Supersingular quantum perturbations

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A perturbation potential is called supersingular whenever generally every matrix element of the perturbation in the unperturbed eigenstates is infinite. It follows that supersingular perturbations do not have conventional perturbation expansions, say for energy eigenvalues. By invoking variational arguments, we determine the asymptotic behavior of the energy eigenvalues for asymptotically small values of the coupling constant of the supersingular perturbation.

I. INTRODUCTION

It is quite common to study an interacting quantum system by perturbation theory which expresses various properties of the full Hamiltonian $\mathcal{H}=\mathcal{H}_0+\lambda V$ in terms of those of the unperturbed Hamiltonian $\mathcal{H}_0$ and perturbation V by power series in the coupling constant λ. In addition to countless problems of practical importance studied in this fashion, significant theoretical knowledge and insight is available for the quartic anharmonic oscillator defined in suitable units by the Hamiltonian

$$\mathcal{H}=p^2+x^2+\lambda x^4, \tag{1}$$

where $p=-i\partial/\partial x$. In particular, it is known that the power series in λ that defines the energy levels for $\lambda>0$ is asymptotic, even though it is not convergent, and, moreover, that certain resummation techniques (e.g., Padé approximants) lead to convergent expressions for the energy levels.[1] Essential for any such calculation are the matrix elements of the potential in the unperturbed eigenstates, and for perturbations that leave discrete levels isolated from the continuum it has been stated by Kato that the series solution is asymptotic whenever the matrix elements of the potential can be defined.[2]

In the present paper we consider quite a different situation. By a supersingular potential we mean a potential so singular that generally *every* matrix element of the potential is infinite unless prohibited by symmetry considerations. As an example of this type we mention the Hamiltonian

$$\mathcal{H}=p^2+x^2+\lambda e^{x^4} \tag{2}$$

for which no conventional perturbation series exists. Our aim in this paper is to determine the leading asymptotic behavior of the energy levels of supersingular perturbations as a function of asymptotically small (positive) λ.

Quite generally, let us suppose that u denotes an eigenstate of $\mathcal{H}_0$ with energy E and ψ denotes an eigenstate of $\mathcal{H}=\mathcal{H}_0+\lambda V$ with energy $E(\lambda)$. We assume that ψ is associated with u in the sense that ψ goes to u and not, for example, to another eigenstate of $\mathcal{H}_0$ as $\lambda\to 0$. In a one-dimensional space such as we consider, it is necessary that u satisfy Dirichlet boundary conditions at the singular point of V, i.e., $u=0$ at the singular point. Then for some $\lambda_0>0$ and all λ, $0\le\lambda\le\lambda_0$, $\int u^*\psi\,dx\neq 0$, and it follows that

$$\begin{aligned} e(\lambda)&\equiv E(\lambda)-E \\ &=\frac{\int u^*(\mathcal{H}-\mathcal{H}_0)\psi\,dx}{\int u^*\psi\,dx} \\ &=\frac{\lambda\int u^*V\psi\,dx}{\int u^*\psi\,dx}. \end{aligned} \tag{3}$$

Consequently, whenever $\int u^*Vu\,dx/\int u^*u\,dx=\infty$ the leading behavior of $e(\lambda)$ is stronger as $\lambda\to 0$ than λ [i.e., $e(\lambda)/\lambda\to\infty$], but is not stronger than 1 [i.e., $e(\lambda)\not\to\infty$]. In fact, since we have imposed Dirichlet boundary conditions we will be concerned with continuous perturbations for which $e(\lambda)\to 0$.[3]

