Solutions of the nonsymmetric unified field theory*

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The field equations in a formulation of Einstein's nonsymmetric unified field theory are solved exactly for the case of a static, spherically symmetric point singularity. The equations also yield the correct equations of motion in the lowest nontrivial order of approximation using the methods of Einstein, Infeld, and Hoffmann. When a universal constant k vanishes, the theory reduces to the Einstein-Maxwell equations and the solution found here becomes the Reissner-Nordström solution. A coordinate singularity occurs in the metric when $r = m + (m^2 - Q^2/2)^{1/2}$, as in the Reissner-Nordström solution. It is shown that this singularity is due to the choice of coordinates by performing a Kruskal-Szerkeres-type transformation. Further, the exact solutions which are generated by a Hermitian tensor, rather than a real nonsymmetric tensor, are given. Finally, the gauge invariance and possible renormalization of the theory are discussed.

I. INTRODUCTION

In Einstein's original 1915 theory of gravitation, the electromagnetic field was incorporated in the field equations

$$G_{\mu\nu} = -8\pi T_{\mu\nu}, \tag{1.1}$$

where $G_{\mu\nu}$ is the contracted Riemann-Christoffel tensor, and $T_{\mu\nu}$ is the energy tensor of the electromagnetic field $F_{\mu\nu}$, which satisfies Maxwell's equations

$$F^{\mu\nu}{}_{;\nu} = 0, \tag{1.2}$$

$$_{[\sigma}F_{\mu\nu]} = \partial_\sigma F_{\mu\nu} + \partial_\mu F_{\nu\sigma} + \partial_\nu F_{\sigma\mu} = 0. \tag{1.3}$$

Here, the subscript semicolon denotes covariant differentiation with respect to the Christoffel symbols $\{{}^{\lambda}_{\mu\nu}\}$.

Many attempts have been made to combine the gravitational and electromagnetic fields so that electromagnetism, as well as gravitation, appears as a property of the space-time continuum, rather than as a separate physical phenomenon. One of the most important attempts is Einstein's unified field theory[1] based on the nonsymmetric field, which he proposed over 25 years ago.[2,3] He considered the nonsymmetric theory the most natural generalization of his gravitational theory, for it incorporated the electromagnetic field into the fundamental tensor $g_{\mu\nu}$. The tensor $g_{\mu\nu}$ is split into its symmetric and antisymmetric parts by

$$g_{\mu\nu} = g_{(\mu\nu)} + g_{[\mu\nu]}, \tag{1.4}$$

where $g_{(\mu\nu)} = \frac{1}{2}(g_{\mu\nu} + g_{\nu\mu})$ and $g_{[\mu\nu]} = \frac{1}{2}(g_{\mu\nu} - g_{\nu\mu})$. If $g_{[\mu\nu]}$ is pure imaginary, then the tensor $g_{\mu\nu}$ not only contains antisymmetric terms but may also be Hermitian. Since there is some question as to whether the tensor is real nonsymmetric or Hermitian,[4] we will investigate the consequences of both possibilities in our calculations. The relation defining the $g^{\mu\nu}$ from the $g_{\mu\nu}$ is given by

$$g^{\mu\nu} g_{\sigma\nu} = \delta^\mu{}_\sigma. \tag{1.5}$$

We note that the order of the indices is important.[3] The nonsymmetric affine connection is defined by

$$\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{(\mu\nu)} + \Gamma^\lambda_{[\mu\nu]}. \tag{1.6}$$

In Einstein's work,[1] the connection $\Gamma^\lambda_{\mu\nu}$ is determined by the 64 equations

$$\partial_\alpha g_{\mu\nu} - g_{\sigma\nu}\Gamma^\sigma_{\mu\alpha} - g_{\mu\sigma}\Gamma^\sigma_{\alpha\nu} = 0, \tag{1.7}$$

where we note the order of the subscripts in the last term of Eq. (1.7). These equations are regarded as the natural generalization of the equations $g_{\mu\nu;\alpha} = 0$ in general relativity.

Einstein proposed two sets of equations[1] both of which include the set (1.7). They are given by set I,

$$\Gamma^\alpha_{[\mu\alpha]} = 0, \tag{1.8}$$

$$R_{\mu\nu} = 0, \tag{1.9}$$

and set II,

$$\Gamma^\alpha_{[\mu\alpha]} = 0, \tag{1.10}$$

$$R_{(\mu\nu)} = 0, \tag{1.11}$$

$$_{[\sigma}R_{[\mu\nu]]} = 0. \tag{1.12}$$

In these equations, the contracted curvature tensor is given by

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha_{\mu\nu} - \tfrac{1}{2}(\partial_\nu \Gamma^\alpha_{(\mu\alpha)} + \partial_\mu \Gamma^\alpha_{(\nu\alpha)}) - \Gamma^\alpha_{\mu\sigma}\Gamma^\sigma_{\alpha\nu} + \Gamma^\alpha_{\mu\nu}\Gamma^\sigma_{\alpha\sigma}. \tag{1.13}$$

