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Regge surfaces and elastic unitarity*

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General singular Regge surfaces $j = \alpha(t)$ of the continued partial-wave amplitude are considered in connection with the requirements of elastic t-channel unitarity. In the absence of shielding cuts, threshold branch points of the trajectory function $\alpha(t)$ can be used for ordinary and for complex Regge-pole surfaces in order to obtain compliance with the unitarity condition. For complex pole surfaces, there are important constraints concerning the way in which the threshold must appear in $\alpha(t)$. Branch-point surfaces are discussed in detail. It is shown that hard branch-point trajectories $\alpha(t)$ with t-independent character generally cannot be made compatible with elastic unitarity by using threshold branch points of $\alpha(t)$. Shielding cuts in the j plane are a natural requirement for these singular surfaces. The notions of "shielding cuts" and "hiding cuts" are defined briefly.

I. INTRODUCTION

The requirement that the continued partial-wave amplitude $F(t,j)$ satisfies t-channel unitarity is a very important constraint for models of high-energy diffraction scattering.[1] Recently, much attention has been focused on multiparticle unitarity, which is relevant for the generation of conventional Regge cuts. The Reggeon calculus[2] provides a field-theoretic method for finding amplitudes which satisfy the j-plane discontinuity equations resulting from multiparticle t-channel unitarity. In particular, renormalization-group methods are very helpful for the infrared problem associated with the Pomeron trajectory near $(j,t)=(1,0)$. It is interesting to see that there are strong-coupling solutions where the renormalized Pomeron trajectory is a singular Regge surface which, as a function of t, has a branch point at $t=0$.[3] Pomeron trajectories of this type have been proposed earlier in connection with rising cross sections and for other purposes.[4,5]

It is important to note that the Reggeon calculus does not take into account two-particle unitarity in the t channel. In the strong-coupling case, the resulting renormalized Pomeron trajectory is a hard Regge surface which, *a priori*, violates two-particle unitarity.[1,6] In previous papers,[7] we have shown how this unitarity condition can be satisfied with the help of shielding cuts.[7,8] These cuts are due to soft branch-point trajectories which are closely correlated with the Pomeron trajectory itself, and which coincide with it at the threshold $t=t_0$. In their secondary sheets, they provide a refuge for the hard Regge surface as we continue the amplitude through the two-particle branch cut. Without the shielding mechanism, the Regge surface would have to change its character suddenly during the continuation, and this it cannot do as a singular surface of an analytic function of two complex variables (continuity theorem).[7]

Although shielding cuts can be constructed for all kinds of hard singular surfaces of the amplitude $F(t,j)$, this mechanism is not the one an ordinary Regge pole surface uses in order to comply with elastic t-channel unitarity. Rather, a pole trajectory $\alpha(t)$ acquires a branch point at the threshold $t=t_0$, so that for $t \geq t_0$, $t \to t_0$

$$\alpha(t+i0)-\alpha(t-i0)=iC(t-t_0)^{\alpha(t_0)+1/2}+\cdots . \tag{1}$$

This threshold behavior is in accordance with the resonance character of the corresponding t-plane poles for $\mathrm{Re}\,t>t_0$, which are in second sheets of the t plane.[1,9]

It is the purpose of this paper to explore general singular Regge surfaces $\alpha(t)$ with branch points at the elastic threshold $t=t_0$. We want to see to what extent this mechanism can be used in order to obtain compliance with t-channel analyticity and unitarity requirements without the introduction of shielding cuts in the complex j plane. As we have pointed out, this is the conventional unitarization method for ordinary Regge-pole trajectories. We show that it can also be used for complex Regge-pole surfaces. These surfaces have a left-hand branch line in $\alpha(t)$ starting at the crossover point of their branches. However, we find important restrictions as to the way in which the threshold cut for $t \geq t_0$ must be introduced into the complex trajectory functions. Simple models are often excluded.

Then we consider hard branch-point trajectories with t-independent character. Here we find that it is generally *not* possible to satisfy the unitarity and analyticity requirements by the introduction of a threshold branch point into the trajectory. This result shows that, for these singular surfaces, it is necessary to have shielding cuts in the j plane of the amplitude.

II. REGGE POLES AND SHIELDING AND HIDING CUTS

For completeness, and in order to introduce our notation, let us briefly review the unitarity properties of an ordinary Regge pole trajectory $j=\alpha(t)$.[9,1] We write the amplitude F in the form

$$F(t,j)=(t-t_0)^j G(t,j)\ , \tag{2}$$

with

$$G(t,j)=\frac{A(t)}{j-\alpha(t)}+B(t,j)\ , \tag{3}$$

where $B(t,j)$ does not have a pole at $j=\alpha(t)$. The elastic unitarity condition is then given by

$$G^{-1\mathrm{II}}(t,j)-G^{-1}(t,j)=2i\rho(t)(t-t_0)^j\ , \tag{4}$$

with

$$i\rho(t)=-\left(\frac{t_0-t}{t}\right)^{1/2}\ ,\qquad \arg(t_0-t)^{1/2}=0 \text{ for } t\leq t_0\ . \tag{5}$$

By the superscript II we denote the continuation into the second sheet reached from the physical sheet through the elastic cut in the interval $t_0\leq t<t_i$; t_i is the first inelastic threshold. In particular, we have

$$G^{\mathrm{II}}(t\pm i0,j)=G(t\mp i0,j)$$

in this interval.