For the particular case of Eq. (2) we have shown that the energy deviation $e(\lambda)$ appropriate to the ground state is given for asymptotically small λ by the relation

$$e(\lambda)\simeq K(-\ln\lambda)^{1/4}\exp[-(-\ln\lambda)^{1/2}], \tag{4}$$

with an undetermined constant K, $0<K<\infty$. In principle, our method determines the constant K, but generally we only obtain an upper bound. More precisely, an equation such as (4) means that as $\lambda\to 0$,

$$e(\lambda)\exp[(-\ln\lambda)^{1/2}]/(-\ln\lambda)^{1/4}\to K, \tag{5}$$

but we prefer the heuristic form (4). It is noteworthy that relations (4) or (5) apply only to the ground state of Eq. (2); different asymptotic expressions apply for each of the excited states.[4]

In Sec. II we study a rather large class of unperturbed Hamiltonians and supersingular poten-

tials of the general form

$$\mathcal{H} = p^2 + w(x) + \lambda V(x), \tag{6}$$

each of which has a discrete spectrum. We consider various expressions for $w(x)$ and for $V(x)$ that may have supersingularities either at $x=\infty$ or at $x=0$ (which is representative of a supersingularity at any finite x), and we determine expressions for the asymptotic form of the energy deviation in each case.

In Sec. III we report the results of a numerical study that provides a satisfactory numerical confirmation of the asymptotic energy deviation for the ground state in the case that $w(x)=x^2$ and $V(x)=|x|^{-\alpha}$, $\alpha \geq 3$, and when we restrict x so that $0<x<\infty$. This calculation also yields values for the constant of proportionality [analog of K in Eq. (4)] that may be compared with our estimates in this case.

II. ANALYTICAL RESULTS

In evaluating the right-hand side of Eq. (3), one might attempt to approximate ψ by a WKB-type solution[5] of the form

$$\psi = [\lambda V(x)]^{-1/4} \exp\left\{ - \left| \int^x [\lambda V(x)]^{1/2}\, dx \right| \right\}, \tag{7}$$

where $w(x)$ and $E(\lambda)$ have been neglected relative to $\lambda V(x)$ since a solution is required in the region where $\lambda V(x)$ is very large. The condition for the validity of this solution is

$$[\lambda V(x)]^{1/2} \gg \frac{V'(x)}{V(x)}. \tag{8}$$

However, if one adopts the WKB solution the integrand $u^* V \psi$ is, for small λ, a strongly peaked function having its maximum generally where

$$[\lambda V(x)]^{1/2} = \frac{3}{4}\frac{V'(x)}{V(x)}. \tag{9}$$

It is therefore clear from condition (8) that WKB does not provide a good solution in the region of interest.

A more useful approach to determining ψ is provided by variational methods. We will assume the ground state solution ψ is of the form $\psi(x) = \eta(x)u(x)$, where $u(x)$ is the solution to the unperturbed problem, and we will determine $\eta(x)$ so as to minimize the energy $E(\lambda)$. The amount of information derived from this method depends on the choice of parameters used to describe $\eta(x)$. We will use a simple expression for η which allows us to determine the *functional form* $f(\lambda)$ of the λ dependence for the leading term in the expansion of $e(\lambda)$, but gives only an *upper limit* on the proportionality constant K [i.e., $e(\lambda) \leq Kf(\lambda)$].

To choose an appropriate form for $\eta(x)$ we need to know something about the effect of our perturbing potential λV on the solution. Characteristic of the supersingular potentials is an extremely steep region near the singularity and a negligible slope some distance away from the singularity. The details of this steep region depend on the particular potential but its *existence* is common to all such potentials. Therefore one is led to approximate these potentials by adding an infinite wall to the problem whose location x_0 is determined by the potential. Noting that to be dimensionally consistent with the ever-present p^2 part of the Hamiltonian $\mathcal{H}_0$, λV must have the dimensions (length)$^{-2}$, we determine the point x_0 at which the perturbation begins to look like an infinite wall from the condition

$$\lambda V(x_0) = 1/x_0^2. \tag{10}$$

The problem is now reduced to finding the eigenvectors of the Hamiltonian $\mathcal{H} = \mathcal{H}_0 + \lambda V$, with an infinite wall introduced at the point x_0. As discussed above, we will assume a solution ψ to this problem of the form

