Dynamics of a small black hole in a background universe*

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A technique involving matched asymptotic expansions is used to investigate the dynamics of a black hole surrounded by an external background universe, according to the Einstein field equations. The approximation scheme is valid provided that the background curvature is small compared to the curvature near the event horizon of the black hole. In this case the background produces only small perturbations of the basic Kerr geometry near the black hole, while the black hole only affects the background metric slightly at distances of the order of the background length scale. These two perturbation expansions are matched in some common region of validity. It is then shown that the black hole moves approximately along a timelike geodesic in the background, and that its spin is approximately parallel transported along the geodesic. The largest effects of the black hole on the background and the largest distortions of the Kerr geometry caused by the background are analyzed in some detail. The background curvature induces distortions of a quadrupole nature in the black hole; these then slow down the rotation, so that the basic structure of the Kerr black hole changes over long time scales. A similar approach is used to describe the behavior of a small black hole in a background, under the Brans-Dicke field equations. In particular, it is shown that a black hole moves on a geodesic in the Einstein conformal frame; this confirms a conjecture by Hawking.

I. INTRODUCTION

In recent studies of black holes in general relativity, most attention has been fixed on the Kerr family of solutions to Einstein's equations. One attempts to give a physically realistic description of a black hole by superimposing linear perturbations on an asymptotically flat Kerr solution while ignoring the presence of the rest of the universe. However, one sometimes needs to understand the way in which a black hole fits into the surrounding spacetime; for example, it may be interacting with matter of comparable or much greater mass, in which case the perturbation approach just mentioned is inadequate, and one should take into account the motion of the black hole relative to the matter. We present here a technique for dealing with this type of situation.

Let us suppose that the true universe $(\mathfrak{M}, g_{ab})$ can be regarded as approximately a background universe $(\mathfrak{M}_0, g_{ab}^{(0)})$ on which a black hole of approximate mass M has been superimposed in some nonlinear fashion. We require that M be small, in the sense that a typical length scale M associated with the black hole $\ll$ a typical length scale of the background universe, as specified by its Riemann tensor invariants, say. This condition will only fail to be satisfied in rather extreme situations, where the shape of the black hole is strongly affected by external gravitational fields, and can be seen as part of a reasonable interpretation of our assumption that a background spacetime can be separated out from $(\mathfrak{M}, g_{ab})$. We suppose that g_{ab} and $g_{ab}^{(0)}$ satisfy the Einstein equations with matter, except that $(\mathfrak{M}, g_{ab})$ is empty in a neighborhood of the event horizon, and that $(\mathfrak{M}_0, g_{ab}^{(0)})$ is empty in the region where the superimposed field is large; thus we are not concerned with accretion problems, nor with rings of matter orbiting near the event horizon. We must also assume that $(\mathfrak{M}, g_{ab})$ and $(\mathfrak{M}_0, g_{ab}^{(0)})$ can each be given some sort of null infinity so that the notion of a black hole is meaningful. Our discussion will be concerned almost entirely with the behavior of the gravitational field in a neighborhood of the black hole, and we deliberately avoid any mention of global questions. A rigorous treatment of the problem is far beyond the scope of this work, and we merely suppose that the true and background universes are sufficiently well behaved in the large that our local considerations are valid.

At points which are separated from the black hole by distances of the order of the background length scale, the presence of the black hole will make little difference to the background metric $g_{ab}^{(0)}$. This suggests that we treat the effect of the black hole as a linear perturbation on $g_{ab}^{(0)}$, at least far from the strong-field region with length scale M. Clearly a different description is needed in the strong-field region. To motivate this description, we remark that the spacetime curvature in the strong-field region is of order M^{-2}, which is much greater than the background curvature in a neighborhood of the black hole. If we accept the indications of recent work that the Kerr solutions are the only stationary vacuum black-hole spacetimes, and are stable against small perturbations, then we are led to describe the strong-field region as a Kerr solution which is perturbed by the presence of the background. It then becomes necessary to relate these two descriptions of the gravitational field on the background length scale and on the black-hole length scale. One achieves this by using the technique of matched asymptotic expansions. This technique was first used in general relativity by Burke[1] in order to relate the near-field region and the wave zone in the problem of slow motion radiation, and is a standard method of applied mathematics for dealing with problems involving more than one length or time scale.

In order to make mathematically sensible statements about a perturbation problem of this type, we must allow the "mass" M of the black hole to become a small parameter, varying in some interval $[0, \kappa)$. Thus, instead of making statements about a fixed spacetime $(\mathfrak{M}, g_{ab})$, we make asymptotic statements about the metric as $M \to 0$, which one can then attempt to interpret in $(\mathfrak{M}, g_{ab})$. Following Geroch,[2] we consider a five-dimensional manifold with boundary, $\mathfrak{N}$, which is built up from the spacetimes $(\mathfrak{M}_M, g_{ab}(M))$ for $M \in [0, \kappa)$, where $g_{ab}(M)$ is the metric corresponding to "mass" M, and $g_{ab}(0) = g_{ab}^{(0)}$. More precisely, when $M=0$ we should exclude from $\mathfrak{M}_0$ a smooth timelike world line l_0 in $(\mathfrak{M}_0, g_{ab}^{(0)})$, which will later be seen to be a zeroth-order approximation to the position of the black hole. It is required that the contravariant metrics $g^{ab}(M)$ define a smooth tensor field on $\mathfrak{N}$; this expresses the fact that the black hole is only a small perturbing influence on $g_{ab}^{(0)}$ far from the event horizon.

We shall introduce a coordinate system (τ, x, y, z, M) in an open subset of $\mathfrak{N}$ such that the excised line l_0 is given locally by $x=y=z=M=0$. Here τ is a temporal coordinate and x, y, z are spatial. Then as $\tau, x, y, z \to$ constants, $M \to 0$ we may give $g_{ab}(M)$ an asymptotic expansion appropriate to the perturbation of $g_{ab}^{(0)}$ by the far field of a Kerr black hole; this procedure will be referred to as the external (approximation) scheme. We shall also require that as $(\tau-\tau_0)M^{-1}$, xM^{-1}, yM^{-1}, $zM^{-1} \to$ constants, $M \to 0$ for any time τ_0, the metric approximates that of a Kerr solution of mass M, where the usual Kerr parameter a satisfies $aM^{-1} \to$ constant as $M \to 0$; this will be referred to as the internal (approximation) scheme. One carries out the matching by assuming that the external and internal schemes each have wider domains of validity than those just described which overlap in such a way that both asymptotic expansions can be used in an intermediate region where $(\tau-\tau_0)M^{-1/2}$, $xM^{-1/2}$, $yM^{-1/2}$, $zM^{-1/2} \to$ constants, $M \to 0$ (say). One can then compare terms in the expansions and use the resulting information to impose boundary conditions on the two schemes. Thus, for example, comparison with the external scheme in the matching region will give us information on the internal scheme at large xM^{-1}, yM^{-1}, zM^{-1}.

We have already given the physical motivation behind the internal and external schemes, and it remains to interpret the matching region in terms of the fixed universe $(\mathfrak{M}, g_{ab})$ as a domain in which a transition is effected from the black-hole regime to the background, on length scales between M and the background length scale. This physical point of view was adopted from the start by Manasse,[3] who studied a special case of the problem considered here, where the background $(\mathfrak{M}_0, g_{ab}^{(0)})$ is a Schwarzschild metric, and a small Schwarzschild-type black hole falls approximately along a radial geodesic l_0 of the background.

In Sec. II we shall discuss the asymptotic expansions and matching in some detail. This leads in Secs. III and IV to the conclusions that the zeroth-order world line l_0 is a geodesic in the background universe and that, at the lowest order, the spin of the black hole is parallel propagated along l_0 in the background. We consider the distorting effects of the background on the internal structure of the black hole in Sec. V; these cause the basic parameters of the black hole to change over a sufficiently long time scale. Then in Sec. VI we consider the situation where the small black hole is essentially nonrotating, in which case more explicit information on the internal structure can be found. Section VII is devoted to the perturbations of the background caused by the far field of the black hole. In Sec. VIII we apply the same methods to the case where a small black hole moves in a background universe, under the field equations of the Brans-Dicke theory of gravitation.

II. ASYMPTOTIC EXPANSIONS AND MATCHING

As remarked in the Introduction, we introduce a coordinate system (τ, x, y, z, M) in an open subset of the five-dimensional manifold $\mathfrak{N}$, such that a certain preferred timelike line l_0 in $(\mathfrak{M}_0, g_{ab}^{(0)})$ is given by $x=y=z=M=0$. Since the contravariant metrics $g^{ab}(M)$ are assumed to define a smooth tensor field on $\mathfrak{N}$, we may make the asymptotic expansion

$$g_{ab}(M)(\tau, x, y, z) = g_{ab}^{(0)}(\tau, x, y, z) + M g_{ab}^{(1)}(\tau, x, y, z) + M^2 g_{ab}^{(2)}(\tau, x, y, z) + O(M^3) \tag{2.1}$$

of the one-parameter family of metrics in this coordinate system. This expansion is valid at least as $\tau, x, y, z \to$ constants, $M \to 0$, provided the limiting field point does not lie on l_0, and is to be called the external (approximation) scheme. In the external scheme, l_0 is singled out by the fact that the perturbations $g_{ab}^{(1)}, g_{ab}^{(2)}$, etc. diverge as $(x, y, z) \to (0, 0, 0)$, and for this reason l_0 must be removed from $\mathfrak{M}$.

We assume that each metric $g_{ab}(M)$ satisfies the Einstein equations with matter, but that as $(\tau, x, y, z) \to (\tau_0, x_0, y_0, z_0) \neq (\tau_0, 0, 0, 0)$, $M \to 0$ with (x_0, y_0, z_0) sufficiently small, then for sufficiently small M we have empty space. In particular, we assume that the background is empty in a neighborhood of l_0,

$$R_{ab}^{(0)} = 0 \tag{2.2}$$

near l_0, where $R_{ab}^{(0)}$ is the Ricci tensor[4] of $g_{ab}^{(0)}$. Note that each $(\mathfrak{M}_M, g_{ab}(M))$ may still contain matter for (x, y, z) sufficiently small, even though the one-parameter family of spacetimes satisfies the condition above.

At present we do not impose any further conditions on the components $g_{ab}^{(0)}, g_{ab}^{(1)}, g_{ab}^{(2)}$, etc. However, any use of matched asymptotic expansions in general relativity is bound to be heavily coordinate-dependent, since one needs to fix a coordinate system in which to accomplish the matching; in our case (τ, x, y, z, M) will be this common coordinate system. So if we choose a convenient coordinate system for the internal expansion, then not only genuine physical information, but also information about the coordinate system will be communicated to the external scheme by means of the matching. Here, for example, some extra conditions on the components of $g_{ab}^{(0)}$ near l_0 will later be seen to be enforced by our coordinate choice for the internal expansion.

