# Quantum field theory in Schwarzschild and Rindler spaces*

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The problem of defining a scalar quantum field in the space-times described by the Schwarzschild and Rindler metrics is discussed. The matrix elements of the field operators are found by calculating the Green's functions for the fields. The requirement of positive frequencies for *asymptotic* timelike separations combined with a careful analysis of the continuity conditions at the event horizons yields a unique prescription for the Green's function. This in turn defines the vacuum state. In the Schwarzschild space the vacuum is shown to be stable and the lowest-energy state. In the Rindler space the quantization procedure yields the same results as quantization in Minkowski coordinates.

## I. INTRODUCTION

The problem of defining a quantum field theory in an arbitrary Riemannian space-time has been the object of some controversy over the years.[1] To the extent that only excitations which are well localized compared to the characteristic scale of the curvature in the region studied are considered, conventional treatments must hold when locally Minkowskian coordinates are used; however, attention cannot be restricted to such situations and, in general, an excitation which is of positive frequency with respect to one coordinate system will not be so with respect to others. This is most strikingly exhibited in the case of the coordinates describing Rindler space which are just comoving coordinates for an observer uniformly accelerated with respect to the usual (flat) Minkowski space: There the imposition of positive-frequency boundary conditions on the Green's function does not yield the usual flat-space quantization.[2] An analogous problem arises in the case of the Schwarzschild metric which describes a space with explicit time dependence of the metric in the regions inside the event horizons. Further, the metric develops a singularity at $r=0$ which is in the future (or past) of the exterior region. In this case the metric does not, globally, become asymptotically static, so that the excitations and vacuum state may be defined relative to a fixed background metric.

Recently, Hawking[3] has published a calculation of the radiation from a spherically symmetric black hole which indicates that there is a steady flux of particles apparently coming from the final stages of the collapse process. The purpose of this work is to show how to calculate the Green's functions and, thereby, the matrix elements of a scalar quantum field. In the process, the vacuum is defined and is then proved to be stable and the lowest-energy state. There are, therefore, no particles emitted by a primordial black hole. It is also shown that no source acting within the future event horizon produces any particles. Thus, the radiation predicted by Hawking, although independent of the details of the collapse process, depends upon the existence of the collapse.

As a check on the method, it is applied to the quantization of a scalar field in Minkowski space described by Rindler coordinates. There, in striking contrast to the results of a naive application of positive-frequency boundary conditions, it yields the usual quantization.

In both the Schwarzschild coordinates and Rindler coordinates, one starts with a coordinate patch which describes only a portion of the full space-time and has an event horizon at the boundary of the coordinate patch. In Minkowski space-time the operator ring whose support is restricted to *any* open space-time region provides a complete operator basis; hence quantization in the original patch might be expected to determine the quantization in the complete extension of the space. This does not hold for the Schwarzschild space-time because the second exterior region is both an infinite number of wavelengths away from the first and necessary to determine the quantization in the interior region. It is, therefore, necessary to consider the quantization in the maximal analytic extension of the original space-time. In the case of Rindler coordinates, it is also necessary to consider the full space so that an infinite time separation can be considered.

Two problems then arise in attempting to construct the Green's functions: (a) The boundary conditions across the event horizons must be found, and (b) the analog of the positive-frequency boundary conditions in Minkowski space must be found. The boundary conditions across the event horizons are fairly straightforward to discuss; there are no real singularities there, only coordinate singularities, and, in terms of coordinates which are valid across the event horizon, the field must satisfy the wave equation. The "positive-frequency" boundary conditions are much more subtle since it is their imposition which defines the Green's functions (matrix elements of the field) and, therefore, the vacuum state itself. In both the maximally extended Schwarzschild and Rindler spaces it turns out that limits may be taken in which the separation of the two points of the Green's function is definitely timelike and arbitrarily large. Thus, an unambiguous positive-frequency specification may be made, and the resulting Green's function may be defined to be the positive-frequency Green's function which, in turn, defines *both* the vacuum and the field matrix elements.

From this Green's function, the particle behavior may be calculated and, in addition, the probability of the initial vacuum developing into the final vacuum *without* the appearance of any particles may be calculated; it is 1, i.e., no particles are created.

Further, for the case of an otherwise free field the expectation value of its stress-energy tensor density, $T^{\mu\nu}$, may be calculated. Just as in Minkowski space, it is divergent and must be renormalized. The counterterms required to yield a finite stress-energy tensor are all real and prevent any conclusions being drawn as to the positivity of the stress-energy tensor or the validity of the singularity theorems of Hawking and Penrose,[4] but do not affect the imaginary part of the matrix elements of $T^{\mu\nu}$. The imaginary part of the matrix element cannot be changed by the renormalization and can be calculated directly; it vanishes, establishing that the vacuum defined by the quantization process is stable. Further, the energy, relative to a spacelike surface passing through the exterior regions of additional particles in the vacuum, can be calculated and is shown to be positive; thus the vacuum is also the lowest-energy state of the system. Finally, the radiation produced by a source acting solely within the future event horizon is shown to vanish.

## II. THE WAVE EQUATIONS

### A. Schwarzschild space

For the Schwarzschild space, the metric in Schwarzschild coordinates is given by (units in which $\hbar = c = G = 1$ are used)

$$ds^2 = -(1-2M/r)dt^2 + dr^2/(1-2M/r) + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{2.1}$$

for the entire space-time. There is a coordinate singularity at $r = 2M$ and a real singularity at $r = 0$. The maximal analytic extension of the metric has been given by Kruskal[5] and the full space-time is shown in the Kruskal diagram of Fig. 1. Region I is the ordinary exterior region, $r > 2M$, while F is the future interior region, $r < 2M$, P is the past interior region, and II is a second exterior region, every point of which is spacelike with respect to every point of I.

The metric has the same functional form in each of the four regions and $t$ runs from $-\infty$ to $+\infty$ in each region. In region I (II), $t$ is a timelike coordinate and the direction of increasing $t$ is toward later (earlier) proper times. In regions F (P) $r$ is the timelike coordinate and decreasing $r$ is the direction of later (earlier) times; the $t$ coordinate is the spacelike coordinate. In terms of the coordinates $\tau = t \pm [2M \ln(|r-2M|/2M) + r]$ and $r$, the metric is regular across the lines $r = 2M$, $t = \pm\infty$, respectively, and any physical quantity expressed in these coordinates must be continuous across the corresponding event horizon.

The wave equation for a scalar field of mass $m$ is

$$\begin{aligned}[-\partial_\mu g^{\mu\nu}\sqrt{-g}\,\partial_\nu + m^2\sqrt{-g}]\,\phi(x) &= 0 \\ &= \left[\frac{\partial^2}{\partial t^2}\frac{r^3\sin\theta}{r-2M} - \frac{\partial}{\partial r}r(r-2M)\sin\theta\frac{\partial}{\partial r} - \frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} - \frac{\partial^2}{\partial\phi^2} + m^2r^2\sin\theta\right]\phi(r,\theta,\phi,t).\end{aligned} \tag{2.2}$$

This may be expanded in spherical harmonics, and Fourier-transformed with respect to $t$,

$$\phi(x) = \sum_{l,m} Y_l^m(\theta,\phi)\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega t}\phi(r,\omega,l,m), \tag{2.3}$$

and $\phi(r,\omega,l,m)$ must obey

$$\left[-\frac{\partial}{\partial r}r(r-2M)\frac{\partial}{\partial r} + l(l+1) + m^2r^2 - \frac{\omega^2r^3}{r-2M}\right]\phi(r,\omega,l) = 0 \tag{2.4}$$

in each of the four regions I, II, P, and F.[6] This

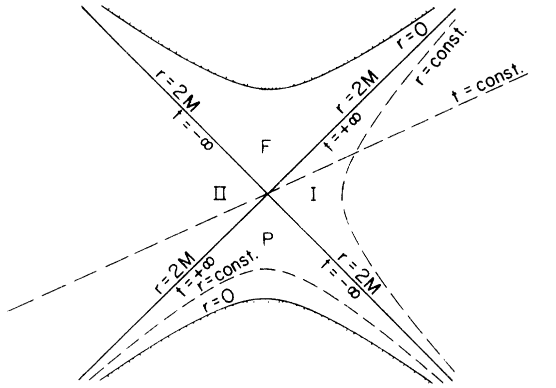


FIG. 1. Kruskal diagram for the Schwarzschild metric. All 45° lines are lightlike. Straight lines passing through the origin are $t$ =constant surfaces, and $r$ =constant surfaces are hyperbolas whose asymptotes are the $r=2M$ lines.

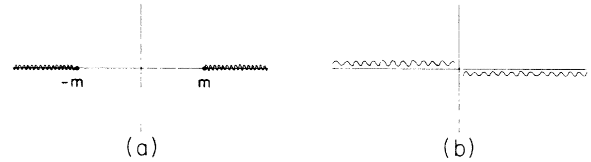


FIG. 2. The cut $\omega$ plane in which the solutions are analytic.

equation, as $r\to\infty$ in either I or II, becomes equivalent to the Schrödinger equation for a charged particle moving in the Coulomb field of a nucleus, and the solutions either behave as

$$\phi^l(r,\omega^2)\underset{r\to\infty}{\sim}\frac{\exp(+i\{qr+[(2\omega^2-m^2)M/q]\ln r\})}{qri^{l+1}}, \tag{2.5}$$

or as the complex conjugate, where $q^2=\omega^2-m^2$. The equation is analytic in $\omega$, and the function $q\equiv(\omega^2-m^2)^{1/2}$ is analytic in the cut plane shown in Fig. 2(a) with Im$q>0$ on the first sheet. Thus $\phi^l(r,\omega^2)$, defined as the solution to the wave equation, (2.4), which behaves as indicated in Eq. (2.5), goes to zero as $r\to\infty$.

For complex $\omega$, the other solution becomes infinite as $r\to\infty$; it is most conveniently characterized in terms of its behavior near $r=2M$, which is a regular singular point. The solution must behave as

$$\psi^l(r,\omega)\underset{r\to 2M}{\sim}(|r-2M|/2M)^{-2i\omega M}. \tag{2.6}$$

Since the equation is analytic and even in $\omega$, $\psi(r,-\omega)$ must also be a solution. The two solutions are linearly independent; hence, since $\phi^l(r,\omega^2)$ is even in $\omega$,

$$\phi^l(r,\omega^2)=\alpha^l(-\omega)\psi^l(r,\omega)+\alpha^l(\omega)\psi^l(r,-\omega). \tag{2.7}$$

Given any two solutions of the wave equation, the Wronskian,

$$W(\phi^1,\phi^2)\equiv r(r-2M)\left(\phi^1\frac{d}{dr}\phi^2-\frac{d\phi^1}{dr}\phi^2\right), \tag{2.8}$$

must be a constant. It is straightforward to calculate

$$W(\psi^l(r,\omega),\psi^l(r,-\omega))=4i\omega M(2M) \tag{2.9}$$

and

$$W(\psi^l(r,\omega),\phi^l(r,\omega^2))=2i\omega(2M)^2\alpha^l(\omega). \tag{2.10}$$

The complex conjugate of the wave equation is the wave equation for $\omega^{*2}$ rather than $\omega^2$; hence $[\phi^l(r,\omega^{*2})]^*=(-1)^l\phi^l(r,\omega^2)$, $[\psi^l(r,\omega^*)]^*=\psi^l(r,-\omega)$, and $[\alpha^l(\omega^*)]^*=(-1)^l\alpha^l(-\omega)$. For real $\omega$, $q(\omega-i\epsilon)=-q(\omega+i\epsilon)$ and

$$W(\phi(r,[\omega-i\epsilon]^2),\phi(r,[\omega+i\epsilon]^2))=2i(-1)^l/q$$
$$=2i\omega(2M)^2[\alpha^l(-\omega+i\epsilon)\alpha^l(\omega+i\epsilon)-\alpha^l(\omega-i\epsilon)\alpha^l(-\omega-i\epsilon)],$$

or

$$|\alpha^l(\omega+i\epsilon)|^2=[\omega q(\omega)(2M)^2]^{-1}+|\alpha^l(-\omega-i\epsilon)|^2. \tag{2.11}$$

In the interior regions, P and F, there are the two solutions $\psi^l(r,\pm\omega)$; the equation has a regular singular point at $r=0$, but the appropriate boundary conditions there are not known. In F (P) it is a singularity in the future (past) and it is not possible to specify, *a priori*, what the wave function should do there; it also turns out to be unnecessary. (Persides[6] chooses the regular solution; this is wrong.)

The boundary conditions across the event horizons are simple in terms of $\phi(r,t)$; however, the Fourier transform $\tilde{\psi}(r,\omega)$ is the integral of $\phi(r,t)$ over all $t$; hence it contains information about *both* boundaries. In order to recover the boundary conditions, consider a wave packet

$$\phi_f^{\pm}(r,t)=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega t}\psi^l(r,\pm\omega)\tilde{f}(\omega), \qquad (2.12)$$

where

$$\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega t}\tilde{f}(\omega)=f(t)e^{-i\omega_0 t}$$

and $f(t)$ forms a well-localized wave packet. Then, for $r\simeq 2M$,

$$\phi_f^{\pm}(r,t)\simeq \exp\left[-i\omega_0\left(t\pm 2M\ln\frac{|r-2M|}{2M}\right)\right] \times f\left(t\pm 2M\ln\frac{|r-2M|}{2M}\right),$$

and the $\psi^l(r,\pm\omega)$ solution describes a wave packet which propagates across the $r=2M$, $t=\pm\infty$ event horizon but vanishes at the other event horizon.