$$\psi(x) = \eta(x)u(x), \tag{11}$$

where u is an eigenstate of $\mathcal{H}_0$, and we will determine η by variational means. Far away from x_0 we expect $\psi \simeq u$ and $\eta \simeq 1$. At x_0, $\psi(x_0) = \eta(x_0) = 0$ because of the infinite wall, so $\eta(x)$ must drop from 1 to 0 over some region Δ near x_0. We distinguish two cases, according to whether the singularity is at $x=0$ or $x=\infty$. In the first case, the problem is approximated by adding to the Hamiltonian a left wall $w_L(x/x_0)$ of the form

$$\begin{aligned} w_L(x/x_0) &= \infty, \quad x < x_0 \\ &= 0, \quad x > x_0 \end{aligned} \tag{12}$$

and assuming η has the form

$$\begin{aligned} \eta(x) &= 0, \quad x \leq x_0 \\ &= 1, \quad x \geq x_0 + \Delta. \end{aligned} \tag{13}$$

A singularity at plus infinity is treated similarly by adding a right wall $w_R(x/x_0)$ to the Hamiltonian, where

$$\begin{aligned} w_R(x/x_0) &= 0, \quad x < x_0 \\ &= \infty, \quad x > x_0 \end{aligned} \tag{14}$$

and requiring

$$\begin{aligned} \eta(x) &= 1, \quad x \leq x_0 - \Delta \\ &= 0, \quad x \geq x_0. \end{aligned} \tag{15}$$

With these modifications the argument in the two cases is similar. We discuss the singularity at $x=0$ in detail, pointing out the corresponding results for a singularity at $x=\infty$ where appropriate.

Our objective is to evaluate the leading term of the λ dependence for the expression

$$e(\lambda) = E(\lambda) - E \simeq \frac{\int_0^\infty \psi^*[\mathcal{H} + w_L(x/x_0)]\psi dx}{\int_0^\infty \psi^*\psi\, dx} - E\,. \tag{16}$$

We first consider the integral in the numerator of Eq. (16) and take ψ to be real:

$$\int_0^\infty \psi^*[\mathcal{H} + w_L(x/x_0)]\psi\, dx = \int_{x_0}^{x_0+\Delta} [(\psi')^2 + \psi^2 w]dx + \int_{x_0+\Delta}^{\infty} [(u')^2 + u^2 w]dx + \int_{x_0}^{\infty} \psi^2 \lambda V dx\,. \tag{17}$$

Here we have used the definition of η to separate the domain of integration into appropriate regions. Since $x_0 + \Delta$ is a small number, the second integral is very nearly the unperturbed eigenvalue E. In the last integral the integrand becomes very small far away from x_0 and we may replace the upper limit by some appropriate "large" constant c. To evaluate the first integral we note that, by our definition of η,

$$\eta' \simeq 1/\Delta\,, \qquad \eta \simeq 1 \tag{18}$$

in the region between x_0 and $x_0 + \Delta$, giving

$$\int_{x_0}^{x_0+\Delta} [(\psi')^2 + \psi^2 w]dx = \int_{x_0}^{x_0+\Delta} (\eta^2 u'^2 + 2\eta\eta' uu' + \eta'^2 u^2 + \eta^2 u^2 w)\, dx$$

$$\simeq \Delta[A\,u'(x_0)^2 + 2B(1/\Delta)u(x_0)u'(x_0) + C(1/\Delta^2)u(x_0)^2 + A\,u(x_0)^2 w(x_0)]\,, \tag{19}$$

where we have introduced the constants A, B, and C to take into account the details of the shape of η^2. (Although the optimum shape of η could in principle also be determined by variational method, fixing the values of A, B, and C, we have considered only a linear dependence of η on x. This gives an upper limit of $\frac{1}{3}$ for the value of A, $\frac{1}{2}$ for B, and 1 for C. We will therefore choose $A = B = C = 1$ for the remainder of our discussion, noting that we have thus determined an upper limit on the proportionality constant K in what follows.)