Using Eq. (3), we may take the limit $j\to\alpha(t)$ in Eq. (4) and obtain

$$\alpha(t)-\alpha^{\mathrm{II}}(t)=2i\rho(t)(t-t_0)^{\alpha(t)}C^{\mathrm{II}}(t)\ , \tag{6}$$

with

$$C^{\mathrm{II}}(t)=A^{\mathrm{II}}(t)\,[1-2i\rho(t)(t-t_0)^{\alpha(t)}B^{\mathrm{II}}(t,\alpha(t))]^{-1} \tag{7}$$

and

$$B^{\mathrm{II}}(t,\alpha(t))=\lim_{j\to\alpha(t)}B^{\mathrm{II}}(t,j)\ . \tag{8}$$

The limit $j\to\alpha^{\mathrm{II}}(t)$ in Eq. (4) gives the conjugate of Eq. (6) with $\alpha\leftrightarrow\alpha^{\mathrm{II}}$, $\rho\to-\rho$, $A^{\mathrm{II}}\to A$, $B^{\mathrm{II}}\to B$. If we take the limit $t\to t_0$ in Eq. (6), we see that $\alpha(t)$ has a branch point at $t=t_0$ as indicated in Eq. (1). We are interested in cases where $\alpha(t_0)>0$ and $C^{\mathrm{II}}(t_0)=C(t_0)$ is finite.

In deriving Eq. (6) we have assumed that there are no other singular surfaces of the amplitude $G(t,j)$ which coincide with the Regge pole $\alpha(t)$ at $t=t_0$. The presence of additional trajectories of this kind could, of course, change the threshold behavior of $\alpha(t)$. As we know from previous work on shielding cuts, we can then even arrange that $\alpha(t)$ is regular at the threshold.

In order to give a different and simple example, consider the function

$$G^{-1}(t,j)=\frac{-1}{\sqrt{t}\,\cos\pi j}\left(\frac{j-d(t)}{c(t)}\right)^{1/\kappa}+\frac{(t_0-t)^{j+1/2}}{\sqrt{t}\,\cos\pi j}\ , \tag{9}$$

where $d(t)$ and $c(t)$ are appropriate real analytic functions which have no singularities at $t=t_0$. We ignore the singularity of the second term at $t=0$, which should not be present in the physical sheet and which can easily be removed.

The amplitude (9) has a branch point at $j=d(t)$. We define the physical sheet of the j plane by drawing the cut to the left and demanding that $(j-d)^{1/\kappa}$ is real where d is real and $\arg(j-d)=0$. In addition to the branch point, we have a pole surface $\alpha(t)$ which satisfies the equation

$$\alpha(t)=d(t)+c(t)[(t_0-t)^{\kappa}]^{\alpha(t)+1/2}\ . \tag{10}$$

This pole is in the physical sheet of the j plane for $\kappa|\arg(t_0-t)|<\pi$, assuming that $\alpha(t)$ in Eq. (10) is defined as the branch which is real for real $t\leq t_0$. The residue of the pole in $G(t,j)$ is given by

$$R(t)=-\frac{\kappa c(t)}{\sqrt{t}\,\cos\pi\,\alpha(t)}[(t_0-t)^{\kappa-1}]^{\alpha(t)+1/2}\ , \tag{11}$$

which shows that the simple pole is modified for $t\to t_0$, where it coincides with the branch-point trajectory. We obtain with $\alpha(t_0)=d(t_0)$

$$G(t_0,j)=-\sqrt{t_0}\cos\pi\,\alpha(t_0)\left(\frac{c(t)}{j-\alpha(t_0)}\right)^{1/\kappa}\ .$$

At the threshold $t=t_0$, the trajectory function $\alpha(t)$ has a branch point like

$$\alpha(t)=\alpha(t_0)+c(t_0)(t_0-t)^{\kappa[\alpha(t_0)+1/2]}+\cdots\ . \tag{12}$$

The cut in the function $G(t,j)$ given in Eq. (9) is a special case of a "hiding cut."[4] In order to explain this, and for future reference, we give here a brief definition of the concepts of "shielding cut" and "hiding cut."

(1). Shielding cut. This is a soft branch-point trajectory $j=\alpha_s(t)$ associated with a hard singular surface $j=\alpha(t)$ of the amplitude $G(t,j)$. It is the purpose of this trajectory $\alpha_s(t)$ to remove the singularity at $j=\alpha(t)$ into a secondary Riemann sheet of the j plane as we continue the amplitude from the physical sheet of the t plane through the two-particle branch cut starting at $t=t_0$ into the second sheet. The trajectories $\alpha(t)$ and $\alpha_s(t)$ must coincide at the threshold $t=t_0$, and the character of the shielding surface α_s is dependent upon that of the surface α.[7]

(2). Hiding cut. This is a soft branch-point surface $j=\alpha_h(t)$, which is called for if the amplitude $G(t,j)$ has singular trajectories $\alpha(t)$ with branch points in t which must not be present in $G(t,j)$. Generally, the branch point of $\alpha(t)$ is not inherited by $G(t,j)$ if all branches of $\alpha(t)$ associated with the Riemann surface of this branch point appear in G in a completely symmetric fashion so that we have full uniformization. If the branch point of $\alpha(t)$ at $t=t_c$ is not algebraic, we have an infinite number of branches which cannot be all in the physical sheet of the j plane. It is the task of the hiding cut to remove all but a finite number of branches into its secondary Riemann sheets.