If we define $\mathfrak{g}^{\mu\nu} = \sqrt{-g}\, g^{\mu\nu}$, then the set (1.7) is equivalent to

$$\partial_\alpha \mathfrak{g}^{\mu\nu} + \mathfrak{g}^{\sigma\nu}\Gamma^\mu_{\sigma\alpha} + \mathfrak{g}^{\mu\sigma}\Gamma^\nu_{\alpha\sigma} - \mathfrak{g}^{\mu\nu}\Gamma^\sigma_{(\sigma\alpha)} = 0. \tag{1.14}$$

Contracting Eq. (1.14), we find

$$\partial_\alpha \mathfrak{g}^{[\mu\alpha]} + \tfrac{1}{2}\mathfrak{g}^{(\mu\sigma)}(\Gamma^\alpha_{\alpha\sigma} - \Gamma^\alpha_{\sigma\alpha}) = 0 . \tag{1.15}$$

Thus, Eqs. (1.8) and (1.10) are equivalent to

$$\partial_\alpha \mathfrak{g}^{[\mu\alpha]} = 0 . \tag{1.16}$$

When $g_{[\mu\nu]}$ is identified with the electromagnetic field, neither I nor II lead to physical static spherically symmetric solutions for a point singularity.[5] Infeld[6] and Callaway[7] showed that neither set of equations gives the correct equations of motion for electric charges. Several papers in recent literature have considered this problem within the context of the weak-field approximation.[4]

II. THE FIELD EQUATIONS

We shall now investigate an alternative approach to the problem considered some time ago by Bonnor[8] and Kurşunoğlu.[9] Bonnor derived a set of field equations from a variational principle

$$\delta \int \mathcal{H}^* d\tau = 0 , \tag{2.1}$$

where

$$\mathcal{H}^* = \mathcal{H} + p^2 \mathfrak{g}^{ik} g_{[ki]} . \tag{2.2}$$

Moreover, $\mathcal{H} = \mathfrak{g}^{\mu\nu} R_{\mu\nu}$ and p^2 is a constant. He then applied the solution of the variational problem to generate the proper equations of motion. We shall write our field equations in a different form. Let us define the contracted curvature tensor

$$R^*_{\mu\nu} = R_{\mu\nu} + I_{\mu\nu} , \tag{2.3}$$

where $R_{\mu\nu}$ is given by Eq. (1.13) and

$$I_{\mu\nu} = -\frac{1}{2k^2}\left(g_{\mu\sigma} g^{[\sigma\rho]} g_{\rho\nu} + \tfrac{1}{2} g_{\mu\nu} g_{\sigma\rho} g^{[\sigma\rho]} + g_{[\mu\nu]}\right) . \tag{2.4}$$

Here, k is a constant to be determined later and may be imaginary. Our field equations are

$$\partial_\alpha g_{\mu\nu} - g_{\mu\sigma}\Gamma^\sigma_{\alpha\nu} - g_{\sigma\nu}\Gamma^\sigma_{\mu\alpha} = 0 , \tag{2.5}$$

$$\Gamma^\alpha_{[\mu\alpha]} = 0 , \tag{2.6}$$

$$R^*_{(\mu\nu)} = 0 , \tag{2.7}$$

$$_{[\sigma} R^*_{[\mu\nu]]} = 0 . \tag{2.8}$$

These field equations are equivalent to those found by Bonnor by varying the modified Hamiltonian. However, our physical interpretation of the theory will be quite different from that of both Bonnor and Kurşunoğlu.

Splitting $I_{\mu\nu}$ into its symmetric and skew parts, we find after some simplifying

$$I_{(\mu\nu)} = -\frac{1}{2k^2}\left(g_{(\mu\rho)} g^{[\rho\sigma]} g_{[\sigma\nu]} + g_{(\nu\rho)} g^{[\rho\sigma]} g_{[\sigma\mu]} + \tfrac{1}{2} g_{(\mu\nu)} g_{[\sigma\rho]} g^{[\sigma\rho]}\right) \tag{2.9}$$

and

$$I_{[\mu\nu]} = -\frac{1}{2k^2}\left(g_{[\mu\rho]} g^{[\rho\sigma]} g_{[\sigma\nu]} + g_{(\mu\rho)} g^{[\rho\sigma]} g_{(\sigma\nu)} + \tfrac{1}{2} g_{[\mu\nu]} g_{[\rho\sigma]} g^{[\rho\sigma]} + g_{[\mu\nu]}\right) . \tag{2.10}$$

III. THE STATIC SPHERICALLY SYMMETRIC SOLUTIONS

As stated above, there appear to be two possible forms for the metric tensor. If $g_{(\mu\nu)}$ contains only diagonal elements, then a real $g_{[\mu\nu]}$ will lead to a real nonsymmetric tensor, while an imaginary $g_{[\mu\nu]}$ will lead to a Hermitian tensor. We will derive the results in detail for the former case and simply state the results for the latter at the end of this section.

We represent the real nonsymmetric tensor by

$$g_{\mu\nu} = \begin{bmatrix} -\alpha & 0 & 0 & w \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2\sin^2\theta & 0 \\ -w & 0 & 0 & \gamma \end{bmatrix} , \tag{3.1}$$

where w is real. We note for use later that the Hermitian tensor may be obtained by the substitution $w \to iw$ in this and subsequent equations. The line element corresponding to Eq. (3.1) is then

$$ds^2 = \gamma\, dt^2 - \alpha\, dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) . \tag{3.2}$$

We shall assume that the off-diagonal element w describes the static electric force due to a spherically symmetric point charge, situated at the origin of the coordinates.

