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Model for chiral SU(4)⊗SU(4) breaking based on the (15, 15) representation*

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We investigate (15, 15) breaking for SU(4) ⊗ SU(4) symmetry breaking as an alternative to the (4*, 4) model. The form for the Hamiltonian is so chosen that pions obey the PCAC (partial conservation of axial-vector current) equation, and the parameters in the Hamiltonian are then determined by meson dominance. The Hamiltonian is found to be SU(3) ⊗ SU(3)-invariant to a very good approximation, permitting charmed mesons to be massive and thus escape detection. We also evaluate π-π scattering lengths and the decay $X \longrightarrow \eta\pi\pi$ in this model. Other choices of Hamiltonians within (15, 15) breaking are also considered, but shown to be unsatisfactory.

I. INTRODUCTION

Recent observation of neutral currents[1] in neutrino experiments gives a great deal of credence to the Weinberg-Salam[2,3] type of unified gauge theory. Incorporating the hadrons in this model thus becomes of great importance. As is well known, the usual quark triplet scheme cannot be incorporated because it leads to strangeness changing neutral currents, in contradiction with experiment. The simplest way out of this difficulty is to introduce a fourth quark, carrying a new quantum number (charm), as proposed by Glashow, Iliopoulos, and Maiani.[4]

The four-quark scheme suggests naturally the group SU(4) as an approximate symmetry group for hadrons. Extension to chiral symmetry then leads to the group SU(4)⊗SU(4). We can conclude a great deal from the observed spectrum of hadrons as to how this group is broken. The fact that the particle spectra seem to follow recognizable patterns on the basis of an SU(3) classification, rather than SU(4) or SU(4)⊗SU(4), strongly suggests that SU(4)⊗SU(4) symmetry of the Hamiltonian should be realized as a Nambu-Goldstone symmetry with the vacuum invariant under SU(3). Thus we have the usual nonet of pseudoscalar mesons π, K, η, and X, and, in addition, all the charmed mesons, appearing as Goldstone particles. Explicit symmetry breaking must be introduced such that the SU(3)⊗SU(3) group is a much better symmetry than the group SU(4) or SU(4) ⊗SU(4). This enables the charmed Goldstone mesons to acquire large masses compared to the pseudoscalar nonet. The large masses of these charmed particles would explain why they have escaped detection. The SU(3)⊗SU(3) group is further broken so that only SU(2) is an exact invariance group, and the pseudoscalar nonet acquires the observed masses. The smallness of the pion mass suggests the approximate validity of the SU(2)⊗SU(2) group, although as we shall see, this is not necessarily a desirable feature.

The direct generalization of the (3*, 3) model proposed by Gell-Mann, Oakes, and Renner[5] for the SU(3)⊗SU(3) group is the (4*, 4) model, where the symmetry-breaking terms transform as quark masses. This model has been investigated recently[6,7] and found to be unsatisfactory. The basic problem can be seen by writing the symmetry-breaking Hamiltonian in the following form:

$$H = u_0 + e\sqrt{3}\, u_{15} + \alpha(1+e)(\tfrac{3}{2})^{1/2} u_8 . \qquad (1.1)$$

This Hamiltonian can be decomposed into three parts:

$$H \equiv H_1 + H_2 + H_3 , \qquad (1.2)$$

where

$$H_1 = \left(\frac{1-3e}{4}\right)(u_0 - \sqrt{3}\, u_{15}) , \qquad (1.3)$$

$$H_2 = \frac{(1+e)(1-2\alpha)}{4}\left(u_0 + \frac{u_{15}}{\sqrt{3}} - \frac{2\sqrt{2}}{\sqrt{3}}\, u_8\right), \qquad (1.4)$$

$$H_3 = \frac{(1+e)(1+\alpha)}{2}\left[u_0 + \frac{u_{15}}{\sqrt{3}} + (\tfrac{2}{3})^{1/2} u_8\right] . \qquad (1.5)$$

This decomposition has the property that H_1 breaks SU(4)⊗SU(4) and SU(4) symmetry, but is invariant under the SU(3)⊗SU(3) subgroup. H_2 breaks SU(3) and SU(3)⊗SU(3) symmetry, but is invariant under SU(2)⊗SU(2) symmetry. H_3 breaks SU(2) ⊗SU(2) while preserving SU(2). From our previous discussion, the pattern of symmetry breaking should be

$$H_1 \gg H_2 \gg H_3 . \qquad (1.6)$$

This implies that the parameters e and α take the values $e \approx -1$ and $\alpha \approx -1$. The solution to the model in terms of the known pseudoscalar masses, however, leads to a value of $e \approx -0.6$.[6] Such a value is insufficient to raise the masses of the charmed particles. Prediction on X and η decay rates into two photons and $X\to\eta\pi\pi$ are also found to be unsatisfactory.[6,8] Thus an alternative to the $(4^*, 4)$ model seems desirable.

Another feature that the $(4^*, 4)$ model shares with the $(3^*, 3)$ model is the smallness of the σ term, leading to discrepancies with the values deduced from π-π scattering and the πN scattering data.[9] This difficulty arises because of the approximate $SU(2)\otimes SU(2)$ invariance of the Hamiltonian, a property shared by $(3^*, 3)$ and $(4^*, 4)$ breaking models. Thus it seems desirable to study alternative models for symmetry breaking, and in particular models that allow $(3^*, 3)$ breaking plus additional terms like $(8, 8)$ breaking. In the next section we shall consider different symmetry-breaking schemes that are possible before selecting $(15, 15)$ as the most likely alternative. In Sec. III we analyze the most general $(15, 15)$ Hamiltonian possible, and then select a particular form which admits PCAC (partially conserved axial-vector current) for pions. Sections IV and V are devoted to calculations of the parameters of the model and of mass formulas for the charmed mesons. The form of the Hamiltonian selected in Sec. III turns out to be approximately $U(3)\otimes U(3)$-invariant, and thus it is possible to make the charmed mesons much more massive than the nonet of pseudoscalar mesons. In Sec. VI pion scattering and X decay are calculated. In Sec. VII other possible choices of $(15, 15)$ Hamiltonians are discussed and in Sec. VIII our results and conclusions are summarized.

II. ALTERNATIVE FORMS OF BREAKING SU(4)⊗SU(4)

In this section we list some of the simpler alternatives to the $(4^*, 4)$ model of symmetry breaking. As a criterion in choosing the Hamiltonian, we shall demand that the transformation property of the Hamiltonian under the $SU(3)\otimes SU(3)$ subgroup include $(3^*, 3)$ and $(8, 8)$ terms. The reason for this is that the study of $SU(3)\otimes SU(3)$ breaking reveals that σ terms are rather small in $(3^*, 3)$ breaking and that a small admixture of $(8, 8)$ rather than $(6^*, 6)$ or $(1, 8)\oplus(8, 1)$ improves the agreement with the experiment.[9] To see what forms are available we list the SU(3) decomposition[10] of the low-dimensional representations of SU(4):

$$\begin{aligned} 4 &= 3\oplus 1\,, \\ 4^* &= 3^*\oplus 1\,, \\ 6 &= 3^*\oplus 3\,, \\ 10 &= 6\oplus 3\oplus 1\,, \\ \overline{10} &= 6^*\oplus 3^*\oplus 1\,, \\ 15 &= 3\oplus 3^*\oplus 8\oplus 1\,. \end{aligned} \tag{2.1}$$

From these decompositions, we can deduce the $SU(3)\otimes SU(3)$ content of the $SU(4)\otimes SU(4)$ Hamiltonians. Some simple models are

$$\begin{aligned} &(a):\ (4^*, 4)\oplus(4, 4^*) = (3, 3^*)\oplus(3^*, 3)\oplus(1, 1)\,, \\ &(b):\ (1, 15)\oplus(15, 1) = (1, 8)\oplus(8, 1)\oplus(1, 1)\,, \\ &(c):\ (6, 6) = (3, 3^*)\oplus(3^*, 3)\,, \\ &(d):\ (10, \overline{10})\oplus(\overline{10}, 10) = (6, 6^*)\oplus(6^*, 6)\oplus(3, 3^*) \oplus(3^*, 3)\oplus(1, 1)\,, \\ &(e):\ (6, 10)\oplus(\overline{10}, 6) = (6, 3)\oplus(3^*, 6^*)\oplus(3^*, 3)\oplus(3, 3^*) \\ &(f):\ (15, 15) = (3, 3^*)\oplus(3^*, 3)\oplus(8, 8)\oplus(1, 8)\oplus(8, 1)\,. \end{aligned} \tag{2.2}$$