As an example, suppose

$$\alpha(t)=\alpha_0(t)+\beta_0(t)(t-t_c)^\kappa \tag{13}$$

is the trajectory of $G(t,j)$ with κ being irrational. The functions $\alpha_0(t)$ and $\beta_0(t)$ are real, analytic, and regular at $t=t_0$. Then the amplitude $G(t,j)$ may depend upon these functions in the combination

$$\left(\frac{j-\alpha_0(t)}{\beta_0(t)}\right)^{n/\kappa}-(t-t_c)^n\ , \tag{14}$$

where n is a positive integer. The hiding cut is associated with the branch point at $j=\alpha_0(t)$, and it is drawn to the left. The physical sheet of the j plane is defined so that $(j-\alpha_0)^{n/\kappa}$ is real for real $\alpha_0(t)$ and $\arg(j-\alpha_0)=0$. We find then that only those branches α_ν of α are null surfaces of Eq. (14) in a given region of the t plane, for which

$$|\arg(t-t_c)+2\pi\nu|\leqslant\frac{\pi n}{\kappa}\ ,$$

$$\nu=0,\ \pm1,\ \pm2,\ \ldots\ . \tag{15}$$

As we see, the hiding cut is hiding two things: the branch point in t of $\alpha(t)$ and almost all branches of $\alpha(t)$. Of course, hiding cuts can also be used in cases where κ is rational. Then we have a finite number of branches, and a hiding cut can remove some of these from the physical sheet of the j plane.[4,10]

In the example given by Eq. (9), the branch point of $\alpha(t)$ at $t=t_c$ coincides with the threshold t_0. The character of the cut in $\alpha(t)$ is different from what is required by the unitarity condition for $G(t,j)$, and the hiding cut adjusts this character as needed.

III. COMPLEX POLE TRAJECTORIES

In this section we consider complex pole surfaces. We are interested in particular in the case of two crossing trajectories $\alpha_+(t)$ and $\alpha_-(t)$ associated with a square-root branch point at $t=t_c$. The amplitude G is then of the form

$$G(t,j)=\frac{A(t,j)}{[j-\alpha_+(t)][j-\alpha_-(t)]}$$

$$=\frac{A_+(t)}{j-\alpha_+(t)}+\frac{A_-(t)}{j-\alpha_-(t)}+B(t,j)\ , \tag{16}$$

where

$$A_\pm(t)=\pm\frac{A(t,\alpha_\pm(t))}{\alpha_+(t)-\alpha_-(t)} \tag{17}$$

and $B(t,j)$ is the rest which is regular at $j=\alpha_\pm(t)$. We now insert the ansatz (16) into the unitarity condition (4) and take the limits $j\to\alpha_\pm(t)$ and $j\to\alpha_\pm^{\rm II}(t)$. The resulting conditions for $\alpha_\pm(t)$ are given by

$$[\alpha_\pm(t)-\alpha_+^{\rm II}(t)][\alpha_\pm(t)-\alpha_-^{\rm II}(t)]$$

$$=\frac{2i}{\sqrt{t}}(t-t_0)^{\alpha_\pm(t)+1/2}A^{\rm II}(t,\alpha_\pm(t)) \tag{18}$$

and

$$[\alpha_\pm^{\rm II}(t)-\alpha_+(t)][\alpha_\pm^{\rm II}(t)-\alpha_-(t)]$$

$$=-\frac{2i}{\sqrt{t}}(t-t_0)^{\alpha_\pm^{\rm II}(t)+1/2}A(t,\alpha_\pm^{\rm II}(t))\ . \tag{19}$$

We are interested in the threshold properties of the trajectories $\alpha_\pm(t)$ which are required by these equations. It is reasonable to distinguish two cases which are characterized by $\alpha_+(t_0)\neq\alpha_-(t_0)$ and $\alpha_+(t_0)=\alpha_-(t_0)$, respectively.

Case 1. Let us assume that $t_c\neq t_0$ and

$$\alpha_+(t_0)\neq\alpha_-(t_0),\quad \alpha_\pm^{\rm II}(t_0)=\alpha_\pm(t_0)\ . \tag{20}$$

Then we have

$$\lim_{t\to t_0}\frac{A(t,\alpha_\pm^{\rm II}(t))}{\alpha_\pm^{\rm II}(t)-\alpha_\mp(t)}=\lim_{t\to t_0}A_\pm(t)\ , \tag{21}$$

where the functions $A_\pm(t)$ are the residues defined in Eqs. (16) and (17). We find that the conditions (18) and (19) can be satisfied with finite residues

$$A_\pm(t_0)=A_\pm^{\rm II}(t_0)\ , \tag{22}$$

and with the discontinuity given by

$$\alpha_\pm(t)-\alpha_\pm^{\rm II}(t)=2i\rho(t)(t-t_0)^{\alpha_\pm(t)}A_\pm(t_0) \tag{23}$$

for $t\geqslant t_0$, $t\to t_0$.

