Three-point functions, chiral symmetry, and the Wilson expansion*

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We have extended a previous analysis of two-point functions, which was based on the $(3, \bar{3})$ model of chiral symmetry breaking and the Wilson short-distance expansion, to the three-point functions of the model. We obtain additional consistency requirements which now determine all parameters of the model in terms of the pseudoscalar masses plus $F_K/F_\pi f_+(0)$, as determined from K_{l3}, K_{l2}, and π_{l2} decay. λ_0 (the slope of the scalar K_{l3} form factor) is predicted to be $\lambda_0 \simeq 0.016$. From this solution, we calculate the rate for $\eta' \to \eta\pi\pi$, but find a small result, in accord with previous calculations. We also find one relation between masses of the two scalar, isoscalar particles of the model.

I. INTRODUCTION

In a previous paper[1] we utilized the Wilson short-distance expansion[2] to reanalyze the various sum rules in the $(3,\bar{3})$ model of chiral symmetry breaking from a unified point of view. In addition to confirming the validity of the older results,[3] we found that a new set of sum rules should be valid. By using these plus some plausible assumptions about two remaining parameters, we found excellent agreement with the experimental values of $F_K/F_\pi f_+(0)$ and λ_0 (the slope of the scalar form factor in K_{l3} decay). We were also able to estimate $\Gamma(\eta' \to 3\pi)$ and $\Gamma(\eta' \to 2\gamma)$ and found results within the present experimental bounds on these numbers.

The present paper extends our analysis to include the three-point functions. In order to construct a consistent solution, we find that an additional constraint must be satisfied so that our solution is now completely determined by the pseudoscalar masses and $F_K/F_\pi f_+(0)$. In Sec. II we review the notation used and summarize the previous sum rules. In Sec. III we consider the $(\kappa K\pi)$ vertex to illustrate the methods used and also to determine the leading asymptotic behavior of two-point functions. This will be needed in the subsequent sections. Section IV is an analysis of the vertex relevant to the η-η' σ term from which we can calculate $\Gamma(\eta' \to \eta\pi\pi)$. At this point we assume zero wave-function mixing, which was our approximation in the previous paper.[1] However, in Sec. V, by analyzing the $(\eta\kappa K)$ vertex, we find an additional constraint on this mixing, which, when combined with the value of $F_K/F_\pi f_+(0)$, yields small but nonzero mixing. In Sec. VI we look again at the analysis of Sec. IV and find constraints on the remaining scalar-meson masses. Finally, in Sec. VII we summarize our results and discuss some difficulties of the model along with possible extensions of it.

II. NOTATION AND SUM RULES

We use the following definitions of the PCAC (partial conservation of axial-vector current) constants and the wave-function normalization constants:

$$\begin{aligned}&\langle 0|A_\mu^{1-3}|\pi\rangle = ip_\mu F_\pi\,, \quad \langle 0|A_\mu^{4-7}|K\rangle = ip_\mu F_K\,,\\&\langle 0|A_\mu^{8}|\eta\rangle = ip_\mu F_y\cos\chi\,, \quad \langle 0|A_\mu^{8}|\eta'\rangle = -ip_\mu F_y\sin\chi\,,\\&\langle 0|V_\mu^{4}|\kappa\rangle = ip_\mu F_\kappa\,,\end{aligned} \tag{1}$$

and

$$\begin{aligned}&\langle 0|v_{1-3}|\pi\rangle = \sqrt{Z_\pi}\,, \quad \langle 0|v_{4-7}|K\rangle = \sqrt{Z_K}\,,\\&\langle 0|v_8|\eta\rangle = \sqrt{Z_8}\cos\varphi\,, \quad \langle 0|v_8|\eta'\rangle = -\sqrt{Z_8}\sin\varphi\,,\\&\langle 0|v_0|\eta\rangle = \sqrt{Z_0}\sin\theta\,, \quad \langle 0|v_0|\eta'\rangle = \sqrt{Z_0}\cos\theta\,,\\&\langle 0|u_{4-7}|\kappa\rangle = \sqrt{Z_\kappa}\,.\end{aligned} \tag{2}$$