The relation (1.5) shows that

$$g^{\mu\nu} = \begin{bmatrix} -\dfrac{\gamma}{\alpha\gamma - w^2} & 0 & 0 & \dfrac{-w}{\alpha\gamma - w^2} \\ 0 & -\dfrac{1}{r^2} & 0 & 0 \\ 0 & 0 & -\dfrac{1}{r^2\sin^2\theta} & 0 \\ \dfrac{w}{\alpha\gamma - w^2} & 0 & 0 & \dfrac{\alpha}{\alpha\gamma - w^2} \end{bmatrix} . \tag{3.3}$$

In the limit $w \to 0$ this just reduces to the familiar expression for $g^{\mu\nu}$ in the Schwarzschild metric.

The 64 homogeneous linear equations (2.5) have been solved by Papapetrou,[5] Wyman,[5] and Bonnor[5] for the static electric case, and by Bonnor[5] for the general static electric and magnetic fields, using Einstein's equations I and II. For our purposes, we shall quote only the results for the electric case:

$$\begin{aligned}
\Gamma^1_{11} &= \frac{\alpha'}{2\alpha},\\
\Gamma^1_{22} &= \csc^2\theta\,\Gamma^1_{33} = -\frac{r}{\alpha},\\
\Gamma^1_{14} &= -\Gamma^1_{41} = \frac{2\alpha\gamma w' - w(\alpha'\gamma + \alpha\gamma')}{2\alpha(w^2 - \alpha\gamma)}\\
&= \frac{w}{2\alpha}\left[\ln\left(1 - \frac{\alpha\gamma}{w^2}\right)\right]',\\
\Gamma^1_{44} &= \frac{4ww'\alpha\gamma - 2w^2\alpha'\gamma - (w^2 + \alpha\gamma)\alpha\gamma'}{2\alpha^2(w^2 - \alpha\gamma)},\\
\Gamma^2_{12} &= \Gamma^2_{21} = 1/r,\\
\Gamma^2_{33} &= -\sin\theta\cos\theta,\\
\Gamma^2_{24} &= -\Gamma^2_{42} = -\frac{w}{r\alpha},\\
\Gamma^3_{13} &= \Gamma^3_{31} = 1/r,\\
\Gamma^3_{23} &= \Gamma^3_{32} = \cot\theta,\\
\Gamma^3_{34} &= -\Gamma^3_{43} = -\frac{w}{r\alpha},\\
\Gamma^4_{14} &= \Gamma^4_{41} = \frac{2ww'\alpha - w^2\alpha' - \alpha^2\gamma'}{2\alpha(w^2 - \alpha\gamma)}.
\end{aligned} \tag{3.4}$$

Here the prime denotes differentiation with respect to r. Substituting these results into Eq. (1.13), we find

$$\begin{aligned}
R_{11} &= \frac{\alpha'}{r\alpha} + \Gamma^4_{14}\left(\frac{\alpha'}{2\alpha} - \Gamma^4_{14}\right) - \Gamma^4_{14,1},\\
R_{22} &= \csc^2\theta R_{33}\\
&= -\left(\frac{r}{\alpha}\right)' - \frac{r}{2\alpha}[\ln(w^2 - \alpha\gamma)]' + 1,\\
R_{44} &= \Gamma^1_{44,1} + (\Gamma^1_{14})^2 + \frac{2w^2}{r^2\alpha^2} + \Gamma^1_{44}\left(\frac{\alpha'}{2\alpha} - \Gamma^4_{14} + \frac{2}{r}\right),\\
R_{14} &= -R_{41} = \Gamma^1_{14,1} + \frac{2w}{r^2\alpha} + \frac{2}{r}\Gamma^1_{14}.
\end{aligned} \tag{3.5}$$

From the definition of the tensor $g_{\mu\nu}$, we obtain from Eqs. (2.9) and (2.10) the results for $I_{\mu\nu}$:

$$\begin{aligned}
I_{11} &= \frac{1}{2k^2}\frac{\alpha w^2}{\alpha\gamma - w^2},\\
I_{22} &= \csc^2\theta I_{33}\\
&= -\frac{1}{2k^2}\frac{r^2w^2}{\alpha\gamma - w^2},\\
I_{44} &= -\frac{1}{2k^2}\frac{\gamma w^2}{\alpha\gamma - w^2},\\
I_{[14]} &= -\frac{1}{2k^2}\left(\frac{\alpha\gamma w}{\alpha\gamma - w^2} + w\right),
\end{aligned} \tag{3.6}$$

and all other $I_{(\mu\nu)}$ and $I_{[\mu\nu]}$ vanish.