The model based on (a) has been extensively discussed in the literature. The model based on $(6, 6)$ breaking is not satisfactory because there is no way to break SU(4) while retaining approximate $SU(3)\otimes SU(3)$ invariance. The models (d) and (e) both are admissible, but since they do not admit $(8, 8)$ breaking, we shall not consider them here. Thus we are left with the $(15, 15)$ representation. The SU(4) decomposition of this representation is[10]

$$15\times 15 = 1^+\oplus 15^+\oplus 15^-\oplus 20''^+\oplus 84^+\oplus 45^-\oplus\overline{45}^-\,. \tag{2.3}$$

Here + and − refer to even and odd parity of the operators. The SU(3) decomposition of these SU(4) representations is

$$\begin{aligned} 1 &= 1\,, \\ 15 &= 8\oplus 1\oplus 3\oplus 3^*\,, \\ 20'' &= 6^*\oplus 8\oplus 6\,, \\ 84 &= 6\oplus 3\oplus 15\oplus 1\oplus 8\oplus 27\oplus 3^*\oplus\overline{15}\oplus 6^*\,, \\ 45 &= 15\oplus 8\oplus 10\oplus 3^*\oplus 6\oplus 3\,, \\ \overline{45} &= \overline{15}\oplus 8\oplus\overline{10}\oplus 3\oplus 6^*\oplus 3^*\,. \end{aligned} \tag{2.4}$$

The most general Hamiltonian that preserves SU(2) and the conservation of Y and C (charm), and also incorporates octet dominance, can be constructed from the following operators:

$$H = 1\oplus 15_{(\text{singlet})}\oplus 15_{(\text{octet})}\oplus 84_{(\text{singlet})}\oplus 84_{(\text{octet})}\oplus 20''_{(\text{octet})}\,. \tag{2.5}$$

In this next section we shall limit the form of this Hamiltonian by imposing additional physical requirements.

III. HAMILTONIAN IN THE (15,15) REPRESENTATION

We start by writing a general two-index tensor for the $(15, 15)$ representation

$$P_{ij}=\frac{\delta_{ij}}{\sqrt{15}}u_0+\frac{D^{\alpha}_{ij}}{\sqrt{3}}u_{\alpha}+i\,\frac{F^{\alpha}_{ij}}{2}v_{\alpha}+\sum_{45,\overline{45},20'',84}\frac{T^{\theta}_{ij}}{\sqrt{N_{\theta}}}u_{\theta}\,. \tag{3.1}$$

The u's and v's are the Hermitian scalar and pseudoscalar fields. The D^{α}_{ij} and F^{α}_{ij} are the (15×15) matrices of the two $\underline{15}$ representations

$$(D_i)_{jk}=d_{ijk},\quad (F_i)_{jk}=-if_{ijk}\,. \tag{3.2}$$

These are normalized by

$$\mathrm{Tr}D^{\alpha}D^{\beta}=3\delta^{\alpha\beta},\quad \mathrm{Tr}F^{\alpha}F^{\beta}=4\delta^{\alpha\beta}\,. \tag{3.3}$$

All the other representations are included in the last term of (3.1). The matrices are normalized by

$$\mathrm{Tr}T^{\theta}T^{\Gamma}=N_{\theta}\delta^{\theta\Gamma}, \tag{3.4}$$

where, strictly for convenience, we allow the normalization to be different even within a representation.

The commutation relations for the two $\underline{15}$ representations of SU(4) are given by

$$[Q^{+}_{\alpha},P_{ij}]=if_{\alpha ik}P_{kj}\,, \tag{3.5a}$$

$$[Q^{-}_{\alpha},P_{ij}]=if_{\alpha jk}P_{ik}\,. \tag{3.5b}$$

In terms of the vector and axial-vector charges these become

$$[Q_{\alpha},P_{ij}]=if_{\alpha ik}P_{kj}-iP_{ik}f_{\alpha kj}=-[F^{\alpha},P]_{ij}\,, \tag{3.6a}$$

$$[Q^{5}_{\alpha},P_{ij}]=if_{\alpha ik}P_{kj}+iP_{ik}f_{\alpha kj}=-\{F^{\alpha},P\}_{ij}\,, \tag{3.6b}$$

where we have used (3.2).

The most general symmetry-breaking Hamiltonian would have to include all the terms which conserve parity, isospin, hypercharge, and charm. This Hamiltonian would have many more parameters than we could hope to determine. We will therefore drop the part of the $\underline{84}$ representation which transforms under SU(3) as $\underline{27}$ since we expect it to be small and we will also quite arbitrarily drop the SU(3) octet piece which comes from $\underline{20}''$. We are then left with a singlet term, plus singlet and octet terms from both $\underline{15}$ and $\underline{84}$:

$$H=\epsilon_0\frac{\delta_{ij}}{\sqrt{15}}P_{ji}+\epsilon_8\frac{D^{8}_{ij}}{\sqrt{3}}P_{ji}+\epsilon_{15}\frac{D^{15}_{ij}}{\sqrt{3}}P_{ji}+\epsilon'_8\frac{T^{8}_{ij}}{(720)^{1/2}}P_{ji}+\epsilon'_{15}\frac{T^{15}_{ij}}{(360)^{1/2}}P_{ji}\,. \tag{3.7}$$

The D are given by (3.2), while T^8 and T^{15} are the octet and singlet pieces of $\underline{84}$ and are given by

$$T^{8}=\begin{bmatrix}
2\\
&2\\
&&2\\
&&&-1\\
&&&&-1\\
&&&&&-1\\
&&&&&&-1\\
&&&&&&&-2&&&&&&&10\sqrt{2}\\
&&&&&&&&-5\\
&&&&&&&&&-5\\
&&&&&&&&&&-5\\
&&&&&&&&&&&-5\\
&&&&&&&&&&&&10\\
&&&&&&&&&&&&&10\\
&&&&&&&10\sqrt{2}&&&&&&&0
\end{bmatrix}, \tag{3.8}$$

$$T^{15}=\begin{bmatrix}
1\\
&1\\
&&1\\
&&&1\\
&&&&1\\
&&&&&1\\
&&&&&&1\\
&&&&&&&1\\
&&&&&&&&-4\\
&&&&&&&&&-4\\
&&&&&&&&&&-4\\
&&&&&&&&&&&-4\\
&&&&&&&&&&&&-4\\
&&&&&&&&&&&&&-4\\
&&&&&&&&&&&&&&16
\end{bmatrix}. \tag{3.9}$$

Now given the H of (3.7) we can easily calculate the divergences of the currents

$$\partial_{\mu}J^{\mu}_{\alpha}=-i[Q_{\alpha},H]\,. \tag{3.10}$$

The divergence of the axial-vector current is proportional to pseudoscalar densities from both the $\underline{15}$ and the $\underline{45}$ representations:

$$\begin{aligned}
-\partial_{\mu}A^{\mu}_{\alpha}=&\frac{\epsilon_0 v^{15}_{\beta}}{\sqrt{15}}\mathrm{Tr}F^{\alpha}F^{\beta}\\
&+\frac{\epsilon_8 v^{15}_{\beta}}{\sqrt{3}}\mathrm{Tr}D^{8}F^{\alpha}F^{\beta}+\frac{\epsilon_{15}v^{15}_{\beta}}{\sqrt{3}}\mathrm{Tr}D^{15}F^{\alpha}F^{\beta}\\
&+\frac{\epsilon'_8 v^{15}_{\beta}}{\sqrt{720}}\mathrm{Tr}T^{8}F^{\alpha}F^{\beta}+\frac{\epsilon'_{15}v^{15}_{\beta}}{\sqrt{360}}\mathrm{Tr}T^{15}F^{\alpha}F^{\beta}\\
&+\frac{\epsilon_8 v^{45}_{\theta}}{\sqrt{3N}}\mathrm{Tr}\{D^{8},F^{\alpha}\}T^{\theta}_{45}\\
&+\frac{\epsilon_{15}v^{45}_{\theta}}{\sqrt{3N}}\mathrm{Tr}\{D^{15},F^{\alpha}\}T^{\theta}_{45}\\
&+\frac{\epsilon'_8 v^{45}_{\theta}}{(720N)^{1/2}}\mathrm{Tr}\{T^{8},F^{\alpha}\}T^{\theta}_{45}\\
&+\frac{\epsilon'_{15}v^{45}_{\theta}}{(360N)^{1/2}}\mathrm{Tr}\{T^{15},F^{\alpha}\}T^{\theta}_{45}\,.
\end{aligned} \tag{3.11}$$

T^{θ}_{45} are the matrices for the $\underline{45}$ representation and N is the $\underline{45}$ normalization constant.