In terms of these definitions, three sets of sum rules can be obtained. From PCAC and Eqs. (1) and (2) only, we find

$$\begin{aligned}&\frac{m_\pi^2F_\pi}{\sqrt{Z_\pi}} - \frac{m_K^2F_K}{\sqrt{Z_K}} = \frac{m_\kappa^2F_\kappa}{\sqrt{Z_\kappa}}\,,\\&\frac{m_\kappa^2F_\kappa}{\sqrt{Z_\kappa}} = \frac{3}{2\sqrt{2}}\frac{F_y}{\sqrt{Z_0}\cos(\theta-\varphi)}\\&\qquad\times(m_\eta^2\cos\chi\sin\varphi - m_{\eta'}^2\sin\chi\cos\varphi)\,,\\&\frac{4m_K^2F_K}{\sqrt{Z_K}} - \frac{m_\pi^2F_\pi}{\sqrt{Z_\pi}} = \frac{3F_y}{\sqrt{Z_8}\cos(\theta-\varphi)}\\&\qquad\times(m_\eta^2\cos\chi\cos\theta + m_{\eta'}^2\sin\chi\sin\theta)\,.\end{aligned} \tag{3}$$

By removing the $(3,\bar{3})\oplus(\bar{3},3)$ combination from $\langle 0|T(J_\mu^\alpha(x)\varphi^\beta(0))|0\rangle$, we obtain

$$\sqrt{Z_\pi}\,F_\pi-\sqrt{Z_K}\,F_K=\sqrt{Z_\kappa}\,F_\kappa\,,$$
$$\sqrt{Z_\kappa}\,F_\kappa=\frac{3}{2\sqrt{2}}F_y\sqrt{Z_0}\,\sin(\theta-\chi)\,, \tag{4}$$
$$4\sqrt{Z_K}\,F_K-\sqrt{Z_\pi}\,F_\pi=3F_y\sqrt{Z_8}\,\cos(\varphi-\chi)\,.$$

Finally by removing the (1, 1) and $(3,\bar{3})\oplus(\bar{3},3)$ combinations from $\langle 0|T(\varphi^\alpha(x)\varphi^\beta(0))|0\rangle$, it follows that

$$4Z_\pi-Z_K+6Z_0=9Z_\kappa\,,$$
$$Z_K-Z_\pi=\frac{3}{\sqrt{2}}\sqrt{Z_0}\,\sqrt{Z_8}\,\sin(\theta-\varphi)\,, \tag{5}$$
$$4Z_K-Z_\pi=3Z_8\,.$$

It should be recalled that Eqs. (4) and (5) involve single-particle saturation of exact sum rules. For a detailed derivation of Eqs. (4) and (5) we refer the reader to Ref. 1. In addition to Eq. (5), the wave-function normalization constants for the other scalar particles can be written in terms of Z_K, Z_π, Z_0, Z_8, Z_κ, θ and φ. These results will be quoted when needed in Sec. VI where we discuss the remaining scalar mesons. Finally, we recall that in the approximation where $\theta\simeq\varphi\simeq 0$, Eqs. (3), (4), and (5) have the solutions

$$\frac{Z_K}{Z_\pi}=\frac{Z_8}{Z_\pi}=1\,,\ \frac{Z_0}{Z_\pi}=0.86\,,\ \frac{Z_\kappa}{Z_\pi}=0.91\,,$$
$$\frac{F_K}{F_\pi}=1.28\,,\ \frac{F_y}{F_\pi}=1.40\,,\ \frac{F_\kappa}{F_\pi}=-0.29\,,$$
$$\chi=11.6^\circ,\ m_\kappa=985\text{ MeV}\,.$$

From these one can calculate

$$\frac{F_K}{F_\pi f_+(0)}\simeq 1.28\,,\ \lambda_0\simeq 0.017\,.$$

For the conventional measures of symmetry breaking, this solution gives $(H=H_0+\epsilon_0 u_0+\epsilon_8 u_8)$

$$C\equiv\epsilon_8/\epsilon_0=-1.29$$

and

$$C_V\equiv\langle 0|u_8|0\rangle/\langle 0|u_0|0\rangle=-0.22\,.$$

III. $\kappa K\pi$ VERTEX

The advantage of treating the $(\kappa K\pi)$ vertex first is that in addition to using the short-distance expansions, we have low-energy theorems available in all three variables. We define

$$S(q^2,k^2,p^2)\equiv\int d^4x\,d^4y\,e^{iqx}e^{-iky}\times\langle 0|T(v_3(x)v_4(y)u_4(0))|0\rangle\,. \tag{6}$$