A quick inspection of Eq. (2.8), using the results for $R_{[\mu\nu]}$ and $I_{[\mu\nu]}$, shows that it is satisfied. Similarly, Eq. (2.6) is satisfied identically, except for $\Gamma^s_{[4s]}$. For this case, we find

$$\Gamma^1_{14} = \frac{2w}{r\alpha}. \tag{3.7}$$

By substituting for Γ^1_{14} from Eq. (3.4), we have

$$\left[\ln\left(1 - \frac{\alpha\gamma}{w^2}\right)\right]' = \frac{4}{r}. \tag{3.8}$$

Upon integration, this becomes

$$\left(1 - \frac{\alpha\gamma}{w^2}\right) = r^4 e^{c'}, \tag{3.9}$$

where c' is a constant of integration. Since $e^{c'}$ is not necessarily positive, we can rewrite Eq. (3.9) as

$$\left(1 - \frac{\alpha\gamma}{w^2}\right) = -\frac{r^4}{k^2l^2}, \tag{3.10}$$

where l is a constant to be determined. This choice of sign is necessary so that at large r the product $\alpha\gamma$ approaches unity. Thus,

$$\alpha\gamma = w^2\left(\frac{r^4 + k^2l^2}{k^2l^2}\right). \tag{3.11}$$

Upon substituting Eq. (3.11) into Eq. (3.6), we find

$$\begin{aligned}
I_{11} &= \frac{\alpha l^2}{2r^4},\\
I_{22} &= \csc^2\theta I_{33}\\
&= -\frac{l^2}{2r^2},\\
I_{44} &= -\frac{\gamma l^2}{2r^4}.
\end{aligned} \tag{3.12}$$

We can now proceed to solve Eq. (2.7). Let us first consider R_{22}. Upon substituting Eq. (3.8) into Eq. (3.5), we find that R_{22} simplifies to

$$R_{22} = \frac{1}{\alpha}\left[-3 + r\left(\frac{\alpha'}{\alpha} - \frac{w'}{w}\right) + \alpha\right]. \tag{3.13}$$

We can similarly simplify R_{11} so that it contains only the unknown functions α and w. Combining Eq. (3.4) and Eq. (3.7), Γ^4_{14} can be rewritten as

$$\begin{aligned}
\Gamma^4_{14} &= -\frac{\alpha'}{2\alpha} + \tfrac{1}{2}\left[\ln\left(1 - \frac{\alpha\gamma}{w^2}\right)\right]' + \tfrac{1}{2}(\ln w^2)'\\
&= \frac{\alpha'}{2\alpha} + \frac{2}{r} - \left(\frac{\alpha'}{\alpha} - \frac{w'}{w}\right).
\end{aligned} \tag{3.14}$$

Further, substitution of Eqs. (3.13) and (3.12) into Eq. (2.7) yields the relation

$$\frac{\alpha'}{\alpha}-\frac{w'}{w}=\frac{1}{r}\left(3-\alpha+\frac{\alpha l^2}{2r^2}\right). \tag{3.15}$$

This implies that

$$\Gamma^4_{14}=\frac{\alpha'}{2\alpha}+\frac{\alpha}{r}\left(1-\frac{l^2}{2r^2}\right)-\frac{1}{r}. \tag{3.16}$$

Substituting this result into the equation for R_{11} yields an expression which is free from w and γ. We write a solution for α in the form

$$\frac{1}{\alpha}=1+\frac{a}{r}+\frac{b}{r^2}, \tag{3.17}$$

where a and b are coefficients to be determined. We find that

$$\frac{1}{\alpha^2}R_{11}=\frac{1}{r^4}\left(2b-\frac{3l^2}{2}\right)+\frac{1}{r^5}\left(\tfrac{5}{2}ab-\tfrac{7}{4}al^2\right)+\frac{1}{r^6}\left(2b^2-bl^2-\frac{l^2}{4}\right). \tag{3.18}$$

We also know that

$$\frac{1}{\alpha^2}I_{11}=\frac{l^2}{2r^4}+\frac{al^2}{2r^5}+\frac{bl^2}{2r^6}. \tag{3.19}$$

Since the equation $R^*_{11}=0$ applies for any r, then the coefficients of r^{-4} to r^{-6}, found from the sum of the last two equations, can be set equal to zero. The three equations so obtained all yield the same solution:

$$b=+\frac{l^2}{2}. \tag{3.20}$$

Thus, we have a *unique* solution for α, namely,

$$\frac{1}{\alpha}=1+\frac{a}{r}+\frac{l^2}{2r^2}. \tag{3.21}$$

This solution can be substituted into Eq. (3.15) and solved for w. We find

$$\frac{w'}{w}=-\frac{2}{r}, \tag{3.22}$$

so that

$$w=\pm kL/r^2, \tag{3.23}$$

where L is an integration constant. Since we demand that $\alpha\gamma\to 1$ as $r\to\infty$, then Eqs. (3.23) and (3.11) imply that L and l are equal.

Finally, we turn out attention to R_{44}. Using the previous results shown in Eqs. (3.7) and (3.16), we find that R_{44} can be written in the form

$$R_{44}=\Gamma^1_{44,1}+\frac{6w^2}{\alpha^2r^2}+\Gamma^1_{44}\left(\frac{\alpha'}{\alpha}-\frac{w'}{w}\right). \tag{3.24}$$

To evaluate this, we need an expression for Γ^1_{44}. After some algebra, we find

$$\Gamma^1_{44}=\frac{w^2}{\alpha^2}\left(\frac{4}{r}\right)+\frac{\gamma'}{2\alpha}. \tag{3.25}$$