The Hamiltonian (3.7) still has too many parameters for us to determine. Further, if we are to be able to calculate quantities of experimental interest we will need to be able to use PCAC for pions. This means we would like $\partial_\mu A_3^\mu$ to be proportional to only one representation. Thus, as a further restriction on H, we require, from (3.11), that

$$\frac{\epsilon_8}{\sqrt{3}}\{F^3, D^8\} + \frac{\epsilon_8'}{(720)^{1/2}}\{F^3, T^8\} + \frac{\epsilon_{15}}{\sqrt{3}}\{F^3, D^{15}\} + \frac{\epsilon_{15}'}{(360)^{1/2}}\{F^3, T^{15}\} \quad (3.12)$$

be equal to ϵF^3 with no T_{45} term. This gives two conditions:

$$\epsilon_8' = -\sqrt{20}\,\epsilon_8\,,$$
$$\epsilon_{15}' = 2\sqrt{10}\,(\epsilon_8 - \epsilon)\,. \quad (3.13)$$

Using these we reduce the H in (3.7) to

$$H = \frac{\epsilon_0}{\sqrt{15}}\,\delta_{ij}P_{ji} + \frac{\epsilon_8}{\sqrt{3}}\,D_{ij}^8 P_{ji} - \frac{\epsilon_8}{6}\,T_{ij}^8 P_{ji} + \frac{1}{\sqrt{6}}(5\epsilon - 2\epsilon_8)D_{ij}^{15}P_{ji} + \tfrac{1}{3}(\epsilon_8 - \epsilon)\,T_{ij}^{15}P_{ji}\,. \quad (3.14)$$

This is the H we will use in the remainder of the paper (except for Sec. VII). It has a manageable number of parameters and has PCAC for pions built in. Most importantly, we will see that it is a Hamiltonian which can leave SU(3)⊗SU(3) as a good group while breaking SU(4)⊗SU(4) badly.

IV. SPECTRAL SUM RULES AND VALUES FOR PARAMETERS

Using the final form of H as given in Eq. (3.14) we can calculate the divergences of the vector and axial-vector currents (the traces of matrices which are needed are given in the Appendix):

$$\partial_\mu V_4^\mu = \frac{4\epsilon_0}{\sqrt{15}}\,\frac{1}{\sqrt{3}}\,a u_5^{15} + \cdots\,, \quad (4.1a)$$

$$\partial_\mu V_9^\mu = \frac{4\epsilon_0}{\sqrt{15}}\,u_{10}^{15}\left(-\frac{1}{\sqrt{3}}\,a - \frac{5}{2\sqrt{3}}\,e\right) + \cdots\,, \quad (4.1b)$$

$$\partial_\mu V_{13}^\mu = \frac{4\epsilon_0}{\sqrt{15}}\,u_{14}^{15}\left(-\frac{2}{\sqrt{3}}\,a - \frac{5}{2\sqrt{3}}\,e\right) + \cdots\,. \quad (4.1c)$$

The u_i^{15} are the scalar densities in the $\underline{15}$ representation. The terms omitted in (4.1) are proportional to scalar densities in the $\underline{84}$ representation. Also we have introduced the notations

$$a = -\frac{3\sqrt{15}}{8}\,\frac{\epsilon_8}{\epsilon_0}\,, \quad (4.2a)$$

$$e = \frac{\sqrt{15}}{2}\,\frac{\epsilon}{\epsilon_0}\,. \quad (4.2b)$$

The omitted term in (4.1a) has a coefficient $-\frac{8}{3}\epsilon_0 a$.

The divergences of the axial-vector currents are

$$\partial_\mu A_3^\mu = -\frac{4\epsilon_0}{\sqrt{15}}\,v_3^{15}(1+e)\,, \quad (4.3)$$

$$\partial_\mu A_4^\mu = -\frac{4\epsilon_0}{\sqrt{15}}\,v_4^{15}(1+a+e) + \cdots\,, \quad (4.4a)$$

$$\partial_\mu A_8^\mu = -\frac{4\epsilon_0}{\sqrt{15}}\,v_8^{15}(1+\tfrac{4}{3}a+e) + \cdots\,, \quad (4.4b)$$

$$\partial_\mu A_{15}^\mu = -\frac{4\epsilon_0}{\sqrt{15}}\,v_{15}^{15}(1+\tfrac{8}{3}a+e) + \cdots\,. \quad (4.4c)$$

The terms omitted in (4.4) are proportional to the pseudoscalar densities in the $\underline{45}$ representation. It is easy to check that the $\underline{45}$ densities have coefficients proportional to a. This is very important since it means that each of these currents is conserved when $e=-1$, $a=0$. Finally,

$$\partial_\mu A_9^\mu = -\frac{4\epsilon_0}{\sqrt{15}}\,v_9^{15}(1-a-\tfrac{3}{2}e) + \cdots\,, \quad (4.5a)$$

$$\partial_\mu A_{13}^\mu = -\frac{4\epsilon_0}{\sqrt{15}}\,v_{13}^{15}(1-2a-\tfrac{3}{2}e) + \cdots\,, \quad (4.5b)$$

where the coefficients of the omitted terms are proportional to both a and e.

Next we wish to calculate the so-called σ terms defined as

$$K_{\alpha\beta} = \langle 0|[Q_\alpha, [Q_\beta, H]]|0\rangle\,, \quad (4.6a)$$

$$I_{\alpha\beta} = \langle 0|[Q_\alpha^5, [Q_\beta^5, H]]|0\rangle\,. \quad (4.6b)$$

These are expressed in terms of the vacuum expectation values of the scalar densities for the $\underline{1}$, $\underline{15}$, and $\underline{84}$ representations. We use the notation

$$\gamma = \tfrac{16}{15}\epsilon_0\langle u_0\rangle_0\,, \quad (4.7a)$$

$$\langle u_8^{15}\rangle_0 = \frac{6}{\sqrt{15}}\,b\langle u_0\rangle_0\,, \quad (4.7b)$$

$$\langle u_{15}^{15}\rangle_0 = \frac{6\sqrt{2}}{\sqrt{15}}\,f\langle u_0\rangle_0\,, \quad (4.7c)$$

$$\langle u_8^{84}\rangle_0 = -8\sqrt{3}\,g\langle u_0\rangle_0\,, \quad (4.7d)$$

$$\langle u_{15}^{84}\rangle_0 = -8\sqrt{6}\,h\langle u_0\rangle_0\,. \quad (4.7e)$$

We neglect any contribution from the $\underline{20''}$ representation or from the SU(3) $\underline{27}$ piece of $\underline{84}$. With this notation the σ terms are

$$K_{44} = -\gamma a(\tfrac{3}{4}b + 30g)\,, \quad (4.8a)$$

$$K_{99} = -\gamma[a(\tfrac{1}{4}b + f + 2g + 100h) + e(\tfrac{5}{8}b + \tfrac{5}{2}f + 25g + 100h)]\,, \quad (4.8b)$$

$$K_{13,13} = \gamma[a(-b + 2f - 136g + 200h) + e(-\tfrac{5}{4}b + \tfrac{5}{2}f - 50g + 100h)]\,, \quad (4.8c)$$

$$I_{33}=\gamma[(1+e)(1+b+f+g+h)], \quad (4.9a)$$

$$I_{44}=\gamma[(1+a+e)(1-\tfrac{1}{2}b+f-\tfrac{1}{2}g+h) +a(\tfrac{3}{4}b-3f+\tfrac{41}{2}g+15h)], \quad (4.9b)$$

$$I_{88}=\gamma[(1+\tfrac{4}{3}a+e)(1-b+f-g+h) +a(-\tfrac{4}{3}b-4f-\tfrac{76}{3}g+20h)], \quad (4.9c)$$

$$I_{15,15}=\gamma[(1+\tfrac{8}{3}a+e)(1-2f+16h) -\tfrac{16}{3}a(b+10g)], \quad (4.9d)$$

$$I_{8,15}=\sqrt{2}\,\gamma[(1+e)(\tfrac{1}{2}b+5g) +a(-\tfrac{4}{3}+\tfrac{8}{3}b+\tfrac{8}{3}f+\tfrac{80}{3}g-\tfrac{64}{3}h)], \quad (4.9e)$$

$$I_{99}=\gamma[(1-a-\tfrac{3}{2}e)(1+\tfrac{1}{2}b-f-\tfrac{5}{2}g-4h) +a(-\tfrac{9}{4}b+6f-\tfrac{45}{2}g+60h) +e(-\tfrac{5}{8}b+5f-\tfrac{25}{4}g+50h)], \quad (4.9f)$$

$$I_{13,13}=\gamma[(1-2a-\tfrac{3}{2}e)(1-b-f+5g-4h) +a(b+4f+40h) +e(\tfrac{5}{4}b+5f+\tfrac{25}{2}g+50h)]. \quad (4.9g)$$

It is very instructive to consider Eqs. (4.1), (4.3), (4.4), (4.5), (4.8), and (4.9) in terms of the variables α and δ rather than a and γ, where

$$a=\alpha(1+e), \quad (4.10a)$$

$$\delta=-\gamma(1+e). \quad (4.10b)$$

We see that the dependence on e disappears from the U(3)⊗ U(3) subset of equations. Physically, SU(3)⊗SU(3) must be a good symmetry relative to SU(4)⊗SU(4) and we are completely free to adjust e to satisfy this. In the next section we will show the dependence of the masses of the charmed particles on e. Since e also disappears from (4.4c) and (4.9d) it is possible to boost the masses of the charmed mesons without also raising the mass of the observed X meson.