Using the PCAC relations, we obtain the following three low-energy theorems:

$$S(0,l^2,l^2)=\frac{1}{2}\frac{\sqrt{Z_\pi}}{m_\pi^2F_\pi}[D^v_{44}(l^2)-D^u_{44}(l^2)]\,,$$
$$S(l^2,0,l^2)=\frac{1}{2}\frac{\sqrt{Z_K}}{m_K^2F_K}[D^v_{33}(l^2)-D^u_{44}(l^2)]\,, \tag{7}$$
$$S(l^2,l^2,0)=\frac{1}{2}\frac{\sqrt{Z_\kappa}}{m_\kappa^2F_\kappa}[D^v_{44}(l^2)-D^v_{33}(l^2)]\,,$$

where

$$D^v_{ij}(l^2)=i\int d^4x\,e^{ilx}\langle 0|T(v_i(x)v_j(0))|0\rangle\,,$$

and (8)

$$D^u_{ij}(l^2)=i\int d^4x\,e^{ilx}\langle 0|T(u_i(x)u_j(0))|0\rangle\,.$$

We now consider the limits on $S(q^2,k^2,p^2)$, where ($q^2\to\infty$, k^2 fixed), ($k^2\to\infty$, q^2 fixed), and ($q^2\to\infty$, p^2 fixed), which imply ($x\sim 0$), ($y\sim 0$), and ($x\sim y$), respectively. These limits involve the short-distance expansion of products like $u_i(x)v_j(0)$. Since these were analyzed in Ref. 1, we quote the results here.

In general we can write

$$u_i(x)u_j(0)=\delta_{ij}A(x)+\tfrac{1}{8}D_{ijk}U_k(x)+\cdots\,,$$
$$v_i(x)v_j(0)=\delta_{ij}A(x)-\tfrac{1}{8}D_{ijk}U_k(x)+\cdots\,, \tag{9}$$
$$v_i(x)u_j(0)=-\tfrac{1}{8}D_{ijk}V_k(x)+\cdots\,,$$
$$u_i(x)v_j(0)=-\tfrac{1}{8}D_{ijk}V_k(x)+\cdots\,,$$

where

$$D_{ijk}=2d_{ijk}-3(\tfrac{2}{3})^{1/2}(\delta_{i0}\delta_{jk}+\delta_{j0}\delta_{ik}+\delta_{k0}\delta_{ij})+9(\tfrac{2}{3})^{1/2}\delta_{i0}\delta_{j0}\delta_{k0}\,.$$

Equation (9) is simply a group-theory expansion of the products. That is,

$$A(x)=\tfrac{1}{18}\sum_i[u_i(x)u_i(0)+v_i(x)v_i(0)]$$

and is the (1, 1) combination of the u's and v's. Likewise, $U_i(x)$ and $V_i(x)$ represent the $(3,\bar{3})\oplus(\bar{3},3)$ combinations, and the "$+\cdots$" stands for the other possible representations. Equation (9) is an algebraic identity. Now, however, the Wilson analysis[2] is a procedure for determining the short-distance behavior of $A(x)$, $U_i(x)$, etc. In the $(3,\bar{3})\oplus(\bar{3},3)$ model, these take the form

$$A(x)\sim g(x)I+\sum_i\alpha_i(x)u_i(0)+\cdots\,,$$
$$U_i(x)\sim\beta_i(x)I+f(x)u_i(0)+\cdots\,, \tag{10}$$
$$V_i(x)\sim\gamma_i(x)I+f(x)v_i(0)+\cdots\,,$$

where the leading singularities of the coefficients are

$$g(x) \sim \frac{a}{x^{2\Delta}}, \quad \alpha_i(x) \sim \frac{\alpha_i}{x^{2\Delta-4}},$$
$$\beta_i(x) \sim \frac{\beta_i}{x^{3\Delta-4}}, \quad \gamma_i(x) \sim \frac{\gamma_i}{x^{3\Delta-4}}, \quad f(x) \sim \frac{b}{x^{\Delta}}, \tag{11}$$

and Δ is the dimension of the $\{u_i, v_i\}$ fields. Terms which have been neglected in the expansion are less singular. The sum rules of Eq. (5) are valid provided that $\Delta < \frac{5}{2}$, and we shall assume that this inequality is satisfied.