In order to investigate the behavior of the wave packet as $r\to\infty$, write

$$\psi^l(r,\omega)=\omega(2M)^2q(\omega)\{[\alpha^l(-\omega-i\epsilon)]^*\phi^l(r,(\omega+i\epsilon)^2)-[\alpha^l(-\omega+i\epsilon)]^*\phi^l(r,(\omega-i\epsilon)^2)\}, \qquad (2.13)$$

then

$$\begin{aligned}\phi_f^l(r,t)\underset{r\to\infty}{\simeq}&[\omega_0(2M)^2/i^{l+1}q_0r]\\ &\times\Big\{[\alpha^l(-\omega_0-i\epsilon)]^*\exp(i\{q_0r-\omega_0t-M[(m^2+2q_0^2)/q_0]\ln r\})f\left(t-(\omega_0/q_0)r^*-\frac{d}{d\omega_0}\arg[\alpha^l(-\omega_0-i\epsilon)]^*\right)\\ &-(-1)^l[\alpha^l(-\omega_0+i\epsilon)]^*\exp(-i\{q_0r+\omega_0t-M[(m^2+2q_0^2)/q_0]\ln r\})\\ &\times f\left(t+(\omega_0/q_0)r^*-\frac{d}{d\omega_0}\arg[\alpha^l(-\omega_0+i\epsilon)]^*\right)\Big\}, \qquad (2.14)\end{aligned}$$

where

$$r^*=r+2M[1-(m^2/2q_0^2)]\ln r.$$

In the distant past the second term, for $\omega_0>0$, consists solely of a wave propagating in from infinity; the probability flux is given by

$$j^\mu(\vec{r},t)=\sqrt{-g}\;g^{\mu\nu}\tfrac{1}{2}\phi^*\frac{1}{i}\overleftrightarrow{\partial}_\nu\phi\,; \qquad (2.15)$$

hence to have unit probability, the wave should be normalized so that the total flux from infinity is

$$\int_{-\infty}^{\infty}dt\,|f(t)|^2\frac{2\omega_0^2(2M)^4}{q_0}|\alpha(\omega_0+i\epsilon)|^2=1, \qquad (2.16)$$

and that out at infinity is

$$\int_{-\infty}^{\infty}dt\,|f|^2\frac{2\omega_0^2(2M)^4}{q_0}|\alpha(-\omega_0-i\epsilon)|^2 =\frac{|\alpha(-\omega_0-i\epsilon)|^2}{|\alpha(\omega_0+i\epsilon)|^2}. \qquad (2.17)$$

Thus, the probability of the wave's reemerging at infinity is $|\alpha(-\omega_0-i\epsilon)|^2/|\alpha(\omega_0+i\epsilon)|^2$.

The flux in through the event horizon is, then,

$$(2M)^2 2\omega_0\int_{-\infty}^{\infty}dt\,|f(t)|^2=[q_0\omega_0(2M)^2|\alpha(\omega_0+i\epsilon)|^2]^{-1}, \qquad (2.18)$$

and the Wronskian, Eq. (2.11), guarantees that the total probability is 1. Thus, the solution $\psi(r,\omega)$ describes a wave propagating in from infinity with nothing coming through the past event horizon, scattering back with the probability given in Eq. (2.17), and entering the future event horizon with the probability given by Eq. (2.18).

In region F (P), $\psi^l(r,\omega)$ describes a wave moving through the event horizon connecting F (P) to I (II), while the $\psi^l(r,-\omega)$ solution describes a wave moving through the other event horizon. If a solution is known in, say, I, then, to calculate the continuation into, say, F, write the solution as a superposition of $\psi^l(r,\pm\omega)$. The $\psi^l(r,+\omega)$ must be continuous across the event horizon because

$$e^{-i\omega t}\psi(r,\omega)\sim\exp\left[-i\omega\left(t+2M\ln\frac{|r-2M|}{2M}\right)\right]=e^{-i\omega\tau},$$

which is continuous and finite. The other solution, when used to form a wave packet, vanishes; there is *no* condition on that solution and any amount of it may be added; it describes a wave having entered F from II.

### B. Rindler space

The Rindler space metric is given by

$$ds^2=-Z^2d\tau^2+dZ^2+dx^2+dy^2, \qquad (2.19)$$

and the complete analytic extension is given by $z=Z\cosh\tau$, $t=Z\sinh\tau$, and the metric becomes

$$ds^2=-dt^2+dz^2+dx^2+dy^2, \qquad (2.20)$$

the usual Minkowski space. There is a singularity at $Z=0$ and the original coordinates, $Z>0$, only map region I of the space, which is shown for fixed $x$ and $y$ in Fig. 3. In the other regions, the corresponding coordinates are

$$\begin{aligned} z &= Z\sinh\tau,\quad t = Z\cosh\tau,\quad z,t\in \text{F} \\ z &= -Z\cosh\tau,\quad t = -Z\sinh\tau,\quad z,t\in \text{II} \end{aligned} \tag{2.21}$$

and

$$z = -Z\sinh\tau,\quad t = -Z\cosh\tau,\quad z,t\in \text{P}$$

where $Z>0$ in each region. The metric becomes

$$ds^2 = \pm(-Z^2 d\tau^2 + dz^2) + dx^2 + dy^2\,, \tag{2.22}$$

where the + (−) holds in regions I and II (P and F). The metric is, in each region, independent of $\tau$ and there is a Killing vector[7] associated with the symmetry. This vector is timelike in I and II but spacelike in P and F. In terms of the Minkowski space, it is easy to see that the symmetry is

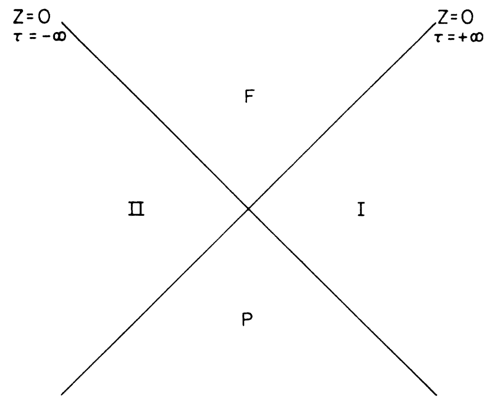


FIG. 3. Complete analytic extension of Rindler space (better known as Minkowski space).

just that of invariance under Lorentz transformations along the $z$ axis; the parameter $\tau$ has nothing to do with time.

The wave equation then becomes

$$[-\partial_\mu\sqrt{-g}\,g^{\mu\nu}\partial_\nu + m^2\sqrt{-g}\,]\,\phi(x) = \left[\pm\left(\frac{1}{Z}\frac{\partial^2}{\partial\tau^2} - \frac{\partial}{\partial Z}Z\frac{\partial}{\partial Z}\right) - \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)Z + m^2 Z\right]\phi = 0\,,$$

where the + (−) sign applies in regions I and II (P and F), as in (2.22).

The equation is translationally invariant with respect to $x$, $y$, and $\tau$; hence

$$\phi(x) = \int \frac{d\nu\,dk_1 dk_2}{(2\pi)^3}\, e^{i(k_1 x + k_2 y - \nu\tau)}\,\phi(Z,\nu,k_1,k_2)\,, \tag{2.23}$$

and the Fourier transform must satisfy

$$\left[-\frac{1}{Z}\frac{\partial}{\partial Z}Z\frac{\partial}{\partial Z} - \frac{\nu^2}{Z^2} \pm (k_1{}^2 + k_2{}^2 + m^2)\right]\phi(Z,\nu) = 0\,. \tag{2.24}$$

This is Bessel's equation of imaginary order, $i\nu$, and imaginary (real) argument. In I and II the argument is imaginary and as $Z\to\infty$ the solution either grows or vanishes exponentially; the vanishing solution is the one which is well behaved at $Z\to\infty$ spatial infinity, $K_{i\nu}(qZ)$, where $q = (m^2 + k_1{}^2 + k_2{}^2)^{1/2}$. As $Z\to 0$, the two possible solutions are

$$I_{\pm i\nu}(qZ) \sim \left(\frac{qZ}{2}\right)^{\pm i\nu}\frac{1}{\Gamma(1\pm i\nu)}\,. \tag{2.25}$$

In order to interpret the solution, again consider a wave packet as $Z\to 0$,

$$\int\frac{d\nu}{2\pi}e^{-i\nu\tau}\,\Gamma(1\pm i\nu)I_{\pm i\nu}(qZ)\tilde{f}(\nu) \simeq e^{-i\nu_0\tau}\left(\frac{qZ}{2}\right)^{\pm i\nu_0} f\left(\tau \pm \ln\frac{qZ}{2}\right)\,. \tag{2.26}$$

Thus $I_{\pm i\nu}(qZ)$ describes, as $Z\to 0$, a wave packet moving through the event horizon at $Z=0$, $\tau=\pm\infty$. The Wronskian,

$$W(\phi^1,\phi^2) = Z\left(\phi^1\frac{d}{dZ}\phi^2 - \frac{d\phi^1}{dZ}\phi^2\right)\,, \tag{2.27}$$

is a constant and

$$W(I_{-i\nu}(qZ), I_{i\nu}(qZ)) = \frac{2i\sinh\pi\nu}{\pi}\,,$$

while

$$K_{i\nu}(qZ) = \frac{i\pi}{2}\frac{[I_{-i\nu}(qZ) - I_{i\nu}(qZ)]}{\sinh\pi\nu}\,; \tag{2.28}$$

hence

$$W(I_{\pm i\nu}(qZ), K_{i\nu}(qZ)) = 1\,.$$

In regions P and F, the equations are the same except that the Bessel functions are of real argument, $J_{\pm i\nu}(qZ)$ or $H^{1,2}_{i\nu}(qZ)$. As the timelike coordinate, $Z$, goes to infinity

$$H^{1,2}_{i\nu}(qZ) \underset{Z\to\infty}{\sim} \frac{2}{\pi qZ}e^{\pm iqZ}e^{\pm i\pi/4}e^{\pm\pi\nu/2}\,, \tag{2.29}$$

while, as $Z\to 0$,

$$J_{\pm i\nu}(qZ) \underset{Z\to 0}{\sim} \frac{\frac{1}{2}qZ^{\pm i\nu}}{\Gamma(1\pm i\nu)}\,,$$

with

$$H^2_{i\nu}(qZ) = \frac{J_{-i\nu}(qZ) - e^{-\pi\nu}J_{i\nu}(qZ)}{\sinh\pi\nu}\,.$$

The Wronskians are, then,

$$W(J_{i\nu}(qZ), J_{-i\nu}(qZ)) = \frac{2}{i\pi}\sinh\pi\nu\,,$$

$$W(H^2_{i\nu}(q\,Z), J_{-i\nu}(qZ)) = \frac{2i}{\pi}e^{-\pi\nu}\,, \tag{2.30}$$

and

$$W(H^2_{i\nu}(q\,Z), J_{i\nu}(q\,Z)) = \frac{2i}{\pi}\,.$$

In order to find the continuity conditions across the event horizons, coordinates which are continuous must be used, e.g., $z$, $t$. Then the solution of the wave packet in I and II, Eq. (2.26), becomes as $Z\to 0$

$$\exp[\pm i\nu_0 \ln\tfrac{1}{2}q(z\mp t)]f(\mp\ln\tfrac{1}{2}q\,(z\mp t))\,,$$

which may be continued across the $Z=\mp t$ event horizons to yield

$$\exp[\pm i\nu_0\ln\tfrac{1}{2}qZe^{\mp\tau}]f(\ln\tfrac{1}{2}qZe^{\mp\tau})$$

in P or F. These, in turn, may be written as

$$\int\frac{d\nu}{2\pi}\,e^{-i\nu\tau}\Gamma(1+i\nu)J_{\pm i\nu}(qZ)\tilde{f}(\nu)\,,$$

and the $I_{\pm i\nu}(qZ)$ continues over into $J_{\pm i\nu}(qZ)$ across the $Z=0$, $\tau\to\mp\infty$ event horizons. The other solutions vanish at the event horizon, do not continue, and must be otherwise specified. For example, if a solution is known in I, then the continuity conditions determine the coefficient of $J_{-i\nu}$ in F, and $J_{+i\nu}$ in P but *not* conversely. The $J_{+i\nu}$ in F and $J_{-i\nu}$ in P components must be determined either in terms of some knowledge about F (P) or about region II.

## III. THE GREEN'S FUNCTIONS

In Minkowski space, the Green's function for the scalar field is the time-ordered product

$$\begin{aligned}G(x-x') &= i\langle 0|\,T(\phi(x)\phi(x'))\,|0\rangle/\langle 0|0\rangle\\ &\equiv[\theta(x^0-x'^0)i\langle 0|\phi(x)\phi(x')|0\rangle\\ &\quad+\theta(x'^0-x^0)i\langle 0|\phi(x')\phi(x)|0\rangle]/\langle 0|0\rangle\,,\end{aligned} \tag{3.1}$$

which, because the field $\phi$ satisfies the homogeneous wave equation $(-\partial^2+m^2)\phi=0$, and the equal-time commutator

$$[\dot\phi(\vec{r},t), \phi(\vec{r}\,',t)] = -i\delta^{(3)}(\vec{r}-\vec{r}\,'), \tag{3.2}$$

must satisfy

$$(-\partial^2+m^2)G(x-x') = \delta^{(4)}(x-x')\,, \tag{3.3}$$

and, because $|0\rangle$ is *assumed* to be the lowest-energy state of the system, all the intermediate states in $\langle 0|\phi(x)\phi(x')|0\rangle$ must be of higher energy and the frequencies which appear are positive. These conditions lead to a unique specification of the Green's function,

$$G(x-x') = \int\frac{d^4p}{(2\pi)^4}\,\frac{e^{ip\cdot(x-x')}}{m^2+p^2-i\epsilon}\,, \tag{3.4}$$

where the $-i\epsilon$ determines the contour to be followed in performing the integrations and, therefore, the boundary conditions on the Green's functions.