To specify the behavior of the metric near the event horizon in the internal approximation scheme, we use certain coordinates (V, R, θ, ϕ', M) which will later be related to the matching coordinates (τ, x, y, z, M). The internal expansion is then

$$ds^2 = M^2\{\Sigma d\theta^2 - 2\chi \sin^2\theta\, dR\, d\phi' + 2dR\, dV + \Sigma^{-1}[(R^2+\chi^2)^2 - \Delta\chi^2\sin^2\theta]\sin^2\theta\, d\phi'^2 - 4\chi\Sigma^{-1}R\sin^2\theta\, d\phi'\, dV$$
$$- (1 - 2R\Sigma^{-1})\, dV^2\} + M^3 h_{ab}^{(1)}(MV, R, \theta, \phi')\, dx^a dx^b + M^4 h_{ab}^{(2)}(MV, R, \theta, \phi')\, dx^a dx^b + O(M^5)\,, \tag{2.3}$$

Here χ is some constant, $0 \leq \chi < 1$. Also

$$\Sigma(R, \theta) = R^2 + \chi^2\cos^2\theta,$$

$$\Delta(R) = R^2 - 2R + \chi^2\,.$$

This expansion is assumed to be valid at least when $(V, R, \theta, \phi') \to (V_0, R_0, \theta_0, \phi_0')$, $M \to 0$, and $R_0 \geq \alpha R_+$, where $R_+ = 1 + (1-\chi^2)^{1/2}$ and $0 < \alpha < 1$. We note that the lowest-order part $M^2\{\;\}$ of the internal metric is just the Kerr solution, describing a black hole of mass M and angular momentum χM^2 rotating about the axis $\theta = 0$, where MV is an ingoing Kerr null coordinate, MR is the usual radial coordinate, and θ, ϕ' are Kerr angular coordinates.

The other important feature of this expansion is the dependence of the perturbation terms $h_{ab}^{(1)}$, $h_{ab}^{(2)}$, etc. on the variables V and R. We see that the internal perturbations are allowed to vary on the same length scale as the dominant Kerr metric, through the variable R. But the time dependence of these perturbations, through the variable MV, is in some sense "slow motion" with respect to the Kerr metric; it is necessary to impose this time dependence in order to match to the external scheme. We prefer to remove a constant conformal factor M^2 from $g_{ab}(M)$ and to regard the internal scheme as describing perturbations of a unit-mass Kerr solution of the form

$$Mh_{ab}^{(1)}(0, R, \theta, \phi') + M^2 V \frac{\partial}{\partial\omega} h_{ab}^{(1)}(\omega, R, \theta, \phi')\Big|_{\omega=0} + M^2 h_{ab}^{(2)}(0, R, \theta, \phi') + O(M^3)\,,$$

where we have given the perturbation terms a Taylor expansion. When viewed in this way, the internal expansion will be called the quasistationary scheme (QS scheme). The removal of the conformal factor need not concern us until we come to computations of surface area.

We assume that as $(V, R, \theta, \phi') \to (V_0, R_0, \theta_0, \phi_0')$, $M \to 0$ with $R_0 \geq \alpha R_+$, then for sufficiently small M each $g_{ab}(M)$ is locally a vacuum metric; as stated in the Introduction, we do not allow the black hole to interact with matter too near the event horizon. But we do tolerate matter in the region $R < \alpha R_+$, for example the matter involved in a stellar collapse, if the black hole is the remnant of a heavy star.

In order to relate the coordinates (V, R, θ, ϕ', M) to the matching coordinates (τ, x, y, z, M), we first define T and ϕ by

$$dV = dT + (R^2 + \chi^2)\Delta^{-1} dR \,, \qquad d\phi' = d\phi + \chi\Delta^{-1} dR \,. \tag{2.4}$$

When we transform to coordinates (T, R, θ, ϕ, M), the lowest-order term in the internal expansion becomes a Kerr solution expressed in Boyer-Lindquist coordinates. Moreover, when $(T, R, \theta, \phi) \to (T_0, R_0, \theta_0, \phi_0)$, $M \to 0$ with R_0 sufficiently large, we are still in the region of validity of the internal scheme. In fact we have a QS scheme in these coordinates, where a unit-mass Kerr solution is given perturbations of the form

$$M j^{(1)}_{ab}(0, R, \theta, \phi) + M^2 T \frac{\partial}{\partial \omega} j^{(1)}_{ab}(\omega, R, \theta, \phi)\Big|_{\omega=0} + M^2 j^{(2)}_{ab}(0, R, \theta, \phi) + O(M^3) \,.$$

This particular scheme is the most convenient for computations of internal perturbations, since we may use the work of Teukolsky[5] on perturbations of the Kerr solution, which is expressed in Boyer-Lindquist coordinates.

We then require that $\tau = \tau_0 + MT$ for all $R > \beta R_+$, some β, where $R > \beta R_+$ implies that the point (T, R, θ, ϕ) is outside the ergosurface of the unperturbed Kerr solution; here τ_0 is some background time. For $\alpha R_+ \leqslant R \leqslant \beta R_+$, we choose $\tau = \tau(V, R, M)$ such that $\{\tau = \text{const}\}$ labels a spacelike hypersurface of $(\mathfrak{M}_M, g_{ab}(M))$ which enters inside the unperturbed event horizon $\{R = R_+\}$, and such that the surfaces $\{\tau = \text{const}\}$ are carried into one another by the integral flow of the time-translation Killing vector of the unperturbed Kerr solution. The behavior of τ for $R < \alpha R_+$ need not be specified. We also require that $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r\cos\theta$, where $r = MR$; this completes the rules for the transformation between (V, R, θ, ϕ', M) and (τ, x, y, z, M).

We see that to each background time τ_0 there corresponds a whole internal scheme. Also the M dependence of the transformation $\tau = \tau_0 + MT$ $(R > \beta R_+)$ explains the quasistationarity of the internal perturbations. More physically, we have QS internal perturbations because gravitational waves only need a time of order M to cross the black hole, whereas the background is changing on a time scale of order 1. Thus the black hole can adjust its gravitational field on what it feels to be a long time scale in order to cope with the tidal field of the background.

For convenience, we chose that the rotation axis of the black hole should in some sense "tie in" to the direction $\partial/\partial z$ at time τ_0 on the world line l_0. But we do allow the rotation axis at another time τ_1 to "tie in" to another direction in the background, $\lambda(\partial/\partial x) + \mu(\partial/\partial y) + \nu(\partial/\partial z)$. This can be expressed simply by rotating the dominant Kerr part of the internal metric. We do not allow the lowest order "mass" and "scalar angular momentum" of the black hole to vary with τ, for reasons to be seen in Sec. V.

The matching of the two expansions is carried out in some intermediate region, say where $M^{-1/2}(\tau - \tau_0), M^{-1/2}x, M^{-1/2}y, M^{-1/2}z \to$ constants, $M \to 0$. It will not be necessary to examine this region explicitly, for we shall assume that all functions in the expansions are so well behaved that they can be developed in power series in the appropriate radial coordinate R or r, for large R or small r, respectively. Thus a typical expression in the internal scheme appropriate to time τ_0 (with conformal factor M^2 removed) can be analyzed into terms of the form $M^i R^j f(\tau_0, MT, \theta, \phi)$, where i and j are integers, $i \geqslant 0$, and we refer the metric to coordinates $T, M^{-1}x, M^{-1}y, M^{-1}z$. Similarly, we decompose the expressions in the external scheme into terms of the form $M^k r^l g(\tau, \theta, \phi)$, where k, l are integers, $k \geqslant 0$. The matching then equates the functions of time and angle, while the exponents are related by $k = i - j$, $l = j$. The internal and external expansions can be regarded as grouping these basic terms with power-law dependence on M and a radial coordinate in two different ways; one grouping may be much more convenient than the other for the purposes of a specific computation.

The conditions $i \geqslant 0$, $k \geqslant 0$ show that $k + l \geqslant 0$, $i - j \geqslant 0$. Conditions are also imposed on the external scheme by the fact that the dominant internal behavior is Kerr-type. This appears in terms with $k + l = 0$, i.e., in the part of $g^{(k)}_{ab}(\tau, x, y, z)$ which is most singular as $(x, y, z) \to (0, 0, 0)$, for each k. For example, we find by examining the Kerr metric at large R that the most singular parts of $g^{(0)}_{ab}, g^{(1)}_{ab}, g^{(2)}_{ab}$ are given by

$$\begin{aligned}
&g^{(0)}_{ab}\, dx^a dx^b \to -d\tau^2 + dx^\alpha dx^\alpha \,, \\
&g^{(1)}_{ab}\, dx^a dx^b \sim 2r^{-1} d\tau^2 + 2r^{-3} x^\alpha x^\beta dx^\alpha dx^\beta , \\
&g^{(2)}_{ab}\, dx^a dx^b \sim -4r^{-3}\epsilon_{\alpha\beta\gamma}\chi^\beta x^\gamma dx^\alpha d\tau + [(4r^{-4} - 2\chi^\gamma\chi^\gamma r^{-4}) x^\alpha x^\beta - \chi^\alpha\chi^\beta r^{-2} + (x^\gamma\chi^\gamma)(x^\alpha\chi^\beta + \chi^\alpha x^\beta) r^{-4} \\
&\qquad + \chi^\gamma\chi^\gamma r^{-2}\delta_{\alpha\beta}]\, dx^\alpha dx^\beta
\end{aligned} \tag{2.5}$$

as $(x, y, z) \to (0,0,0)$. Here Greek indices run from 1 to 3, $x^1 = x$, $x^2 = y$, $x^3 = z$, $\epsilon_{\alpha\beta\gamma}$ is the three-dimensional alternating symbol, $\delta_{\alpha\beta}$ is the Kronecker symbol, and we use a Cartesian summation convention for Greek indices. Also χ^α is the three-dimensional vector representing the lowest-order spin of the black hole; thus $\chi^\alpha \equiv (0,0,\chi)$ when $\tau = \tau_0$. The condition on the components of the background metric for $(x, y, z) = (0,0,0)$ is just a coordinate condition which follows, via the matching, from our coordinate choice for the dominant Kerr part of the internal metric. But the conditions on $g^{(1)}_{ab}$ and $g^{(2)}_{ab}$ contain physical information about the far field of the black hole.

Similarly, the background metric determines those internal terms with $i - j = 0$; these are the terms which grow most rapidly as $R \to \infty$ at each order in the internal scheme.

III. THE LOWEST-ORDER INTERNAL PERTURBATIONS

In this section we shall show that the $O(M)$ QS perturbations are essentially trivial. This will allow us to simplify both internal and external expansions by choice of coordinates.