Since the identity operator (I) does not contribute to the three-point expansions and the next leading term is $f(x)$, we find from Eq. (6) that with l^2 large and k^2 fixed

$$S(l^2, k^2, l^2) \sim \frac{i}{8} D^v_{44}(k^2) \tilde{f}(l^2),$$
$$S(k^2, l^2, l^2) \sim \frac{i}{8} D^v_{33}(k^2) \tilde{f}(l^2), \tag{12}$$
$$S(l^2, l^2, k^2) \sim \frac{i}{8} D^u_{44}(k^2) \tilde{f}(l^2),$$

where $\tilde{f}(l^2)$ is the Fourier transform of $f(x)$.

We are now in a position to compare Eq. (7) and Eq. (12). Evidently, if we set $k^2 = 0$ in Eq. (12) and let l^2 get large in Eq. (7), they should agree. Referring to Eq. (7) and Eq. (9), we see that the $A(x)$ term will not contribute to the limit because Eq. (7) always involves differences of diagonal terms. Also, since we take vacuum expectation values, only $\langle 0|u_0|0\rangle \equiv C_0$ and $\langle 0|u_8|0\rangle \equiv C_8$ contribute.[4] Using $D^v_{44}(0) = Z_K/m_K^2$, $D^v_{33}(0) = Z_\pi/m_\pi^2$, $D^u_{44}(0) = Z_\kappa/m_\kappa^2$, which are consistent with our saturation approximation, and comparing Eq. (7) and Eq. (12), we find

$$\sqrt{Z_\pi}\, F_\pi = (\tfrac{2}{3})^{1/2} C_0 + \frac{1}{\sqrt{3}} C_8,$$
$$\sqrt{Z_K}\, F_K = (\tfrac{2}{3})^{1/2} C_0 - \frac{1}{2\sqrt{3}} C_8, \tag{13}$$
$$\sqrt{Z_\kappa}\, F_\kappa = \frac{\sqrt{3}}{2} C_8.$$

From Eq. (13) it follows that

$$C_0 = \frac{1}{\sqrt{6}} (\sqrt{Z_\pi}\, F_\pi + 2\sqrt{Z_K}\, F_K),$$
$$C_8 = \frac{2}{\sqrt{3}} (\sqrt{Z_\pi}\, F_\pi - \sqrt{Z_K}\, F_K). \tag{14}$$

Note that the relation among $\sqrt{Z_\pi}\, F_\pi$, $\sqrt{Z_K}\, F_K$, and $\sqrt{Z_\kappa}\, F_\kappa$ which is implied by Eq. (13) agrees with Eq. (4), which was obtained by an entirely different approach.

Thus we find that our low-energy theorems are consistent with the short-distance expansion for the $(\kappa K \pi)$ vertex and lead to a determination of C_0 and C_8 which will be needed in the succeeding section. Our aim will be to achieve a form like Eq. (7) for other three-point functions in cases where we do not have all three low-energy theorems. From equations like Eq. (7), a complete low-energy expansion of the form

$$S(q^2, k^2, p^2) \simeq \frac{a + bq^2 + ck^2 + dp^2}{(q^2 - m_\pi^2)(k^2 - m_K^2)(p^2 - m_\kappa^2)} \tag{15}$$

can be constructed.

IV. THE η-η' VERTICES

In this section we analyze the vertices

$$\int e^{iqx} d^4x\, e^{-iky} d^4y \langle 0| T(v_{8,0}(x) v_{8,0}(y) u_{8,0}(0))|0\rangle, \tag{16}$$

which we label S_{888}, S_{880}, ... etc. Since our only low-energy theorem involves $\partial_\mu A^\mu{}_8 = \alpha v_8(x) + \beta v_0(x)$, we cannot obtain a complete set of equations like Eq. (7) from low-energy theorems alone. Evaluated at $(0, q^2, q^2)$, the low-energy results are