The process of specifying the boundary conditions also defines the vacuum state. In the case of a curved space, the significance of the coordinates is not clear and it is, in general, not obvious what the time coordinate is with respect to which the positive-frequency condition must be applied.

In any case, the action for a scalar field in an arbitrary metric space is (no $R\phi^2$ term is included because the metric is source-free except at $r=0$)

$$W = \int d^4x\sqrt{-g}\,[-\tfrac{1}{2}(\partial_\mu\phi)g^{\mu\nu}(\partial_\nu\phi)-\tfrac{1}{2}m^2\phi^2]\,. \tag{3.5}$$

The momentum conjugate to $\phi$ is

$$\frac{\delta W}{\delta\dot\phi} = n_\mu\sqrt{-g}\;g^{\mu\nu}\partial_\nu\phi\,,$$

where $\dot\phi = n^\mu\partial_\mu\phi$ and $n^\mu g_{\mu\nu}n^\nu=-1$. The timelike vector $n^\mu$ defines (locally) a spacelike surface. Locally the coordinates may always be chosen to be Minkowskian, in which case the commutation relations read

$$\begin{aligned}\delta(x^0-x'^0)[\dot\phi(x), \phi(x')] &= i\delta^{(4)}(x-x')\,,\\ &= \delta(n\cdot(x-x'))[\sqrt{-g}\,n^\mu\partial_\mu\phi(x), \phi(x')]\\ &= i\delta^{(4)}(x-x')\,,\end{aligned} \tag{3.6}$$

and

$$[\phi(x), \phi(x')] = 0,\ \text{for } x, x' \text{ spacelike}\,. \tag{3.7}$$

These relations imply that

$$G(x,x') = i\langle 0|T(\phi(x)\phi(x))|0\rangle/\langle 0|0\rangle$$

$$\equiv\begin{cases} i\langle 0|\phi(x)\phi(x')|0\rangle/\langle 0|0\rangle, & x\in J^+(x')\\ i\langle 0|\phi(x')\phi(x)|0\rangle/\langle 0|0\rangle, & x'\in J^+(x)\\ i\langle 0|\phi(x)\phi(x')|0\rangle/\langle 0|0\rangle = i\langle 0|\phi(x')\phi(x)|0\rangle/\langle 0|0\rangle, & (x,x')^\mu \text{ spacelike}\,,\end{cases} \tag{3.8}$$

where $J^+(x)$ is the causal future of $x$. The function is, then, symmetric under the interchange of $x$ and $x'$, and the vacuum state is yet to be defined. This is, for the Minkowski space, the requirement that the vacuum be the lowest-energy state and that $\phi$ operating on it increase rather than decrease the energy. In the general case, such a definition is ambiguous because there is no globally conserved energy; but, in the two cases at hand, it is possible to generalize the definition of the vacuum and, further, to show that the vacuum so defined is stable and that, in the asymptotically flat regions, the generalizations correspond to the positive-energy conditions. The generalizations are different in the two cases at hand, so I shall not attempt to define them more carefully here.

It is convenient to record here the equation for $G(x, x')$ which follows from the Lagrangian given,

$$(-\partial_\mu g^{\mu\nu}\sqrt{-g}\,\partial_\nu + m^2\sqrt{-g})G(x, x') = \delta^{(4)}(x - x'), \quad (3.9)$$

and the stress-energy tensor density,

$$\begin{aligned}\mathcal{T}^{\mu\nu}(x) &= 2\frac{\delta}{\delta g_{\mu\nu}(x)} W \\ &= g^{\mu\lambda}(\partial_\lambda\phi)(\partial_\sigma\phi)g^{\sigma\nu}\sqrt{-g} \\ &\quad - \frac{g^{\mu\nu}\sqrt{-g}}{2}(g^{\lambda\sigma}\partial_\lambda\phi\partial_\sigma\phi + m^2\phi^2). \quad (3.10)\end{aligned}$$

### A. Schwarzschild space

The differential operator in the equation for the Green's function, Eq. (3.9), is the same as that for $\phi$, Eq. (2.2), and $G(x, x')$ may be expanded just as $\phi$,

$$G(x, x') = \sum_{l,m} Y_l^m(\theta, \phi) Y_l^{m*}(\theta', \phi') \times \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t')} G^l(r, r'; \omega), \quad (3.11)$$

where $G^l(r, r'; \omega)$ and the precise contour are yet to be determined. The symmetry of $G(x, x')$ implies that $G^l(r, r'; \omega) = G^l(r', r; -\omega)$ and that the contour is symmetric under $\omega \to -\omega$. The appearance of $t - t'$ requires some comment; for $x, x'$ both in the same quadrant, the $t$ translational invariance implies that only the difference of the coordinates may appear. When $x$ and $x'$ are in different quadrants, say I and F, the result still holds because the generator $P_t \equiv \int d\sigma_\mu \mathcal{T}^\mu{}_t$ is a conserved scalar independent of the spacelike surface on which it is defined. The vacuum state is *defined* to be invaraint under $t$ translations; hence

$$\begin{aligned}0 &= \langle 0 | [P_t, \phi(x)\phi(x')] | 0\rangle \\ &= i\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial t'}\right)\langle 0|\phi(x)\phi(x')|0\rangle. \quad (3.12)\end{aligned}$$

Then, $G^l(r, r'; \omega)$ must satisfy

$$\left[-\frac{\partial}{\partial r} r(r - 2M)\frac{\partial}{\partial r} + l(l+1) + m^2 r^2 - \frac{\omega^2 r^3}{r - 2M}\right] G^l(r, r'; \omega) = \delta(r - r'). \quad (3.13)$$

Usually, the solution is just, for complex $\omega$, the product of the solution regular at infinity evaluated at the larger coordinate times the solution regular at the inner boundary (here $2M$) evaluated at the smaller coordinate and divided by the Wronskian of the two solutions. Here the boundary condition at the "inner boundary" is not well defined.

For $r$ and $r'$ in I, the situation is essentially the same as the usual flat-space situation. The standard positive-frequency boundary conditions may be obtained by taking

$$G^l(r, r'; \omega) = \frac{i\psi^l(r_<, \sqrt{\omega^2})\phi^l(r_>, \omega^2)}{2\sqrt{\omega^2}(2M)^2\alpha^l(\sqrt{\omega^2})}. \quad (3.14)$$

The Green's function is then analytic in the cut plane shown in Fig. 2(b). This analyticity is chosen because as $r \to \infty$ the space is asymptotically flat and the metric static; hence the appropriate boundary conditions are the usual ones and the integration contour must be the usual one. There are two differences here: (1) The boundary condition at the event horizon is not one of regularity but of specifying the flux out through the horizon in the past, and (2) there is no condition on the frequency components for $\omega < m$ because these drop off exponentially in any case. Because of these differences, a homogeneous solution which is well behaved at infinity, $\phi^l(r, \omega^2)$, must be added and its coefficient determined. If $\alpha^l(\omega)$ possessed zeros in the upper half plane, the form given in Eq. (3.14) would have complex poles on the first sheet; in Appendix A, I prove that $\alpha^l(\omega)$ has no zeros in the upper half plane. Then, for $r, r' \in I$,

$$G^l(r, r', \omega) = \frac{i\psi^l(r_<, \sqrt{\omega^2})\phi^l(r_>, \omega^2)}{2\sqrt{\omega^2}(2M)^2\alpha^l(\sqrt{\omega^2})} + iB(\omega^2)\phi^l(r, \omega^2)\phi^l(r', \omega^2), \quad (3.15)$$

where $B$ is taken to be analytic in the cut plane and is yet to be determined.

In order to determine $B$, let $x$ approach spatial infinity in the future and consider the terms which describe an outgoing wave. This wave must be associated with either (1) the production of a particle existing in the final state, (2) the annihilation of a particle existing in the initial state, or (3) the annihilation of the particle created by $\phi(x')$. If I *define* the initial and final states so that the first two cases do not occur, then only case three can obtain. But if $x'$ is then allowed to approach the event horizon, only the wave entering from the past event horizon can contribute to the production of the particle annihilated at $x$. For $t>t'$, the Green's function becomes

$$G^l(r,t;r',t')=i\int_0^\infty \frac{d\omega}{2\pi}e^{-i\omega(t-t')}\left\{\frac{1}{2\omega(2M)^2}\left[\frac{\psi^l(r',\omega)\phi^l(r,(\omega+i\epsilon)^2)}{\alpha^l(\omega+i\epsilon)}+\frac{\psi^l(r',-\omega)\phi^l(r,(\omega-i\epsilon)^2)}{\alpha^l(-\omega+i\epsilon)}\right]\right.$$
$$+B((\omega+i\epsilon)^2)\phi^l(r,(\omega+i\epsilon)^2)\phi^l(r',(\omega+i\epsilon)^2)$$
$$\left.-B((\omega-i\epsilon)^2)\phi^l(r,(\omega-i\epsilon)^2)\phi^l(r',(\omega-i\epsilon)^2)\right\}, \tag{3.16}$$

and the outgoing wave at infinity is described by the $\phi^l(r,(\omega+i\epsilon)^2)$ terms. Then, the $B$ term describes, as $x'\to$ F, a wave traveling into F and the particle at $x$ is not necessarily the particle created by $\phi(x')$. Thus, for $\omega>m$, $B(\omega^2)=0$, and, if it is an analytic function, it must vanish identically.

Now, $x'$ may be continued into P and

$$G(x,x')=i\int_0^\infty \frac{d\omega\, e^{-i\omega(t-t')}\psi^l(r',\omega)\phi^l(r,(\omega+i\epsilon)^2)}{2\pi 2\omega(2M)^2\alpha^l(\omega+i\epsilon)}$$
$$+i\int_{-\infty}^\infty \frac{d\omega}{2\pi}e^{-i\omega(t-t')}\psi^l(r',-\omega)\phi^l(r,\omega^2)C(\omega), \tag{3.17}$$

with the second term coming from the arbitrary additional solution. The particle associated with $x'$ propagates from P to II and cannot be annihilated by $\phi(x)$ for $x$ in I; hence $C(\omega)$ must also vanish for $|\omega|>m$ because otherwise $\phi(x)$, as $r\to\infty$, would describe the creation or annihilation of a particle which was not created or annihilated by $\phi(x')$. For $\omega^2<m^2$, continue $x'$ into II; then, the solution must be

$$G(x,x')=i\int_{-m}^m \frac{d\omega}{2\pi}e^{-i\omega(t-t')}\phi^l(r',\omega^2)\phi^l(r,\omega^2)C(\omega) \tag{3.18}$$

because the solution damped at infinity must be used. Now, the Green's function may be viewed, in I, as either creating a particle which propagates into F or annihilating a particle which emerged from P. In Minkowski space such a phenomenon occurs: Two field operators at spacelike separation, as here, have some probability of respectively creating and annihilating an excitation. There, however, the probability decreases exponentially with increasing distance, the scale being given by the Compton wavelength $1/m$. Here the scale is not the same, and this particular term is not dictated by any boundary conditions. I therefore take $C(\omega)=0$ for all $\omega$; it must be zero for $\omega^2>m^2$ to avoid the description of preexisting particles, and for $\omega^2<m^2$ the last argument indicates that it must vanish there also. (Note that, although the distance along a $t=$constant surface from a point in I to one in II is finite,

$$\int_{2M}^r \frac{dr'}{(1-2M/r')^{1/2}}<\infty\ ,$$

the phase of a wave is given by

$$\exp\left\{i\omega\left[r+2M\ln\left(\frac{r-2M}{2M}\right)\right]\right\}$$

and a point in II is an infinite number of wavelengths away; thus the regions are disjoint as far as the wave equation is concerned.)

Thus, for $r\in$ I, $r'\in$ P,

$$G^l(r,r';\omega)=i\theta(\omega)[\psi^l(r',\omega+i\epsilon)\phi^l(r,(\omega+i\epsilon)^2)/2\omega(2M)^2\alpha^l(\omega+i\epsilon)], \tag{3.19}$$

and $G^l$ vanishes for $r\in$ I, $r'\in$ II. These results as well as results of a similar analysis for $r'\in$ F are recorded in Table I. The terms for $r'\in$ I and $r$ anywhere else are obtained from the symmetry of $G$ under $r\leftrightarrow r'$, $\omega\to-\omega$. If $r$ and $r'$ are in II, then everything is the same as for I *except* that the coordinates $t$ and $t'$ measure the negative of the time; hence, in terms of these coordinates a negatively time-ordered product is required; however, the positive-frequency condition is then relative to $(-t)$ rather than $t$. The combination of these two changes results in the same conditions and the same Green's function as in region I.