With these substitutions, Eq. (17) becomes

$$\int_0^\infty \psi^*[\mathcal{H} + w_L(x/x_0)]\psi\, dx \leq E + \Delta[u'(x_0)^2 + u(x_0)^2 w(x_0)] + 2u(x_0)u'(x_0) + \frac{1}{\Delta}u(x_0)^2 + \lambda\int_{x_0}^{c} u^2 V dx\,. \tag{20}$$

Choosing Δ to minimize this expression we find

$$\frac{1}{\Delta^2} = \left[\frac{u'(x_0)}{u(x_0)}\right]^2 + w(x_0)\,. \tag{21}$$

When similarly treated, the normalization integral

$$\int_0^\infty \psi^*\psi\, dx \tag{22}$$

yields a lower-order correction and may therefore be ignored in the above treatment.

Substituting Eq. (20) into (16), and using Eqs. (10) and (21) to express x_0 and Δ as functions of λ, we now have the desired expression for $e(\lambda)$,

$$e(\lambda) \leq 2u(x_0)\{|u'(x_0)| + [u'(x_0)^2 + u(x_0)^2 w(x_0)]^{1/2}\} + \lambda\int_{x_0}^{c} u^2 V dx\,. \tag{23}$$

[The absolute value sign on $u'(x_0)$ has been introduced to emphasize that this solution is independent of whether the point of singularity is a left or a right boundary of the domain under consideration.]

In general, the contribution of the final term in Eq. (23) is $O(\lambda)$, which is negligible relative to the first term and it may therefore be dropped. For most of the remainder of this section we assume that this is the case. However, in Eqs. (31) and (32) we discuss a particular $V(x)$ for which this term plays a significant role.

For the case of a singularity at ∞ the sign of η' changes and the last term in Eq. (20) becomes

$$\lambda\int_{c}^{x_0} u^2 V dx\,, \tag{24}$$

where c is now some appropriate "small" number [again this term is $O(\lambda)$ and in general negligible]. Repeating the above argument and noting that $-u'(x_0) = |u'(x_0)|$ yields

$$e(\lambda) \leq 2u(x_0)\{|u'(x_0)| + [u'(x_0)^2 + u(x_0)^2 w(x_0)]^{1/2}\} + \lambda \int_c^{x_0} u^2 V dx . \quad (25)$$

For an infinite square well we can compare the predictions of Eq. (23) with the exact calculations. We let $w(x)$ be

$$w(x) = 0, \quad 0 < x < L$$
$$= \infty, \quad x < 0, \ x > L \quad (26)$$

and ask for the change in the ground-state energy level when we introduce an infinite wall $w_L(x/x_0)$ near $x = 0$. The exact calculation gives

$$\frac{\pi^2}{(L - x_0)^2} - \frac{\pi^2}{L^2} \simeq \frac{2\pi^2}{L^3} x_0 . \quad (27)$$

From Eq. (23), taking $u(x_0) = (2\pi/L)^{1/2} \sin(\pi x_0/L) \simeq (2\pi/L)^{1/2}(\pi x_0/L)$, we find

$$8 \frac{\pi^3}{L^3} x_0 = K x_0 . \quad (28)$$

This is in agreement with the form of Eq. (27), and gives an upper limit for K equal to 4π times the true value.

We now consider the more general case where $w(x) = |x|^\xi$ for various values of ξ and perturbations singular at the origin or infinity. Since the exact form of $u(x)$ is known for $\xi = 2$ we will treat that case first. For a perturbation singular at the origin [e.g., $V(x) = 1/|x|^\alpha$, $V(x) = e^{1/x^2}$], Dirichlet boundary conditions require that we use the first excited state of the harmonic oscillator, $u(x) = (2/\pi^{1/4}) x e^{-x^2/2}$ for our unperturbed ground state. Substituting this expression into Eqs. (21) and (23) and using the approximations, valid for small x_0, $u(x_0) \simeq (2/\pi^{1/4}) x_0$, $u'(x_0) \simeq 2/\pi^{1/4}$, we find $\Delta = x_0$ and thus