To solve for the parameters we first take the matrix element of (4.1a), (4.3), and (4.4a) between vacuum and one-particle states, assuming that the scalar κ meson and the pseudoscalar pion and K meson belong to the <u>15</u> representations. Assuming

$$\langle 0|v_3^{15}|\pi\rangle=\langle 0|v_4^{15}|K\rangle \quad (4.11)$$

we have an equation for α,

$$\frac{f_K m_K{}^2}{f_\pi m_\pi{}^2}=1+\alpha \quad (4.12)$$

in terms of the π- and K-meson decay constants defined by

$$\langle 0|A_a^\mu(0)|P_a\rangle=ip^\mu f_a \quad (4.13)$$

and the physical masses. Next assuming

$$\langle 0|v_4^{15}|K\rangle=\langle 0|u_5^{15}|\kappa\rangle \quad (4.14)$$

we have an equation for the κ decay constant, f_κ, defined by analogy to (4.13),

$$\frac{f_\kappa}{f_K}=-\frac{1}{\sqrt{3}}\,\frac{\alpha}{1+\alpha}\,\frac{m_K{}^2}{m_\kappa{}^2}. \quad (4.15)$$

We will later relax assumptions (4.11) and (4.14) by allowing f_K/f_π and m_κ to vary.

If we take the one-particle intermediate-state approximation for K_{44}, I_{33}, and I_{44} we have three more equations in terms of the same masses and decay constants. Unfortunately, these three equations still depend on five unknown parameters, δ, b, f, g, and h. Thus we will first assume that the vacuum is not broken by the <u>84</u> representation, i.e., $g=h=0$. With this approximation

$$f_\pi{}^2 m_\pi{}^2=\delta(1+b+f), \quad (4.16a)$$

$$f_K{}^2 m_K{}^2=\delta[(1+\alpha)(1-\tfrac{1}{2}b+f)+3\alpha(\tfrac{1}{4}b-f)], \quad (4.16b)$$

$$f_\kappa{}^2 m_\kappa{}^2=-\tfrac{3}{4}\delta\alpha b, \quad (4.16c)$$

which, when taken together with (4.12) and (4.15), we can solve for δ, b, and f.

When the equations were actually solved we allowed m_κ and f_K/f_π to vary between the limits

$$1050\le m_\kappa\le 1500\text{ MeV}, \quad 1.05\le f_K/f_\pi\le 1.40. \quad (4.17)$$

The solutions for α, b, and f are

$$13.2\le\alpha\le 17.9, \quad -0.106\le b\le -0.045, \quad -0.103\le f\le -0.004. \quad (4.18)$$

If we set $b=f=0$ and allow the vacuum to be broken instead by the <u>84</u> representation we find

$$-3.9\times10^{-3}\le g\le -1.5\times10^{-3}, \quad 5.6\times10^{-3}\le h\le 3.8\times10^{-2} \quad (4.19)$$

for the same range of m_κ and f_K/f_π. Since, in either case, the vacuum breaking is very small we will work with solution (4.18) in the remainder of the paper.

Let us now move on to the (8, 15) system of equations. If we define the decay constants for the currents A_μ^8 and A_μ^{15} with the states of the mesons η and X(958 MeV) as

$$\langle 0|A_\mu^8(0)|\eta(p)\rangle=if_\eta^8 p_\mu, \quad (4.20a)$$

$$\langle 0|A_\mu^8(0)|X(p)\rangle=if_X^8 p_\mu, \quad (4.20b)$$

$$\langle 0|A_\mu^{15}(0)|\eta(p)\rangle=if_\eta^{15} p_\mu, \quad (4.20c)$$

$$\langle 0|A_\mu^{15}(0)|X(p)\rangle=if_X^{15} p_\mu, \quad (4.20d)$$

then the meson approximation is

$$I_{88} = -[(f_\eta^8)^2 m_\eta{}^2 + (f_X^8)^2 m_X{}^2], \qquad (4.21a)$$

$$I_{8,15} = -(f_\eta^8 f_\eta^{15} m_\eta{}^2 + f_X^8 f_X^{15} m_X{}^2), \qquad (4.21b)$$

$$I_{15,15} = -[(f_\eta^{15})^2 m_\eta{}^2 + (f_X^{15})^2 m_X{}^2]. \qquad (4.21c)$$

These give a system of three equations in the four unknown f's. To solve for the f's we assume the following octet relation among them:

$$4f_K{}^2 - f_\pi{}^2 = 3[(f_\eta^8)^2 + (f_X^8)^2]. \qquad (4.22)$$

We will need to know the f's when we consider the decay $X \to \eta\pi\pi$ in Sec. VI.

It is interesting to notice that the mixing assumption of exact SU(3) symmetry for the matrix elements of A_μ^8 and A_μ^{15},

$$\langle 0|A_\mu^8|P_{15}\rangle = 0, \quad \langle 0|A_\mu^{15}|P_8\rangle = 0, \qquad (4.23)$$

is completely inconsistent with positivity of the σ terms for either $g=h=0$ or $f=b=0$. To see this, recall that the states η and X are the physical mixtures of P_8 and P_{15},

$$|\eta\rangle = \cos\theta|P_8\rangle + \sin\theta|P_{15}\rangle, \qquad (4.24a)$$

$$|X\rangle = -\sin\theta|P_8\rangle + \cos\theta|P_{15}\rangle, \qquad (4.24b)$$

where θ is given by the mass relation

$$4m_K{}^2 - m_\pi{}^2 = 3(m_\eta{}^2\cos^2\theta + m_X{}^2\sin^2\theta). \qquad (4.25)$$

The mixing assumption then gives two equations on the f's,

$$f_\eta^{15} = f_X^{15}\tan\theta, \quad f_X^8 = -f_\eta^8\tan\theta. \qquad (4.26a)$$

Using these in (4.21) we have

$$\frac{I_{88}I_{15,15}}{I_{8,15}^2} = \frac{(m_\eta{}^2 + m_X{}^2\tan^2\theta)(m_X{}^2 + m_\eta{}^2\tan^2\theta)}{\tan^2\theta(m_X{}^2 - m_\eta{}^2)^2}. \qquad (4.27)$$

This combination is ≥ 4 for $\tan^2\theta \geq 0$. But the relations

$$\frac{I_{44}}{I_{33}} = \frac{f_K{}^2 m_K{}^2}{f_\pi{}^2 m_\pi{}^2} \qquad (4.28)$$

and

$$\frac{I_{33}}{K_{44}} \geq 0 \qquad (4.29)$$

put bounds on b and f $(g=h=0)$ or g and h $(b=f=0)$ such that the combination (4.27) is never as large as 4.