The remaining sectors are for $r \in$ P and $r' \in$ F and vice versa; these are straightforward to calculate by taking, say, $r' \in$ F and continuing $r$ into P from I and from II. The combined results determine both the positive- and negative-frequency parts. The same procedure applies for $r, r' \in$ P or $r, r' \in$ F and yields the results shown in Table I. An elementary consistency check is provided by the fact that the resultant Green's function then satisfies the correct inhomogeneous equation in every region, although it was only imposed for $r, r' \in$ I and $r, r' \in$ II.

The resultant Green's function is, at first look, strange: As an event horizon, say $t \to +\infty$, $r \to 2M$, is approached from one side, the solution $e^{-i\omega t}\psi(r, -\omega)$ appears while its continuation on the other side does not. I have shown that this solution describes a wave moving through the *past* horizon at $t = -\infty$ and have dropped it. Rather paradoxically, a wave packet constructed from this solution may vanish at the event horizon without its derivative vanishing there. To see this, use the coordinates $\tau \equiv t + r + 2M \ln|(r - 2M)/2M|$ and $r$ which are nonsingular coordinates at the event horizon. Then the metric becomes

$$ds^2 = -(1 - 2M/r)d\tau^2 + 2d\tau dr + r^2(d\theta^2 + \sin^2\theta\, d\phi^2),$$

and the wave function becomes

$$\int \frac{d\omega}{2\pi} e^{-i\omega t}\psi(r, -\omega)\tilde{f}(\omega) \underset{r \to 2M}{\simeq} f\left(\tau - 2M - 4M \ln\left|\frac{r - 2M}{2M}\right|\right),$$

with $\lim_{t \to \infty} f(t) = 0$. However,

$$\frac{\partial}{\partial r} f \to \left(1 + \frac{4M}{r - 2M}\right) f',$$

which need not vanish as $r \to 2M$. At first sight there should be a requirement of continuity of the derivative across the event horizon which can be met only if $f$ vanishes. This is not the case for $\tau$ finite and $r \sim 2M$, if $f$ obeys the wave equation

$$\left[-2\frac{\partial}{\partial \tau}\left(r^2 \frac{\partial}{\partial r} + r\right) - \frac{\partial}{\partial r} r(r - 2M)\frac{\partial}{\partial r} + l(l+1) + m^2 r^2\right] f = 0,$$

and

$$f \sim f\left(\tau - 2r - 4M \ln\left|\frac{r - 2M}{2M}\right|\right)$$

near the event horizon. Then, $\theta(r - 2M) f(\cdots)$ is an equally good solution because the additional

TABLE I. Green's function, $G(r, r'; \omega)$, in Schwarzschild space-time.

| | $r' \in$ I | $r' \in$ P | $r' \in$ II | $r' \in$ F |
|---|---|---|---|---|
| $r \in$ I | $\frac{i\psi^l(r_<, (\omega^2+i\epsilon)^{1/2})\,\phi^l(r_>, \omega^2+i\epsilon)}{2(\omega^2+i\epsilon)^{1/2}(2M)^2\alpha^l((\omega^2+i\epsilon)^{1/2})}$ | $\frac{i\theta(\omega)\psi^l(r', \omega)\phi^l(r, \omega^2+i\epsilon)}{2\omega(2M)^2\alpha^l(\omega+i\epsilon)}$ | 0 | $\frac{i\theta(-\omega)\psi^l(r', -\omega)\phi^l(r, \omega^2+i\epsilon)}{2(-\omega)(2M)^2\alpha^l(-\omega+i\epsilon)}$ |
| $r \in$ P | $\frac{i\theta(-\omega)\psi^l(r, -\omega)\phi^l(r', \omega^2+i\epsilon)}{2(-\omega)(2M^2)\alpha^l(-\omega+i\epsilon)}$ | $\frac{i\psi^l(r_>, -(\omega^2+i\epsilon)^{1/2})\psi^l(r_<, (\omega^2+i\epsilon)^{1/2})}{2(\omega^2+i\epsilon)^{1/2}(2M)^2}$ | $\frac{i\theta(\omega)\psi^l(r, \omega)\phi^l(r', \omega^2+i\epsilon)}{2\omega(2M)^2\alpha^l(\omega+i\epsilon)}$ | $\frac{i\psi^l(r, (\omega^2+i\epsilon)^{1/2})\psi^l(r', (\omega^2+i\epsilon)^{1/2})\alpha^l(-(\omega^2+i\epsilon)^{1/2})}{2(\omega^2+i\epsilon)^{1/2}(2M)^2\alpha^l((\omega^2+i\epsilon)^{1/2})}$ |
| $r \in$ II | 0 | $\frac{i\theta(-\omega)\psi^l(r', -\omega)\phi^l(r, \omega^2+i\epsilon)}{2(-\omega)(2M)^2\alpha^l(-\omega+i\epsilon)}$ | $\frac{i\psi^l(r_<, (\omega^2+i\epsilon)^{1/2})\phi^l(r_>, \omega^2+i\epsilon)}{2(\omega^2+i\epsilon)^{1/2}(2M)^2\alpha^l((\omega^2+i\epsilon)^{1/2})}$ | $\frac{i\theta(\omega)\psi^l(r', \omega)\phi^l(r, \omega^2+i\epsilon)}{2\omega(2M)^2\alpha^l(\omega+i\epsilon)}$ |
| $r \in$ F | $\frac{i\theta(\omega)\psi^l(r, \omega)\phi^l(r', \omega^2+i\epsilon)}{2\omega(2M)^2\alpha^l(\omega+i\epsilon)}$ | $\frac{i\psi^l(r, (\omega^2+i\epsilon)^{1/2})\psi^l(r', (\omega^2+i\epsilon)^{1/2})\alpha^l(-(\omega^2+i\epsilon)^{1/2})}{2(\omega^2+i\epsilon)^{1/2}(2M)^2\alpha^l((\omega^2+i\epsilon)^{1/2})}$ | $\frac{i\theta(-\omega)\psi^l(r, -\omega)\phi^l(r', \omega^2+i\epsilon)}{2(-\omega)(2M)^2\alpha^l(-\omega+i\epsilon)}$ | $\frac{i\psi^l(r_<, (\omega^2+i\epsilon)^{1/2})\psi^l(r_>, -(\omega^2+i\epsilon)^{1/2})}{2(\omega^2+i\epsilon)^{1/2}(2M)^2}$ |

terms arising from the step function are

$$-2r^2\delta(r-2M)\left[\frac{\partial}{\partial\tau}+\left(1-\frac{2M}{r}\right)\frac{\partial}{\partial r}\right]f \\ -\frac{\partial}{\partial r}[r(r-2M)\delta(r-2M)f]\,;$$

the first terms vanish as $r$ approaches $2M$ while the second term vanishes as long as $(r-2M)f$ does. Thus, the wrong solution does vanish across the event horizon and continues to satisfy the wave equation in spite of the apparent discontinuity in the derivative. Although the Green's function possesses a cusp at the event horizon, it still satisfies the wave equation there, provided the $e^{-i\omega t}\psi(r,-\omega)$ contributions vanish at the event horizon.

The $e^{-i\omega t}\psi(r,-\omega)$ contribution to the Green's function as the future event horizon is approached depends on the low-frequency behavior and is dominated by the $\omega\sim 0$ dependence. The relevant terms are

$$i\int_0 \frac{d\omega}{4\pi\omega}\,\frac{e^{-i\omega(t-t')}}{(2M)^2}\,\frac{\psi(r,-\omega)}{\alpha(-\omega)}\,\phi(r',0)\,.$$

For small $\omega$, $\alpha(\omega)\sim a-ib/\omega$ and

$$\phi(r,0)\,{}_{r}\!\underset{2M}{\simeq}\,a-b\,2M\ln\left|\frac{r-2M}{2M}\right|\,.$$

As long as $b$ does not vanish, the $\omega$ integral is well defined and the expression vanishes; however, $b$ cannot vanish because, in terms of $r^*\equiv r+2M\ln|(r-2M)/2M|$, $r\phi(r,0)$ is concave upward and must go to infinity as $r\to 2M$. For $m=0$, the argument is more complicated because the cut in $\phi$ extends down to $\omega=0$ and $\phi$ becomes singular as $\omega\to 0$ $(\phi\sim\tilde{\phi}/\omega^{l+1})$. But $\alpha^l(\omega)\sim(1/\omega^{l+1})(a-ib/\omega)$, and the same results obtain. (If the calculation is performed in one space dimension so that $\sqrt{-g}=1$, then $e^{\pm i\omega r^*}$ is an exact solution and $b$ does vanish; in that case these considerations do not hold because the Green's function has a logarithmic singularity.)

This Green's function may now be used to discover the properties of the particles in the exterior region. There are two states for $\omega>m$: The particle may come in from $r=\infty$ or emerge through the past event horizon, and it can either escape to infinity or reenter the black hole. In general for $\omega$ very large the two eigenstates are a particle proceeding directly from infinity into the black hole and a particle emerging from the past event horizon to escape at infinity, while for $\omega\gtrsim m$ the particle will usually be reflected from the effective potential so as to return to its source,

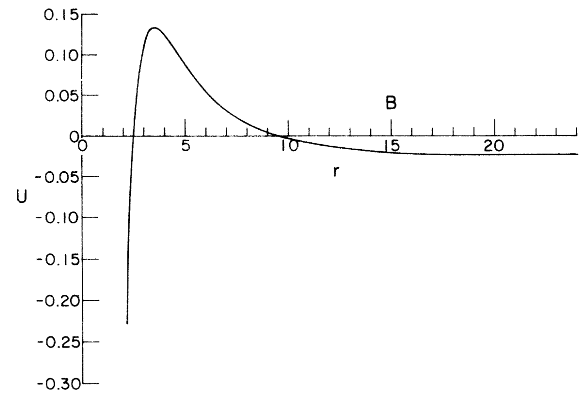


FIG. 4. The effective potential, $U=1/r+L^2/r^2(1-2/r)$, for a classical particle moving in a Schwarzschild metric $U$ is measured in units of $mc^2$, $r$ in units of $\frac{1}{2}$ the Schwarzschild radius, and the angular momentum is chosen so that the potential minimum is at $r=20.5$.

as shown in Fig. 4. For $0<\omega<m$, there is a single solution consisting of particles emerging from the past event horizon and reentering the black hole (the "bound" states eventually penetrate the barrier to enter the black hole).

All the above properties as well as the reduction formulas for the creation and annihilation of various types of particles in the different regions are derived in Appendix A.

### B. Rindler space

Just as in the case of the Schwarzschild space, the differential operator for the Green's function may be expanded in terms of the same set of basis functions as the solutions of the homogeneous equation

$$G(x,x')=\int\frac{d\nu\,dk_1dk_2}{(2\pi)^3}e^{i[k_1(x-x')+k_2(y-y')-\nu(\tau-\tau')]}\times G(Z,Z';\nu,k)\,,\qquad(3.20)$$

where $k\equiv(k_1^2+k_2^2)^{1/2}$, and $G(Z,Z';\nu,k)$ must obey

$$\left[-\frac{1}{Z}\frac{\partial}{\partial Z}Z\frac{\partial}{\partial Z}-\frac{\nu^2}{Z^2}\pm(k^2+m^2)\right]G(Z,Z';\nu,k)=\delta(Z-Z')\,,\qquad(3.21)$$

where $\delta(Z-Z')$ is defined to be zero if $Z$ and $Z'$ lie in different quadrants and the $+(k^2+m^2)$ applies if $Z$ lies in I or II, while $-(k^2+m^2)$ applies if $Z$ lies in P or F.

For $Z,Z'\in$ I, the Green's function must be the product of two solutions of the homogeneous equation divided by their Wronskian; it must be well behaved at infinity and symmetric under $Z\leftrightarrow Z'$, $\nu\to-\nu$. The most general such solution is

$$G(Z,Z';\nu,k)=\tfrac{1}{2}K_{i\nu}(qZ_>)[e^{\pi\nu}I_{-i\nu}(qZ_<)-e^{-\pi\nu}I_{i\nu}(qZ_<)]/\sinh\pi\nu+B(\nu^2)K_{i\nu}(qZ)K_{i\nu}(qZ')\,.\qquad(3.22)$$

There is no asymptotically free regime here, i.e., a region where there is an asymptotically flat space and the particles have infinite time separation, and, in fact, no singularities in $\nu$; $G$ is an entire function of $\nu$ which, being neither constant nor vanishing, must be infinite in some direction as $\nu\to\infty$ in the complex $\nu$ plane. There is nothing more to be said here; the integral is constrained to run along the real axis. I have, naturally, chosen the term which is the solution to the inhomogeneous equation so that $B(\nu^2)$ vanishes; the inhomogeneous solution term will always have the asserted properties, and I take $B(\nu^2)$ to share them.

Now continue $Z'$ into P; the continuity conditions then require that $I_{-i\nu}(qZ')$ continue over into $J_{-i\nu}(qZ')$ with an unknown amount of $J_{+i\nu}(qZ')$; hence, for $Z\in$ I, $Z'\in$ P

$$G(Z,Z';\nu,k)=-\left[\frac{e^{\pi\nu}-\pi iB(\nu^2)}{2\sinh\pi\nu}\right]K_{i\nu}(qZ)J_{-i\nu}(qZ')+K_{i\nu}(qZ)X(\nu)J_{i\nu}(qZ')\,,$$

where $X$ is an unknown function of $\nu$.