It may be useful to give an example which satisfies the assumptions (20). We write

$$G^{-1}(t,j)=-[j-a_+(t)][j-a_-(t)]c^{-1}(t)$$

$$+\frac{(t_0-t)^{j+1/2}}{\sqrt{t}\cos\pi j}\ , \tag{24}$$

with

$$a_\pm(t)=a_c(t)\pm(t-t_c)^{1/2}b_c(t)\ ,$$

$$c(t)=(t-t_c)c_c(t)\ , \tag{25}$$

where we ignore again the singularity at $t=0$. The functions a_c, b_c, and c_c are regular at $t=t_c$ and $t=t_0$, and their analytic properties are as required otherwise. The amplitude (24) has complex Regge-pole trajectories which satisfy the equations

$$\alpha_\pm(t) = a_c(t) \pm \left[b_c{}^2(t)(t-t_c) + \frac{c(t)}{\sqrt{t}\,\cos\pi\,\alpha_\pm(t)}\,(t_0-t)^{\alpha_\pm(t)+1/2}\right]^{1/2} . \tag{26}$$

By expanding the square root, we see that the branch points of $\alpha_\pm(t)$ at the threshold $t=t_0$ are characterized by the discontinuity relation (23), with the residues given by

$$A_\pm(t_0) = \mp\,\frac{c(t_0)}{2(t_0-t_c)^{1/2}b_c(t_0)} . \tag{27}$$

We note that, in general, the inequality of $\alpha_+(t_0)$ and $\alpha_-(t_0)$ together with the centrifugal j dependence of the amplitude imply a rather complicated structure of the pole surfaces $\alpha_\pm(t)$ at the threshold $t=t_0$. If we write

$$\alpha_\pm(t) = \alpha_c(t) \pm (t-t_c)^{1/2}\beta_c(t) , \tag{28}$$

we find that the branch point at $t=t_0$ must be present in $\alpha_c(t)$ as well as in $\beta_c(t)$, at least within the framework of the assumptions we have made. In this connection, it is of interest to consider complex trajectories of the frequently used form

$$\alpha_\pm(t) = \alpha_0(t) \pm (at)^{1/2} , \tag{29}$$

with $a=\text{const}$. As we have learned, in this case we cannot obtain compatibility with elastic unitarity by having a threshold branch point in $\alpha_0(t)$ and constant residues $A_\pm(t_0)$. Rather, in order to satisfy the relations (18) and (19) without shielding cuts, the functions $A_\pm(t)$ must themselves have branch points at $t=t_0$. Specifically, we have the requirement that for $t\to t_0$

$$A_\pm(t) \propto (t-t_0)^{\mp(at_0)^{1/2}} ,$$

provided $B(t, \alpha^{\mathrm{II}}(t))$ is sufficiently nonsingular in this limit. These branch points of the residue are not present in the amplitude $G(t,j)$ itself, and in Eq. (16) they must be canceled by contributions from $B(t,j)$. Generally, such compensations are done with appropriate j-plane branch-point surfaces which coincide with $\alpha_\pm(t)$ at $t=t_0$. But under these circumstances it is usually much simpler to use shielding cuts in order to satisfy elastic unitarity without a threshold branch point in $\alpha_0(t)$.

Case 2. From our previous discussion of complex pole trajectories with $\alpha_+(t_0)\neq\alpha_-(t_0)$, it is plausible that compliance with Eqs. (18) and (19) may be more straightforward for pole surfaces of the form (28) which coincide at $t=t_0$. In the following we assume that

$$\alpha_+(t_0) = \alpha_-(t_0) = \alpha_c(t_0), \quad \text{i.e.,} \quad \beta_c(t_0)=0 , \qquad \alpha_c^{\mathrm{II}}(t_0) = \alpha_c(t_0) . \tag{30}$$

Then we obtain the conditions

$$[\alpha_c(t) - \alpha_c^{\mathrm{II}}(t)]^2_{t\to t_0} = \frac{2i}{\sqrt{t_0}}\,(t-t_0)^{\alpha_c(t_0)+1/2} \times \lim_{t\to t_0} A^{\mathrm{II}}(t, \alpha_c(t)) \tag{31}$$

and

$$\lim_{t\to t_0} \frac{A^{\mathrm{II}}(t, \alpha_c(t))}{A(t, \alpha_c^{\mathrm{II}}(t))} = -1 . \tag{32}$$

A simple way to satisfy these equations, which also avoids a double pole of $G(t,j)$ at $j=\alpha_\pm(t_0) = \alpha_c(t_0)$, is obtained by choosing the function $A(t,j)$ of Eq. (16) for our case as

$$A(t,j) = [j-\alpha_c(t)]C(t,j) , \tag{33}$$

with

$$C^{\mathrm{II}}(t_0, \alpha_c(t_0)) = C(t_0, \alpha_c(t_0)) . \tag{34}$$

Then we have

$$A(t, \alpha^{\mathrm{II}}(t)) = [\alpha_c^{\mathrm{II}}(t) - \alpha_c(t)]C(t, \alpha^{\mathrm{II}}(t)) , \tag{35}$$

and the discontinuity relation (31) reduces to

$$\alpha_c(t) - \alpha_c^{\mathrm{II}}(t) = \frac{2i}{\sqrt{t_0}}\,(t-t_0)^{\alpha_c(t_0)+1/2} \times C(t_0, \alpha_c(t_0)) \tag{36}$$

for $t\to t_0$.