We proceed by examining the linearized Einstein equations for the QS scheme, and first recall the results of Teukolsky[5] on perturbations of the Kerr metric. In the (unit mass) Kerr metric, expressed in Boyer-Lindquist coordinates (T, R, θ, ϕ) as

$$\begin{aligned} ds^2 = &-(1-2R\Sigma^{-1})dT^2 - 4\chi R\sin^2\theta\,\Sigma^{-1}dT\,d\phi \\ &+ \Sigma\Delta^{-1}dR^2 + \Sigma d\theta^2 \\ &+ \sin^2\theta(R^2+\chi^2+2\chi^2 R\Sigma^{-1}\sin^2\theta)d\phi^2\,, \end{aligned} \tag{3.1}$$

where Σ, Δ are defined as before, we consider the Kinnersley null tetrad $l^a, n^a, m^a, \overline{m}^a$, where

$$\begin{aligned} l^a &\equiv ((R^2+\chi^2)\Delta^{-1}, 1, 0, \chi\Delta^{-1}), \\ n^a &\equiv (R^2+\chi^2, -\Delta, 0, \chi)/2\Sigma\,, \\ m^a &\equiv (i\chi\sin\theta, 0, 1, i/\sin\theta)/2^{1/2}(R+i\chi\cos\theta)\,. \end{aligned} \tag{3.2}$$

Here $l^a n_a = -1$, $m^a \overline{m}_a = 1$ and all other inner products are zero. We define the Newman-Penrose quantity

$$\Psi_0 = C_{abcd}\, l^a m^b l^c m^d\,, \tag{3.3}$$

where C_{abcd} is the Weyl tensor of any spacetime perturbed about the Kerr solution. Then Ψ_0 has unperturbed value zero, and that part of Ψ_0 corresponding to the lowest-order perturbations is gauge-independent, i.e., independent of infinitesimal coordinate transformations.

The field equation for the lowest-order part of Ψ_0 is completely separable with respect to T, R, θ, ϕ. For stationary vacuum perturbations the angular dependence in any mode is of the form $e^{im\phi}S(\theta)$, where

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dS}{d\theta}\right) + \left(\lambda + 2 - \frac{m^2}{\sin^2\theta} - \frac{4m\cos\theta}{\sin^2\theta} - 4\cot^2\theta\right)S = 0\,. \tag{3.4}$$

This equation has eigenvalues $\lambda = (L-2)(L+3)$ for $L \geq 2$, $L \geq |m|$, and eigenfunctions $S(\theta) = {}_2Y^m_L(\theta)$, the spherical harmonics[6] of spin weight 2. The radial equation is

$$\Delta\frac{d^2W}{dR^2} + 6(R-1)\frac{dW}{dR} + \{[\chi^2 m^2 + 4i\chi m(R-1)]\Delta^{-1} - \lambda\}W = 0\,. \tag{3.5}$$

This is a hypergeometric equation, with regular singular points at $R = R_- = 1 - (1-\chi^2)^{1/2}$, R_+, and ∞. As $R \to \infty$, a solution of the radial equation behaves either as R^{L-2} or as R^{-L-3}.

We apply this to the perturbations at $O(M)$ in the QS scheme. As follows from the remarks on matching near the end of the previous section, the $O(M)$ QS metric perturbations cannot grow any faster than R as $R \to \infty$, when referred to coordinates T, $R\sin\theta\cos\phi = X$, $R\sin\theta\sin\phi = Y$, $R\cos\theta = Z$. Moreover, the part growing as R is determined by the $M^0 r^1$ terms in the external scheme, i.e., by the first derivatives of the background metric on l_0. Provided the functions involved are sufficiently regular, the $O(M)$ part of Ψ_0, due to the lowest-order QS perturbations, will then be $O(R^{-1})$ as $R \to \infty$. Hence in each mode, specified by (L, m) the radial function $W^m_L(R)$ must be $O(R^{-L-3})$ as $R \to \infty$.

The argument now takes a different course according as $m \neq 0$ or $m = 0$. Suppose first that $m \neq 0$. Then it may be verified that

$$W^m_L(R) = \text{const} \times (R - R_-)^{-2-\gamma_m}(R - R_+)^{-L-1+\gamma_m} \times F\left[L-1, L+1-2\gamma_m; 2L+2; \left(\frac{R_- - R_+}{R - R_+}\right)\right]\,, \tag{3.6}$$

where $\gamma_m = im\chi/(R_+ - R_-)$ and $F[\,,\,;\,;\,]$ is the standard notation for the hypergeometric function.[7] Before imposing the boundary condition that the perturbations be regular at $R = R_+$, we remark that the Kinnersley tetrad becomes singular as $R \to R_+$. A tetrad which is regular near $R = R_+$ has been described by Hartle[8]; in particular

$$l_H^a = \frac{\Delta}{2(R^2+\chi^2)}\, l^a\,,$$
$$m_H^a = m^a - \frac{i\chi\sin\theta}{\sqrt{2}\,(R+i\chi\cos\theta)}\, l^a\,, \tag{3.7}$$

where $l_H^a, n_H^a, m_H^a, \bar{m}_H^a$ is the Hartle tetrad. We transform the $O(M)$ part of Ψ_0 to this tetrad, and find[8] that in the (L,m) mode it behaves as

$$e^{im\phi'}\,{}_2Y_L^m(\theta)\left(\frac{\Delta}{4R_+}\right)^2\left(\frac{R-R_-}{R-R_+}\right)^{\gamma_m} W_L^m(R) \text{ as } R\to R_+\,,$$

where (θ,ϕ') are Kerr angles which are well behaved near $R=R_+$. To find the behavior of $W_L^m(R)$ as $R\to R_+$, we use the relation[7]

$$F[A,B;C;\zeta]/\Gamma(C) = \frac{\Gamma(B-A)}{\Gamma(B)\Gamma(C-A)}(-\zeta)^{-A}F[A,1-C+A;1-B+A;\zeta^{-1}] + \frac{\Gamma(A-B)}{\Gamma(A)\Gamma(C-B)}(-\zeta)^{-B}F[B,1-C+B;1-A+B;\zeta^{-1}]\,, \tag{3.8}$$

which is valid for application to (3.6) as $R\to R_+$. Altogether, this shows that the $O(M)$ part of Ψ_0 in this regular tetrad contains a term which behaves as $(R-R_+)^{2-2\gamma_m}$ in the (L,m) mode as $R\to R_+$, unless $\Psi_0\equiv 0$ in this mode. Since the QS perturbation must be regular at $R=R_+$, and $m\neq 0$, we must have $\Psi_0\equiv 0$ in this mode at $O(M)$.

Now suppose that $m=0$ and that the perturbation is regular at $R=R_+$. The boundary condition as $R\to\infty$ again implies that $W_L^0(R)$ is given by (3.6), where now $\gamma_m=0$. Suppose that the constant in (3.6) is nonzero. Then instead of (3.8) one may use a more complicated formula[7] to show that Ψ_0 will be nonzero on the horizon at $O(M)$ in the Hartle tetrad. But arguments which will be elaborated in Sec. V show that the horizon shear and hence Ψ_0 on the horizon must be zero for stationary axisymmetric perturbations. We conclude that $\Psi_0\equiv 0$ at $O(M)$ is the axisymmetric modes also.

Hence $\Psi_0\equiv 0$ at $O(M)$ in the QS scheme. By a slight modification of Wald's theorem,[9] which states that Ψ_0 gives an almost complete description of vacuum perturbations of the Kerr metric, we find that the $O(M)$ QS metric perturbations can be described, after a coordinate transformation, merely by changes in the mass (unperturbed value 1) and angular momentum (unperturbed value χ, directed along $\theta=0$) of the Kerr solution. We shall suppose that these coordinate transformations, one of which exists for each background time τ_0 labeling a QS scheme, can be fitted together so as to define a coordinate transformation $(\tau,x,y,z,M)\to(\tau',x',y',z',M)$ on the five-dimensional manifold $\mathfrak{M}$, which preserves the properties of the asymptotic expansions that we have so far assumed. It will follow from our considerations in Sec. V that the $O(M)$ perturbations in the QS mass and QS scalar angular momentum are independent of the background time τ' along the world line l_0. We can now choose to ignore these perturbations, since in a physical context they would correspond to a small error in our estimate of mass and angular momentum, which can easily be reabsorbed.

Thus we can assume that (τ,x,y,z,M) have been chosen so that all $O(M)$ QS perturbations have been eliminated. The QS perturbations are now $M^2 j_{ab}^{(2)}(0,R,\theta,\phi)+O(M^3)$; we shall see in Sec. V that $j_{ab}^{(2)}$ in general describes nontrivial internal perturbations.

IV. EQUATIONS OF MOTION

The elimination of the $O(M)$ QS perturbations in the previous section has also the merit of removing all terms in the external scheme which behave as $M^k r^l g(\tau,\theta,\phi)$, where $k+l=1$. In particular, we see that the background metric is $\mathrm{diag}(-1,1,1,1)+O(r^2)$ near l_0, so that l_0 must be a timelike geodesic in $(\mathfrak{M}_0, g_{ab}^{(0)})$, i.e., the lowest-order approximation to the path of the small black hole in the background is always a geodesic, regardless of the black hole's rotation. The argument in Sec. III can be regarded as showing that were the world line nongeodesic, then the structure of the gravitational field near the unperturbed event horizon $\{R=R_+\}$ would be disastrously altered by the perturbing effects of the background.

We mention two other types of argument which might be used to show that l_0 is a geodesic. One might assume that the spacetimes $(\mathfrak{M}_M, g_{ab}(M))$ each contain matter in a region $\{R<\text{some constant}\}$, such that the energy-momentum tensor $T^{ab}(x^c)$ can be approximated by a distribution $\int d\tau M u^a(\tau)u^b(\tau)\,\delta(x^c,y^c(\tau))$, where $l_0=\{y^c(\tau)\}$ and $u^b(\tau)=(d/d\tau)y^b(\tau)$. The geodesic property then follows on using the conservation equation $T^{ab}{}_{|b}=0$, where the subscript vertical bar denotes covariant differentiation with respect to the background; this argument was given by Robertson[10] in a discussion of the motion of test particles.

Alternatively, one may work with the empty space field equations for the external scheme. If one examines those parts of the $O(M)$ linearized external field equations which are most singular as $r \to 0$, one again finds that l_0 must be a geodesic. An argument of this type was first given by Infeld and Schild.[11]

We remark that, in order to produce a "pole-dipole" test particle, which deviates from geodesic motion as a result of spin forces, according to the Papapetrou equations,[12] one expects to have to scale angular momentum as $O(M)$ rather than $O(M^2)$ as here in the limit $M \to 0$. This scaling is not appropriate to rotating black holes, since the Kerr parameter a (in the usual notation) should remain less than M. In some sense, the rotation of the black hole will affect its motion with respect to the background at $O(M)$ and higher orders. But at these orders the internal structure of the black hole cannot be ignored, and it does not seem feasible to give precise geometrical expression to the statement that "the black hole deviates from geodesic motion."