In P, as $Z'\to\infty$ the time separation goes to infinity and it is now possible to impose a positive-frequency condition on the bona fide time coordinate $Z'$; as $Z'\to\infty$, it is moving into the distant past and for fixed $\tau$ is $(-t')$; hence the positive-frequency requirement is that the solution goes as $e^{i\omega t'}=e^{-i\omega(-t')}\sim e^{-i\omega Z'}$. But the $H^{1,2}_{i}(qZ')$ solutions go as $e^{\pm iqZ}$; hence the $H^{2}_{i}(qZ')$ solution is required, and for $Z\in$ I, $Z'\in$ P

$$G(Z,Z';\nu,k)=-\frac{e^{\pi\nu}-\pi iB(\nu^2)}{2}K_{i\nu}(qZ)H^{2}_{i\nu}(qZ')\,. \tag{3.23}$$

The continuation of $Z'$ to II is straightforward and yields ($G$ must vanish as $Z'\to\infty$)

$$G(Z,Z',\nu,k)=-[1-\pi iB(\nu^2)e^{-\nu\pi}]\frac{K_{i\nu}(qZ)K_{i\nu}(qZ')}{i\pi} \tag{3.24}$$

for $Z\in$ I, $Z'\in$ II.

The continuation of $Z'$ in F from II, together with the positive-frequency condition for $Z'$ in F, namely that $H^{2}_{i\nu}(qZ')$ appears, yields

$$G(Z,Z',\nu,k)=-\tfrac12[1-i\pi B(\nu^2)e^{-\pi\nu}]K_{i\nu}(qZ)H^{2}_{i\nu}(qZ')\,,$$

while a direct continuation from I yields

$$G(Z,Z',\nu,k)=-\tfrac12[1-i\pi B(\nu^2)e^{\pi\nu}]K_{i\nu}(qZ)H^{2}_{i\nu}(qZ')\,.$$

The two can be equal only if $B=0$; hence

$$G(Z,Z';\nu,k)=-\tfrac12K_{i\nu}(qZ)H^{2}_{i\nu}(qZ')\,. \tag{3.25}$$

The Green's functions for all sectors are displayed in Table II. This Green's function is pre-

TABLE II. Green's function. $G(Z,Z';\omega)$. in Rindler space-time.

| | $Z'\in$ I | $Z'\in$ P | $Z'\in$ II | $Z'\in$ F |
|---|---|---|---|---|
| $Z\in$ I | $-\frac{K_{i\nu}(qZ_>)}{2\sinh\pi\nu}[e^{\pi\nu}I_{-i\nu}(qZ_<)-e^{-\pi\nu}I_{i\nu}(qZ_<)]$ | $-\frac12 e^{\pi\nu}K_{i\nu}(qZ)H^{(2)}_{i\nu}(qZ')$ | $-\frac{1}{2\pi i}K_{i\nu}(qZ)K_{i\nu}(qZ')$ | $-\frac12 K_{i\nu}(qZ)H^{(2)}_{i\nu}(qZ')$ |
| $Z\in$ P | $-\frac12 H^{(2)}_{i\nu}(qZ)K_{i\nu}(qZ')$ | $-\frac{i\pi H^{(2)}_{i\nu}(qZ_>)}{2\sinh\pi\nu}[J_{-i\nu}(qZ_<)-e^{\pi\nu}J_{i\nu}(qZ_<)]$ | $-\frac12 e^{\pi\nu}H^{(2)}_{i\nu}(qZ)K_{i\nu}(qZ')$ | $-\frac12 i\pi e^{\pi\nu}H^{(2)}_{i\nu}(qZ)H^{(2)}_{i\nu}(qZ')$ |
| $Z\in$ II | $-\frac{1}{2\pi i}K_{i\nu}(qZ)K_{i\nu}(qZ')$ | $-\frac12 K_{i\nu}(qZ)H^{(2)}_{i\nu}(qZ')$ | $-\frac{K_{i\nu}(qZ_>)}{2\sinh\pi\nu}[e^{\pi\nu}I_{-i\nu}(qZ_<)-e^{-\pi\nu}I_{i\nu}(qZ_<)]$ | $-\frac12 e^{\pi\nu}K_{i\nu}(qZ)H^{(2)}_{i\nu}(qZ')$ |
| $Z\in$ F | $-\frac12 e^{\pi\nu}H^{(2)}_{i\nu}(qZ)K_{i\nu}(qZ')$ | $-\frac12 i\pi e^{\pi\nu}H^{(2)}_{i\nu}(qZ)H^{(2)}_{i\nu}(qZ')$ | $-\frac12 H^{(2)}_{i\nu}(qZ)K_{i\nu}(qZ')$ | $-\frac{i\pi H^{(2)}_{i\nu}(qZ_>)}{2\sinh\pi\nu}[J_{-i\nu}(qZ_<)-e^{\pi\nu}J_{i\nu}(qZ_<)]$ |

cisely that of the ordinary Minkowski space, expressed in Rindler coordinates and Fourier-transformed with respect to $\tau$; the equality is demonstrated in Appendix B.

Since the Green's functions in the Minkowski and Rindler spaces are equal, there is nothing more to be said: The quantizations are equivalent.

## IV. STABILITY OF THE VACUUM

In the preceding section, I *assumed* that there was no flux of particles out through the past event horizon or into the future event horizon and found a consistent quantization for both the Schwarzschild and Rindler spaces. The latter was precisely the usual Minkowski space quantization, so there is nothing more to be said. The Schwarzschild space quantization is much more complicated and the time ($r$) dependence of the metric inside the event horizons is real: A real singularity develops. Thus, the question arises as to whether I have, by choice of boundary conditions, chosen the "vacuum" states (past and future) to be just those states for which the time ($r$) variation of the metric absorbs (produces) all the particles initially (finally) present but the vacuum state is in fact not stable; i.e., the past vacuum does not develop into the future vacuum and is not the lowest-energy state.

The question of the stability of the vacuum can be answered from the properties of the Green's function already derived. In the process, I shall calculate the vacuum matrix elements of the stress-energy tensor, Eq. (3.10); it is no more well defined than in Minkowski space but, just as there, it can be renormalized. The counterterms necessary to define the stress-energy may be inferred by considering the problem of coupling the scalar field to a more general gravitational field; they must appear as invariant terms in the Lagrangian; hence the allowed terms are

$$\delta W = \int dx\left[\delta\lambda\sqrt{-g} + \delta\left(\frac{1}{16\pi G}\right)\sqrt{-g}R + \delta\alpha\sqrt{-g}R^2 + \delta\beta\sqrt{-g}R_{\mu\nu}R^{\mu\nu}\right], \tag{4.1}$$

and the corresponding contributions to the stress tensor are

$$\begin{aligned}\delta\langle \mathcal{T}^{\mu\nu}\rangle &= 2\frac{\delta}{\delta g_{\mu\nu}}\delta W \\ &= \delta\lambda g^{\mu\nu}\sqrt{-g}\;\delta\left(\frac{1}{8\pi G}\right)\sqrt{-g}G^{\mu\nu} + \delta\alpha\sqrt{-g}[4(g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda} - 2g^{\mu\nu}g^{\lambda\sigma})R_{,\lambda;\sigma} - (2R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}R)R] \\ &\quad - \delta\beta\sqrt{-g}[2(R^{\mu\alpha;\nu}{}_{;\alpha} + R^{\nu\alpha;\mu}{}_{;\alpha} - R^{\mu\nu;\sigma}{}_{;\sigma} - g^{\mu\nu}R^{;\alpha}{}_{;\alpha}) - 4(g^{\mu\lambda}g^{\nu\sigma} - \tfrac{1}{4}g^{\mu\nu}g^{\lambda\sigma})R_{\lambda\alpha}R_{\sigma}{}^{\alpha}]\,. \end{aligned} \tag{4.2}$$

In the case at hand, only the renormalization of the cosmological term, $\delta\lambda\sqrt{-g}$, can occur because the Ricci tensor, $R^{\mu\nu}$, vanishes everywhere (except at $r = 0$) for the Schwarzschild metric. These divergences are a purely local reflection of the singularity of the theory at short distances and may be calculated, at any given point, by taking coordinates which are locally Minkowskian. The counterterms must be and, when explicitly calculated,[8] are purely real if they are not to destroy the Hermiticity of the Hamiltonian. The presence of the divergences and the necessity of subtracting the stress tensor means that nothing can be said regarding the positivity of the energy density in the "vacuum" or the validity of the singularity theorems of Hawking and Penrose[4]; it does not, however, prevent a calculation of the stability of the vacuum or of the energy of a particle state relative to that of the vacuum.

The calculation of the Green's function does not, in itself, yield the vacuum persistence amplitude, $\langle 0\text{ out}|0\text{ in}\rangle$, where the specifications "out" and "in" are added because the equivalence of the two states is to be proved. However, the change in any matrix element due to an infinitesimal change in some external parameter, $\lambda$, is

$$\delta\langle A|B\rangle = i\langle A|\,\delta W\,|B\rangle\,, \tag{4.3}$$

where $\delta W$ is the change in the action due to the change in parameter. This is most easily seen from a functional integral formulation in which matrix elements are defined by the functional integral:

$$\langle T(F[\phi,\lambda])\rangle \equiv \int [d\phi]\,e^{iW[\phi,\lambda]}F[\phi,\lambda]\,; \tag{4.4}$$

hence

$$\delta\langle F[\phi,\lambda]\rangle = i\langle T(\delta W[\phi,\lambda]F[\phi,\lambda])\rangle + \langle T(\delta F[\phi,\lambda])\rangle\,. \tag{4.5}$$

The result of varying the background metric, $g_{\mu\nu}$, in the action, $W$, is

$$\delta W[\phi, g] = \int d^4x\, \tfrac{1}{2}\delta g_{\mu\nu}\, T^{\mu\nu}(\phi, g, x)$$
$$= \tfrac{1}{2}\int d^4x\, \delta g_{\mu\nu}\sqrt{-g}\,[(g^{\mu\lambda}g^{\nu\sigma} - \tfrac{1}{2}g^{\mu\nu}g^{\lambda\sigma})(\partial_\lambda\phi)(\partial_\sigma\phi) - \tfrac{1}{2}m^2 g^{\mu\nu}\phi^2]\,; \tag{4.6}$$

hence the change in the vacuum persistence amplitude is

$$\delta\langle 0\text{ out}|0\text{ in}\rangle = i\int d^4x \tfrac{1}{2}\delta g_{\mu\nu}\sqrt{-g}\,[(g^{\mu\lambda}g^{\nu\sigma} - \tfrac{1}{2}g^{\mu\nu}g^{\lambda\sigma})\langle 0\text{ out}|(\partial_\lambda\phi)(x)(\partial_\sigma\phi)(x)|0\text{ in}\rangle - \tfrac{1}{2}m^2 g^{\mu\nu}\langle 0\text{ out}|\phi^2(x)|0\text{ in}\rangle]$$
$$= \tfrac{1}{2}\int d^4x\, \delta g_{\mu\nu}\sqrt{-g}\,[(g^{\mu\lambda}g^{\nu\sigma} - \tfrac{1}{2}g^{\mu\nu}g^{\lambda\sigma})(\partial_{x\lambda}\partial_{x'\sigma}G(x,x';g))_{x=x'} - \tfrac{1}{2}m^2 g^{\mu\nu}G(x,x;g)]\langle 0\text{ out}|0\text{ in}\rangle\,. \tag{4.7}$$

It is convenient to write this as an equation for the logarithm of the vacuum persistence amplitude and to recognize that

$$G(x,x';g) = \langle x|(-\partial_\lambda g^{\lambda\sigma}\sqrt{-g}\,\partial_\sigma + m^2\sqrt{-g}\,)^{-1}|x'\rangle\,; \tag{4.8}$$

hence

$$\delta\ln\langle 0\text{ out}|0\text{ in}\rangle = -\tfrac{1}{2}\int dx\,dx'\langle x'|\delta(-\partial_\lambda g^{\lambda\sigma}\sqrt{-g}\,\partial_\sigma + m^2\sqrt{-g}\,)|x\rangle\langle x|(-\partial_\lambda g^{\lambda\sigma}\sqrt{-g}\,\partial_\sigma + m^2\sqrt{-g}\,)^{-1}|x'\rangle$$
$$= -\tfrac{1}{2}\,\mathrm{tr}\,\delta G^{-1}G\,, \tag{4.9}$$

where the trace is over the space-time coordinates. The trace is the definition of the variation of the Fredholm determinant of $G$; hence

$$\delta\{\langle 0\text{ out}|0\text{ in}\rangle(\det G[g])^{-1/2}\} = 0\,,$$

or (4.10)

$$\langle 0\text{ out}|0\text{ in}\rangle = \det{}^{1/2}(G[g]/G[\eta])\,,$$

and the vacuum persistence amplitude is determined by the Green's function.