An explicit model for case 2 with $\alpha_+(t_0)=\alpha(t_0)$ can be obtained from Eq. (24) with $b_c(t)\to 0$ and $c(t)\to 0$ for $t\to t_0$, but $c(t)/b_c(t)\to$ finite.

IV. BRANCH-POINT SURFACES

First, we consider briefly soft branch-point trajectories in connection with elastic t-channel unitarity. It is convenient to introduce the function[1,9]

$$\psi(t,j) = G^{-1}(t,j) + i\rho(t)\,\frac{(t_0-t)^j}{\cos\pi j} , \tag{37}$$

which has no branch point at $t=t_0$. If the inverse amplitude G^{-1} has a branch-point surface $j=\alpha(t)$, it is evident from Eq. (37) that this same surface is also present in $\psi(t,j)$, and vice versa. As a rather general example of a weak cut, we consider the expression

$$\psi(t,j) = a(t,j)[j-\alpha(t)]^\beta + b(t,j) , \tag{38}$$

where β is real, and the functions $a(t,j)$ and $b(t,j)$ are regular at $j=\alpha(t)$. With the function ψ as given in Eq. (38), the amplitude $G(t,j)$ has a soft branch point at $j=\alpha(t)$, provided

$$b(t,\alpha(t)) \neq i\rho(t)\,\frac{(t_0-t)^{\alpha(t)}}{\cos\pi\,\alpha(t)}\ . \tag{39}$$

The elastic unitarity condition is satisfied with the ansatz (38), provided the trajectory $j=\alpha(t)$ as well as the functions $a(t,j)$ and $b(t,j)$ have *no* branch point at the threshold $t=t_0$. Note that these conditions actually imply the requirement (39) as long as there are no shielding cuts present. With shielding, the limits $j\to\alpha(t)$ and $t\to t_0$ are no longer interchangeable in general. For example, a term in $\psi(t,j)$ of the form

$$[j-\alpha_s(t)]^{j+1/2}\ , \tag{40}$$

with

$$\alpha_s(t)=\alpha(t)+c(t-t_0)\ , \tag{41}$$

can generate the branch point $(t_0-t)^{\alpha(t)+1/2}$ for $j\to\alpha(t)$.[7,8]

We now turn to the question of hard branch-point surfaces. From Eqs. (37) and (38) we can already see the difficulties which arise if we want to make hard cuts compatible with elastic unitarity using only threshold singularities of $\alpha(t)$, but no shielding cuts. With $\beta>0$, we have

$$G^{-1}(t,j)\propto[j-\alpha(t)]^{\beta}\quad\text{for } j\to\alpha(t) \tag{42}$$

if

$$\lim_{j\to\alpha(t)}\ [j-\alpha(t)]^{-\beta}\left[b(t,j)-i\rho(t)\,\frac{(t_0-t)^j}{\cos\pi j}\right]=0\ . \tag{43}$$

Since the trajectory $\alpha(t)$ has a branch point at $t=t_0$, we have $\alpha^{\rm II}(t)\neq\alpha(t)$. Furthermore, the surface $\alpha^{\rm II}(t)$ satisfies the equation

$$G^{-1\,\rm II}(t,\alpha^{\rm II}(t))=0 \tag{44}$$

unless there are intervening additional j-plane cuts which could move $\alpha^{\rm II}$ as a null surface into a secondary sheet of the j plane. Since

$$\psi(t,j)=G^{-1\,\rm II}(t,j)-i\rho(t)\,\frac{(t_0-t)^j_{\rm II}}{\cos\pi j}\ , \tag{45}$$

with

$$(t_0-t)^j_{\rm II}=-(t_0-t)^j+2\cos\pi j\,(t-t_0)^j\ , \tag{46}$$

the branch-point surface $j=\alpha^{\rm II}(t)$ is also present in ψ and hence in G^{-1}, although it is not a hard branch point of these functions.

By construction, the function $\psi(t,j)$ is regular at $t=t_0$. Hence the threshold singularities of the branch-point surfaces α and $\alpha^{\rm II}$ must not be inherited by ψ. If these singularities are not algebraic, this requires the existance of a hiding cut, as we have explained in Sec. II. For example, if we write

$$\alpha(t)=\alpha_0(t)+c(t)(t_0-t)^{\kappa}\ , \tag{47}$$

with $\alpha_0(t)$ and $c(t)$ being regular at $t=t_0$, we could express $\psi(t,j)$ in the form

$$\psi(t,j)=a(t,j)\left[\left(\frac{j-\alpha_0(t)}{c(t)}\right)^{1/\kappa}-(t_0-t)\right]^{\beta}+b(t,j)\ . \tag{48}$$