V. MASS FORMULA

The most important consideration for a model of SU(4)⊗SU(4) symmetry breaking is whether the masses of the charmed particles can be made sufficiently large. In our model this is possible because the divergences and σ terms with nonzero charm depend on the additional parameter e. To see this define

$$\langle 0|u_i(0)|S\rangle \equiv Z_i^S, \quad \langle 0|v_i(0)|P\rangle \equiv Z_i^P. \qquad (5.1)$$

The vacuum-to-one-particle matrix element of (4.1b) gives

$$f_9^S[m_9(S)]^2 = -\frac{4\epsilon_0}{\sqrt{15}}\frac{1}{\sqrt{3}}(a + \tfrac{5}{2}e)Z_9^S, \qquad (5.2)$$

where $m_9(S)$ is the mass of the charmed scalar particle and f_9^S is the decay constant defined as in (4.13). The ratio of (5.2) with the similar relation for the K meson gives

$$\frac{f_9^S[m_9(S)]^2}{f_K m_K{}^2} = \frac{1}{\sqrt{3}}\frac{\alpha + \frac{5}{2}e/(1+e)}{1+\alpha}\frac{Z_9^S}{Z_K} \quad (Z_K \equiv Z_4^P). \qquad (5.3)$$

In the one-meson-saturation approximation the ratio of (4.8b) and (4.9b) gives an equation for

$$\frac{(f_9^S)^2[m_9(S)]^2}{f_K{}^2 m_K{}^2}.$$

Using this equation and (5.3) we have

$$\frac{[m_9(S)]^2}{m_K{}^2} = \frac{1}{3}\frac{\alpha + \frac{5}{2}e/(1+e)}{(1+\alpha)^2}\frac{(1+\alpha)(1-\frac{1}{2}b+f) + 3\alpha(\frac{1}{4}b - f)}{\frac{1}{4}b + f}\left(\frac{Z_9^S}{Z_K}\right)^2. \qquad (5.4)$$

The same procedure with (4.1c) and (4.8c) gives

$$\frac{[m_{13}(P)]^2}{m_K{}^2} = \frac{1}{3}\frac{2\alpha + \frac{5}{2}e/(1+e)}{(1+\alpha)^2}\frac{(1+\alpha)(1-\frac{1}{2}b+f) + 3\alpha(\frac{1}{4}b - f)}{f - \frac{1}{2}b}\left(\frac{Z_{13}^S}{Z_K}\right)^2. \qquad (5.5)$$

Therefore e can be chosen (≈ -1) such that the masses of the charmed scalars are very large. The only unknowns in (5.4) and (5.5) are the ratios of the Z's. In fact the Z's are equal if the chiral SU(3)⊗SU(3) group generated by

$$Q_4, Q_5, \tfrac{1}{2}(Q_3 + \sqrt{3}\,Q_8), Q_9, Q_{10}, Q_{13}, Q_{14}, \frac{1}{2\sqrt{3}}Q_3 - \tfrac{1}{6}Q_8 + \tfrac{2}{3}\sqrt{2}\,Q_{15}$$

and the corresponding axial charge octet satisfy an asymptotic symmetry in the sense that

$$\lim_{q\to\infty} q^2[\Delta^P_{\alpha\beta}(q) - \Delta^S_{\alpha\beta}(q)]=0\,.$$

This is discussed in Ref. 6.

In the same way we can find the pseudoscalar masses

$$\frac{[m_9(P)]^2}{m_K{}^2} = \frac{[1-\alpha-\frac{5}{2}e/(1+e)]^2}{(1+\alpha)^2}\;\frac{(1+\alpha)(1-\frac{1}{2}b+f)+3\alpha(\frac{1}{4}b-f)}{[1-\alpha-\frac{5}{2}e/(1+e)](1+\frac{1}{2}b-f)+\alpha(-\frac{9}{4}b+6f)+[e/(1+e)](-\frac{5}{8}b+5f)}\left(\frac{Z^P_9}{Z_K}\right)^2, \tag{5.6}$$

$$\frac{[m_{13}(P)]^2}{m_K{}^2} = \frac{[1-2\alpha-\frac{5}{2}e/(1+e)]^2}{(1+\alpha)^2}\;\frac{(1+\alpha)(1-\frac{1}{2}b+f)+3\alpha(\frac{1}{4}b-f)}{[1-2\alpha+\frac{5}{2}e/(1+e)](1-b-f)+\alpha(b+4f)+5[e/(1+e)](\frac{1}{4}b+f)}\left(\frac{Z^P_{13}}{Z_K}\right). \tag{5.7}$$

Again the Z's are equal if we assume asymptotic symmetry for some subgroup.[6] However, as before, we are only concerned with the fact that if the parameter e is close to -1 then the charmed pseudoscalar masses will be large.

VI. PION-PION SCATTERING LENGTHS AND $X\to\eta\pi\pi$ DECAY

The π-π s-wave scattering lengths, $a_0^{(I)}$, are given by[11]

$$a_0^{(0)} = \frac{1}{96\pi m_\pi}\left(5A - 16\frac{m_\pi{}^2}{f_\pi{}^2}\right), \tag{6.1a}$$

$$a_0^{(2)} = \frac{1}{48\pi m_\pi}\left(A - 4\frac{m_\pi{}^2}{f_\pi{}^2}\right), \tag{6.1b}$$

where

$$A = -\frac{1}{f_\pi{}^4}\langle 0|[Q_3^5,[Q_3^5,[Q_3^5,[Q_3^5,H]]]]|0\rangle\,. \tag{6.2}$$

Since our model has PCAC for the pions we can legitimately calculate A. This involves traces of six matrices, four factors of F^3, and combinations of $I, D^8, D^{15}, T^8, T^{15}$ taken two at a time. These traces are given in the Appendix. The result is

$$A = -\frac{1}{f_\pi{}^4}\,\tfrac{5}{2}\gamma[(1+e)(1+\tfrac{8}{5}b+\tfrac{8}{5}f-2g-2h)]\,. \tag{6.3}$$

Replacing γ by its value from the one-meson approximation to I_{33}, Eq. (4.9a), A becomes

$$A = +\frac{5}{2}\,\frac{m_\pi{}^2}{f_\pi{}^2}\,\frac{1+\frac{8}{5}b+\frac{8}{5}f-2g-2h}{1+b+f+g+h}\,. \tag{6.4}$$

Again using (4.18) with $g=h=0$ we have

$$A\approx(2.1\sim2.4)\frac{m_\pi{}^2}{f_\pi{}^2}, \tag{6.5}$$

which is somewhat better than the Weinberg value $A=m_\pi{}^2/f_\pi{}^2$.

In any model of this type it is of interest to calculate $X\to\eta\pi\pi$. In previous work[8] we have calculated this decay by assuming PCAC for the η and X as well as the π. This large mass extrapolation undoubtedly introduces a significant error, but we will only ask whether we get the right order of magnitude. In that case the decay width is related to $\bar{A}$ defined as

$$\bar{A} = \frac{1}{f_\pi{}^2 f_\eta f_X}\{\langle 0|[Q_3^5,[Q_\eta^5,[Q_3^5,[Q_X^5,H]]]]|0\rangle + \langle 0|[Q_X^5,[Q_3^5,[Q_3^5,[Q_\eta^5,H]]]]|0\rangle + \langle 0|[Q_\eta^5,[Q_X^5,[Q_3^5,[Q_3^5,H]]]]|0\rangle\} \tag{6.6}$$

by

$$\Gamma\ (\text{keV}) = \frac{1}{3}\left|\frac{m_\pi{}^2}{f_\pi{}^2}\bar{A}\right|^2, \tag{6.7}$$

where

$$\frac{1}{f_\eta}Q_\eta^5 = -\frac{f_X^8}{N}Q_{15}^5 + \frac{f_X^{15}}{N}Q_8^5, \tag{6.8a}$$

$$\frac{1}{f_X}Q_X^5 = \frac{f_\eta^8}{N}Q_{15}^5 - \frac{f_\eta^{15}}{N}Q_8^5, \tag{6.8b}$$

$$N = f_X^{15}f_\eta^8 - f_X^8 f_\eta^{15}, \tag{6.9}$$

with the f's defined by (4.17). These definitions of η and X follow from requiring orthogonality for the physical η and X fields,

$$\langle 0|\phi_\eta|X\rangle = 0\,,$$

$$\langle 0|\phi_X|\eta\rangle = 0\,.$$

After some algebra we find

$$\bar{A} = +\frac{m_\pi{}^2}{N^2}\,\frac{1}{1+b+f}\left[f_X^{15}f_\eta^{15}(\tfrac{5}{6}-\tfrac{2}{3}b+\tfrac{4}{3}f)-(f_X^{15}f_\eta^8+f_X^8f_\eta^{15}-f_X^8f_\eta^8 2\sqrt{2})\frac{1}{3\sqrt{2}}(1+b-2f)\right], \tag{6.10}$$

where we have set $g=h=0$ as before. The f's can be determined from (4.21) and (4.22).

Because of a quadratic equation in determining the f's, there are two possible solutions. If we again relax (4.11) and (4.14) by allowing m_K and f_K/f_π to vary as in (4.17), we find

$$\Gamma = 0.02 \sim 0.14 \text{ MeV} \tag{6.11a}$$

or

$$\Gamma = 0.12 \sim 0.28 \text{ MeV} \tag{6.11b}$$

for the two solutions.