So far we have not considered the time dependence of the 3-vector χ^α which represents the spin of the black hole in relation to the background. We now show that the 4-vector χ^a, with components $(0, \chi^\alpha)$ in coordinates (τ, x, y, z), is parallel propagated along the geodesic l_0 in the background. For let us consider the empty space Einstein equations, linearized up to order M^2 in the external scheme:

$$\begin{aligned}0 = R_{ab} = R^{(0)}_{ab} &+ \tfrac{1}{2} M\big[g_a^{(1)d}{}_{|bd} + g_b^{(1)d}{}_{|ad} - g_{ab}^{(1)|d}{}_{|d} - g^{(1)d}{}_{d|ab}\big] \\ &+ M^2\Big\{\tfrac{1}{2}\big[g_a^{(2)d}{}_{|bd} + g_b^{(2)d}{}_{|ad} - g_{ab}^{(2)|d}{}_{|d} - g^{(2)d}{}_{d|ab}\big] \\ &\quad + \tfrac{1}{2}\big[g^{(1)de} g^{(1)}_{ab|de} + g^{(1)de} g^{(1)}_{de|ab} - g^{(1)de} g^{(1)}_{ad|be} - g^{(1)de} g^{(1)}_{bd|ae}\big] \\ &\quad + \tfrac{1}{4}\big[2 g^{(1)de}{}_{|d} g^{(1)}_{ab|e} - 2 g^{(1)de}{}_{|d} g^{(1)}_{ae|b} - 2 g^{(1)de}{}_{|d} g^{(1)}_{be|a} \\ &\qquad - g_{ab}^{(1)|d} g_e^{(1)e}{}_{|d} + g_a^{(1)d}{}_{|b} g_e^{(1)e}{}_{|d} + g_b^{(1)d}{}_{|a} g_e^{(1)e}{}_{|d} + g^{(1)}_{de|a} g^{(1)de}{}_{|b} \\ &\qquad - 2 g_{ae}^{(1)|d} g_{bd}^{(1)|e} + 2 g_{ae}^{(1)|d} g_b^{(1)e}{}_{|d}\big]\Big\} + O(M^3)\,. \end{aligned} \tag{4.1}$$

Here $R^{(0)}_{ab}$ is the background Ricci tensor ($=0$ locally) and we raise indices of varied quantities with the background metric. Thus

$$g_a^{(1)d} = g^{(0)de} g^{(1)}_{ae}\,, \text{ etc.} \tag{4.2}$$

This expression for variations in the Ricci tensor can be derived using Eqs. (7.3) and (7.4) of Ref. 13.

Referring to our Eq. (2.5), we see that the most singular terms in the $O(M^2)$ external field equations behave as r^{-4} as $r \to 0$. The $O(r^{-4})$ part of these equations is already satisfied, since covariant differentiation may be replaced by partial differentiation in considering these terms, which only involve the most singular parts of $g^{(1)}_{ab}, g^{(2)}_{ab}$. So for these purposes we may replace the background by Minkowski space, and take the spacetimes $(\mathfrak{M}_M, g_{ab}(M))$ to be exactly Kerr solutions, while preserving Eq. (2.5); i.e., our $O(M^2)O(r^{-4})$ external equations agree with those for an exact Kerr family where they are satisfied.

In considering the $O(M^2)O(r^{-3})$ part of the external field equations, we use the fact that the $O(M)$ QS perturbations have been removed. This shows that the next terms beyond Eq. (2.5) in asymptotic expansions of $g^{(0)}_{ab}, g^{(1)}_{ab}, g^{(2)}_{ab}$ as $r \to 0$ are respectively $O(r^2), O(r), O(1)$. Then the $O(M^2)O(r^{-3})$ part of Eq. (4.1), with $a=0$, $b=\beta$, gives

$$\begin{aligned}0 = r^{-6}(&-2\dot\chi^\alpha \chi^\alpha x^\beta r^2 - \dot\chi^\alpha x^\alpha \chi^\beta r^2 - \chi^\alpha x^\alpha \dot\chi^\beta r^2 \\ &+ 8\,\dot\chi^\alpha x^\alpha \chi^\gamma x^\gamma x^\beta)\end{aligned} \tag{4.3}$$

$\forall x^\alpha$, with $\dot\chi^\alpha = (d/d\tau)\chi^\alpha(\tau)$. Hence

$$\dot\chi^\alpha = 0\,. \tag{4.4}$$

Thus χ^a has constant components, say $(0, 0, 0, \chi)$, in coordinates (τ, x, y, z), and is parallel propagated along l_0 with respect to the background. The remaining $O(M^2)O(r^{-3})$ external equations are now satisfied by an argument like that in the previous paragraph.

V. EFFECT OF BACKGROUND ON INTERNAL STRUCTURE

We now examine the QS perturbations given by $M^2 j^{(2)}_{ab}(0, R, \theta, \phi)$. Since we have removed the $O(M)$ QS terms, $j^{(2)}_{ab}$ describes the largest deviations of the structure of the gravitational field near the event horizon from that of a Kerr solution, due to the distorting effect of the background universe. We shall see that $j^{(2)}_{ab}$ is essentially determined by the Riemann tensor of the background on l_0.

Let us decompose the background Riemann tensor on l_0 into its electric and magnetic parts[14] with respect to u^a, where u^a is the velocity vector of l_0, $\equiv(1,0,0,0)$ in (τ, x, y, z) coordinates on $\mathfrak{M}_0$. Thus we define the electric tensor

$$E_{ac} = C^{(0)}_{abcd} u^b u^d = E_{ca} \tag{5.1}$$

and magnetic tensor

$$H_{ac} = \tfrac{1}{2}\eta_{ab}{}^{gh} C^{(0)}_{ghcd} u^b u^d = H_{ca}, \tag{5.2}$$

where $C^{(0)}_{abcd}$ is the background Weyl tensor on l_0, and η^{abcd} is the totally antisymmetric tensor such that $\eta^{0123} = (-g^{(0)})^{-1/2}$, where $g^{(0)} = \det(g^{(0)}_{ab})$. Then

$$E_{ac}u^c = H_{ac}u^c = 0 \tag{5.3}$$

so that E_{ac}, H_{ac} define three-dimensional symmetric tensors $E_{\alpha\gamma}, H_{\alpha\gamma}$ which give a complete description of the background Riemann tensor on l_0 in vacuo. We now decompose

$$E_{\alpha\gamma} = \alpha_0 \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} + \alpha_1(\tfrac{3}{2})^{1/2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & i \\ 1 & i & 0 \end{pmatrix} + \alpha_2(\tfrac{3}{2})^{1/2} \begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$+ \alpha_{-1}(\tfrac{3}{2})^{1/2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -i \\ 1 & -i & 0 \end{pmatrix} + \alpha_{-2}(\tfrac{3}{2})^{1/2} \begin{pmatrix} 1 & -i & 0 \\ -i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{5.4}$$

where $\alpha_{-1} = \bar{\alpha}_1$, $\alpha_{-2} = \bar{\alpha}_2$, and the α_m are, of course, functions of τ. Similarly, we decompose $H_{\alpha\gamma}$ in terms of coefficients β_m $(-2 \leq m \leq 2)$. This provides an analysis of $C^{(0)}_{abcd}$ into parts with convenient rotational properties.

Now the arguments of Sec. II show that the $O(M^2)$ QS metric perturbations are $O(R^2)$ as $R \to \infty$ when referred to coordinates (T, X, Y, Z), and that the matching determines the M^2R^2 parts in terms of those parts of the background metric which behave as r^2 as $r \to 0$. We then find that at $O(M^2)$ in the QS scheme

$$\Psi_0 = \sum_{m=-2}^{+2} \tfrac{1}{2}(\alpha_m + i\beta_m)\,{}_2Y_2^m(\theta,\phi) + O(R^{-1}) \tag{5.5}$$

as $R \to \infty$, where

$${}_2Y_2^0(\theta,\phi) = 6\sin^2\theta,$$

$${}_2Y_2^{\pm1}(\theta,\phi) = -2\sqrt{6}\,\sin\theta(\cos\theta \mp 1)e^{\pm i\phi},$$

$${}_2Y_2^{\pm2}(\theta,\phi) = 2\sqrt{6}\,(2 - \sin^2\theta \mp 2\cos\theta)e^{\pm 2i\phi}$$

are spin-weight-2 spherical harmonics. Note that the limiting value $\sum_{m=-2}^{+2} \frac{1}{2}(\alpha_m + i\beta_m)\,{}_2Y_2^m(\theta,\phi)$ is just what would be found by "matching out" the Kinnersley tetrad to large R, giving in (τ, x, y, z) coordinates

$$l^a \equiv (1, \sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta),$$

$$n^a \equiv \tfrac{1}{2}(1, -\sin\theta\cos\phi, -\sin\theta\sin\phi, -\cos\theta), \tag{5.6}$$

$$m^a = \frac{1}{\sqrt{2}}(0, \cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta) + \frac{i}{\sqrt{2}}(0, -\sin\phi, \cos\phi, 0),$$

and computing Ψ_0 for the background.

Hence the $L=2$ modes will in general be present in Ψ_0 at $O(M^2)$. Now when $m \neq 0$, it may be verified that a solution of the radial equation (3.5) for $L=2$ which satisfies the boundary conditions of Sec. III at $R = R_+$ is

$$W_2^m(R) = (R - R_-)^{-3-\gamma_m}(R - R_+)^{-2+\gamma_m} \times F\left[1, 3 + 2\gamma_m; -1 + 2\gamma_m; \left(\frac{R - R_+}{R - R_-}\right)\right]. \tag{5.7}$$

When $m = 0$, two linearly independent solutions of the radial equation are

$$W \equiv 1, \tag{5.8}$$

$$W = \int_R^\infty \frac{d\omega}{(\omega^2 - 2\omega + \chi^2)^3}. \tag{5.9}$$

For the first,

$$\Psi_0 = 0, \tag{5.10}$$

and for the second,

$$\Psi_0 = \text{const} \times \sin^2\theta \tag{5.11}$$

at $O(M^2)$ in the Hartle tetrad of Sec. III, on the unperturbed horizon $\{R = R_+\}$. In fact, the boundary conditions on the horizon imply that we only need consider the first solution. To find the boundary conditions, we follow Hawking and Hartle[15] and consider the Newman-Penrose[16] equation

$$\frac{d\sigma}{ds} = 2\rho\sigma + (3\epsilon - \bar{\epsilon})\sigma + \Psi_0. \tag{5.12}$$

Here

$$\rho = -l_{a;b} m^a \bar{m}^b \tag{5.13}$$

is the convergence of the null geodesic generators of the event horizon, with tangent vector $l^a = dx^a/ds$. Also

$$\sigma = -l_{a;b} m^a m^b \tag{5.14}$$

is the shear of the horizon, and

$$\epsilon = -\tfrac{1}{2}(l_{a;b} n^a l^b - m_{a;b} \bar{m}^a l^b). \tag{5.15}$$

The vectors $l^a, n^a, m^a, \bar{m}^a$ form a null tetrad. We assume that m^a and $\bar{m}^a$ are parallel propagated along the horizon generators, so that $\epsilon = \bar{\epsilon}$. These conditions are satisfied by the Hartle tetrad for the unperturbed solution, so that we may take

$$l^a = l^a_H + O(M^2), \quad n^a = n^a_H + O(M^2), \quad m^a = m^a_H + O(M^2). \tag{5.16}$$