The hiding cut is due to the new branch point at $j=\alpha_0(t)$ with the branch line drawn to the left in the j plane. For real $\alpha_0(t)$, the physical sheet of $\psi(t,j)$ is defined by the requirement that $(j-\alpha_0)^{1/\kappa}$ is real for $\arg(j-\alpha_0)=0$. Since the main branch $\alpha(t)$ of the trajectory (47) is defined by $(t_0-t)^{\kappa}$ being real for $\arg(t_0-t)=0$, with a cut along the real axis for $t\geq t_0$, we see that this branch $j=\alpha(t)$ is a null surface of

$$\left[\left(\frac{j-\alpha_0(t)}{c(t)}\right)^{1/\kappa}-(t_0-t)\right]^{\beta} \tag{49}$$

in the region defined by

$$\kappa|\arg(t_0-t)|\leq\pi\ . \tag{50}$$

For values of t where the inequality (50) is not satisfied, the trajectory $\alpha(t)$ becomes a null surface of (49) in a secondary sheet of the j plane reached through the hiding cut. If $\kappa<1$, the entire branch $\alpha(t)$ is a null surface in the physical sheet of the j plane. The second branch $\alpha^{\rm II}(t)$ of Eq. (47) is a real analytic function with

$$\kappa\leq|\arg(t_0-t)|\leq 2\pi\ .$$

Under the previous conditions, it is a null surface of Eq. (49) at least for points with $|\arg(t_0-t)|=\pi$, provided $\kappa<1$. We have, of course, for $t\geq t_0$,

$$\alpha^{\rm II}(t\pm i0)=\alpha(t\mp i0)\ ,$$

and $\alpha^{\rm II}$ has a cut to the left of $t=t_0$ ($\kappa=\frac{1}{2}$ is an exception).

The question is whether the threshold in $\alpha(t)$ is of any help in satisfying Eq. (43). Suppose we have $\beta<1$, so that a simple zero of the factor in square brackets in Eq. (43) is sufficient. We know that the function $b(t,j)$ has no branch point at $t=t_0$. Hence the singularity of

$$b(t,\alpha(t))=i\rho(t)\,\frac{(t_0-t)^{\alpha(t)}}{\cos\pi\,\alpha(t)} \tag{51}$$

must be introduced into b via the surface $\alpha(t)$. For the dominant term in the limit $t\to t_0$, we can achieve this with the help of the branch point in $\alpha(t)$ as given in Eq. (47) by writing

$$b(t,j)=\frac{-1}{\sqrt{t}\,\cos\pi j}\left(\frac{j-\alpha_0(t)}{c(t)}\right) \tag{52}$$

and equating κ with $\alpha(t_0)+\frac{1}{2}$. However, this choice does not help for other values of t. We cannot choose $\kappa=\alpha(t)+\frac{1}{2}$ in Eq. (47), because $\psi(t,j)$ depends upon κ, and this dependence would introduce a branch point at $t=t_0$ in the function ψ.

As is well known, what is called for here is a shielding cut.[7,8] For example, we may write

$$b(t,j) = -\frac{1}{\sqrt{t}\cos\pi j}\left(\frac{j-\alpha_0(t)}{c(t)}\right)^{(j+1/2)/\kappa}, \tag{53}$$

with the branch line drawn again to the left. Then $b(t,\alpha(t))$ satisfies Eq. (51) for $\kappa|\arg(t_0-t)| \leq \pi$, and hence is a hard branch-point surface of $G(t,j)$ as required in Eq. (42). The compliance with the unitarity condition is due to the shielding cut and has nothing to do with the threshold introduced into $\alpha(t)$. The latter only complicates the model, and we may simply set $\kappa=1$. More general amplitudes with shielding cuts may be found in Ref. 7.

In order to see the difficulties of hard branch-point surfaces with elastic unitarity from another point of view, it is of interest to follow the argumentation we have used for pole trajectories in Secs. II and III. For this purpose we write the continued partial-wave amplitude in the form

$$G(t,j) = \frac{A(t,j)}{[j-\alpha(t)]^\beta} + B(t,j), \tag{54}$$

where it is assumed that $\beta>0$ and

$$\lim_{j\to\alpha(t)} [j-\alpha(t)]^\beta B(t,j) = 0. \tag{55}$$

In general, the function B may still have a branch point at $j=\alpha(t)$, but it should be less hard than the dominant one. Branch-point surfaces of logarithmic character can be handled analogously, and we will not discuss them here explicitly.

With the ansatz (54) and the unitarity condition (4), we obtain in the limit $j\to\alpha(t)$

$$[\alpha(t)-\alpha^{\rm II}(t)]^\beta = 2i\rho(t)(t-t_0)^{\alpha(t)}C^{\rm II}(t), \tag{56}$$

with

$$C^{\rm II}(t) = A^{\rm II}(t)[1-2i\rho(t)(t-t_0)^{\alpha(t)}B^{\rm II}(t,\alpha(t))]^{-1}. \tag{57}$$

We assume here that there are no additional branch-point surfaces present in $G(t,j)$ which coincide with $\alpha(t)$ at $t=t_0$. In particular, there should be no shielding cuts. Then we obtain from the unitarity relation in the limit $j\to\alpha^{\rm II}(t)$

$$[\alpha^{\rm II}(t)-\alpha(t)]^\beta = -2i\rho(t)(t-t_0)^{\alpha^{\rm II}(t)}C(t), \tag{58}$$

with

$$C(t) = \frac{A(t)}{1+2i\rho(t)(t-t_0)^{\alpha^{\rm II}(t)}B(t,\alpha^{\rm II}(t))}. \tag{59}$$