Our answer is somewhat smaller than the width expected for this decay, but that may be entirely due to our use of PCAC for the η and X particles. As expected, our result lies between the value derived in the $(3^*, 3)$ model of SU(3)⊗SU(3), which is ~0.2 keV,[12] and the answer of ~1 MeV that comes from a pure (8, 8) model.[13]

VII. OTHER APPROXIMATIONS FOR H

The Hamiltonian (3.14) was derived from the more general form by insisting upon PCAC for pions. There is no really compelling reason for using (3.14), *a priori*, and we now want to mention some other possibilities. By analogy to octet dominance of SU(3) we might assume $\underline{15}$ dominance of SU(4)⊗SU(4), i.e.,

$$H = \frac{\epsilon_0}{\sqrt{15}}\,\delta_{ij}P_{ji} + \frac{\epsilon_8}{\sqrt{3}}\,D^8_{ij}P_{ji} + \frac{\epsilon_{15}}{\sqrt{3}}D^{15}_{ij}P_{ji}\,. \tag{7.1}$$

With this H it is convenient to change our definitions of a and e from (4.2) to

$$a = \frac{\sqrt{15}}{6}\,\frac{\epsilon_8}{\epsilon_0}\,, \tag{7.2a}$$

$$e = \frac{\sqrt{15}}{6\sqrt{2}}\,\frac{\epsilon_{15}}{\epsilon_0}\,. \tag{7.2b}$$

The divergences of the vector currents can be expressed in terms of the scalar densities of the $\underline{15}$ representation only,

$$\partial_\mu V^\mu_4 = -\frac{2}{\sqrt{5}}\,\epsilon_0 a u_5\,, \tag{7.3a}$$

$$\partial_\mu V^\mu_9 = -\frac{\epsilon_0}{\sqrt{5}}(a+4e)u_{10}\,, \tag{7.3b}$$

$$\partial_\mu V^\mu_{13} = +\frac{2}{\sqrt{5}}\,\epsilon_0(a-2e)u_{14}\,, \tag{7.3c}$$

but the divergences of the axial-vector currents depend on the pseudoscalar densities from both the $\underline{15}$ and the $\underline{45}$ representations:

$$\partial_\mu A^\mu_3 = -\frac{4}{\sqrt{15}}\,\epsilon_0(1+a+e)\,v^{15}_3 + \cdots\,, \tag{7.4a}$$

$$\partial_\mu A^\mu_4 = -\frac{4}{\sqrt{15}}\,\epsilon_0(1-\tfrac{1}{2}a+e)\,v^{15}_4 + \cdots\,, \tag{7.4b}$$

$$\partial_\mu A^\mu_8 = -\frac{4}{\sqrt{15}}\,\epsilon_0\left[(1-a+e)\,v^{15}_8 + \frac{1}{\sqrt{2}}\,av^{15}_{15}\right] + \cdots\,, \tag{7.4c}$$

$$\partial_\mu A^\mu_9 = -\frac{4}{\sqrt{15}}\,\epsilon_0(1+\tfrac{1}{2}a-e)\,v^{15}_9 + \cdots\,, \tag{7.4d}$$

$$\partial_\mu A^\mu_{13} = -\frac{4}{\sqrt{15}}\,\epsilon_0(1-a-e)\,v^{15}_{13} + \cdots\,, \tag{7.4e}$$

$$\partial_\mu A^\mu_{15} = -\frac{4}{\sqrt{15}}\,\epsilon_0[(1-2e)\,v^{15}_{15} + \sqrt{2}\,av^{15}_8] + \cdots\,, \tag{7.4f}$$

On the right-hand sides we have omitted the terms from the $\underline{45}$ representation, but for $i = 3, 4, 8$ each divergence in (7.4) is of the form

$$\partial_\mu A^\mu_i \sim (1+\alpha_i a + e)\,v^{15}_i + (\alpha'_i a + \epsilon_i e)\,v^{45}_i\,. \tag{7.5}$$

It is easy to see that there is no value of a and e for which SU(3)⊗SU(3) is a good symmetry or even a much better symmetry than SU(4)⊗SU(4). This is because (7.3a) requires $a=0$, while, in (7.5), it is not possible to choose a value for e that will make the coefficient of both v^{15}_i and v^{45}_i zero.

We could equally as well assume an $\underline{84}$ dominance for H

$$H = \frac{\epsilon_0}{\sqrt{15}}\,\delta_{ij}P_{ji} + \frac{\epsilon_8}{(720)^{1/2}}\,T^8_{ij}P_{ji} + \frac{\epsilon_{15}}{(360)^{1/2}}\,T^{15}_{ij}P_{ji}\,, \tag{7.6}$$

where T^8 and T^{15} are given in (3.8) and (3.9). Here again the divergences of the SU(3)⊗SU(3) currents are of the form (7.3) and (7.5) and it is not possible for the SU(3)⊗SU(3) symmetry to be much better than the SU(4)⊗SU(4) symmetry.

It is possible to construct more general models that would remedy this defect. For example, consider the model defined by the Hamiltonian:

$$H = \frac{\epsilon_0}{\sqrt{15}}\,\delta_{ij}P_{ji} + \frac{\epsilon'_{15}}{(360)^{1/2}}\,T^{15}_{ij}P_{ji} + \frac{\epsilon_{15}}{\sqrt{3}}\,D^{15}_{ij}P_{ji} + \frac{\epsilon_8}{\sqrt{3}}\,D^8_{ij}P_{ji}\,. \tag{7.7}$$

We can write this in the more convenient form

$$\frac{\sqrt{15}}{\epsilon_0}H = \left(\frac{1-e'}{2}\right)\left(\delta_{ij} - \frac{10}{\sqrt{6}}\,D^{15}_{ij} + \tfrac{2}{3}\,T^{15}_{ij}\right)P_{ij} + \left(\frac{1+e'}{2}\right)\left(\delta_{ij} + 2\sqrt{3}\,aD^8_{ij} + 2\sqrt{6}\,eD^{15}_{ij}\right)P_{ij}\,. \tag{7.8}$$

It can now be seen that in the limit $e' \to -1$ the Hamiltonian is U(3)⊗U(3)-invariant. The divergence of currents in the noncharmed subsector of SU(4) is identical in form to the $\underline{15}$ dominance model, i.e., Eqs. (7.3a), (7.4a), (7.4c), and (7.4f) are the same except for an over-all multiplicative factor $(1+e')$. The charmed masses are then found to behave as

$$m^2(\text{charmed mesons}) \propto \frac{1-e'}{1+e'} \;. \tag{7.9}$$

Thus these masses can be very large as $e' \to -1$. The parameter e' has no constraint from the known pseudoscalar nonet, a property shared with the model discussed in Sec. III. The essential difference between the two models is that in the present model pions do not obey strict PCAC. The parameters a and e can be determined using, for example, the equations analogous to (4.3) and (4.9). Using orthogonal mixing for η and X, we find solutions, all of which yield rather large values for the quantity A [see Eq. (6.2)]. The smallest value is

$$A \cong 22 \frac{m_\pi^{\,2}}{f_\pi^{\,2}} \;. \tag{7.10}$$

The data on π-π scattering would thus seem to rule this model out.[14] A different model with a T^8 instead of D^8 is also ruled out for a similar reason. When both T^8 and D^8 terms are present we have, of course, too many unknown parameters. It may be possible to determine all these using the π-π scattering length and the $X \to \eta\pi\pi$ decay rate, but the model would then have little predictive power.

VIII. SUMMARY AND DISCUSSION

We have investigated the symmetry breaking of the chiral group SU(4)⊗SU(4) in a model where the breaking transforms as (15, 15). The solutions which emerge from using meson dominance for the σ commutators are much more satisfactory than was the case in the (4*, 4) model. In particular, only the equations with charm depend upon the parameter e and it can be chosen such that the Hamiltonian is approximately U(3)⊗U(3)-invariant. This means the charmed mesons have large masses compared to the observed nonet of pseudoscalar mesons. This is a major success since there does not seem to be any way in the (4*, 4) model to sufficiently boost the masses of the charmed particles.

The π-π scattering length that emerges is characterized by the parameter A, whose value is

TABLE I. Traces of matrices defined in Eq. (A5).