In order to establish that $|\det(G[g]/G[\eta])| = 1$, return to the form for the variation of the vacuum persistence amplitude, Eq. (4.7), and consider variations of $M$. Then,

$$\delta g_{\mu\nu} = (2\delta M/r)\{\delta^t_\mu\delta^t_\nu + \delta^r_\mu\delta^r_\nu/[1-(2M/r)]^2\} \tag{4.11}$$

and

$$\tfrac{1}{2}\delta g_{\mu\nu}T^{\mu\nu} = (\delta M/r)r^2\sin\theta\left[\left(\frac{r}{r-2M}\right)^2\left(\frac{\partial\phi}{\partial t}\right)^2 + \left(\frac{\partial\phi}{\partial r}\right)^2\right]; \tag{4.12}$$

hence,

$$\delta\ln\langle 0\text{ out}|0\text{ in}\rangle = \left(\int_{\mathrm{I}} + \int_{\mathrm{II}} + \int_{\mathrm{P}} + \int_{\mathrm{F}}\right)d^4x\,\delta M r\sin\theta\left\{\left[\left(\frac{r}{r-2M}\right)^2\frac{\partial}{\partial t}\frac{\partial}{\partial t'} + \frac{\partial}{\partial r}\frac{\partial}{\partial r'}\right]G(x,x')\right\}_{x'\to x}$$
$$= i2\delta M\sum_l(2l+1)\int_{-\infty}^{\infty}dt\left\{\int_{2M}^{\infty}r\,dr\left[\int_0^m d\omega\left(\left|\frac{r\omega\chi^{l(b)}(r,\omega)}{r-2M}\right|^2 + \left|\frac{\partial\chi^{l(b)}(r,\omega)}{\partial r}\right|^2\right)\right.\right.$$
$$\left. + \int_0^\infty\frac{q^2dq}{2\omega}\sum_{j=1}^{2}\left(\left|\frac{\omega r\chi_j^{l(\mathrm{in})}(r,\omega)}{r-2M}\right|^2 + \left|\frac{\partial\chi_j^{l(\mathrm{in})}(r,\omega)}{\partial r}\right|^2\right)\right]$$
$$\left. + \int_0^{2M}r\,dr\left[\int_0^\infty\frac{d\omega}{\omega}\left(\left|\frac{r\omega\psi^l(r,\omega)}{r-2M}\right|^2 + \left|\frac{\partial\psi^l(r,\omega)}{\partial r}\right|^2\right)\frac{1}{2\pi(2M)^2}\right]\right\}. \tag{4.13}$$

Thus, the variation of $\ln\langle 0\,\text{out}|0\,\text{in}\rangle$ is purely imaginary and the probability of ending in the state $\langle 0\,\text{out}|$ if the system started in the state $|0\,\text{in}\rangle$ is unity; i.e., the vacuum is stable. To reiterate, the imaginary part of $\delta W$ is finite and not renormalized; it vanishes here, and hence a change in $M$ does not change the absolute value of $\langle 0|0\rangle$ and the probability that the initial state ends in the final state.

In order to calculate the matrix elements of the stress-energy tensor in a single-particle state, the four-point Green's function,

$$-\langle 0|T(\phi(x)\phi(y)\phi(y')\phi(x'))|0\rangle = G(x,x')G(y,y') + G(x,y)G(y',x') + G(x,y')G(y,x')\,, \tag{4.14}$$

is required. Then,

$$\langle 0|T(\phi(x)\mathcal{T}^{\mu\nu}(y)\phi(x'))|0\rangle - \langle 0|T(\phi(x)\phi(x'))|0\rangle\langle 0|\mathcal{T}^{\mu\nu}(y)|0\rangle = +G(x,y)\{\overleftarrow{\partial}_{y\lambda}(g^{\mu\lambda}(y)g^{\nu\sigma}(y)+g^{\mu\sigma}(y)g^{\nu\lambda}(y)-g^{\mu\nu}(y)g^{\lambda\sigma}(y))[-g(y)]^{1/2}\overrightarrow{\partial}_{y\sigma}+g^{\mu\nu}(y)[-g(y)]^{1/2}m^2\}G(y,x')\,. \tag{4.15}$$

The reduction formulas from Appendix A, Eqs. (A13) and (A16), together with the expression for the Green's function, Eq. (A6), imply that, for $y \in$ I, II and the unbound states,

$$\langle l',m',j',q',\,\text{in}|\mathcal{T}^{\mu\nu}(y)|l,m,j,q,\,\text{in}\rangle - \langle\mathcal{T}^{\mu\nu}(y)\rangle\langle l',m',j',q',\text{in}|l,m,j,q,\,\text{in}\rangle = (g^{\mu\lambda}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\lambda}-g^{\mu\nu}g^{\lambda\sigma})\sqrt{-g}\,[\partial_\lambda(Y_{l'}^{m'}\chi_j^{l'(\text{in})}e^{-i\omega t})^*\partial_\sigma(Y_l^m\chi_j^{l(\text{in})}e^{-i\omega t})] - m^2\sqrt{-g}\,g^{\mu\nu}(Y_{l'}^{m'}\chi_j^{l'(\text{in})}e^{-i\omega t})^*(Y_l^m\chi_j^{l(\text{in})}e^{-i\omega t})\,; \tag{4.16}$$

similar equations hold for the bound states. The diagonal matrix elements of $\mathcal{T}^{00}$ are manifestly positive-definite, and the integrated energy is also. (The positivity of the energy density, even in flat space, does not hold for all states. Even for a free field, if the state is a superposition of states with different numbers of quanta the energy density is indefinite.[9]) Thus, the vacuum is the lowest-energy state, as measured in the exterior regions and any other choice of initial state would have had a greater energy.

This definition of the Green's function has guaranteed that there is no flux of particles at infinity and that the vacuum is the stable lowest-energy state of the system. This was achieved by assuming no flux of particles out from the primordial white-hole region P. These results have no direct bearing on the radiation from a black hole which results from the collapse of a star; however, the radiation predicted by Hawking is strikingly independent of the details of the precollapse evolution of the star, suggesting that the radiation mechanism is related to intrinsic properties of the metric itself. This calculation shows that the Schwarzschild metric alone is not responsible for the radiation.

Furthermore, a source inside the future event horizon produces no particles whatsoever in the exterior region.

To see this, calculate the radiation which appears at future infinity $I^+$ in I due to an external source acting in F. Then,

$$\langle 0\,\text{out}|\phi(x)|0\,\text{in}\rangle^\eta = \int_F dx'\,G(x,x')\eta(x')\,. \tag{4.17}$$

Use the reduction formula (A12) and the Green's function for $x\in$ I, $x'\in$ F from Table I,

$$G(x,x') = i\sum_{l,m} Y_l^m(\theta,\phi)Y_l^{m*}(\theta',\phi')\int_{-\infty}^0 \frac{d\omega\,\phi^l(r,\omega^2+i\epsilon)\psi^l(r',-\omega)e^{-i\omega(t-t')}}{2\pi 2(-\omega)(2M)^2\alpha^l(-\omega+i\epsilon)}\,,$$

which has only *negative* frequencies in $t$; hence the reduction formula yields

$$\langle l,m,1,q,\text{I}\,\text{out}|0\,\text{in}\rangle^\eta = 0\,, \tag{4.18}$$

and there is *no* radiation.

The situation is very strange. A quantum field coupled to the Schwarzschild-Kruskal metric does nothing violent to the basic properties of the system: There is still a stable vacuum and, for the black hole in its ground state, there is no flux of particles at infinity. Thus, an isolated primordial black hole can exist and is a stable entity (however, in the presence of surrounding matter it accumulates matter along the past event horizon.[10] It is quite likely that the formation of the new event horizon would yield a flux of particles at late times just as does a collapsing star). On the other hand, Hawking's result indicates that a collapsing star produces, in the final stages of its collapse, a burst of radiation which, because of the large red-shift, appears at infinity as blackbody radiation from an object of temperature $kT = 1/8\pi M$.

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## APPENDIX A

In region I, a function which is $C^\infty$ and square integrable may be used to construct the probability current, Eq. (2.15); the normalization is then

$$\int d\sigma_\mu j^\mu = \int_0^{2\pi} d\phi \int_0^\pi d\theta \int_{2M}^\infty dr \sqrt{-g}\, g^{00} \phi^*(x) \frac{1}{i} \partial_0 \phi(x) = \text{constant}. \tag{A1}$$

This relationship, in combination with the Green's function

$$G(x,x') = \sum_{l,m} Y_l^m(\theta,\phi) Y_l^{m*}(\theta',\phi') \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \frac{i}{2} \frac{\psi^l(r_<, (\omega^2+i\epsilon)^{1/2}) \phi^l(r_>, \omega^2+i\epsilon)}{(\omega^2+i\epsilon)^{1/2}(2M)^2 \alpha^l((\omega^2+i\epsilon)^{1/2})}, \tag{A2}$$

will yield the matrix elements of the field and the reduction formulas for the various states.

First, the Green's functions may be written as, assuming that $\alpha^l(z)$ has no zeros for $\operatorname{Im} z > 0$,

$$G(x,x') = i \sum_{l,m} Y_l^m(\theta,\phi) Y_l^{m*}(\theta',\phi') \int_0^\infty \frac{d\omega}{2\pi} \frac{e^{-i\omega|t-t'|}}{2\omega(2M)^2} \left[ \frac{\psi^l(r_<,\omega)\phi^l(r_>, (\omega+i\epsilon)^2)}{\alpha^l(\omega+i\epsilon)} + \frac{\psi^l(r_<,-\omega)\phi^l(r_>, (\omega-i\epsilon)^2)}{\alpha^l(-\omega+i\epsilon)} \right]. \tag{A3}$$

For $\omega > m$, the factor in square brackets is *not* a product, $f(r)g(r')$, but may be written in terms of $\psi^l(r,\omega)$ and $\phi^l(r,(\omega+i\epsilon)^2)$ using

$$\psi^l(r,-\omega) = [\phi(r,(\omega+i\epsilon)^2) - \alpha^l(-\omega-i\epsilon)\psi^l(r,\omega)]/\alpha^l(\omega+i\epsilon)$$

and

$$(-1)^l \phi^l(r,(\omega-i\epsilon)^2) = \{[\alpha^l(-\omega-i\epsilon)]^*/\alpha^l(\omega+i\epsilon)\}\phi^l(r,(\omega+i\epsilon)^2) + [q(\omega)\omega(2M)^2\alpha^l(\omega+i\epsilon)]^{-1}\psi^l(r,\omega).$$

For $0<\omega<m$, $\phi(\omega+i\epsilon) = \phi(\omega-i\epsilon)$ and the Green's function becomes

$$G(x,x') = i\sum_{l,m} Y_l^m(\theta,\phi)Y_l^{m*}(\theta',\phi') \left\{ \int_0^m \frac{d\omega\, e^{-i\omega|t-t'|}[\phi^l(r,\omega^2)]^*\phi^l(r',\omega^2)}{4\pi\omega(2M)^2|\alpha^l(\omega)|^2} + \int_m^\infty \frac{d\omega}{4\pi\omega(2M)^2} \left| \frac{[\phi^l(r,(\omega+i\epsilon)^2)]^*\phi^l(r,(\omega+i\epsilon)^2)}{|\alpha^l(\omega+i\epsilon)|^2} + \frac{[\psi^l(r',\omega)]^*\psi^l(r,\omega)}{\omega(2M)^2q(\omega)|\alpha^l(\omega+i\epsilon)|^2} \right| \right\}. \tag{A4}$$

The symmetry of $G(x,x')$ now implies that it is also symmetric under $r \leftrightarrow r'$, hence,

$$[\phi^l(r',(\omega+i\epsilon)^2)]^*\phi^l(r,(\omega-i\epsilon)^2) + [\omega q(\omega)(2M)^2]^{-1}[\psi^l(r',\omega)]^*\psi^l(r,\omega) = [\phi^l(r',(\omega-i\epsilon)^2)]^*\phi^l(r,(\omega-i\epsilon)^2) + [\omega q(\omega)(2M)^2]^{-1}[\psi^l(r',-\omega)]^*\psi^l(r,-\omega).$$

In Minkowski space only $\omega > m$ appears, and the conventional sum over states is

$$\int_0^\infty \frac{q^2 dq}{2\omega(q)} = \int_m^\infty \frac{q(\omega)d\omega}{2};$$

with this normalization, the normalized wave functions are

$$\chi_2^{l(\text{in})}(r,\omega) = \frac{\phi^l(r,(\omega+i\epsilon)^2)}{[2\pi\omega q(\omega)(2M)^2]\alpha^l(\omega+i\epsilon)}$$

and

$$\chi_1^{l(\text{in})}(r,\omega) = \frac{\psi^l(r,\omega)}{\sqrt{2\pi}\,\omega q(\omega)(2M)^2\alpha^l(\omega+i\epsilon)}, \tag{A5}$$

or

$$\chi_i^{l(\text{out})}(r,\omega) = [\chi_i^{l(\text{in})}(r,\omega)]^*.$$

Either the "in" or the "out" functions may be used, and each pair provides a complete set. The function $\chi_1^{l(\text{in})}$ describes a normalized wave propagating in from $\infty$, partially entering the event horizon and partially being reflected to infinity, while $\chi_2^{l(\text{in})}$ describes the corresponding wave emerging from the past event horizon, partially propagating to infinity and partially being reflected back into the future event horizon. For $\omega<m$, I shall use the normalization $\int_0^m d\omega$; there is only a continuum with no stable bound states. Then,

$$\chi^{l(b)}(r,\omega)=\phi^l(r,\omega^2)/[4\pi\omega(2M)^2]^{1/2}\alpha^l(\omega),$$

and the Green's function becomes

$$G(x,x')=i\sum_{l,m}Y_l^m(\theta,\phi)Y_l^{m*}(\theta',\phi')\left[\int_0^m d\omega\,\chi^{l(b)}(r,\omega)[\chi^{l(b)}(r',\omega)]^*e^{-i\omega|t-t'|}+\int_0^\infty\frac{q^2dq}{2\omega(q)}\sum_{j=1}^2[\chi_j^{l(\text{in})}(r',\omega(q))]^*\chi_j^{l(\text{in})}(r,\omega(q))e^{-i\omega(q)|t-t'|}\right].\tag{A6}$$

The Green's function obeys its equation, (3.3), hence the completeness relation

$$\delta(r-r')=\frac{r^3}{r-2M}\int_0^\infty 2\omega'd\omega'[\chi^{l(b)}(r',\omega')]^*\chi^{l(b)}(r',\omega')+\int_0^\infty q'^2dq'\sum_{j=1}^2[\chi_j^{l(\text{in})}(r',\omega(q'))]^*\chi_j^{l(\text{in})}(r,\omega(q')).\tag{A7}$$

From this relation and the linear independence of $\chi_1$ and $\chi_2$, the orthonormality relations

$$\int_{2M}^\infty\frac{r^3dr}{r-2M}\chi^{l(b)}(r,\omega)[\chi^{l(b)}(r,\omega')]^*=\frac{\delta(\omega-\omega')}{2\omega}$$

and

$$\int_{2M}^\infty\frac{r^3dr}{r-2M}[\chi_j^{l(\text{in})}(r,\omega(q))]^*\chi_{j'}^{l(\text{in})}(r,\omega(q'))=\delta_{jj'}\frac{\delta(q-q')}{q^2}\tag{A8}$$

follow immediately.