From the limit $t\to t_0$ of Eqs. (56) and (58) we obtain the condition

$$\lim_{t\to t_0}\frac{[\alpha(t)-\alpha^{\rm II}(t)]^\beta}{[\alpha^{\rm II}(t)-\alpha(t)]^\beta} = \lim_{t\to t_0}\left(-\frac{C^{\rm II}(t)}{C(t)}\right). \tag{60}$$

For general $\beta>0$, this requirement is clearly not compatible with simple forms of the function $C(t)$, for example $C(t_0)=C^{\rm II}(t_0)=$ const. Roughly, we must have for $t\to t_0$

$$\frac{C^{\rm II}(t)}{C(t)} \to -(-1)^\beta, \tag{61}$$

which indicates that a β-dependent branch point of $C(t)$ at $t=t_0$ is required.

In order to solve Eqs. (56) and (58) explicitly, we make the ansatz

$$\begin{aligned}&\lim_{t\to t_0}[\alpha(t)-\alpha^{\rm II}(t)] \simeq \frac{2ia_0}{\sqrt{t_0}}(t-t_0)^{\lambda+1/2},\\ &\lim_{t\to t_0} C(t) \simeq C_0(t-t_0)^\sigma(t_0-t)^\gamma,\end{aligned} \tag{62}$$

so that for $t\to t_0$, $t\geq t_0$

$$C(t\pm i0) = C_0(t-t_0)^{\sigma+\gamma}e^{\mp i\pi\gamma}.$$

Then Eqs. (56) and (58) reduce to

$$\left(\frac{2a_0}{\sqrt{t_0}}\right)^\beta e^{\pm i\pi\beta/2}(t-t_0)^{(\lambda+1/2)\beta} = \frac{2C_0}{\sqrt{t_0}}e^{\pm i\pi(\gamma+1/2)}(t-t_0)^{\alpha(t_0)+1/2+\sigma+\gamma}, \tag{63}$$

which implies, with $n=0,\pm1,\pm2,\ldots,$

$$\begin{aligned}\gamma &= \tfrac{1}{2}(\beta-1)+n,\\ \lambda &= -\tfrac{1}{2}+\frac{1}{\beta}[\alpha(t_0)+\tfrac{1}{2}+\sigma+\gamma]\\ &= [\alpha(t_0)+\sigma+n]\beta^{-1},\\ \frac{2C_0}{\sqrt{t_0}} &= e^{i\pi n}\left(\frac{2a_0}{\sqrt{t_0}}\right)^\beta.\end{aligned} \tag{64}$$

It is relevant that the parameter γ is determined by the character of the branch-point surface $j=\alpha(t)$. Hence the function $C(t)$ must have a β-independent branch point at the threshold $t=t_0$ with a cut drawn to the right. The parameter σ is relatively free, *a priori*. It allows changing the character of the threshold in $\alpha(t)$ at the cost of a left-hand branch line in $C(t)$. No such left-hand branch line has been introduced in the ansatz for $\alpha(t)$, since this would correspond to allowing additional branch-point surfaces in $G(t,j)$ which coincide at $t=t_0$.

Let us suppose, at first, that $B(t,\alpha^{\rm II}(t))$ is not too singular for $t\to t_0$ so that the properties of $C(t)$ are those of $A(t)$ in this limit. Now

$$A(t) = \lim_{j\to\alpha(t)}[j-\alpha(t)]^\beta G(t,j), \tag{65}$$

and since $\alpha(t)$ and $G(t,j)$ do not have left-hand cuts starting at $t=t_0$, we expect $\sigma=0$ for $A(t)$. Then the equations (64) require

$$A(t) \simeq A_0(t_0 - t)^{(\beta-1)/2+n} ,$$
$$\alpha(t) \simeq \alpha(t_0) + \text{const} \times (t_0 - t)^{[\alpha(t_0)+n]/\beta+1/2} . \quad (66)$$

Since $G(t,j)$ has no β-dependent branch points at $t=t_0$, these aspects of the cuts in $A(t)$ and $\alpha(t)$ must cancel in G. Generally, such cancellations require the help of shielding and hiding cuts, as we have seen in the first part of this section.

The properties of $C(t)$ required by unitarity may also depend upon $B(t, \alpha^{\text{II}}(t))$, provided this function is sufficiently singular for $t \to t_0$. A detailed discussion shows again that a complicated branch-point structure of the amplitude is necessary for $t \to t_0$, which generally requires shielding. (The special case $\beta = 2$, corresponding to a double pole, will be discussed elsewhere by S. Paranjape.)

V. CONCLUDING REMARKS

There is no problem with elastic t-channel unitarity if only ordinary Regge-pole trajectories and soft branch-point surfaces are present in the complex angular momentum plane of the amplitude $G(t,j)$. The first-order pole surfaces have threshold branch points and move into secondary Riemann sheets as expected for resonance poles. The Regge cuts are generated by the pole surfaces via inelastic t-channel unitarity as demonstrated by the weak-coupling limit of the Reggeon calculus.[11] These branch-point trajectories are soft and have nothing to do with elastic unitarity.