This implies that

$$s = V + O(M^2) \tag{5.17}$$

(apart from a constant), where V is the null Kerr coordinate which measures "group time" along the null generators of the unperturbed Killing horizon. We write

$$\rho = \rho^{(0)} + M\rho^{(1)} + M^2\rho^{(2)} + \cdots \tag{5.18}$$

for the perturbed convergence, and similarly for σ, ϵ, Ψ_0. Then

$$\rho^{(0)} = 0, \quad \sigma^{(0)} = 0, \quad \epsilon^{(0)} = \frac{(1-\chi^2)^{1/2}}{4R_+}. \tag{5.19}$$

Since the perturbations are $O(M^2)$, we have

$$\rho^{(1)} = \sigma^{(1)} = \epsilon^{(1)} = 0. \tag{5.20}$$

Then Eq. (5.12) gives

$$\frac{d\sigma^{(2)}}{ds} = 2\epsilon^{(0)}\sigma^{(2)} + \Psi_0^{(2)}. \tag{5.21}$$

Now it is already implicit in our approximations that the position and shape of the event horizon are in some sense differentiable functions of the spacetime metric. We shall further suppose that the structure of the perturbed event horizon in the QS scheme can be determined from the purely local metric perturbations. This is plausible at least at order M^2, where the perturbations are stationary, since in a stationary metric an event horizon has a local characterization as a stationary null surface. At order M^i ($i>2$), the properties of the event horizon should not vary at a rate faster than V^{i-2} as $V\to\infty$, in order to comply with the requirement of quasistationarity, and so should also be determined locally. Hence for an $L=2$, $m=0$ mode at $O(M^2)$, the horizon should undergo a stationary, axisymmetric perturbation. But the null generator

$$l^a = K^a + \omega\tilde{K}^a + O(M^2), \tag{5.22}$$

where $\omega = \chi/2R_+$ is the angular velocity of the Kerr horizon, K^a is the time-translation Killing vector, and $\tilde{K}^a$ is the rotational Killing vector of the Kerr solution. Since ρ and σ represent the variations in the intrinsic horizon metric as the generator l^a winds around the horizon, $\rho^{(2)}$ and $\sigma^{(2)}$ must be zero for our $m=0$ mode. Equation (5.21) then shows that $\Psi_0^{(2)}=0$ on the horizon. Thus only the solution $W\equiv$ const of the $L=2$, $m=0$ radial equation is acceptable.

We can now write the $L=2$ parts of Ψ_0 in the Kinnersley tetrad at $O(M^2)$ in the QS scheme to be

$$\Psi_0 = \tfrac{1}{2}(\alpha_0 + i\beta_0)\,{}_2Y_2^0(\theta,\phi) + \sum_{\{m=-2,-1,1,2\}} \frac{(\alpha_m + i\beta_m)\,\Gamma(3+2\gamma_m)(R_+ - R_-)^5}{48\Gamma(-1+2\gamma_m)}\,{}_2Y_2^m(\theta,\phi)(R-R_-)^{-3-\gamma_m}(R-R_+)^{-2+\gamma_m} \times F\left[1, 3+2\gamma_m; -1+2\gamma_m; \left(\frac{R-R_+}{R-R_-}\right)\right]. \tag{5.23}$$

Here we have used Ref. 7 to find the asymptotic behavior as $R\to\infty$ of the hypergeometric function in Eq. (5.7), together with the boundary conditions (5.5). Further, Eq. (5.5) shows that the $L>2$ parts of Ψ_0 at $O(M^2)$ are $O(R^{-1})$ as $R\to\infty$. By an argument as in Sec. III, these $L>2$ parts must be identically zero. Thus Eq. (5.23), by Wald's uniqueness theorem, gives almost all the information about the $O(M^2)$ QS perturbations. We see that the largest internal perturbations due to the background are caused by conformal curvature on l_0, and are of a quadrupole character.

We may now argue as in Ref. 15 to find the lowest-order rates of change of area and angular momentum of the black hole. First, Eq. (5.23) and the remarks in Sec. III show that the lowest-order Ψ_0 on the horizon is

$$\Psi_0^{(2)}=\frac{(R_+-R_-)^4}{768(R_+)^2}\times\sum_{m=-2}^{+2}(\alpha_m+i\beta_m)\frac{\Gamma(3+2\gamma_m)}{\Gamma(-1+2\gamma_m)}\,{}_2Y_2^m(\theta,\phi')\,. \tag{5.24}$$

Then Eqs. (5.21) and (5.22) show that when $\Psi_0^{(2)}$ has $e^{im\phi'}$ dependence on ϕ',

$$\sigma^{(2)}=\frac{\Psi_0^{(2)}}{im\omega-2\epsilon^{(0)}}\,. \tag{5.25}$$

Hence

$$\sigma^{(2)}=\frac{(R_+-R_-)^3}{192R_+}\times\sum_{m=-2}^{+2}(\alpha_m+i\beta_m)\frac{\Gamma(3+2\gamma_m)}{\Gamma(2\gamma_m)}\,{}_2Y_2^m(\theta,\phi')\,. \tag{5.26}$$

We also need the Newman-Penrose equation (in vacuo)

$$\frac{d\rho}{ds}=\rho^2+\sigma\bar\sigma+(\epsilon+\bar\epsilon)\rho\,. \tag{5.27}$$

Hence

$$\frac{d\rho^{(2)}}{ds}=2\epsilon^{(0)}\rho^{(2)}\,. \tag{5.28}$$

But $\rho^{(2)}$ will be periodic along a generator as it winds around the horizon. Thus

$$\rho^{(2)}=0\,. \tag{5.29}$$

Similarly,

$$\frac{d\rho^{(3)}}{ds}=2\epsilon^{(0)}\rho^{(3)}\,. \tag{5.30}$$

But we expect that $\rho^{(3)}$ will be $O(s)$ as $s\to\infty$, since the $O(M^3)$ QS perturbations are $O(V)$ as $V\to\infty$. Hence also

$$\rho^{(3)}=0\,. \tag{5.31}$$

Then Eq. (5.27) shows

$$\frac{d\rho^{(4)}}{ds}=\sigma^{(2)}\bar\sigma^{(2)}+2\epsilon^{(0)}\rho^{(4)}\,. \tag{5.32}$$

If we consider the area $A(V)$ of the instantaneous horizon at constant V, then

$$\frac{dA(V)}{dV}=-2\int\rho\,dA\,, \tag{5.33}$$

where the integral is taken over the instantaneous 2-surface. When Eq. (5.32) is integrated over this surface, we find

$$\frac{dA(V)}{dV}=\frac{M^4}{\epsilon^{(0)}}\int\sigma^{(2)}\bar\sigma^{(2)}\,dA+O(M^5)\,, \tag{5.34}$$

where the derivative term has vanished since the perturbation is quasistationary. Here $\int dA(\ )$ refers to the unperturbed instantaneous horizon metric[17]

$$ds^2=(R_+^2+\chi^2\cos^2\theta)\,d\theta^2+\frac{(R_+^2+\chi^2)\sin^2\theta}{(R_+^2+\chi^2\cos^2\theta)}\,d\phi'^2\,. \tag{5.35}$$

Using Eq. (5.26), we find in the QS scheme

$$\frac{dA(V)}{dV}=\tfrac{16}{15}\pi M^4(1-\chi^2)^{-1/2}\chi^2\times\sum_{m=-2}^{+2}(|\alpha_m|^2+|\beta_m|^2)m^2[1+\chi^2(m^2-1)]\times[4+\chi^2(m^2-4)]+O(M^5)\,. \tag{5.36}$$

To find the real increase of area, we suppose that the hypersurfaces of constant τ were chosen to coincide with hypersurfaces of constant V near the horizon. The only differences between the real and QS rates of change of area are caused by the time rescaling (by factor M) and the conformal factor M^2 removed from the internal metric perturbations. Hence if ${}^RA(\tau)$ is the real area at time τ, then

$$\frac{d\,{}^RA(\tau)}{d\tau}=M\frac{dA(V)}{dV}\,. \tag{5.37}$$

As written, the rate of area increase depends on the α_m,β_m, which were given a somewhat arbitrary normalization. We can rewrite the expression in terms of invariants, and so remove any such arbitrariness. Recall that u^a is the unit tangent vector to the geodesic l_0 in the background, and let z^a be the unit spacelike vector in the background into which the spin direction ties. Define

$$\begin{aligned}
B_1&=R^{(0)abcd}R^{(0)}_{abcd}\,,\\
B_2&=R^{(0)abcd}u_bu_dR^{(0)}_{aecf}u^eu^f\,,\\
B_3&=R^{(0)abcd}z_cu_dR^{(0)}_{abef}z^eu^f\,,\\
B_4&=R^{(0)abcd}u_bz_cu_dR^{(0)}_{aefg}u^ez^fu^g\,,\\
B_5&=R^{(0)abcd}z_bu_cz_dR^{(0)}_{aefg}z^eu^fz^g\,,
\end{aligned} \tag{5.38}$$

where $R^{(0)}_{abcd}$ is the background Riemann tensor on l_0. Then

$$\frac{dA(V)}{dV} = \tfrac{16}{45}\pi M^4\chi^2(1-\chi^2)^{-1/2}\left[-(1+3\chi^2)B_1+16(1+3\chi^2)B_2-6(1+3\chi^2)B_3-3(8+29\chi^2)B_4-15\chi^2B_5\right]+O(M^5)\,. \tag{5.39}$$

If we suppose that the Kerr solution is stable, then secular effects such as area increase must correspond to variations in the basic parameters of the Kerr solution over a long time scale. For a Kerr solution with mass μ and the usual spin parameter a, the area of the instantaneous event horizon is

$$A = 8\pi\mu[\mu+(\mu^2-a^2)^{1/2}]\,. \tag{5.40}$$

So if we know the lowest-order time dependence of A and μ, say, we also know how a varies at the lowest order. An argument is given in Ref. 15. which shows that the mass is constant at $O(M^4)$ for QS perturbations with our parametrization, provided the perturbations are asymptotically flat, being caused by a distant matter distribution. Clearly, our QS perturbations are not asymptotically flat [see (5.5)], so that the argument cannot be used as it stands. But the mass and angular momentum changes are being caused by purely local effects near the horizon, for which the far field is irrelevant. Since these local effects can presumably be simulated by asymptotically flat perturbations, the mass is constant at $O(M^4)$ in the QS scheme. Hence all the area change at $O(M^4)$ is caused by the change in a. Hence, in the QS scheme

$$\frac{da}{dV} = -\frac{(1-\chi^2)^{1/2}}{8\pi\chi}\frac{dA}{dV}+O(M^5)\,. \tag{5.41}$$

We see that the presence of the background always forces the black hole to lose angular momentum. Also the mass of the black hole changes by a fraction of $O(1)$ on a background time scale of $O(M^{-4})$, while the angular momentum changes by a fraction of $O(1)$ on a τ scale of $O(M^{-3})$; this justifies our assertions in Secs. II and III about the time dependence of low-order mass and spin terms.