$$\alpha S_{888} + \beta S_{088} = \frac{-1}{\sqrt{3}} [D^v_{88}(q^2) - D^u_{88}(q^2) - \sqrt{2}\, D^v_{80}(q^2) + \sqrt{2}\, D^u_{80}(q^2)],$$
$$\alpha S_{808} + \beta S_{008} = (\tfrac{2}{3})^{1/2} [D^V_{00}(q^2) - D^u_{88}(q^2) - (1/\sqrt{2})\, D^v_{80}(q^2)],$$
$$\alpha S_{880} + \beta S_{080} = (\tfrac{2}{3})^{1/2} [D^v_{88}(q^2) - D^u_{00}(q^2) + (1/\sqrt{2})\, D^u_{80}(q^2)],$$
$$\alpha S_{800} + \beta S_{000} = (\tfrac{2}{3})^{1/2} [D^v_{80}(q^2) - D^u_{80}(q^2)], \tag{17}$$

where

$$\alpha = \frac{F_y}{\sqrt{Z_8}\cos(\theta - \varphi)} (m_\eta^2 \cos\chi \cos\theta + m_{\eta'}^2 \sin\chi \sin\theta),$$
$$\beta = \frac{F_y}{\sqrt{Z_0}\cos(\theta - \varphi)} (m_\eta^2 \cos\chi \sin\varphi - m_{\eta'}^2 \sin\chi \cos\varphi).$$

On the other hand, if we let k^2 become large and set $q = 0$ in Eq. (16) and make use of Eq. (9), we find (interchanging k and q in the notation)

$$S_{i88}(0, q^2, q^2) \sim -\frac{i}{8} (\tfrac{2}{3})^{1/2} [D^v_{i0}(0) + \sqrt{2}\, D^v_{i8}(0)] \tilde{f}(q^2),$$
$$S_{i08}(0, q^2, q^2) \sim -\frac{i}{8} (\tfrac{2}{3})^{1/2} D^v_{i8}(0) \tilde{f}(q^2),$$
$$S_{i80}(0, q^2, q^2) \sim -\frac{i}{8} (\tfrac{2}{3})^{1/2} D^v_{i8}(0) \tilde{f}(q^2), \tag{18}$$
$$S_{i00}(0, q^2, q^2) \sim \frac{i}{4} (\tfrac{2}{3})^{1/2} D^v_{i0}(0) \tilde{f}(q^2),$$

where the subscript i assumes the values 0 and 8. Comparison of Eq. (17) and Eq. (18) suggests that we write

$$S_{i88}(0,q^2,q^2)=-\frac{i}{8}(\tfrac{2}{3})^{1/2}[D^v_{i0}(0)+\sqrt{2}D^v_{i8}(0)][D^v_{88}(q^2)-D^u_{88}(q^2)-\sqrt{2}D^v_{80}(q^2)+\sqrt{2}D^u_{80}(q^2)]/N_1\,,$$
$$S_{i08}(0,q^2,q^2)=-\frac{i}{8}(\tfrac{2}{3})^{1/2}D^v_{i8}(0)[D^v_{00}(q^2)-D^u_{88}(q^2)-(1/\sqrt{2})D^v_{80}(q^2)]/N_2\,,$$
$$S_{i80}(0,q^2,q^2)=-\frac{i}{8}(\tfrac{2}{3})^{1/2}D^v_{i8}(0)[D^v_{88}(q^2)-D^u_{00}(q^2)+(1/\sqrt{2})D^u_{80}(q^2)]/N_3\,, \tag{19}$$
$$S_{i00}(0,q^2,q^2)=\frac{i}{4}(\tfrac{2}{3})^{1/2}D^v_{i0}(0)[D^v_{80}(q^2)-D^u_{80}(q^2)]/N_4\,,$$

where the N_j are to be chosen such that, for example,

$$[D^v_{80}(q^2)-D^u_{80}(q^2)]/N_4\to\tilde{f}(q^2)\text{ as }q^2\to\infty\,.$$

Using the asymptotic forms of the $D_{ij}(q^2)$ as determined by Eq. (14) and Eq. (9), we find that

$$N_1=\frac{i}{12}(2\sqrt{Z_K}F_K+\sqrt{Z_\pi}F_\pi)\,,$$
$$N_2=N_3=-\frac{i}{12}(2\sqrt{Z_K}F_K-\tfrac{1}{2}\sqrt{Z_\pi}F_\pi)\,, \tag{20}$$
$$N_4=-\frac{2i\sqrt{2}}{12}(\sqrt{Z_K}F_K-\sqrt{Z_\pi}F_\pi)\,.$$