	(α, β) (3, 3)	(4, 4)	(8, 8)	(9, 9)	(13, 13)	(15, 15)	(8, 15)
$R_8^{(\alpha,\beta)}$	$2/\sqrt{3}$	$-1/\sqrt{3}$	$-2/\sqrt{3}$	$1/\sqrt{3}$	$-2/\sqrt{3}$	0	$2/\sqrt{6}$
$R_{15}^{(\alpha,\beta)}$	$2/\sqrt{6}$	$2/\sqrt{6}$	$2/\sqrt{6}$	$-2/\sqrt{6}$	$-2/\sqrt{6}$	$-4/\sqrt{6}$	0
$S_{8,8}^{(\alpha,\beta)}$	$\frac{5}{6}$	$\frac{23}{24}$	$\frac{1}{2}$	$\frac{5}{8}$	1	$\frac{2}{3}$	$-1/3\sqrt{2}$
$S_{8,15}^{(\alpha,\beta)}$	$\sqrt{2}/6$	$-1/6\sqrt{2}$	$-1/3\sqrt{2}$	0	0	0	$-\frac{1}{3}$
$S_{15,15}^{(\alpha,\beta)}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	1	1	$\frac{2}{3}$	0
$T_{8,8}^{(\alpha,\beta)}$	$\frac{5}{6}$	$-\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{3}$	$-1/3\sqrt{2}$
$T_{8,15}^{(\alpha,\beta)}$	$\sqrt{2}/6$	$-1/6\sqrt{2}$	$-1/3\sqrt{2}$	$-\sqrt{2}/4$	$\sqrt{2}/2$	0	$-\frac{1}{3}$
$T_{15,15}^{(\alpha,\beta)}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	0	0	$\frac{2}{3}$	0
$X_8^{(\alpha,\beta)}$	−2	1	2	5	−10	0	$-10\sqrt{2}$
$X_{15}^{(\alpha,\beta)}$	−1	−1	−1	4	4	−16	0
$Y_{8,8}^{(\alpha,\beta)}$	34	220	78	228	354	200	$-50\sqrt{2}$
$Y_{8,15}^{(\alpha,\beta)}$	23	$-\frac{23}{2}$	−23	$\frac{135}{2}$	−135	0	$40\sqrt{2}$
$Y_{15,15}^{(\alpha,\beta)}$	19	19	19	204	204	64	0
$Z_{8,8}^{(\alpha,\beta)}$	34	−50	78	−90	−270	200	$-50\sqrt{2}$
$Z_{8,15}^{(\alpha,\beta)}$	23	$-\frac{23}{2}$	−23	$-\frac{165}{2}$	165	0	$40\sqrt{2}$
$Z_{15,15}^{(\alpha,\beta)}$	19	19	19	−96	−96	64	0
$U_{8,8}^{(\alpha,\beta)}$	$2/\sqrt{3}$	$5/\sqrt{3}$	$-6/\sqrt{3}$	$-3/\sqrt{3}$	$6/\sqrt{3}$	$-20/\sqrt{3}$	$10/\sqrt{6}$
$U_{8,15}^{(\alpha,\beta)}$	$-1/2\sqrt{3}$	$1/4\sqrt{3}$	$1/2\sqrt{3}$	$9/4\sqrt{3}$	$-9/2\sqrt{3}$	0	$-8/\sqrt{6}$
$U_{15,8}^{(\alpha,\beta)}$	$8/\sqrt{6}$	$-4/\sqrt{3}$	$-8/\sqrt{6}$	0	0	0	$10/\sqrt{3}$
$U_{15,15}^{(\alpha,\beta)}$	$7/\sqrt{6}$	$7/\sqrt{6}$	$7/\sqrt{6}$	$-12/\sqrt{6}$	$-12/\sqrt{6}$	$+16/\sqrt{6}$	0

$$A = (2.1 \sim 2.4)\frac{m_\pi^2}{f_\pi^2}$$

compared to $A = m_\pi^2/f_\pi^2$ in (3*, 3) (see Ref. 15) and (4*, 4) models. This larger value is in better agreement with experiment.[14]

The decay rate $X \to \eta\pi\pi$ still comes out somewhat small. This may be due to our unjustified use of PCAC for the η and X mesons, but the value we get is certainly reasonable since it lies between the value for pure (3*, 3) symmetry breaking and the value of pure (8, 8) breaking. We had hoped that by choosing a model which included both (3*, 3) and (8, 8) breaking we could get physically reasonable values for both the decay rate and pion-pion scattering and this seems to have worked. Our insistence that the Hamiltonian give PCAC did not remove too much of the (8, 8) breaking; in fact, if we give up this assumption, as in the alternative Hamiltonians considered in Sec. VII, we find unacceptably large values for A.

In summary: The Hamiltonian given in (3.14) is the first successful model for the symmetry breaking of SU(4)⊗SU(4) in the sense that it is the first model to give the correct orders for the symmetry breaking of the various subgroups. In addition it gives a better value for π-π scattering than the (3*, 3) model of SU(3)⊗SU(3) and also a better value for the $X \to \eta\pi\pi$ decay rate.

APPENDIX

In working out the various current divergences and σ terms, it is useful to know the traces of different combinations of the matrices involved in the reduction of $\underline{15} \times \underline{15}$ into irreducible representations of SU(4). The 225 linearly independent 15×15 matrices involved are denoted as follows:

$$1, F_\alpha, D_\alpha, T_\alpha^{84}, T_\alpha^{20''}, T_\alpha^{45}, T_\alpha^{\overline{45}}\,.$$

Here the identity transforms as SU(4) singlet, F_α and D_α as antisymmetric $\underline{15}$ and symmetric $\underline{15}$, respectively, T_α^A transform as the representation $\underline{A}$, with α taking values from 1 to A. The 120 symmetric matrices are 1, D_α, T_α^{84}, and $T_\alpha^{20''}$; the rest are antisymmetric. We choose the matrices F and D to be the symmetric and antisymmetric structure functions of SU(4):

$$(F_\alpha)_{ij} = -if_{\alpha ij}\,, \quad \text{(A1a)}$$

$$(D_\alpha)_{ij} = d_{\alpha ij}\,. \quad \text{(A1b)}$$

The values of $f_{\alpha ij}$ and $d_{\alpha ij}$ are tabulated in Ref. 6. The normalizations of these matrices are

$$\mathrm{Tr} F_\alpha F_\beta = 4\delta_{\alpha\beta}\,, \quad \text{(A2a)}$$

$$\mathrm{Tr} D_\alpha D_\beta = 3\delta_{\alpha\beta}\,, \quad \text{(A2b)}$$

$$\mathrm{Tr} T_\alpha^A T_\beta^B = N_\alpha^A \delta_{AB}\delta_{\alpha\beta}\,. \quad \text{(A2c)}$$

Here the normalization of T_α^A is left arbitrary, and values are later chosen for algebraic convenience. The matrices are traceless (except for 1) and orthogonal (i.e., trace of the product of any two different matrices vanishes). Some useful commutation rules[10,16] are

$$[F_\alpha, F_\beta] = if_{\alpha\beta\gamma}F_\gamma\,, \quad \text{(A3a)}$$

$$[F_\alpha, D_\beta] = if_{\alpha\beta\gamma}D_\gamma\,, \quad \text{(A3b)}$$

$$F_\alpha D_\beta + F_\beta D_\alpha = d_{\alpha\beta\gamma}F_\gamma\,, \quad \text{(A3c)}$$

$$D_\alpha F_\beta + D_\beta F_\alpha = d_{\alpha\beta\gamma}F_\gamma\,, \quad \text{(A3d)}$$

$$[D_\alpha, D_\beta]_{ij} = if_{\alpha\beta\gamma}(F_\gamma)_{ij} + \tfrac{1}{2}(\delta_{\alpha j}\delta_{\beta i} - \delta_{\alpha i}\delta_{\beta j})\,, \quad \text{(A3e)}$$

$$\{D_\alpha, D_\beta\}_{ij} + \{F_\alpha, F_\beta\}_{ij} = \delta_{\alpha\beta}\delta_{ij} + 2d_{\alpha\beta\gamma}(D_\gamma)_{ij} - \tfrac{1}{2}(\delta_{i\beta}\delta_{\alpha j} + \delta_{\alpha i}\delta_{\beta j})\,. \quad \text{(A3f)}$$

The following trace theorems then hold:

$$\mathrm{Tr} F_\alpha F_\beta F_\gamma = 2if_{\alpha\beta\gamma}\,, \quad \text{(A4a)}$$

$$\mathrm{Tr} F_\alpha D_\beta D_\gamma = \tfrac{3}{2}if_{\alpha\beta\gamma}\,, \quad \text{(A4b)}$$

$$\mathrm{Tr} D_\alpha F_\beta F_\gamma = 2d_{\alpha\beta\gamma}\,, \quad \text{(A4c)}$$

$$\mathrm{Tr} D_\alpha D_\beta D_\gamma = \tfrac{1}{2}d_{\alpha\beta\gamma}\,. \quad \text{(A4d)}$$

Traces of products of four, five, and six matrices are also necessary for evaluation of σ commutators and π-π scattering length. Since we only

TABLE II. Traces of matrices defined in Eq. (A6). The values for $(\alpha,\beta)=(8,15)$ can be found from (15,15) by the relation $2\sqrt{2}A^{(8,15)} = A^{(15,15)}$, where $A^{(\alpha,\beta)}$ is any one of the elements of the first column.