The secular changes on the horizon at $O(M^4)$ in the QS scheme are determined completely by $\rho^{(4)}$ and $\sigma^{(4)}$. We expect the parameters $(\theta, \phi'-\omega V)$ labeling an unperturbed null generator to change on a V-time scale of order M^{-4}, as the horizon evolves. Because the perturbations are quasistationary secular effects in $\rho^{(4)}$, $\sigma^{(4)}$ are averages of some function of ϕ', taken over many revolutions of the null generator around the horizon. Thus secular effects depend only on θ (at the lowest order). The "rings" of null geodesics at constant θ will slowly be moved through new values of θ. In particular, the polar geodesics at $\theta=0,\pi$ will remain polar at the lowest order. This makes it hard to assign a direction to the angular momentum change of the black hole. If instead one tries to use the external scheme to define the direction of change of angular momentum, no obvious local geometrical definitions are apparent; the situation is similar to that in Sec. IV respecting the deviation from geodesic motion. It may only be possible to make sensible statements about vector angular momentum change under certain extra global conditions.

Our knowledge of $\rho^{(2)}$ and $\sigma^{(2)}$ allows us to write down the perturbed intrinsic horizon metric at $O(M^2)$ in the QS scheme corresponding to the nonaxisymmetric modes. Let us use coordinates $x^1=\theta$, $x^2=\tilde{\phi}$, $x^3=V$ on the horizon, and assume (making a gauge transformation if necessary) that the generator $l^a=(\partial/\partial V)^a$. For the unperturbed horizon, we would have $\tilde{\phi}=\phi-\omega V$. Then the perturbed intrinsic degenerate metric can be written

$$\begin{aligned} ds^2 = {} & \Big\{f_{11}+M^2\Big[{}_0n_{11}+\sum_{\{m=1,2\}}({}_mn_{11}e^{im(\tilde{\phi}+\omega V)}+{}_m\bar{n}_{11}e^{-im(\tilde{\phi}+\omega V)})\Big]\Big\}\,d\theta^2 \\ & +2M^2\Big[{}_0n_{12}+\sum_{\{m=1,2\}}({}_mn_{12}e^{im(\tilde{\phi}+\omega V)}+{}_m\bar{n}_{12}e^{-im(\tilde{\phi}+\omega V)})\Big]\,d\theta\,d\tilde{\phi} \\ & +\Big\{f_{22}+M^2\Big[{}_0n_{22}+\sum_{\{m=1,2\}}({}_mn_{22}e^{im(\tilde{\phi}+\omega V)}+{}_m\bar{n}_{22}e^{-im(\tilde{\phi}+\omega V)})\Big]\Big\}\,d\tilde{\phi}^2+O(M^3)\,. \end{aligned} \tag{5.42}$$

Here the unperturbed terms

$$\begin{aligned} f_{11} &= R_+{}^2+\chi^2\cos^2\theta\,, \\ f_{22} &= \frac{(R_+{}^2+\chi^2)^2\sin^2\theta}{(R_+{}^2+\chi^2\cos^2\theta)}\,, \end{aligned} \tag{5.43}$$

and ${}_mn_{11}$, ${}_mn_{12}$, ${}_mn_{22}$ $(m=0,1,2)$ are functions of θ, with ${}_0n_{11}$, ${}_0n_{12}$, ${}_0n_{22}$ real. To find ${}_mn_{11}$, ${}_mn_{12}$, ${}_mn_{22}$ for $m=1,2$, we recall that ρ and σ on the horizon are quantities depending only on the intrinsic geometry of the null surface[18] and can be computed from Eq.

(5.42) by embedding the null surface in a spacetime with an extra null coordinate V' such that the spacetime metric is given by the addition of an extra term $-2dV\,dV'$ to Eq. (5.42). Examining the Hartle tetrad on the horizon[8] we see that

$$m^\theta = \frac{1}{\sqrt{2}(R_+ + i\chi\cos\theta)} + O(M^2)\,,$$
$$m^{\tilde\phi} = \frac{i(R_+ - i\chi\cos\theta)}{\sqrt{2}\,\sin\theta(R_+{}^2+\chi^2)} + O(M^2)\,, \tag{5.44}$$
$$m^V = O(M^2)\,.$$

Write

$$\sigma^{(2)} = \sum_{\{m=-2,-1,1,2\}} Z_m(\theta)e^{im\phi'}\,. \tag{5.45}$$

Then by computing ρ and σ as above from Eq. (5.42), and considering modes with $m=\pm1$, $m=\pm2$ in isolation, we find that

$$\rho^{(2)} = 0$$
$$\Rightarrow (R_+{}^2+\chi^2)^2\sin^2\theta\; {}_m n_{11} + (R_+{}^2+\chi^2\cos^2\theta)^2\; {}_m n_{22} = 0\,,\quad m=1,2\,. \tag{5.46}$$

Also

$$-\frac{im\omega}{4(R_+ + i\chi\cos\theta)^2}\,{}_m n_{11} + \frac{m\omega(R_+ - i\chi\cos\theta)}{2(R_+ + i\chi\cos\theta)(R_+{}^2+\chi^2)\sin\theta}\,{}_m n_{12} + \frac{im\omega(R_+ - i\chi\cos\theta)^2}{4(R_+{}^2+\chi^2)^2\sin^2\theta}\,{}_m n_{22} = Z_m\,, \tag{5.47}$$

$$\frac{im\omega}{4(R_+ + i\chi\cos\theta)^2}\,{}_m \bar n_{11} - \frac{m\omega(R_+ - i\chi\cos\theta)}{2(R_+ + i\chi\cos\theta)(R_+{}^2+\chi^2)\sin\theta}\,{}_m \bar n_{12} - \frac{im\omega(R_+ - i\chi\cos\theta)^2}{4(R_+{}^2+\chi^2)^2\sin^2\theta}\,{}_m \bar n_{22} = Z_{-m}\,,\quad m=1,2\,. \tag{5.48}$$

Hence, for $m=1,2$,

$${}_m n_{11} = \frac{i(R_+ + i\chi\cos\theta)^2}{m\omega} Z_m + \frac{i(R_+ - i\chi\cos\theta)^2}{m\omega}\bar Z_{-m}\,,$$
$${}_m n_{12} = \frac{(R_+{}^2+\chi^2)(R_+ + i\chi\cos\theta)\sin\theta}{m\omega(R_+ - i\chi\cos\theta)} Z_m - \frac{(R_+{}^2+\chi^2)(R_+ - i\chi\cos\theta)\sin\theta}{m\omega(R_+ + i\chi\cos\theta)}\bar Z_{-m}\,, \tag{5.49}$$
$${}_m n_{22} = -\frac{i(R_+{}^2+\chi^2)\sin^2\theta}{m\omega(R_+ - i\chi\cos\theta)^2} Z_m - \frac{i(R_+{}^2+\chi^2)^2\sin^2\theta}{m\omega(R_+ + i\chi\cos\theta)^2}\bar Z_{-m}\,.$$

One may then use Eq. (5.26) for $\sigma^{(2)}$ to find (slightly lengthy) expressions for ${}_m n_{11}$, ${}_m n_{12}$, ${}_m n_{22}$ ($m=1,2$). This approach does not give us ${}_0 n_{11}$, ${}_m n_{12}$, ${}_m n_{22}$.

VI. THE NONROTATING CASE

In this section we examine the internal structure in the case where the small black hole is Schwarzschild-type. More precisely, we assume that the black hole is rotating so slowly that $\chi=0$ and that the $O(M)$ QS perturbations of the Schwarzschild solution are zero [so that there is no perturbation among the Kerr family at $O(M)$]. We may than apply well-known results on Schwarzschild perturbations to find the QS metric terms $M^2 j^{(2)}_{ab}(0,R,\theta,\phi)$.

If we adopt the gauge condition of Regge and Wheeler,[19] then in Schwarzschild coordinates (T,R,θ,ϕ), stationary perturbations can be classified by angular numbers (L,m) and into modes of even or odd parity, where in an even (L,m) mode

$$j^{(2)}_{ab} = \begin{bmatrix} \left(1-\frac{2}{R}\right)H_0 Y^m_L & H_1 Y^m_L & 0 & 0 \\ H_1 Y^m_L & \left(\frac{R}{R-2}\right)H_2 Y^m_L & 0 & 0 \\ 0 & 0 & R^2 K Y^m_L & 0 \\ 0 & 0 & 0 & R^2\sin^2\theta\, K Y^m_L \end{bmatrix}. \tag{6.1}$$

and in an odd (L,m) mode

$$j^{(2)}_{ab}=\begin{bmatrix} 0 & 0 & -\frac{1}{\sin\theta}h_0\frac{\partial}{\partial\phi}Y_L^m & \sin\theta h_0\frac{\partial}{\partial\theta}Y_L^m \\ 0 & 0 & -\frac{1}{\sin\theta}h_1\frac{\partial}{\partial\phi}Y_L^m & \sin\theta h_1\frac{\partial}{\partial\theta}Y_L^m \\ -\frac{1}{\sin\theta}h_0\frac{\partial}{\partial\phi}Y_L^m & -\frac{1}{\sin\theta}h_1\frac{\partial}{\partial\phi}Y_L^m & 0 & 0 \\ \sin\theta h_0\frac{\partial}{\partial\theta}Y_L^m & \sin\theta h_1\frac{\partial}{\partial\theta}Y_L^m & 0 & 0 \end{bmatrix}. \tag{6.2}$$

Here $Y_L^m(\theta,\phi)$ is the usual spherical harmonic, and H_0, H_1, H_2, K, h_0, h_1 are all functions of R. We shall only be interested in the $L=2$ modes.

Stationary Schwarzschild perturbations have been discussed by Vishveshwara;[20] in the even case the field equations imply $H_0=H_2=H$, $H_1\equiv 0$, where

$$\frac{d^2H}{dR^{*2}}+\frac{2}{R}\left(1-\frac{2}{R}\right)\frac{dH}{dR^*}-\left[\frac{4}{R^4}+\frac{L(L+1)}{R^2}\left(1-\frac{2}{R}\right)\right]H=0 \tag{6.3}$$

and $R^*=R+2\ln(R-2)$. The boundary conditions at the horizon require

$$H\to 0 \text{ as } R\to 2\,. \tag{6.4}$$

Hence

$$H=C_mR(R-2) \tag{6.5}$$

for some constant C_m, when $L=2$. We find K from two of the field equations[21]

$$\frac{dK}{dR}-\frac{dH}{dR}-\frac{2}{R(R-2)}H=0\,,$$
$$(R-1)\frac{dK}{dR}-(R-2)\frac{dH}{dR}+2(H-K)=0\,. \tag{6.6}$$

Hence, when $L=2$,

$$K=C_m(R^2-2)\,. \tag{6.7}$$

For odd stationary perturbations, $h_1\equiv 0$ and

$$\frac{d^2h_0}{dR^{*2}}-\frac{2}{R^2}\frac{dh_0}{dR^*}-\left[\frac{L(L+1)}{R^2}-\frac{4}{R^3}\right]\left(1-\frac{2}{R}\right)h_0=0\,. \tag{6.8}$$

The boundary conditions at the horizon require

$$h_0\to 0 \text{ as } R\to 2\,. \tag{6.9}$$

Hence, when $L=2$,

$$h_0=D_mR^2(R-2) \tag{6.10}$$

for some constant D_m.

