathcal{J}_{j\pm1}$, with special restrictions for $j=(\frac{1}{2}$ or $\mathcal{S})$. Thus, one ends up with an $(\mathcal{S}+\frac{1}{2})\times(\mathcal{S}+\frac{1}{2})$ matrix equation to solve for the $\mathcal{S}+\frac{1}{2}$ individual uncoupled mass states.

To see how this is done, consider the special case $\mathcal{S}=\frac{3}{2}$. The two equations are

$$E\mathcal{J}_{1/2}\psi=[\chi(\alpha_4^{-1}\mathcal{J}_{1/2})+(\alpha_4^{-1}\mathcal{J}_{1/2})(\vec{\partial}\cdot\vec{\alpha})(\mathcal{J}_{1/2}+\mathcal{J}_{3/2})]\psi\ , \tag{6.31}$$

$$E\mathcal{J}_{3/2}\psi=[\chi(\alpha_4^{-1}\mathcal{J}_{3/2})+(\alpha_4^{-1}\mathcal{J}_{3/2})(\vec{\partial}\cdot\vec{\alpha})\mathcal{J}_{1/2}]\psi\ , \tag{6.32}$$

which are easily decoupled to yield the equations

$$E\mathcal{J}_{1/2}\psi=(\alpha_4^{-1}\mathcal{J}_{1/2})\{\chi+(\vec{\partial}\cdot\vec{\alpha})[1+(E-\chi\alpha_4^{-1})^{-1}(\alpha_4^{-1}\mathcal{J}_{3/2})(\vec{\partial}\cdot\vec{\alpha})]\}\mathcal{J}_{1/2}\psi\ , \tag{6.33}$$

$$E\mathcal{J}_{3/2}\psi=(\alpha_4^{-1}\mathcal{J}_{3/2})\{\chi+(\vec{\partial}\cdot\vec{\alpha})[E-\chi\alpha_4^{-1}-\alpha_4^{-1}\mathcal{J}_{1/2}(\vec{\partial}\cdot\vec{\alpha})]^{-1}(\alpha_4^{-1}\mathcal{J}_{1/2})(\vec{\partial}\cdot\vec{\alpha})\}\mathcal{J}_{3/2}\psi\ . \tag{6.34}$$

It is thus clear that, in general, if one is decoupling $(\mathcal{S}+\frac{1}{2})$ mass states, one will have terms in the decoupled equations with powers of $(\vec{\partial}\cdot\vec{\alpha})$ up to order $(\mathcal{S}+\frac{1}{2})$, and that between each of the factors $(\vec{\partial}\cdot\vec{\alpha})$ there will be energy denominator operators, such as those in Eqs. (6.33) and (6.34).

VII. DISCUSSION

The calculations performed in this paper have made clear the mass and spin content of the general Bhabha equations and algebras. Specifically, as discussed in Sec. IV, in a given so(5) representation, the mass eigenstates are decoupled in the rest system and each mass state has in general more than one spin solution. Outside of the rest system, the different possible mass states of a particular Bhabha equation are coupled, and to decouple them one uses the generalization of the Sakata-Taketani reduction, which has been the main theme of this paper.

Further, the physical significance of the built-in subsidiary components for integer-spin systems is understood as being the "infinite mass" solutions to the equations. They are there simply because the algebra matrices for integer spin have some eigenvalues which are zero, as does any integer so(5) angular momentum matrix.

It was also of special interest to find out that when a Sakata-Taketani reduction is made, decoupling these "subsidiary components" from the "particle components," then the separate subsidiary-components Hamiltonian equation is an identity in terms of the particle-components eigenvalue E. That is, all of the physics of the Hamiltonian formulation of integer-spin Bhabha fields is placed in the "particle components" alone by the ST decomposition.

This enlightens an interesting calculation by Iachello.[22] Recall that it has recently been found[23-25] that when there is symmetry breaking (the initial mass m is not equal to the final mass μ) in meson current processes, then the description of the process using the DKP formulation no longer yields the same results as the standard KG description. Iachello[22] calculated some of the same symmetry-breaking meson current quantities using the Sakata-Taketani particle components (i.e., what is commonly called the Feshbach-Villars equation[13]) and found that the same new results obtained using the DKP equation were obtained using the ST equation. In particular, using Eqs. (2.7) and (5.1) in paper I (Ref. 12) for the DKP and ST free solutions, one can verify Iachello's result that the expectation value of the density operator (fourth component of the current) is

$$\begin{aligned}\langle \rho \rangle &= \langle \bar{\psi}' \beta_4 \psi \rangle^{\text{DKP}} \\ &= \langle \psi' \tau_z \psi \rangle^{\text{ST}} \\ &= \left[\frac{1}{2V} \frac{1}{(EE')^{1/2}}\right] \left(\frac{mE' + \mu E}{m^{1/2}\mu^{1/2}}\right) e^{i(p-p')\cdot x} .\end{aligned} \tag{7.1}$$

If one writes this result in the rest frame of the initial particle ($E = m$), then one has that the crucial quantity in the large parentheses is

$$\left(\frac{mE' + \mu E}{m^{1/2}\mu^{1/2}}\right)\bigg|_{E=m} = \left[\frac{m+\mu}{m^{1/2}\mu^{1/2}}\right] \left\{ (m + E') + \left[\frac{(-m+\mu)}{(m+\mu)}\right](m - E') \right\} \left(\frac{1}{2}\right) . \tag{7.2}$$

The two quantities in the square brackets of (7.2) are those which produce the different results for the DKP formulation of K_{l3} decay. Explicitly, they are the quantity $(m+\mu)/(m^{1/2}\mu^{1/2}) \simeq 1.22$ which makes the Cabibbo angle smaller,[24] and the quantity $(-m+\mu)/(m+\mu)$ which adds the factor[23] (-0.57) to what one would have thought was the symmetry-breaking parameter ξ.

We can also comment on the type of ST operators one would have if the Harish-Chandra algebra relation Eq. (1.27) were used. We now can easily see that the relation (1.27) demands that the α_μ are the γ_μ (not $\frac{1}{2}\gamma_\mu$) for the Dirac case, that they are the β_μ for the DKP case, and that for higher spin the relation will have only three eigenvalues. In particular, the eigenvalues will always be 0 and ± 1 (as for DKP). Thus, the two Harish-Chandra ST operators would simply be

$$\mathcal{G}_0^{\text{HC}}(\mathcal{S}) = 1 - \alpha_\mu{}^2 , \tag{7.3}$$

$$\mathcal{G}_1^{\text{HC}}(\mathcal{S}) = \alpha_\mu{}^{2\mathcal{S}-1} . \tag{7.4}$$

$\mathcal{G}_1$ would project out the particle components of a single spin and mass. The high dimensional $\mathcal{G}_0$ would eliminate the equivalent of the rest of the Bhabha multiple mass and spin states (leaving the single one desired) as well as the equivalent of the infinite-mass subsidiary components in the integer-spin case. However, as we have pointed out, it has proven difficult[14, 15] to try to define a consistent arbitrary spin set of equations based on the Harish-Chandra condition (1.27).

Finally, since we have obtained the general Bhabha Hamiltonians, H, for both half-integer spin [Eq. (5.2)] and integer spin [Eq. (5.14)], the other Poincaré generators ($\vec{P}$, $\vec{J}$, and $\vec{K}$) are easily defined for an arbitrary Bhabha field. We can thus now show that these generators satisfy the expected Lie algebra. This will be done in paper III.[16]

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