$$e(\lambda) \simeq K x_0(\lambda) , \quad (29)$$

with $K \leq 16/\sqrt{\pi} \simeq 9$, whenever the last term in (23) can be ignored. For $V(x) = e^{1/x^2}$ this becomes, using the definition of x_0 from Eq. (10),

$$e(\lambda) \simeq K/(-\ln\lambda)^{1/2} . \quad (30)$$

Another example of this case is given by $V(x) = 1/|x|^\alpha$, $\alpha \geq 3$. This case is interesting in that the contribution from the last term in Eq. (23) is not negligible, and in fact for $\alpha = 3$ it is the dominant term. When we evaluate this term for $\alpha = 3$ we get, using the value $x_0 = \lambda$ from (10),

$$e(\lambda) \simeq K x_0 + \lambda \int_{x_0}^{c} u^2 V dx \simeq K x_0 + \lambda \frac{4}{\sqrt{\pi}} \int_{x_0}^{c} \frac{1}{x} dx = -\frac{4}{\sqrt{\pi}} \lambda \ln\lambda + C'\lambda . \quad (31)$$

For $\alpha > 3$, the integral contributes one term of order λ and one which behaves like $x_0 = \lambda^{1/(\alpha-2)}$. Therefore, whenever $\alpha > 3$, Eq. (29) is valid, with a slightly larger upper limit for K and

$$e(\lambda) \simeq K\lambda^{1/(\alpha-2)} . \quad (32)$$

For $2 < \alpha < 3$, the $O(\lambda)$ term dominates, in agreement with Kato's predictions for a perturbation with finite matrix elements.

We remark that inverse power potentials frequently arise in scattering theory[6] in which the Hamiltonian reads $\vec{p}^2 + \lambda/|\vec{x}|^\alpha$. Simple rescaling ($\vec{x} \to S\vec{x}$, $\vec{p} \to S^{-1}\vec{p}$, $S^{\alpha-2} = |\lambda|$) recasts this relation into the equivalent form $|\lambda|^{2/(2-\alpha)}(\vec{p}^2 \pm 1/|\vec{x}|^\alpha)$, which exhibits the complete dependence of the Hamiltonian on λ for any $\alpha \neq 2$.

For a perturbation singular at infinity the appropriate unperturbed solution is $u(x) = \pi^{-1/4} e^{-x^2/2}$, and we find

$$e(\lambda) \simeq K x_0(\lambda) e^{-[x_0(\lambda)]^2} , \quad (33)$$

with $K \leq 2(1 + \sqrt{2})/\sqrt{\pi} \simeq 2.7$. An example of this case has already been given in Eqs. (2) and (4).

In general, it can be shown that whenever $w(x) = x^\xi$, with $\xi > -2$, the lowest-energy unperturbed solution obeying Dirichlet boundary conditions at the origin behaves like $u(x) \simeq x$ for small x. Therefore Eq. (29) holds for perturbations singular at the origin whenever $w(x) = x^\xi$, $\xi > -2$. For $\xi < -2$ we can use the WKB method to obtain an approximate solution near the origin of the form

$$u(x) \simeq x^{-\xi/4} \exp[2/(\xi + 2) x^{(\xi+2)/2}] , \quad (34)$$

giving

$$e(\lambda) \simeq K \exp\{4/(\xi + 2)[x_0(\lambda)]^{(\xi+2)/2}\} . \quad (35)$$

If $w(x) = C/x^2$ ($\xi = -2$), the solution $u(x)$ is a Bessel function which behaves near the origin like $A x^{[(4C+1)^{1/2}+1]/2}$, giving

$$e(\lambda) \simeq K[x_0(\lambda)]^{(4C+1)^{1/2}} . \quad (36)$$

For perturbations singular at infinity, we consider only the case $w(x) = x^\xi$, with $\xi > 2$. For $\xi > 2$ the WKB method gives

$$u(x) \simeq x^{-\xi/4} \exp[-2/(\xi + 2) x^{(\xi+2)/2}], \quad (37)$$

and we have from Eq. (25)

$$e(\lambda) \simeq K \exp[-4/(\xi + 2) x_0^{(\xi+2)/2}]. \quad (38)$$

Other choices of $w(x)$ may be treated in the same manner.