To compare the metric perturbations with the work of the previous section, we first specify our normalization of spherical harmonics:

$$Y_2^0(\theta,\phi)=3\cos^2\theta-1\,,$$
$$Y_2^{\pm1}(\theta,\phi)=\sqrt{6}\sin\theta\cos\theta e^{\pm i\phi}\,, \tag{6.11}$$
$$Y_2^{\pm2}(\theta,\phi)=\sqrt{6}\sin^2\theta e^{\pm 2i\phi}\,.$$

Then for an $(L=2,m)$ even mode, in the Kinnersley tetrad

$$\Psi_0=-\tfrac{1}{2}M^2\frac{H}{R(R-2)}\,{}_2Y_2^m(\theta,\phi)+O(M^3)\,, \tag{6.12}$$

and for an $(L=2,m)$ odd mode,

$$\Psi_0=\frac{i}{2}M^2\left[\frac{1}{R(R-2)}\frac{dh_0}{dR}-\frac{2}{R^2(R-2)^2}h_0\right]{}_2Y_2^m(\theta,\phi)+O(M^3)\,. \tag{6.13}$$

Altogether,

$$\Psi_0=-\frac{M^2}{2}\sum_{m=-2}^{+2}(C_m-3iD_m)\,{}_2Y_2^m(\theta,\phi)+O(M^3)\,. \tag{6.14}$$

Comparing this with the asymptotic behavior of Ψ_0 as $R\to\infty$ in Eq. (5.5), we find

$$C_m=-\alpha_m\,,$$
$$D_m=\tfrac{1}{3}\beta_m\,. \tag{6.15}$$

Thus the electric part of the background Weyl tensor produces even internal perturbations, and the magnetic part produces odd perturbations.

By Wald's theorem, we have already given a complete description of the $O(M^2)$ QS perturbations, apart from the $L=0,1$ modes which represent respectively changes in mass and angular momentum among the Kerr family.[9,20,22] The arguments of Sec. V show that the $L=0,1$ perturbations will be constant over a gackground time scale, as measured by τ.

We see that the $O(M^2)$ QS metric perturbations, when referred to coordinates $(T, R\sin\theta\cos\phi, R\sin\theta\sin\phi, R\cos\theta)$ are $O(R^2)$ as $R\to\infty$; this is consistent with our matching conditions in Sec. II. The M^2R^2 QS terms match onto those background terms which behave as r^2 when $r \to 0$. Of course, the Regge-Wheeler gauge condition on the internal scheme enforces a coordinate condition on these $O(r^2)$ background terms.

VII. SOLUTION OF EXTERNAL PROBLEM

The Green's function approach to general relativity developed by Sciama, Waylen, and Gilman[23] allows us to give a useful representation of the lowest-order external perturbation $g^{(1)}_{ab}$, when the external scheme refers to the vacuum Einstein equations. To show this, we first define a coordinate transformation $(\tau, x, y, z, M) \to (\tau, x', y', z', M)$ on a subset of $\mathfrak{N}$ such that $x'r = r'x$, $y'r = r'y$, $z'r = r'z$, and

$$r = r'\left(1 + \frac{M}{2r'}\right)^2 \tag{7.1}$$

for $r <$ some constant, $r/M >$ some constant; outside these regions the coordinate transformation should be defined so as to be regular. We may assume that the transformation has been defined so that the components of $g^{(0)}_{ab}$ are unchanged. We note that Eq. (2.5) for the most singular part of $g^{(1)}_{ab}$ near l_0 has been replaced by

$$g^{(1)}_{ab} \sim \frac{2}{r'} \operatorname{diag}(1, 1, 1, 1) \tag{7.2}$$

as $r' \to 0$, in (τ, x', y', z') coordinates. In fact the coordinate transformation $r = r'(1 + M/2r')^2$ just gives the transformation from Schwarzschild to isotropic coordinates in the Schwarzschild metric.

Following Ref. 23, we consider

$$\delta g^{ab} = -g^{(0)ac} g^{(0)bd} g^{(1)}_{cd}, \tag{7.3}$$

which gives the first-order variation in g^{ab} in the external scheme. Define

$$\psi_a = \tfrac{1}{2} g^{(0)}_{bc} \delta g^{bc}{}_{|a} - g^{(0)}_{ab} \delta g^{bc}{}_{|c}, \tag{7.4}$$

where we recall that the subscripted vertical bar denotes covariant differentiation with respect to the background. Because of our coordinate conditions on $g^{(0)}_{ab}$ and $g^{(1)}_{ab}$ in (τ, x, y, z) coordinates, we find that ψ_a is $O(1)$ as $r' \to 0$, in (τ, x', y', z') coordinates. By a further coordinate transformation, ψ_a may be removed entirely. Let us solve the equation

$$g^{(0)ab} \xi_{c|ab} = \psi_c \tag{7.5}$$

to find a vector field ξ_c on $\mathfrak{M}_0$. Then perform a coordinate transformation on $\mathfrak{N}$ to coordinates $(\hat{\tau}, \hat{x}, \hat{y}, \hat{z}, M)$, where

$$(\hat{\tau}, \hat{x}, \hat{y}, \hat{z}) = (\tau, x', y', z') + M\xi^a + O(M^2) \tag{7.6}$$

as $M \to 0$, with τ, x', y', z' fixed. This transformation has the effect of adding $\xi^{a|b} + \xi^{b|a}$ to δg^{ab}, so that $\psi_a = 0$ in $(\hat{\tau}, \hat{x}, \hat{y}, \hat{z})$ coordinates. We still have

$$g^{(1)}_{ab} \sim \frac{2}{\hat{r}} \operatorname{diag}(1, 1, 1, 1) \tag{7.7}$$

as $\hat{r} \to 0$, in $(\hat{\tau}, \hat{x}, \hat{y}, \hat{z})$ coordinates, where $\hat{r}^2 = \hat{x}^2 + \hat{y}^2 + \hat{z}^2$.

The harmonic gauge condition $\psi_a = 0$ simplifies the linearized empty-space Einstein equations, to give

$$g^{(0)cd} \delta g^{ab}{}_{|cd} + 2R^{(0)a}{}_c{}^b{}_d \, \delta g^{cd} = 0. \tag{7.8}$$

Let $E^{a'b'}{}_{cd}(x', x)$ be the retarded Green's function for this equation such that

$$g^{(0)ef} E^{a'b'}{}_{cd|ef} + 2R^{(0)e}{}_c{}^f{}_d E^{a'b'}{}_{ef} = g^{(0)a'}{}_{(c} g^{(0)b'}{}_{d)} [g^{(0)}(x') g^{(0)}(x)]^{-1/4} \delta(x, x'). \tag{7.9}$$

Here $E^{a'b'}{}_{cd} = E^{(a'b')}{}_{(cd)}(x', x)$ is a two-point tensor, i.e., it has tensorial properties at x' with respect to the indices a', b', and at x with respect to the indices c, d. Also $g^{(0)}(x) = \det(g^{(0)}_{ab}(x))$, $\delta(x, x')$ is the Dirac distribution, and $g^{(0)a'}{}_c(x', x)$ denotes the two-point vector of geodesic parallel transport (which is only defined locally). Use of Green's identity leads, as in Ref. 23, to a Kirchhoff representation

$$\delta g^{a'b'}(x') = \int_{\partial\Omega} g^{(0)cd} \times [E^{a'b'}{}_{ef|c} \, \delta g^{ef} - E^{a'b'}{}_{ef} \, \delta g^{ef}{}_{|c}] \times [-g^{(0)}(x)]^{1/2} dS_d, \tag{7.10}$$

where Ω is a volume containing x', and dS_d is the outward-directed coordinate surface element on the boundary $\partial\Omega$.

We choose Ω to be bounded by a small tube $\{\hat{r} = \epsilon\}$, with $l_0 \cap \Omega = \emptyset$, and by a surface "near infinity." We assume that the contribution from the large surface tends to zero as Ω increases to fill the whole of $\mathfrak{M}_0 - l_0$. This is in some sense a requirement that the linearized external solution should be completely determined by the black hole, without any additional gravitational radiation incoming from infinity. Assuming that the relevant functions are sufficiently well behaved, our conditions on δg^{ab} near l_0 imply

$$\delta g^{a'b'}(x') = 8\pi \int d\tau \, u^e(\tau) u^f(\tau) E^{a'b'}{}_{ef}(x', x(\tau)) \times [-g^{(0)}(x(\tau))]^{1/2}, \tag{7.11}$$

which is the desired integral representation. When translated back into our original coordinate system (τ, x, y, z), these results show that δg^{ab} is defined via such an integral representation up to the addition of a gauge term $\xi^{a|b}+\xi^{b|a}$.

In general, $E^{a'b'}{}_{ef}$ will include tail effects, i.e., its support will include points x with timelike separation from x'. This implies that the $O(M^3)$ QS perturbations may depend, through the matching, on the whole past history of the black hole, whereas the $O(M^2)$ QS perturbations are caused only by background curvature at one time τ. Roughly speaking, outgoing information about the black-hole field is partially reflected back into the black hole off the conformal curvature of the background.

It might be possible to extend this Green's function approach in order to solve higher-order external equations. One would need to subtract off the most singular parts of $g_{ab}^{(n)}$ as $r\to 0$, so that the integrals involved would converge near l_0; these singular parts would presumably be known already from the matching.

One could also try to consider the field equations with matter in the external scheme, rather than with empty space as in this section. The situation becomes more complicated since effects due to the black hole will change the energy-momentum tensor T_{ab} by $O(M)$, and so will produce extra $O(M)$ changes in the external metric; we shall not discuss this question further.