Substituting this into the previous equation and using the equation $R^*_{44}=0$, we generate a relation involving α, γ, and w. However, we have already solved for the form of α and w, uniquely, and this implies that γ can be determined by the use of the condition $\Gamma^{\alpha}_{[4\alpha]}=0$. Although it will yield nothing new, nevertheless the condition on R^*_{44} must be satisfied. Therefore, we shall choose to eliminate γ from Γ^1_{44}, leaving R_{44} in terms of α and w. We shall substitute for w and solve for α, as before. Eliminating γ', we find Γ^1_{44} is given by

$$\Gamma^1_{44}=\frac{1}{\alpha^2}\left[\frac{2k^2l^2}{r^5}-\frac{\alpha'}{2\alpha}\left(1+\frac{k^2l^2}{r^4}\right)\right]. \tag{3.26}$$

Substitution of Eq. (3.26) into Eq. (3.24) yields, after some algebra,

$$\alpha R_{44}=\frac{b}{r^4}\left(1+\frac{k^2l^2}{r^4}\right). \tag{3.27}$$

Imposing the equation $R^*_{44}=0$ leads immediately to

$$b=\frac{l^2}{2}. \tag{3.28}$$

Hence, we have that R_{44} contains the same solutions as R_{11} and R_{22}. We will show in the equations of motion that k must be real for the real antisymmetric tensor. Then, the identification $g_{[14]}=kE_1$, where E_1 is the electrostatic field strength, leads to the result $l=\pm Q$. Since we also demand that the solution go over to the Schwarzschild solution at $Q=0$, then $a=-2m$ in the units $G=c=1$. Thus, our solutions, written out in full, are

$$\alpha\gamma=w^2\left(\frac{r^4+k^2Q^2}{k^2Q^2}\right), \tag{3.29}$$

$$\alpha=\left(1-\frac{2m}{r}+\frac{Q^2}{2r^2}\right)^{-1}, \tag{3.30}$$

$$w\equiv kE_1=\pm\frac{kQ}{r^2}, \tag{3.31}$$

$$\gamma=\left(1+\frac{k^2Q^2}{r^4}\right)\left(1-\frac{2m}{r}+\frac{Q^2}{2r^2}\right). \tag{3.32}$$

Here, k is a universal constant with dimensions of length. The metric in our solution is then given by

$$ds^2=\left(1-\frac{2m}{r}+\frac{Q^2}{2r^2}\right)\left(1+\frac{k^2Q^2}{r^4}\right)dt^2-\left(1-\frac{2m}{r}+\frac{Q^2}{2r^2}\right)^{-1}dr^2-r^2(d\theta^2+\sin^2\theta\, d\varphi^2). \tag{3.33}$$

We see that our solution goes smoothly over into the Schwarzschild solution when the electric charge goes to zero. Also, Eq. (3.31) yields the familiar result for the static electric field because of a point charge at the origin.

If we chose instead to write the metric tensor as Hermitian, then we would obtain a different result. Aside from the substitution $w \to iw$, we would require Eq. (3.10) to change sign, and k to be imaginary. This gives

$$ds^2 = \left(1 - \frac{2m}{r} + \frac{Q^2}{2r^2}\right)\left(1 - \frac{k^2Q^2}{r^4}\right)dt^2 - \left(1 - \frac{2m}{r} + \frac{Q^2}{2r^2}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) . \tag{3.34}$$

As pointed out by Schrödinger,[3] Eq. (1.5) is not the only method that can be used for raising and lowering indices, although it is the one which he employs. One could also have chosen

$$g^{\mu\nu}g_{\nu\sigma} = \delta^{\mu}{}_{\sigma} . \tag{3.35}$$

The consequence of this is to transpose our expression (3.3) for $g^{\mu\nu}$ which leads, after considerable calculation, to the metric

$$ds^2 = \left(1 - \frac{2m}{r} - \frac{Q^2}{2r^2}\right)\left(1 + \frac{k^2Q^2}{r^4}\right)dt^2 - \left(1 - \frac{2m}{r} - \frac{Q^2}{2r^2}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) , \tag{3.36}$$

for a real nonsymmetric metric tensor. One could, however, choose a Hermitian metric tensor, as stated before. This would lead to the result

$$ds^2 = \left(1 - \frac{2m}{r} - \frac{Q^2}{2r^2}\right)\left(1 - \frac{k^2Q^2}{r^4}\right)dt^2 - \left(1 - \frac{2m}{r} - \frac{Q^2}{2r^2}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) . \tag{3.37}$$

IV. EQUATIONS OF MOTION

Bonnor[8] obtained the equations of motion

$${}^1m\frac{d^2\vec{r}}{dt^2} = -\frac{{}^1m\,{}^2m\vec{r}}{r^3} + p^2q^2\frac{{}^1e\,{}^2e\vec{r}}{r^3} , \tag{4.1}$$

where $\vec{r}$ is the position vector of the first particle relative to an origin which corresponds instantaneously with the position of the second particle. Equation (4.1) was obtained using the field equations (2.5)–(2.8) written in a different form. For completeness, we shall now give a brief derivation of the equations of motion which removes the constants p^2q^2 from Eq. (4.1); the origin and implications of these arbitrary constants are not satisfactorily elaborated in Bonnor's work.[8] We follow the notation of Refs. 6, 10, 11, and 8, and the reader is referred to these papers for further details. The usual convention is adopted that Latin indices run from 1 to 3 and Greek indices from 1 to 4.