III. NUMERICAL RESULTS

In this section we discuss the numerical determination of values of $e(\lambda)$ for the case $w(x) = x^2$,

TABLE I. Ground-state energy levels (less 3) $e(\lambda)$ of the Hamiltonian $p^2+x^2+\lambda/|x|^\alpha$, for $0<x<\infty$, for indicated λ and α values.

λ \ α	3.0	3.5	4.0	4.5	5.0	5.5	6.0
0.01	0.075 83	0.134 86	0.205 27	0.280 32	0.356 80	0.432 33	0.505 74
0.005	0.044 91	0.089 85	0.148 39	0.214 46	0.283 54	0.353 65	0.423 02
0.0025	0.025 88	0.059 12	0.106 70	0.163 45	0.225 38	0.289 27	0.353 95
0.001 25	0.014 53	0.038 30	0.076 24	0.124 26	0.178 70	0.236 84	0.296 19
0.000 625	0.008 16	0.024 02	0.054 01	0.094 03	0.141 43	0.193 47	0.247 88
0.000 312 5	0.004 71	0.014 53	0.037 07	0.070 40	0.111 44	0.157 71	0.207 39

$V(x)=1/|x|^\alpha$ for several values of $\alpha \geq 3$ and compare the results with the analytical expressions given by Eqs. (31) and (32).

Using a computer program kindly supplied by H. Ezawa, we have applied the Milne method[7] to find eigenvalues $E(\lambda)$ of the equation $\mathcal{H}\psi = E(\lambda)\psi$, where $\mathcal{H}=p^2+x^2+\lambda/|x|^\alpha$, for various values of λ and α. The resulting values of $e(\lambda)=E(\lambda)-E$ (with $E=3$ for the ground state) are listed in Table I.

In order to determine values of K and C which best fit the equation

$$e(\lambda)=K\lambda^C \tag{39}$$

predicted analytically (for $\alpha>3$), we used the linear least-squares method to fit the data to the equation

$$\ln e(\lambda)=C\ln\lambda+\ln K. \tag{40}$$

For $\alpha=3$ the data were fitted to the equation

$$e(\lambda)/\lambda=K\ln\lambda+C. \tag{41}$$

Here we have kept the λ term from Eq. (31), although the $\lambda\ln\lambda$ term is dominant asymptotically, because the λ term does not become negligible until $\lambda \lesssim 10^{-5}$, which is smaller than the values we have used.

In Table II the values of K and C numerically obtained for various values of α are displayed, along with the analytically predicted values for C, namely $1/(\alpha-2)$, for $\alpha>3$. We find reasonable agreement between the analytical and numerical values of C (about 1%) and values of K which are well below the upper limits found in Sec. II.

IV. SUMMARY

We have derived a general formula for the asymptotic behavior of the lowest-energy eigenvalue $E(\lambda)$ of a one-dimensional Hamiltonian $\mathcal{H}=\mathcal{H}_0+\lambda V$ (with V being a supersingular perturbation) which may be applied whenever the behavior of the corresponding eigenstate u of $\mathcal{H}_0$ is known near the singularity. This correspondence between the eigenstate ψ of $\mathcal{H}$ and the eigenstate u of $\mathcal{H}_0$ is defined by the requirement that $\psi \to u$ as $\lambda \to 0$. In a one-dimensional space, this requirement is satisfied for ψ, the ground state of $\mathcal{H}$, by choosing u to be the lowest-energy eigenstate of $\mathcal{H}_0$ which satisfies Dirichlet boundary conditions at the point where V is singular.