With these relations in hand, the reduction formula is immediate; first the unordered (Wightman) product (the result of Sec. III, $\langle 0|0\rangle=1$ is assumed here)

$$\begin{aligned}\langle 0|\phi(x)\phi(x')|0\rangle&=\sum_{l,m}Y_l^m(\theta,\phi)\left\{\int_0^m d\omega\,\chi^{l(b)}(r,\omega)e^{i\omega(t-t')}[\chi^{l(b)}(r',\omega)]^*+\int_0^\infty\frac{q^2dq}{2\omega(q)}\sum_{j=1}^2\chi_j^{l(\text{in})}(r,\omega(q))e^{-i\omega(q)(t-t')}[\chi_j^{l(\text{in})}(r',\omega(q))]^*\right\}Y_l^{m*}(\theta',\phi')\\&=\sum_a\langle 0|\phi(x)|a\rangle\langle a|\phi(x')|0\rangle\end{aligned}\tag{A9}$$

yields, with an implicit choice of phase for $|a\rangle$,

$$\langle 0|\phi(x)|l,m,b,\omega\rangle=Y_l^m(\theta,\phi)\chi^{l(b)}(r,\omega)e^{-i\omega t},\quad 0<\omega<m$$

and

$$\langle 0|\phi(x)|l,m,j,q\,\text{in}\rangle=Y_l^m(\theta,\phi)\chi_j^{l(\text{in})}(r,\omega(q))e^{-i\omega(q)t},\quad \omega>m\tag{A10}$$

and the normalization of the sum over states, $\int d\omega$ or $\int q^2dq/2\omega$, implies the nonvanishing matrix elements

$$\langle l',m',b,\omega'|l,m,b,\omega\rangle=\delta(\omega-\omega')\delta_l^{l'}\delta_m^{m'}$$

and

$$\langle l',m',j',q'\,\text{in}|l,m,j,q\,\text{in}\rangle=2\omega(q)\delta(q-q')\delta_{jj'}\delta_l^{l'}\delta_m^{m'}/q^2.\tag{A11}$$

Then the states are created by

$$\begin{aligned}\langle l,m,b,\omega|&=\langle 0|\int_0^{2\pi}d\phi\int_0^\pi d\theta\int_{2M}^\infty\frac{r^3dr}{r-2M}Y_l^{m*}(\theta,\phi)\chi^{l(b)}(r,\omega)e^{i\omega t}\left(\omega+\frac{1}{i}\frac{\partial}{\partial t}\right)\phi(x)\\&=\langle 0|\int d\sigma_\mu\sqrt{-g}\,g^{\mu\nu}\left([\chi(x)]^*\frac{1}{i}\overleftrightarrow{\partial}_\nu\phi(x)\right)\end{aligned}$$

and

$$\langle l,m,j,q\,\text{in}|=\langle 0|\int_0^{2\pi}d\phi\int_0^\pi d\theta\int_{2M}^\infty\frac{r^3dr}{r-2M}Y_l^{m*}(\theta,\phi)[\chi_j^{l(\text{in})}(r,q)]^*e^{i\omega t}\left(\omega+\frac{1}{i}\frac{\partial}{\partial t}\right)\phi(x).\tag{A12}$$

The out states form an alternative basis for which the corresponding formulas are obtained by employing the $\chi_j^{l(\text{out})}$ wave functions.

If the fields are interacting or there is some external perturbation, these formulas cannot be applied for arbitrary $t$ and for nonlocalized waves; instead, a well-localized wave packet must be used, formed by a superposition of the $\chi$'s which I shall denote as $\chi(x)$ and the corresponding state as $\langle\chi|$ or $|\chi\rangle$. The reduction formula, just as in Minkowski space, is then

$$\langle\chi^{(\text{out})}|\phi(x')|0\rangle = -i\lim_{t\to\infty}\int d\sigma_\mu\sqrt{-g}\,g^{\mu\nu}[\chi^{(\text{out})}(x)]^*\frac{1}{i}\overleftrightarrow{\partial}_\nu G(x,x')$$

and (A13)

$$\langle 0|\phi(x)|\chi^{(\text{in})}\rangle = -i\lim_{t\to-\infty}\int d\sigma'_\mu\sqrt{-g}\,g^{\mu\nu}(x')\left(G(x,x')\frac{1}{i}\overleftrightarrow{\partial}'_\nu\chi^{(\text{in})}(x')\right).$$

Then "out" and "in" specification indicates both the usual $t=\pm\infty$ limit and that the states previously denoted "out" and "in" are respectively appropriate to the limits.

As an example, I shall present the calculation of the $S$ matrix for this process:

For $\omega > m$,

$$\langle 0|\phi(x)|l,m,1,q,\text{ in}\rangle = e^{-i\omega t}Y_l^m(\theta,\phi)\psi^l(r,\omega(q))/\sqrt{2\pi}\,(2M)^2\alpha^l(\omega+i\epsilon)\omega q(\omega)$$

$$= Y_l^m(\theta,\phi)\left[\frac{1}{(2M)\sqrt{\omega q}}\,\frac{\chi_2^{l(\text{out})}(r,\omega)}{\alpha^l(\omega+i\epsilon)}-\frac{[\alpha^l(-\omega-i\epsilon)]^*}{\alpha^l(\omega+i\epsilon)}\,\chi_1^{l(\text{out})}(r,\omega)\right]e^{-i\omega t}$$

and (A14)

$$\langle 0|\phi(x)|l,m,2,q,\text{ in}\rangle = e^{-i\omega t}Y_l^m(\theta,\phi)\phi^l(r,\omega^2+i\epsilon)/\sqrt{2\omega}\,q\omega(2M)^2\alpha^l(\omega+i\epsilon)$$

$$= Y_l^m(\theta,\phi)e^{-i\omega t}\left[\frac{\alpha^l(-\omega-i\epsilon)}{\alpha^l(\omega+i\epsilon)}\,\chi_2^{l(\text{out})}(r,\omega)+\frac{1}{\sqrt{q\omega}\,(2M)}\,\frac{\chi_1^{l(\text{out})}(r,\omega)}{\alpha^l(\omega+i\epsilon)}\right].$$

The reduction formula applied to $\phi$ then yields the out states:

$$\langle l',m',j',q',\text{ out}|l,m,j,q,\text{ in}\rangle = 2\omega\,\frac{\delta(q-q')}{q^2}\,\delta_l^{l'}\delta_m^{m'}S_{j'j}\,, \qquad \text{(A15)}$$

where

$$S_{j'j}=\frac{1}{\alpha^l(\omega+i\epsilon)}\begin{pmatrix}-[\alpha^l(-\omega-i\epsilon)]^* & 1/\sqrt{q\omega}\,2M\\ 1/\sqrt{q\omega}2M & \alpha^l(-\omega-i\epsilon)\end{pmatrix}$$

is a unitary matrix, as is easily verified.

Before discussing the $\omega<m$ states, I shall first prove that $\alpha^l(z)$ can have only zeros for $\text{Im}z<0$. For some complex $\omega=\omega_0$, with $\text{Im}\omega_0>0$, suppose that $\alpha^l(\omega_0)=0$. Then $\phi(r,\omega_0^{\,2})$ is regular at both $r=\infty$ and $r=2M$ and

$$0<\int_{2M}^\infty dr\left\{r(r-2M)\left|\frac{d}{dr}\phi(r,\omega_0^{\,2})\right|^2+[l(l+1)+m^2r^2]\,|\phi(r,\omega_0^{\,2})|^2\right\}$$

$$=[\phi(r,\omega_0^{\,2})]^*r(r-2M)\frac{d}{dr}\phi(r,\omega_0^{\,2})\Big|_{2M}^\infty$$

$$+\int_{2M}^\infty dr[\phi(r,\omega_0^{\,2})]^*\left[-\frac{d}{dr}r(r-2M)\frac{d}{dr}+l(l+1)+m^2r^2\right]\phi(r,\omega_0^{\,2})$$

$$=\omega_0^{\,2}\int_{2M}^\infty\frac{r^3dr}{r-2M}|\phi(r,\omega_0^{\,2})|^2.$$

The left-hand side is real and positive, as is the integral on the right-hand side. Hence $\omega_0^{\,2}$ must be real and positive. Thus, $\alpha^l(\omega_0)$ cannot be zero for $\text{Im}\omega_0>0$. For $\text{Im}\omega_0<0$, $\psi$ is singular as $r\to 2M$; hence the integrals do not converge and, in fact, besides the quasibound-state zeros discussed below, $\alpha^l(-in)=0$ for $n$ any integer and $\psi^l(r,\omega)$ has poles at $\omega=-in$. Further, $\alpha^l(\omega)$ cannot be zero for real $\omega<M$ (the proof fails for real $\omega$ because the integrals do not converge at $r=2M$): The reality of $i^l\phi(r,\omega^2)$ implies

that $i^l\alpha(-\omega)$ $[i^l\alpha(\omega)]^*$; hence $\alpha(\omega)=0$ implies $\phi(r,\omega)=0$.

The reducon formulas for the continuum bound states are

$$\langle l,m,b,\omega\ \text{out}| = \lim_{t\to\infty}\int d\sigma_\mu g^{\mu\nu}\sqrt{-g}\left[Y_l^{m*}(\theta,\phi)[\chi^b(r,\omega)]^* e^{i\omega t}\langle 0|(1/i)\overleftrightarrow{\partial}_\nu\phi(\omega)\right]$$

and (A16)

$$|l,m,b,,\ \text{in}\rangle = \lim_{t\to-\infty}\int d\sigma_\mu g^{\mu\nu}\sqrt{-g}\left[\phi(x)|0\rangle(1/i)\overleftrightarrow{\partial}_\nu\chi^b(r,\omega)Y_l^m(\theta,\phi)e^{-i\omega t}\right].$$

Although th·e is only a continuum, it is a familiar aspect of, :y, a star that objects are bound and one does ncexpect this to be altered by quantum mechanics.In fact the quasibound states decay with a long fetime given by the barrier penetration, and $\alpha\omega)$ has zeros on the second sheet very near the re axis. Away from the zero $\alpha^l(\omega)$ must be, fc $\omega<m$, quite large because the probability of fiing the particle in $B$ (Fig. 4) is very small. Thı, $\alpha^l(\omega)\simeq\alpha'(\omega-\omega_0+i\gamma)$, where $\gamma$ is small and ( large; the contribution to the Green's function fro $\omega\sim\omega_0$ is then, approximately,

$$\int_{\omega_0^-}^{\omega_0^+}\frac{d\omega}{4\pi\omega(2)^2}\frac{[\phi(r,\omega^2)]^*\phi(r',\omega^2)e^{-i\omega|t-t'|}}{\alpha'^2[(\omega-\omega_0)^2+\gamma^2]}$$
$$\simeq\frac{\phi(r,\omega_0{}^2)]^*\phi(r',\omega_0{}^2)e^{-i\omega_0|t-t'|}}{4\omega_0\gamma(2M)^2\alpha'^2}, \quad \text{(A17)}$$

and the normalized quasibound-state wave function is

$$\phi(r,\omega_0)/(4\omega_0\gamma)^{1/2}(2M)|\alpha'(\omega_0)|,$$

which will be concentrated in the exterior region.

The preceding discussion has applied only to both field operators operating in the normal exterior region, I; if they act in region II, the same analysis holds, except that the states produced by $\phi$ acting on the vacuum are localized in region II rather than region I. Thus, each state must carry a label as to the region in which it is localized.