We set $g_{(\mu\nu)} = a_{\mu\nu}$ and $g_{[\mu\nu]} = f_{\mu\nu}$, and expand $g_{\mu\nu}$ in a parameter λ:

$$\begin{aligned} a_{44} &= 1 + \lambda^2\,{}_2a_{44} + \lambda^4\,{}_4a_{44} + \cdots , \\ a_{4n} &= \lambda^3\,{}_3a_{4n} + \lambda^5\,{}_5a_{4n} + \cdots , \\ a_{mn} &= -\delta_{mn} + \lambda^2\,{}_2a_{mn} + \lambda^4\,{}_4a_{mn} + \cdots , \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} f_{4n} &= \lambda^3\,{}_3f_{4n} + \lambda^5\,{}_5f_{4n} + \cdots , \\ f_{mn} &= \lambda^3\,{}_3f_{mn} + \lambda^4\,{}_4f_{mn} + \cdots . \end{aligned} \tag{4.3}$$

A calculation gives

$$\begin{aligned} {}_2I_{44} &= {}_3I_{(4n)} = {}_2I_{(mn)} = 0 , \\ {}_4I_{44} &= -\frac{1}{4k^2}\,{}_2f_{st}\,{}_2f_{st} , \\ {}_4I_{(mn)} &= \frac{1}{2k^2}(2\,{}_2f_{ms}\,{}_2f_{sn} + \tfrac{1}{2}\delta_{mn}\,{}_2f_{st}\,{}_2f_{st}) , \\ {}_4I_{rr} &= -\frac{1}{4k^2}\,{}_2f_{st}\,{}_2f_{st} . \end{aligned} \tag{4.4}$$

The field equation (2.6) requires

$${}_2f_{ms,s} = 0 , \quad {}_3f_{4s,s} = 0 . \tag{4.5}$$

Denoting by $P_{\alpha\beta}$ the contracted curvature tensor formed from the Christoffel symbols $\{{}^{\lambda}_{\mu\nu}\}$, then

$${}_2P_{44} = {}_2P_{mn} = {}_3P_{4m} = 0 . \tag{4.6}$$

We assume that only two particles are present. Thus,

$$\phi = {}^1_2\phi + {}^2_2\phi , \tag{4.7}$$

where

$$\begin{aligned} {}^k\phi &= {}^ke/{}^kr , \\ {}^kr^2 &= (x^m - {}^k\xi^m)(x^m - {}^k\xi^m) , \end{aligned} \tag{4.8}$$

${}^k\xi^m$ being the spatial coordinates of ${}^ke$. The solution we adopt for Eq. (4.5) is

$${}_2f_{mn} = \sqrt{2}\,k\epsilon_{mns}\,{}_2\phi_{,s} , \tag{4.9}$$

where ϕ is a harmonic function of r. Recalling Eq. (4.3), if $g_{[\mu\nu]}$ is real then k is real, and similarly for the imaginary case. The equations of motion come from the surface integrals

$${}^k_4C'_m(\tau) = {}^k_4C_m(\tau) - \frac{1}{4\pi}\oint^k 2I^*_{(mr)}n_r dS = 0 , \tag{4.10}$$

where ${}^k_4C'_m(\tau)$ is the result obtained from general relativity[10,11]:

$${}^k_4C_m(\tau) \equiv -\frac{1}{4\pi}\oint 2\,{}_4P^*_{mr}n_r dS , \tag{4.11}$$

and

$${}_4P^*_{mn} = {}_4P_{mn} - \tfrac{1}{2}\delta_{mn}\,{}_4P_{ss} + \tfrac{1}{2}\delta_{mn}\,{}_4P_{44}\,, \qquad (4.12)$$

$${}_4I^*_{mn} = {}_4I_{mn} - \tfrac{1}{2}\delta_{mn}\,{}_4I_{ss} + \tfrac{1}{2}\delta_{mn}\,{}_4I_{44}\,. \qquad (4.13)$$

When the equations

$$P^*_{mr,r} = 0 \qquad (4.14)$$

and

$$I^*_{mr,r} = 0 \qquad (4.15)$$

are satisfied, then the surface integrals will not depend on the shape of the two-dimensional surfaces enclosing the kth particle. A calculation shows that Eq. (4.14) is satisfied, while Eq. (4.13) leads to the relation

$${}_4I^*_{(mn)} = 2\phi_{,m}\phi_{,n} - \delta_{mn}\,\phi_{,s}\phi_{,s}\,, \qquad (4.16)$$

which in turn satisfies Eq. (4.15). A calculation of the surface integral yields the result for the first particle:

$$-\frac{1}{4\pi}\oint^1 2\,{}_4I_{(mr)}n_r dS = 4{}^1e\,{}^2e r^{-3}({}^2\xi^m - {}^1\xi^m)\,, \qquad (4.17)$$

where r is the distance between the two particles. This, combined with Eqs. (4.10) and (4.11), yields the result

$${}^1_2m\,{}^1\ddot{\xi}^m = -{}^1_2m\,{}^2_2m({}^1\xi^m - {}^2\xi^m)r^{-3} + {}^1_2e\,{}^2_2e({}^1\xi^m - {}^2\xi^m)r^{-3}\,, \qquad (4.18)$$