Note that Eq. (19) now satisfies Eq. (18) by choice of the N_j. That it also satisfies Eq. (17) is a nontrivial constraint. However, it can be shown by use of the sum rules, Eq. (4), that, for example,

$$-\frac{i}{8}[\alpha D^v_{88}(0)+\beta D^v_{80}(0)]=N_3$$

as required. In fact, Eq. (17) is completely satisfied.

Since $S_{ijk}(0,q^2,q^2)=S_{jik}(q^2,0,q^2)$ by symmetry, we have only to construct $S_{ijk}(q^2,q^2,0)$ in order to have a complete low-energy specification of $S_{ijk}(q^2,k^2,p^2)$. Rather than construct this function for all possible cases, we now assume that $\theta=\varphi\simeq0$ so that $D^v_{80}(0)=0$, i.e., we have no η-η' wave-function mixing. Similarly, we assume $D^u_{80}(0)=0$. We also look explicitly only at the matrix elements relevant to the η-η' σ term in order to calculate $\Gamma(\eta'\to\eta\pi\pi)$. For this we need

$$\lim_{q^2\to m_{\eta'}^2}\lim_{k^2\to m_\eta^2}(q^2-m_{\eta'}{}^2)(k^2-m_\eta{}^2)[S_{088}(q^2,k^2,p^2)+\sqrt{2}S_{080}(q^2,k^2,p^2)]_{p^2=0}\,,$$

which is proportional to the σ term. However, we note from Eq. (19) that both $S_{080}(0,q^2,q^2)$ and $S_{080}(q^2,0,q^2)$ are proportional to $D^v_{80}(0)$ and hence are zero in this approximation. In addition

$$\lim_{q^2\to\infty,\,p^2\text{ fixed}}S_{080}\sim S_{080}(q^2,q^2,p^2)\sim D^v_{80}(p^2)\sim0\,.$$

Therefore, in our approximation we choose $S_{080}(q^2,k^2,p^2)\simeq0$ in the low-energy region.

To construct $S_{088}(q^2,q^2,0)$, we first note that from the short-distance expansion (q^2 large)

$$S_{088}(q^2,q^2,0)\sim-\frac{i}{8}(\tfrac{2}{3})^{1/2}D^u_{88}(0)\tilde{f}(q^2)\,.$$

From considering the quantum numbers involved, we expect that

$$S_{088}(q^2,q^2,0)=-\frac{i}{8}(\tfrac{2}{3})^{1/2}D^u_{88}(0)\times[D^v_{88}(q^2)-xD^v_{00}(q^2)-yD^v_{80}(q^2)]/N\,.$$

Since we want the $A(x)$ piece of our short-distance expansion to cancel, we choose $x=1$. y and N are determined by the requirements that

1. $[D^v_{88}(q^2)-D^v_{00}(q^2)-yD^v_{80}(q^2)]/N\to\tilde{f}(q^2)$ for large q^2,

2. All three expressions for $S_{088}(0,0,0)$ agree.

From these we find $y=3/\sqrt{2}$ and $N=\frac{1}{12}i(4/\sqrt{Z_K}F_K+\frac{1}{2}\sqrt{Z_\pi}F_\pi)$. Note that although we assume $D^v_{80}(q^2)\simeq0$ for small q^2, we do not neglect its asymptotic contribution to $S_{088}(q^2,q^2,0)$ for large q^2.