	(α,β)		
	(3,3)	(8,8)	(15,15)
$E^{(\alpha,\beta)}$	$\frac{5}{2}$	$\frac{5}{6}$	$\frac{2}{3}$
$I_8^{(\alpha,\beta)}$	$2/\sqrt{3}$	$-\sqrt{3}/9$	$\sqrt{3}/9$
$I_{15}^{(\alpha,\beta)}$	$2/\sqrt{6}$	$\sqrt{6}/9$	$-\sqrt{6}/9$
$J_8^{(\alpha,\beta)}$	$\frac{5}{2}$	$-\frac{7}{6}$	$-\frac{10}{3}$
$J_{15}^{(\alpha,\beta)}$	$\frac{5}{4}$	$\frac{5}{12}$	$-\frac{8}{3}$
$K_{8,8}^{(\alpha,\beta)}$	$\frac{17}{24}$	$\frac{5}{72}$	$\frac{1}{18}$
$K_{8,15}^{(\alpha,\beta)}$	$7\sqrt{2}/24$	$-5/36\sqrt{2}$	$-1/9\sqrt{2}$
$K_{15,15}^{(\alpha,\beta)}$	$\frac{5}{12}$	$\frac{5}{36}$	$\frac{1}{9}$
$L_{8,8}^{(\alpha,\beta)}$	$\frac{29}{2}$	$\frac{17}{6}$	$\frac{50}{3}$
$L_{8,15}^{(\alpha,\beta)}$	$\frac{35}{4}$	$\frac{11}{12}$	$\frac{40}{3}$
$L_{15,15}^{(\alpha,\beta)}$	$\frac{25}{4}$	$\frac{25}{12}$	$\frac{32}{3}$
$M_{8,8}^{(\alpha,\beta)}$	$7/2\sqrt{3}$	$1/6\sqrt{3}$	$-5/3\sqrt{3}$
$M_{8,15}^{(\alpha,\beta)}$	$5/\sqrt{6}$	$-13/24\sqrt{3}$	$-4/3\sqrt{3}$
$M_{15,8}^{(\alpha,\beta)}$	$11/8\sqrt{3}$	$-1/3\sqrt{6}$	$10/3\sqrt{6}$
$M_{15,15}^{(\alpha,\beta)}$	$13/4\sqrt{6}$	$13/12\sqrt{6}$	$8/3\sqrt{6}$

encounter T_8^{84}, T_{15}^{84}, and certain combinations of D_8 and D_{15}, we shall only list these. The matrices T_8^{84} and T_{15}^{84} are defined in Eqs. (3.8) and (3.9), their normalization being 720 and 360, respectively.

We define the following traces:

$$R_i^{(\alpha,\beta)} = \mathrm{Tr}\,F^\alpha F^\beta D^i\,, \tag{A5a}$$

$$S_{ij}^{(\alpha,\beta)} = \mathrm{Tr}\,F^\alpha F^\beta D^i D^j\,, \tag{A5b}$$

$$T_{ij}^{(\alpha,\beta)} = \mathrm{Tr}\,F^\alpha D^i F^\beta D^j\,, \tag{A5c}$$

$$X_i^{(\alpha,\beta)} = \mathrm{Tr}\,F^\alpha F^\beta T_i^{84}\,, \tag{A5d}$$

$$Y_{ij}^{(\alpha,\beta)} = \mathrm{Tr}\,F^\alpha F^\beta T_i^{84} T_j^{84}\,, \tag{A5e}$$

$$Z_{ij}^{(\alpha,\beta)} = \mathrm{Tr}\,F^\alpha T_i^{84} F^\beta T_j^{84}\,, \tag{A5f}$$

$$U_{ij}^{(\alpha,\beta)} = \mathrm{Tr}\,F^\alpha F^\beta D^i T_j^{84}\,, \tag{A5g}$$

$$V_{ij}^{(\alpha,\beta)} = \mathrm{Tr}\,F^\alpha D^i F^\beta T_j^{84}\,. \tag{A5h}$$

The values of those traces involved in our calculations are listed in Table I. For our choice of α, β and i, j, $U_{ij}^{(\alpha,\beta)} = V_{ij}^{(\alpha,\beta)}$, and hence $V_{ij}^{\alpha\beta}$ are not listed separately. In addition, we need the following traces to evaluate the pion scattering length and the $X\to\eta\pi\pi$ decay rate:

$$E^{\alpha\beta} = \mathrm{Tr}\,F^3 F^3 F^\alpha F^\beta\,, \tag{A6a}$$

$$I_i^{\alpha\beta} = \mathrm{Tr}\,F^3 F^3 F^\alpha F^\beta D^i\,, \tag{A6b}$$

$$J_i^{\alpha\beta} = \mathrm{Tr}\,F^3 F^3 F^\alpha F^\beta T_i^{84}\,, \tag{A6c}$$

$$K_{ij}^{\alpha\beta} = \mathrm{Tr}\,F^3 F^3 F^\alpha F^\beta D^i D^j\,, \tag{A6d}$$

$$L_{ij}^{\alpha\beta} = \mathrm{Tr}\,F^3 F^3 F^\alpha F^\beta T_i^{84} T_j^{84}\,, \tag{A6e}$$

$$M_{ij}^{\alpha\beta} = \mathrm{Tr}\,F^3 F^3 F^\alpha F^\beta T_i^{84} D_j\,. \tag{A6f}$$

These traces are listed in Table II.

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[1]A. Benvenuti *et al.*, Phys. Rev. Lett. 32, 800 (1974); F. J. Hasert *et al.*, Phys. Lett. 46B, 138 (1973).

[2]S. Weinberg, Phys. Rev. Lett. 19, 1264 (1967).

[3]A. Salam, in *Elementary Particle Theory: Relativistic Groups and Analyticity (Nobel Symposium No. 8)*, edited by N. Svartholm (Almquist and Wiksell, Stockholm, 1968), p. 367.

[4]S. L. Glashow, J. Iliopoulos, and L. Maiani, Phys. Rev. D 2, 1285 (1970).

[5]M. Gell-Mann, R. J. Oakes, and B. Renner, Phys. Rev. 175, 2195 (1968).

[6]D. A. Dicus and V. S. Mathur, Phys. Rev. D 9, 1003 (1974); Phys. Lett. 49B, 63 (1974).

[7]P. Dittner and S. Eliezer, Phys. Rev. D 8, 1929 (1973); P. Dittner, S. Eliezer, and T. K. Kuo, Phys. Rev. Lett. 30, 1274 (1973).

[8]N. G. Deshpande and D. A. Dicus, Phys. Rev. D 10, 1613 (1974).

[9]N. G. Deshpande and D. A. Dicus, Phys. Rev. D (to be published).

[10]D. Amati, H. Bacry, J. Nuyts, and J. Prentki, Nuovo Cimento 34, 1732 (1964).

[11]S. Weinberg, Phys. Rev. Lett. 17, 616 (1966); A. Sirlin and M. Weinstein, Phys. Rev. D 6, 3588 (1972).

[12]Riazuddin and S. Oneda, Phys. Rev. Lett. 27, 548 (1971); P. Weisz, Riazuddin, and S. Oneda, Phys. Rev. D 5, 2264 (1972).

[13]A. Ali, Phys. Rev. D 8, 1441 (1973).

[14]See, for example, P. R. Auvil, in *Proceedings of the XVI International Conference on High Energy Physics, Chicago-Batavia, Ill., 1972*, edited by J. D. Jackson, and A. Roberts (NAL, Batavia, Ill., 1973), Vol. 2, p. 267.

[15]S. Weinberg, Ref. 11.

[16]L. M. Kaplan and M. Resnikoff, J. Math. Phys. 8, 2194 (1967).