where the constants ${}^1_2m$, ${}^2_2m$, ${}^1_2e$, and ${}^2_2e$ are terms in the expansion of ${}^1m$, ${}^1e$, etc. in powers of λ. Writing $\ddot{\xi}^m = \lambda^{-2}(d^2\xi^m/dx_0{}^2)$ and removing the parameter λ from Eq. (4.18) by multiplying by λ^4, we find

$${}^1m\frac{d^2\vec{r}}{dt^2} = \frac{{}^1m\,{}^2m\vec{r}}{r^3} + \frac{{}^1e\,{}^2e\vec{r}}{r^3}\,, \qquad (4.19)$$

which is free from Bonnor's constant factor p^2q^2.[8] It is expected that in the next higher order of approximation, the term $e\dot{\vec{r}}\times\vec{H}$, in the Lorentz force, will appear.

Now, the important result, from our point of view, is the sign of the Coulomb force term in Eq. (4.19) which is physically correct. This fixes the sign of $I_{(\mu\nu)}$ in Eq. (2.9) and, consequently, the sign of the term $Q^2/2r^2$ in our metric, given by Eq. (3.33). This will play a crucial role in the physical interpretations of our solution.

V. CONSEQUENCES OF THE METRIC

The result (3.33) is similar to the Reissner-Nordström[12] geometry with one exception, namely the factor $(1+k^2Q^2/r^4)$ in g_{44}. This factor is clearly of significance for $r<(Qk)^{1/2}$, which, for a proton, corresponds to a length of about 10^{-34} cm if $k=Q$. It is perhaps not coincidental that this value of k leads to a radius which is of the order of magnitude where gravitational effects are thought to begin.[13] When the radius is of the order of, say, the Compton wavelength, the effect of this extra factor is negligible unless k is very large. We note that the singularity at $r=0$ is now stronger than that of the Reissner-Nordström solution because of this factor. In the Hermitian case, whose solution is given by Eq. (3.34), the same sign is formed for the $Q^2/2r^2$ term, but the factor in g_{44} becomes $(1-k^2Q^2/r^4)$. The significance of (g_{11}/g_{44}) changing sign at $r=(Qk)^{1/2}$ escapes us, ans we will ignore the Hermitian case in the remainder of our discussions.

If we define the contravariant metric tensor by Eq. (3.35), then the solution obtained does not go over to the Reissner-Nordström solution at large r, having a factor $-Q^2/2r^2$ instead of $+Q^2/2r^2$. The consequences of this sign are, however, intriguing. It would imply that the coordinate singularity occurs at

$$r = m + (m^2+\epsilon^2)^{1/2}\,, \qquad (5.1)$$

where

$$m = GM/c^2\,,\qquad \epsilon^2 = \frac{Q^2}{2} = \frac{4\pi Ge^2}{c^4}\,, \qquad (5.2)$$

and where M and e are the physical mass and charge. This has particular significance for charged particles. According to Eq. (3.30), the singular sphere for a proton or electron no longer exists because $\epsilon \gg m$. However, this is not the case for Eq. (5.1) and we find a coordinate singularity at about 10^{-34} cm, which implies that both charged and neutral particles have singularity "structure," a possibility considered by Eddington.[14]

This metric also implies a different behavior for black holes. Whereas Eq. (3.30) predicts that the black hole will cease to exist for $\epsilon > m$, the metric (3.36) allows it to continue to exist for this condition. Indeed, the more charge that is added to the black hole, the stronger it becomes.

Let us return for a moment to Eq. (2.4). Suppose we defined $g_{[\mu\nu]}$ as $kg'_{[\mu\nu]}$. Then $I_{(\mu\nu)}$ would be a function of $g'_{[\mu\nu]}$ only, while $R_{\mu\nu}$ would contain $kg'_{[\mu\nu]}$. When $k\to 0$, $R_{\mu\nu}$ is "decoupled" from $g'_{[\mu\nu]}$ and the resulting form of Eq. (2.7) is *identical* to the old theory given by Eq. (1.1). Inspection of Eq. (2.10) shows that the terms cubic in $g_{[\mu\nu]}$ vanish, leaving only the linear term when $k\to 0$. Since we now have the identification

$$g'_{[\mu\nu]} = F_{\mu\nu}\,, \qquad (5.3)$$

then Eqs. (1.16) and (2.8) lead immediately to Maxwell's equations (1.2) and (1.3) in the $k \to 0$ limit. This would imply that our metric should now be the Reissner-Nordström metric, and quick inspection of Eq. (3.33) shows that it is for $k \to 0$. Thus, our unified theory contains the old theory of gravity and electromagnetism as a special case when the universal constant k vanishes.