The use of this formula has been demonstrated in several particular cases for $\mathcal{H}_0=p^2+|x|^\xi$, with various choices of ξ and perturbations V singular at either $x=0$ or $x=\infty$. In the case of $\mathcal{H}_0=p^2+x^2$, $V=1/|x|^\alpha$, $\alpha \geq 3$, we have confirmed these results by a numerical calculation reported in Sec. III.

Techniques such as those developed here may possibly be extended to higher-dimensional problems and to excited states. It would be interesting to look for a practical iterative procedure to determine higher-order terms in the development of the energy in λ. Application of such results could be made in singular perturbation theory and perhaps in quantum field theory.

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TABLE II. Coefficients K and C for best fit of data of Table I to $e(\lambda)=K\lambda^C$.

α	K	C	$1/(\alpha-2)$
3.0	−2.100 00 [a]	−2.198 39 [a]	···
3.5	2.665 66	0.640 23	0.666 67
4.0	2.006 08	0.491 57	0.500 00
4.5	1.765 68	0.398 05	0.400 00
5.0	1.677 00	0.335 41	0.333 33
5.5	1.650 11	0.290 66	0.285 71
6.0	1.652 55	0.257 16	0.250 00

[a] Fit to $e(\lambda)=K\lambda\ln\lambda+C\lambda$.

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[1]B. Simon, Ann. Phys. (N.Y.) 58, 76 (1970).

[2]T. Kato, J. Fac. Sci., Univ. Tokyo, Sect. I, 6, part 3, 145 (1951). This article has several interesting examples of nonasymptotic perturbations. See also T. Kato, *Perturbation Theory for Linear Operators* (Springer, New York, 1966). This is one of the classic texts dealing with perturbation theory, especially perturbations that are less singular than that represented by the anharmonic oscillator.

[3]If Dirichlet boundary conditions are not imposed on the eigenfunctions of $\mathcal{H}_0$ at the singular point of V, then V generally represents a *discontinuous* perturbation of $\mathcal{H}_0$ in the sense that $\psi \to \psi_0 \neq \boldsymbol{u}$ and $e(\lambda) \to e(0) \neq 0$. Here, ψ_0 retains the Dirichlet boundary conditions forced on ψ by V for all $\lambda > 0$, and $e(0)$ is the eigenvalue appropriate to such boundary conditions. Problems related to discontinuous perturbations are discussed by J. R. Klauder, in *Recent Developments in Mathematical Physics*, proceedings of the XII Schladming conference on nuclear physics, edited by P. Urban (Springer, Berlin, 1973) [Acta Phys. Austriaca Suppl. 11, (1973)], p. 341, and by B. Simon, J. Funct. Anal. 14, 295 (1973), with more emphasis on the mathematics. In this paper we choose to avoid the complications of discontinuous perturbations, and we assume that the unperturbed Hamiltonian $\mathcal{H}_0$ has been chosen to be consistent with any boundary conditions that the introduction of V would require.

[4]V. P. Maslov [*Theorie des Perturbations et Methods Asymptotiques*, translated by J. Lascoux and R. Seneor (Dunod, Paris, 1972)] considers singular perturbations in great detail. In particular, he discusses (p. 8) the example $p^2 + x^2 + x^2(e^{\epsilon x^4} - 1)$ and shows that despite divergent matrix elements, the leading corrections to the (harmonic oscillator) energies are $O(\epsilon)$. This example involves important changes of the functional form of the perturbation, and is thus rather different from the example of Eq. (2).

[5]See any elementary quantum-mechanics text; for example, L. I. Schiff, *Quantum Mechanics* (McGraw-Hill, New York, 1968), 3rd edition, pp. 268–279.

[6]See, e.g., W. M. Frank, D. J. Land, and R. M. Spector, Rev. Mod. Phys. 43, 36 (1971).

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