We are now in a position to find an approximation for $S_{088}(q^2,k^2,p^2)$ except for the unknown function $D^u_{88}(q^2)$. We assume that this function has the form $D^u_{88}\simeq X_8/(m_8{}^2-q^2)$ and note that $X_8/m_8{}^2=D^u_{88}(0)$ is determined by the $S_{088}(0,0,0)$ constraint mentioned above. We are thus left with one free parameter. A straightforward but tedious calculation yields the following result for the σ term:

$$\sigma_{\eta\eta'}/F_\pi{}^2=0.29-(0.34\text{ GeV}^2)/m_8{}^2\,. \tag{21}$$

Using the result[5-8]

$$\Gamma(\eta'\to\eta\pi\pi)\approx14.8(\sigma_{\eta\eta'}/F_\pi^2)^2\text{ keV}\,,$$

we find $2\lesssim\Gamma\lesssim17$ keV for m_8 in the range $500\lesssim m_8\lesssim700$ MeV. Although $\Gamma(\eta'\to\eta\pi\pi)$ is not yet determined experimentally, this value seems rather small for a strong decay width. This prediction of an anomalously small decay width has been a feature of previous calculations based on the $(3,\bar{3})$ model of chiral symmetry breaking as well.[5,7,9,10]

V. THE VERTEX $\langle v_{0,8}v_4u_4\rangle$

This three-point function is of interest for two reasons: First, it has an almost complete set of low-energy theorems, and second, it will show that our approximation of $D^v_{80} \simeq 0$ at low q^2, i.e., $\theta \simeq \varphi \simeq 0$, is not consistent. Since the procedure for constructing approximations to three-point functions has been illustrated in the last two sections, we simply quote our results at this point.

From the low-energy theorems for v_4 and u_4, we find

$$S_8(q^2,0,q^2) = -\frac{\sqrt{Z_K}}{2\sqrt{3}\,m_K{}^2F_K}[D^v_{88}(q^2) - D^u_{44}(q^2) - 2\sqrt{2}\,D^v_{80}(q^2)],$$
$$S_0(q^2,0,q^2) = (2/3)^{1/2}\frac{\sqrt{Z_K}}{m_K{}^2F_K}[D^v_{00}(q^2) - D^u_{44}(q^2) - (1/2\sqrt{2})D^v_{80}(q^2)]; \tag{22}$$

$$S_8(q^2,q^2,0) = \frac{\sqrt{3}}{2}\frac{\sqrt{Z_K}}{m_K{}^2F_K}[D^v_{44}(q^2) - D^v_{88}(q^2)],$$
$$S_0(q^2,q^2,0) = -\frac{\sqrt{3}}{2}\frac{\sqrt{Z_K}}{m_K{}^2F_K}D^v_{80}(q^2). \tag{23}$$

Using the short-distance expansion, plus the low-energy theorem for A^8_μ, we find that

$$S_8(0,q^2,q^2) = -\frac{1}{2\sqrt{3}}\frac{[\sqrt{2}\,D^v_{80}(0) + D^v_{88}(0)]}{\sqrt{Z_\pi}\,F_\pi}\times[D^v_{44}(q^2) - D^u_{44}(q^2)],$$
$$S_0(0,q^2,q^2) = -\frac{1}{2}(2/3)^{1/2}\frac{[D^v_{00}(0) + (1/\sqrt{2})D^v_{80}(0)]}{\sqrt{Z_\pi}\,F_\pi}\times[D^v_{44}(q^2) - D^u_{44}(q^2)], \tag{24}$$

where the q^2 dependence follows from the low-energy theorem and the constant coefficients from the short-distance expansion. We again remark that the relative values of these constants are just such as to satisfy $\partial_\mu A^\mu_8 = \alpha v_8 + \beta v_0$.

If we now equate Eqs. (22), (23), and (24) at $q^2=0$, we obtain four equations in the three unknowns $D^v_{00}(0)$, $D^v_{80}(0)$, and $D^v_{88}(0)$. That these have a consistent solution is not obvious, but in fact they do, with that solution given by

$$D^v_{00}(0) = \frac{2}{3}\frac{(K-4\pi)\kappa}{\kappa+K-4\pi},$$
$$D^v_{80}(0) = -\frac{2\sqrt{2}}{3}\frac{(K-\pi)\kappa}{\kappa+K-4\pi}, \tag{25}$$
$$D^v_{88}(0) = \frac{1}{3}\frac{4K\kappa - 9K\pi - \kappa\pi}{\kappa+K-4\pi},$$