VIII. A SMALL BLACK HOLE IN THE BRANS-DICKE THEORY

The procedure which we have set up in this paper for analyzing the behavior of a small black hole in a background, under the Einstein field equations, can also be applied to the same problem in the Brans-Dicke theory of gravitation. The Brans-Dicke field equations were originally formulated in terms of a metric g^{ab} and scalar field Φ which satisfy

$$\begin{aligned}R_{ab}-\tfrac{1}{2}g_{ab}R&=8\pi\Phi^{-1}T_{ab}\\&\quad+\xi\Phi^{-2}(\Phi_{;a}\Phi_{;b}-\tfrac{1}{2}g_{ab}g^{cd}\Phi_{;c}\Phi_{;d})\\&\quad+\Phi^{-1}(\Phi_{;ab}-g_{ab}g^{cd}\Phi_{;cd})\,,\end{aligned}\tag{8.1}$$

$$g^{cd}\Phi_{;cd}=8\pi(3+2\xi)^{-1}T\,.\tag{8.2}$$

Here R_{ab}, R are the Ricci tensor and scalar of g^{ab}, and a subscripted semicolon denotes covariant differentiation with respect to g^{ab}. Also T_{ab} is the usual energy-momentum tensor which satisfies $T_{ab;c}g^{bc}=0$, and ξ is a constant. In this formulation, we say that the theory is viewed in the Brans-Dicke frame. Alternatively, one may make the conformal transformation $\bar{g}_{ab}=\Phi g_{ab}$, $\bar{T}_{ab}=\Phi^{-1}T_{ab}$. Then the field equations become

$$\begin{aligned}\bar{R}_{ab}-\tfrac{1}{2}\bar{g}_{ab}\bar{R}&=8\pi\bar{T}_{ab}\\&\quad+(3+2\xi)\Phi^{-2}(16\pi)^{-1}\\&\quad\times(\Phi_{|a}\Phi_{|b}-\tfrac{1}{2}\bar{g}_{ab}\bar{g}^{cd}\Phi_{|c}\Phi_{|d})\,,\end{aligned}\tag{8.3}$$

$$\bar{g}^{cd}(\ln\Phi)_{|cd}=8\pi(3+2\xi)^{-1}\bar{T}\,.\tag{8.4}$$

Here $\bar{R}_{ab}, \bar{R}$ are the Ricci scalar and tensor of $\bar{g}^{ab}$, and covariant differentiation with respect to $\bar{g}^{ab}$ is denoted by a stroke. We say that the theory is viewed in the Einstein frame when written in this second way.

A normal test body, with negligible self-gravitation, can be shown to move on a geodesic in the Brans-Dicke frame. But a body with significant gravitational binding energy will, in general, deviate from geodesic motion in the Brans-Dicke frame. A black hole is an example of a body with binding energy of the same order as its rest-mass energy; moreover, if the black hole is exactly stationary, then it has no scalar monopole moment [Hawking (Ref. 24)] unlike an ordinary test body. By arguing that there is no flux of the scalar field through a small world tube surrounding the black hole, and hence no force on the black hole due to the scalar field, Hawking was led to conjecture that a small black hole moves along a geodesic in the Einstein frame with respect to the background universe.

We are able to verify this conjecture by means of our approximation method. Before setting up an internal approximation scheme for the small black hole, we recall the result of Ref. 24: that the stationary vacuum black-hole solutions in the Brans-Dicke theory are identical to the stationary vacuum black-hole solutions of the Einstein equations and have $\Phi\equiv$ const. So we again take the internal metric to be that of a Kerr solution, perturbed by the presence of the background. In the Einstein frame, the internal scheme can be expressed by

$$\begin{aligned}\bar{g}_{ab}&=M^2\bar{\jmath}_{ab}^{(0)}(R,\theta)+M^3\bar{\jmath}_{ab}^{(1)}(MT,R,\theta,\phi)\\&\quad+M^4\bar{\jmath}_{ab}^{(2)}(MT,R,\theta,\phi)+\cdots\,.\end{aligned}\tag{8.5}$$

$$\begin{aligned}\Phi&=\Phi^{(0)I}(MT)+M\Phi^{(1)I}(MT,R,\theta,\phi)\\&\quad+M^2\Phi^{(2)I}(MT,R,\theta,\phi)+\cdots\,.\end{aligned}\tag{8.6}$$

Here $\bar{\jmath}_{ab}^{(0)}$ is a unit-mass Kerr metric in Boyer-Lindquist coordinates (T,R,θ,ϕ). The form of the expansion for Φ is dictated by the requirement that the lowest-order internal solution, found by letting $M\to 0$ with T,R,θ,ϕ constant, should have $\Phi\equiv$ constant, as in the exactly stationary solutions. This condition permits $\Phi^{(0)I}$ to have slow time dependence, but no spatial dependence.

We use matching coordinates (τ,x,y,z,M) which are related to the internal coordinates (T,R,θ,ϕ)

just as before. The external scheme can be written

$$\bar{g}_{ab}=\bar{g}^{(0)}_{ab}(\tau,x,y,z)+M\bar{g}^{(1)}_{ab}(\tau,x,y,z)+M^2\bar{g}^{(2)}_{ab}(\tau,x,y,z)+\cdots\,, \tag{8.7}$$

$$\Phi=\Phi^{(0)E}(\tau,x,y,z)+M\Phi^{(1)E}(\tau,x,y,z)+M^2\Phi^{(2)E}(\tau,x,y,z)+\cdots\,, \tag{8.8}$$

where $\bar{g}^{(0)}_{ab}(\tau,x,y,z)$ and $\Phi^{(0)E}(\tau,x,y,z)$ are the background metric and scalar field in the Einstein frame. The labels I and E have been attached to the expressions $\Phi^{(n)}$ to distinguish internal from external terms.

Since Φ acts as a source in Eq. (8.3) through products of its first derivatives, it only generates curvature of $O(1)$ in $\bar{g}_{ab}$ in the internal region. Hence the lowest-order QS perturbations of $\bar{j}^{(0)}_{ab}$ satisfy the vacuum Einstein equations, linearized about the Kerr solution. The boundary conditions for $\bar{j}^{(1)}_{ab}(0,R,\theta,\phi)$ as $R\to\infty$ are determined by the $O(r)$ part of $\bar{g}^{(0)}_{ab}$ as $r\to 0$, and one may argue just as before to find that $\bar{j}^{(1)}_{ab}$ can be eliminated by choice of coordinates. Then, from the matching,

$$\bar{g}^{(0)}_{ab}(\tau,x,y,z)=\mathrm{diag}(-1,1,1,1)+O(r^2) \tag{8.9}$$

as $r\to 0$, so that the lowest-order world line of the black hole is indeed a geodesic with respect to $\bar{g}^{(0)}_{ab}$.

Further inferences can be made by examining the matching conditions. From the matching for Φ, we find

$$\Phi^{(0)I}(MT)=\Phi^{(0)E}(\tau,0,0,0)\,, \tag{8.10}$$

where $\tau=\tau_0+MT$ is an appropriate background time. Moreover, the fact that $\Phi^{(0)I}$ depends only on time implies that

$$\Phi^{(1)E}=O(1),\quad \Phi^{(2)E}=O(r^{-1}),\quad \Phi^{(3)E}=O(r^{-2}),\ldots \tag{8.11}$$

as $r\to 0$. The condition that $\Phi^{(1)E}$ is bounded near the world line can be regarded as a statement that the black hole has no scalar multipole moments at the lowest order.

The conditions (8.11) allow us to prove that the spin vector of the black hole is parallel transported along the geodesic, with respect to $\bar{g}^{(0)}_{ab}$. We may recall that the corresponding property in the Einstein theory was found from the M^2r^{-3} part of the field equations in the external scheme. Here the parts of $\bar{g}^{(0)}_{ab}$, $\bar{g}^{(1)}_{ab}$, and $\bar{g}^{(2)}_{ab}$ which contribute to this part of the field equations have the same behavior as in $g^{(0)}_{ab}$, $g^{(1)}_{ab}$, and $g^{(2)}_{ab}$ of Sec. IV. Also Eqs. (8.3) and (8.11) show that the scalar field does not contribute to this part of the field equations. The argument then proceeds as in Sec. IV.

The largest internal perturbations of the scalar field and conformal metric can also be analyzed. The lowest-order QS perturbation in Φ is

$$M\left[T\frac{d}{d(MT)}\Phi^{(0)I}(MT)\bigg|_{MT=0}+\Phi^{(1)I}(0,R,\theta,\phi)\right]=MN(T,R,\theta,\phi)\,, \tag{8.12}$$

say. The field equation (8.4) for Φ shows that in vacuo

$$\bar{j}^{(0)cd}N_{|cd}=0\,, \tag{8.13}$$

where the covariant differentiation is with respect to the Kerr background $\bar{j}^{(0)}_{ab}$. Now the part $T[d/d(MT)]\Phi^{(0)I}(MT)|_{MT=0}$ of N is already harmonic in the Kerr background. Thus $\Phi^{(1)I}(0,R,\theta,\phi)$ separately is harmonic in the Kerr background. If one wished, one could compute $\Phi^{(1)I}(0,R,\theta,\phi)$ in terms of hypergeometric functions; as $R\to\infty$, $\Phi^{(1)I}(0,R,\theta,\phi)\sim\alpha X+\beta Y+\gamma Z$ due to the spatial gradient of $\Phi^{(0)E}$ near the black hole, while the boundary condition at the horizon is that $\Phi^{(1)I}(MT,R,\theta,\phi)$ should be regular there. Then one could obtain a Teukolsky equation for the metric perturbation $\bar{j}^{(2)}_{ab}$. This would include source terms arising from products of gradients of N. The boundary conditions on $\bar{j}^{(2)}_{ab}(0,R,\theta,\phi)$ as $R\to\infty$ will be due to the r^2 parts of $\bar{g}^{(0)}_{ab}$ near the world line.

We remark that the close resemblance of the analysis in this section to the earlier arguments about black holes in the pure Einstein theory has only been possible because, in the Einstein frame, Φ appears on the right-hand side of (8.3) in a form involving products of first derivatives. Had we instead tried to repeat the earlier arguments in the Brans-Dicke frame, we should have found that Φ produces curvature of $O(M^{-1})$ in g_{ab} in the internal region, due to the second derivatives of Φ in Eq. (8.1); the $O(M)$ QS metric perturbations could not then be zero.

IX. CONCLUSION

We have seen that a technique involving matched asymptotic expansions can be brought to bear on the problem of a small black hole in a background universe, in the context of the Einstein field equations. In particular, it helps us to understand how the black hole moves with respect to the background, and also how the background gravitational field distorts the internal structure of the black hole. Under our assumptions the black hole moves approximately along a timelike geodesic in the background, and its angular momentum vector is approximately parallel transported along the geodesic in the background metric. The largest deviations from the Kerr geometry near the black hole are of a quadrupole nature, caused by local background curvature. Our technique can also be used in order to understand the behavior of a small black hole in the Brans-Dicke theory.

It would be interesting if these results could be extended to the case of a charged black hole in a background universe containing electromagnetic fields. An extension of the treatment given here might also deal with the situation where the background $(\mathfrak{M}_0, g^{(0)}_{ab})$ contains matter on the zeroth-order world line l_0. At present, the problem of two black holes with relative Newtonian potential and kinetic energies of the same order (e.g., a bound state) is under investigation, using a combination of the methods of this paper and the Einstein-Infeld-Hoffman method.[25]

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[4]In our conventions, the signature is +2, the Riemann tensor is defined by $Z^a{}_{;dc} - Z^a{}_{;cd} = R^a{}_{bcd} Z^b$ for any vector field Z^a, and the Ricci tensor is $R_{bd} = R^a{}_{bad}$.

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