Now consider $x$ in F and $x'$ in I; the Green's function is

$$i\sum Y_l^m(\theta,\phi)Y_l^{m*}(\theta',\phi')\int_0^\infty\frac{d\omega\, e^{-i\omega(t-t')}\psi^l(r,\omega)\phi^l(r',(\omega+i\epsilon)^2)}{2\pi 2\omega(2M)^2\alpha^l(\omega+i\epsilon)}$$
$$=i\sum Y_l^m(\theta,\varphi)Y_l^{m*}(\theta',\phi')\int_0^m d\omega\ e^{-i\omega(t-t')}[4\pi\omega(2M)^2]^{-1/2}\psi^l(r,\omega)\chi^{l(b)}(r',\omega)$$
$$+\int_{2M}^\infty\frac{q^2dq\,e^{-i\omega(t-t')}}{2\omega}\frac{\psi^l(r,\omega)}{[2\pi\omega q(2M)^2]^{1/2}}[\chi_2^{l(\text{out})}(r',\omega)]^*. \quad \text{(A18)}$$

Using the reduction formula on $\phi(x')$ in I, it immediately follows that, for $x\in$ F,

$$\langle 0|\phi(xl,m,b,\omega,\text{I, out}\rangle = Y_l^m(\theta,\phi)e^{-i\omega t}\psi^l(r,\omega)/[4\pi\omega(2M)^2]^{1/2}$$

and (A19)

$$\langle 0|\phi(xl,m,j,\omega,\text{I, out}\rangle = \delta_{j2}Y_l^m(\theta,\phi)e^{-i\omega t}\psi^l(r,\omega)/[2\pi\omega q(2M)^2]^{1/2},$$

and, fromhe corresponding formula for $x'\in$ II,

$$\langle 0|\phi(xl,m,b,\omega,\text{II, out}\rangle = Y_l^m(\theta,\phi)e^{i\omega t}\psi^l(r,\omega)/[4\pi\omega(2M)^2]^{1/2},\quad 0<\omega<m$$

and (A20)

$$\langle 0|\phi(xl,m,j,\omega,\text{II, out}\rangle = \delta_{j2}Y_l^m(\theta,\phi)e^{i\omega t}\psi^l(r,\omega)/[2\pi\omega q(2M)^2]^{1/2},\quad m<\omega.$$

The ortbnormality relation over the *spacelike* surface $r=$ const is then, trivially,

$$\int d\sigma_r^{\mu\nu}\sqrt{-g}\,\chi^{l'm'*}(r,\omega')\frac{1}{i}\overleftrightarrow{\partial}_\nu\chi^{lm}(r,\omega)=\delta_{ll'}\delta_{mm'}\times\begin{cases}\delta(\omega-\omega'), & 0<\omega<m\\ \dfrac{2\omega\delta(q-q')}{q^2}, & m<\omega\end{cases} \quad \text{(A21)}$$

and

$$\langle l,m,b,\omega,\ (^{\mathrm{I}}_{\mathrm{II}}),\ \text{out}| = \int_{r\in\mathrm{F}} d\sigma_\mu g^{\mu\nu}\sqrt{-g}\ Y_l^{m*}(\theta,\phi)e^{\pm i\omega t}\{[\psi^l(r,\omega)]^*/[4\pi\omega(2M)^2]^{1/2}\}\frac{1}{i}\overleftrightarrow{\partial}_\nu\langle 0|\phi(x),$$

and

$$\langle l,m,2,\omega,\ (^{\mathrm{I}}_{\mathrm{II}}),\ \text{out}| = \int_{r\in\mathrm{F}} d\sigma_\mu g^{\mu\nu}\sqrt{-g}\ Y_l^{m*}(\theta,\phi)e^{\pm i\omega t}\{[\psi^l(r,\omega)]^*/[2\pi\omega q(2M)^2]^{1/2}\}\frac{1}{i}\ \overleftrightarrow{\partial}_\nu\langle 0|\phi(x), \tag{A22}$$

and the limit may be taken as $r\to 0$. For completeness, I here record the reduction formulas ithe regions II and P:

$$\langle l,m,b,\omega,\text{II},\ \text{out}| = \lim_{t\to-\infty}\int_{r\ \mathrm{II}} d\sigma_\mu g^{\mu\nu}\sqrt{-g}\ Y_l^{m*}(\theta,\phi)e^{-i\omega t}[\chi^{(b)}(r,\omega)]^*\frac{1}{i}\overleftrightarrow{\partial}_\nu\langle 0|\phi(x),$$

$$\langle l,m,2,\omega,\text{II},\ \text{out}| = \lim_{t\to-\infty}\int_{r\ \mathrm{II}} d\sigma_\mu g^{\mu\nu}\sqrt{-g}\ Y_l^{m}(\theta,\phi)e^{+i\omega t}[\chi_j^{(\text{out})}(r,\omega)]^*\frac{1}{i}\overleftrightarrow{\partial}_\nu\langle 0|\phi(x), \tag{A23}$$

where the functions $\chi$ are defined precisely as in region I; note that the limit is taken as $t\to-\infty$the *future* infinity of region II:

$$|l,m,b,\omega,\ (^{\mathrm{I}}_{\mathrm{II}}),\ \text{in}\rangle = \lim_{r\to 0}\int_{r\in\mathrm{P}} d\sigma_\mu g^{\mu\nu}\sqrt{-g}\left\{\phi(x)|0\rangle\ \frac{1}{i}\ \overleftrightarrow{\partial}_\nu Y_l^m(\theta,\phi)e^{\mp i\omega t}[\psi^l(r,\omega)]^*/[4\pi\omega(2M)^2]^{1/2}\right\}$$

and

$$|l,m,2,\omega,\ (^{\mathrm{I}}_{\mathrm{II}}),\ \text{in}\rangle = \lim_{r\to 0}\int_{r\in\mathrm{P}} d\sigma_\mu g^{\mu\nu}\sqrt{-g}\left\{\phi(x)|0\rangle\frac{1}{i}\ \overleftrightarrow{\partial}_\nu Y_l^m(\theta,\phi)e^{\mp i\omega t}[\psi^l(r,\omega)]^*/[2\omega q(2M)^2]^{1/2}\right\}. \tag{A24}$$

Considerable caution must be exercised in the use of the reduction formulas for the states which are determined by their properties at or interior to the event horizon: If there are other interactions involved, they will, in general, also be effective at the event horizon; hence, the $t\to\pm\infty$ limit from regions I or II need not yield an isolated system, and the free basis states derived here will not be appropriate. Similarly, in the interior region, spatial separation and hence a noninteracting (except with the metric) system will not occur near the event horizon; as $r\to 0$, the distance between points with fixed unequal $t$ does become infinite, and that limit may be taken to yield the correct states.

## APPENDIX B

I shall establish the equivalence of the Rindler and Minkowski space quantizations by showing the equality of the Green's functions for $x$ and $x'$ both in I; the other cases are easier.

The Bessel function $K_{i\nu}(qZ)$ is given by the integral representation[11]

$$\tfrac{1}{2}\int_{-\infty}^{\infty} d\alpha\, e^{-qZ\cosh\alpha}e^{+i\nu\alpha} = K_{i\nu}(qZ), \tag{B1}$$

while

$$I_{i\nu}(qZ) = \frac{1}{2\pi i}\int_C d\alpha\, e^{-qZ\cosh\alpha}e^{i\alpha\nu}e^{\pi\nu}, \tag{B2}$$

where the contour is shown in Fig. 5.

The combination $(e^{\pi\nu}I_{-i\nu}-e^{-\pi\nu}I_{i\nu})/\sin\pi\nu$ appears in the Green's function; this is jιt $H_{i\nu}^{(2)}(iqZ)$. Hence

$$[e^{\pi\nu}I_{-i\nu}(qZ)-e^{-\pi\nu}I_{i\nu}(qZ)]/\sinh\pi\nu$$
$$= -e^{\pi\nu}\frac{i}{\pi}\int_{-\infty+2i\pi}^{\infty} d\alpha\, e^{qZ\cosh\alpha}e^{i\alpha\nu}$$
$$= \frac{1}{\pi i}\int_{-\infty+i\pi}^{\infty-i\pi} d\alpha\, e^{qZ\cosh\alpha}e^{i\alpha\nu}. \tag{B3}$$

The Green's function may be written as

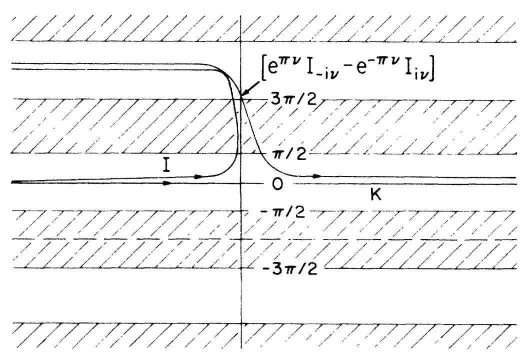


FIG. 5. The complex $\alpha$ plane with the contœrs for the integral representation of the indicated functiιs. The integrand becomes infinite as $|\mathrm{Re}\alpha|\to\infty$ in thshaded regions and zero in the remainder.

$$G(x,x',k)=\int_{-\infty}^{\infty}\frac{d\nu}{2\pi}e^{-i\nu(\tau-\tau')}\tfrac{1}{2}\int_{-\infty}^{\infty}d\alpha\, e^{-qZ_>\cosh\alpha}e^{i\nu\alpha}\frac{i}{2\pi}\int_{-\infty+i\pi}^{\infty-i\pi}d\alpha' e^{qZ_<\cosh\alpha'}e^{i\alpha'\nu}\,. \tag{B4}$$

The end points of the $\alpha'$ integration may be moved to $\pm(\infty - i(\pi/2+0))$ and the integral rewritten as

$$\frac{i}{2(2\pi)^2}\int_{-\infty}^{\infty}d\nu\, e^{-i\nu(\tau-\tau')}\int_{-\infty}^{\infty}d\alpha\, e^{-qZ_>\cosh\alpha}\int_{-\infty+i\pi/2}^{\infty-i\pi/2}d\alpha' e^{qZ_<\cosh(\alpha-\alpha')}e^{i\nu\alpha'}$$

$$=\frac{i}{2(2\pi)^2}\int_{-\infty}^{\infty}d\nu\, e^{-i\nu(\tau-\tau')}\int_{-\infty}^{\infty}d\alpha\, e^{-qZ_>\cosh\alpha}\left[\int_{0}^{\infty-i\pi/2}d\alpha'\, e^{qZ_<\cosh(\alpha-\alpha')}e^{i\nu\alpha'}+\int_{-\infty+i\pi/2}^{0}d\alpha'\, e^{qZ_<\cosh(\alpha+\alpha')}e^{i\nu\alpha'}\right]. \tag{B5}$$

In each term, the $\alpha$ contour may be moved up by $i\pi/2$ and the $\alpha'$ contour shifted until it runs along the real axis, yielding

$$G(x-x';k)=\frac{i}{2(2\pi)^2}\int_{-\infty}^{\infty}d\nu\, e^{-i\nu(\tau-\tau')}\int_{-\infty}^{\infty}d\alpha\, e^{-iqZ_>\sinh\alpha}\left[\int_{0}^{\infty}d\alpha'\, e^{iqZ_<\sinh(\alpha-\alpha')}e^{i\nu\alpha'}+\int_{-\infty}^{0}d\alpha' e^{iqZ_<\sinh(\alpha+\alpha')}e^{i\nu\alpha'}\right]$$

$$=\frac{i}{4\pi}\int_{-\infty}^{\infty}d\alpha\, e^{-iqZ_>\sinh\alpha}e^{iqZ_<\sinh(\alpha-|\tau-\tau'|)}$$

$$=\frac{i}{4\pi}\int_{-\infty}^{\infty}d\alpha\, e^{+iqZ\sinh(\alpha-\epsilon\tau)}e^{-iqZ'\sinh(\alpha-\epsilon\tau')}\,, \tag{B6}$$

where $\epsilon\equiv\epsilon(\tau-\tau')$ and the last step follows because the integral is symmetric under the interchange of $Z_<$ and $Z_>$. Then

$$G(x,x';k)=\frac{i}{4\pi}\int_{-\infty}^{\infty}d\alpha\,\exp[iqZ(\cosh\tau\ \sinh\alpha-\epsilon\ \sinh\tau\ \cosh\alpha)]\exp[-iqZ'(\cosh\tau'\ \sinh\alpha-\epsilon\ \sinh\tau'\ \cosh\alpha)]\,. \tag{B7}$$

Let $k_3=q\ \sinh\alpha$ and the integral becomes

$$G(x,x';k)=\frac{i}{4\pi}\int_{-\infty}^{\infty}\frac{dk_3}{(q^2+k_3{}^2)^{1/2}}\exp\{i[k_3(z-z')-(q^2+k_3{}^2)^{1/2}\epsilon(\tau-\tau')(t-t')]\}. \tag{B8}$$

For $(z-z')^2>(t-t')^2$, the interval is spacelike and the integral is independent of the sign of $\epsilon$. For a timelike interval, the sign of $t-t'$ is the same as that of $\tau-\tau'$; hence this is precisely the Minkowski space Green's function, complete with positive-frequency condition.

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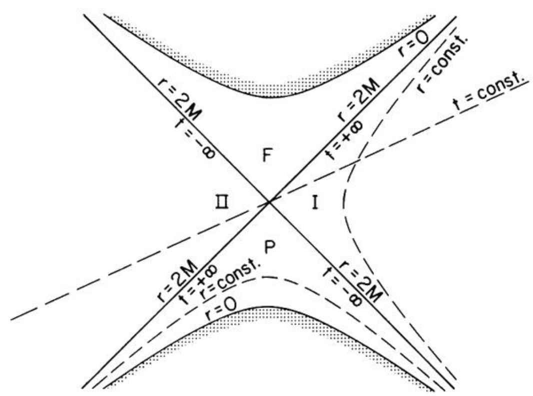


FIG. 1. Kruskal diagram for the Schwarzschild metric. All 45° lines are lightlike. Straight lines passing through the origin are $t$ = constant surfaces, and $r$ = constant surfaces are hyperbolas whose asymptotes are the $r = 2M$ lines.