For pole trajectories with left-hand branch lines due to the crossover of a finite number of branches, we can still satisfy elastic unitarity with the help of thresholds in the trajectory functions, although we have seen that there are important restrictions concerning the detailed properties of the threshold branch points of the complex trajectory functions.

The situation concerning elastic t-channel unitarity is completely different for hard branch-point trajectories, whether they have left-hand cuts or not. For these surfaces a threshold branch point is generally not sufficient to make the amplitude compatible with elastic unitarity. On the contrary, such branch points generate considerable complications with the analytic properties of the amplitude $G(t,j)$. On the other hand, the introduction of shielding cuts in the j plane is a very efficient way of satisfying unitarity.[7] There is then no need for any threshold singularity of the hard branch-point trajectory. We find that elastic t-channel unitarity implies shielding cuts for hard branch-point surfaces, while, similarly, inelastic unitarity generates the familiar multi-Regge cuts and Regge-particle cuts.

It is quite possible that the Pomeron is actually a hard branch-point surface,[4] or a structure involving such branch points.[3] In particular, singularities of this type are to be expected if the total cross section should continue to rise indefinitely. Our results show then that shielding cuts are required. These shielding cuts may well be phenomenologically relevant. (An explicit model will be described elsewhere by S. Paranjape.)

On the other hand, if one assumes that there are *no* shielding cuts, but only the usual branch-point trajectories generated by multiparticle unitarity, then our results indicate that the Pomeron must be a pole trajectory. This situation corresponds to a weak coupling limit in the Reggeon calculus, and it may lead to difficulties with decoupling theorems.

Although we have considered in this paper only elastic unitarity, our discussions can be extended to more general two-particle thresholds and coupled channels. The shielding and factorization problem of hard branch-point surfaces has been discussed previously.[12]

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[1]R. Oehme, in *Strong Interactions and High Energy Physics*, edited by R. G. Moorhouse (Oliver and Boyd, London, 1964), pp. 129–222. (This paper contains further references.)

[2]V. N. Gribov, Zh. Eksp. Teor. Fiz. 53, 654 (1967) [Sov. Phys.—JETP 26, 414 (1968)].

[3]A. A. Migdal, A. M. Polyakov, and K. A. Ter-Martirosyan, Phys. Lett. 48B, 239 (1974); H. D. I. Abarbanel and J. B. Bronzan, *ibid.* 48B, 345 (1974); J. B. Bronzan and J. W. Dash, NAL Report No. 74/53 THY, 1974 (unpublished); M. Baker, Phys. Lett. 51B, 158 (1974).

[4]R. Oehme, lecture presented at the Summer Institute for Theoretical Physics, Hamburg, 1971; in *Springer Tracts in Modern Physics*, edited by G. Höhler (Springer, Berlin, 1972), Vol. 61, pp. 109–119 (this paper contains further references).

[5]H. Cheng and T. T. Wu, Phys. Rev. Lett. 24, 1456

(1970); G. Auberson, T. Kinoshita, and A. Martin, Phys. Rev. D 3, 3185 (1971); J. Finkelstein and F. Zachariasen, Phys. Lett. 34B, 407 (1971); R. Oehme, Nuovo Cimento 25, 183 (1962), Phys. Rev. D 4, 1485 (1971); P. G. O. Freund and R. Oehme, Phys. Rev. Lett. 10, 415 (1963).

[6]R. Oehme, Phys. Rev. Lett. 9, 358 (1962); J. B. Bronzan and C. E. Jones, Phys. Rev. 160, 1494 (1967).

[7]R. Oehme, Phys. Rev. D 9, 2695 (1974); Phys. Rev. Lett. 18, 1222 (1967).

[8]C. E. Jones and V. Teplitz, Phys. Rev. 158, 1271 (1967); S. Mandelstam and L.-L. Wang, *ibid.* 160, 1490 (1967); D. Horn and F. Zachariasen, *Hadron Physics at Very High Energies* (Benjamin, Reading, Mass., 1973); M. Creutz, F. E. Paige, and L.-L. Wang, Phys. Rev. Lett. 30, 343 (1973); D. Branson, Nuovo Cimento 15A, 217 (1973); R. Oehme, in *Springer Tracts in Modern Physics*, edited by G. Höhler (Springer, Berlin, 1971), Vol. 57, pp. 132–157.

[9]V. N. Gribov and I. Ya. Pomeranchuk, Nucl. Phys. 38, 516 (1962); A. O. Barut and D. E. Zwanziger, Phys. Rev. 127, 974 (1962); J. R. Taylor, *ibid.* 127, 2257 (1962).

[10]R. Oehme, Phys. Rev. D 2, 801 (1970).

[11]V. N. Gribov and A. A. Migdal, Yad. Fiz. 8, 1002, (1968) [Sov. J. Nucl. Phys. 8, 583 (1969)]; R. Jengo, Phys. Lett. 51B, 143 (1974).

[12]R. Oehme and S. Paranjape, Phys. Rev. D 10, 2229 (1974).