Finally, we examine the nature of the singularity at $r = m + (m^2 - \epsilon^2)^{1/2}$. To do this, we transform our metric into a set of Kruskal-Szerkeres-type[12,15] coordinates given by

$$r' = T\left(\frac{r^2}{2mr - \epsilon^2} - 1\right)^{1/2} \exp\left(\frac{r^2}{4mr - 2\epsilon^2}\right) \cosh\left(\frac{t}{4M'}\right), \tag{5.4}$$

and

$$t' = T\left(\frac{r^2}{2mr - \epsilon^2} - 1\right)^{1/2} \exp\left(\frac{r^2}{4mr - 2\epsilon^2}\right) \sinh\left(\frac{t}{4M'}\right), \tag{5.5}$$

where T is a constant. The constant M' is defined by

$$M' = \tfrac{1}{2}[m + (m^2 - \epsilon^2)^{1/2}]. \tag{5.6}$$

In the limit that the charge vanishes, $M' = m$. Combining Eqs. (5.4) and (5.5) yields the relations

$$r'^2 - t'^2 = T^2\left(\frac{r^2}{2mr - \epsilon^2} - 1\right) \exp\left(\frac{r^2}{2mr - \epsilon^2}\right), \tag{5.7}$$

and

$$\frac{2r't'}{r'^2 + t'^2} = \tanh\left(\frac{t}{2M'}\right). \tag{5.8}$$

We see from Eq. (5.7) that at $r = 0$,

$$r' = (t'^2 - T^2)^{1/2} \tag{5.9}$$

and we obtain the usual two-sheeted space with a branch point at $r = 0$. We transform the metric into these coordinates by means of the formula

$$g'_{\alpha\beta} = \frac{\partial x^\rho}{\partial x'^\alpha} \frac{\partial x^\sigma}{\partial x'^\beta} g_{\rho\sigma}. \tag{5.10}$$

This gives the result

$$ds^2 = \frac{32m^3}{rT^2} \exp\left(-\frac{r}{2m(1+\rho)}\right)[f_-(\rho)dt'^2 - f_+(\rho)dr'^2] - r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{5.11}$$

where

$$f_\pm(\rho) = \frac{(1+\rho)}{4}\left\{2\left[\frac{(1+\rho)^4}{(1+2\rho)^2} + \mu^2\left(1 + \frac{k^2Q^2}{r^4}\right)\right] \pm \cosh\left(\frac{t}{2M'}\right)\left[\frac{(1+\rho)^4}{(1+2\rho)^2} - \mu^2\left(1 + \frac{k^2Q^2}{r^4}\right)\right]\right\}, \tag{5.12}$$

and

$$\rho = \frac{-\epsilon^2}{2mr}, \tag{5.13}$$

$$\mu = \frac{M'}{m} = \tfrac{1}{2}[1 + (1 - \epsilon^2/m^2)]^{1/2}. \tag{5.14}$$

In the limit that $Q \to 0$, we see that $\rho \to 0$ and $\mu \to 1$, so that

$$f_\pm(0) = 1,$$

and Eq. (5.11) becomes the usual Kruskal-Szerkeres coordinate transformation for the Schwarzschild solution. Examination of Eqs. (5.11) and (5.12) shows that the line element possesses a singularity only at $r = 0$ and $r = \epsilon^2/2m$. Thus, the singularity at $r = m + (m^2 - \epsilon^2)^{1/2}$ is a coordinate singularity.

VI. CONCLUSIONS

We have solved a set of field equations for Einstein's nonsymmetric unified field theory and found solutions that appear physically reasonable. In particular, we recover the usual result for a static electric field of a point charge. The class of solutions that this theory generates contains the old theory of gravitation and electromagnetism as a special case. We also investigated other possible metrics and their implications to black-hole and particle physics. The field equations also give the correct equations of motion for charged masses to the order of approximation considered. It is the combination of both of these results which is striking, and could have far-reaching consequences in our understanding of gravitational and electromagnetic phenomena.

Although, formally, the modification of Einstein's field equations used is similar to that proposed by Bonnor and Kurşunoğlu, our physical interpretation of the theory is different, and we are actually able to obtain exact solutions for the metric. The present theory is more satisfactory than the 1915 theory of general relativity incorporating the electromagnetic fields into the energy momentum tensor, because in the nonsymmetric theory, both the gravitational and the electromagnetic fields appear as a unified property of space-time, in the manner originally envisaged by Einstein.

If we perform the transformation on the affine connection

$$\Gamma^{\lambda\prime}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} + \delta^{\lambda}_{\mu}\partial_{\nu}\lambda\,, \tag{6.1}$$

where λ is an arbitrary function of the coordinates, then the nonsymmetric curvature tensor $R^{\lambda}_{\mu\nu\sigma}(\Gamma')$ formed by replacing Γ' by the right-hand side of (6.1) gives

$$R^{\lambda}_{\mu\nu\sigma}(\Gamma') = R^{\lambda}_{\mu\nu\sigma}(\Gamma)\,, \tag{6.2}$$

$$R_{\mu\nu}(\Gamma') = R_{\mu\nu}(\Gamma)\,. \tag{6.3}$$

As was emphasized by Einstein,[1] this "gauge invariance" of the theory could be of fundamental importance. Einstein's gravitational theory and the Einstein-Maxwell theory are not invariant under a transformation such as (6.1), because contrary to a coordinate transformation, the transformation (6.1) produces a nonsymmetric Γ' from a Γ that is symmetric in μ and ν.

It could be that the main significance of the λ-gauge invariance lies in the fact that it may influence the renormalizability of the theory. It has recently been shown[16] that Einstein's gravitational theory is not, in general, renormalizable. It is possible that the unified field theory described here is renormalizable, because of its invariance under the extended gauge group of transformations.

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