where we have abbreviated $Z_K/m_K{}^2$ by K, etc. for simplicity. Since these three results each contain θ and/or φ, we seem to have three new constraints to be used in addition to Eqs. (3), (4), and (5). However, simultaneous solution of Eqs. (3), (4), (5), and (25) shows that only one of the three relations in Eq. (25) is independent when used in conjunction with Eqs. (3), (4), and (5). Hence, this group of equations can now be solved given the pseudoscalar masses plus one additional piece of information. We choose this to be $F_K/F_\pi f_+(0) = 1.25 \pm 0.05$.[11,12] The results are given in Table I. We see that the mixing angles θ and φ are small ($\theta \approx -3°$, $\varphi \approx +3°$), the κ mass is in the range 950–1100 MeV, and the slope of the scalar K_{l3} form factor is $\lambda \approx 0.016$. This last result is in excellent agreement with the measurement of Donaldson *et al.*,[13] who find $\lambda_0 = 0.019 \pm 0.004$.

Clearly we should now return to Sec. IV with our new results and recalculate the η-η' σ term. Although now $D^v_{80}(q^2) \neq 0$ and thus $S_{080} \neq 0$, it is apparent that these quantities will be of order $\sin\theta$ or $\sin\varphi$. Since our new solution is quite close to the old one where $\theta = \varphi = 0$, these small changes cannot affect the order of magnitude of our previous result. Thus, we expect $\Gamma(\eta' \to \eta\pi\pi)$ to remain small. However, there is another reason for reinvestigating the equations of Sec. IV. We can find constraints on the remaining scalar-meson masses.

VI. SCALAR-MESON MASSES

In Sec. IV we used the symmetry of the arguments to find $S_{800}(q^2,0,q^2)$ from $S_{080}(0,q^2,q^2)$, but we did not investigate the implications of the fact that this demands

TABLE I. Simultaneous solution of Eqs. (3), (4), (5), and (25) as a function of the parameter $\alpha \equiv F_K/F_\pi f_+(0)$. The experimental value of $\alpha = 1.25 \pm 0.05$.

α	θ	φ	χ	m_κ (MeV)	λ_0	$f_+(0)$	$\frac{F_K}{F_\pi}$	$\frac{F_\kappa}{F_\pi}$	$\frac{F_y}{F_\pi}$	$\frac{Z_K}{Z_\pi}$	$\frac{Z_\kappa}{Z_\pi}$	$\frac{Z_0}{Z_\pi}$	$\frac{Z_8}{Z_\pi}$	C	C_V
1.21	−2°	5.09°	8.85°	1082	0.014	1.005	1.216	−0.188	1.30	0.874	0.533	0.279	0.832	−1.291	−0.118
1.25	−3°	3.35°	9.02°	1010	0.016	1.005	1.259	−0.233	1.36	0.874	0.578	0.347	0.832	−1.295	−0.149
1.30	−4°	1.58°	9.14°	952	0.018	1.005	1.307	−0.283	1.43	0.878	0.627	0.421	0.837	−1.299	−0.184

$$S_{800}(0,0,0)=S_{080}(0,0,0)$$

and (26)

$$S_{808}(0,0,0)=S_{088}(0,0,0)\,.$$

These equations involve $D^{v}_{88}(0)$, $D^{v}_{80}(0)$, and $D^{v}_{00}(0)$, which we now know, and, in addition, $D^{u}_{88}(0)$, $D^{u}_{80}(0)$, and $D^{u}_{00}(0)$. Equation (26) provides two constraints on these quantities. If we parameterize them in a way similar to the D^{v}'s (i.e., assuming two mixed isoscalar scalar mesons), then we would write

$$\langle 0|u_8|S\rangle=\sqrt{X_8}\cos\Phi\,,\ \langle 0|u_8|S'\rangle=-\sqrt{X_8}\sin\Phi\,,$$
$$\langle 0|u_0|S\rangle=\sqrt{X_0}\sin\Theta\,,\ \langle 0|u_0|S'\rangle=\sqrt{X_0}\cos\Theta\,, \tag{27}$$

so that

$$D^{u}_{00}(0)=X_0\left(\frac{\cos^2\Theta}{m'^2}+\frac{\sin^2\Theta}{m^2}\right),\ \text{etc}\,.$$

In addition, from the extension of Eq. (5) we can express X_0, X_8, Θ, and Φ in terms of previously determined quantities:

$$X_0=Z_K+Z_\kappa-Z_0\,,$$
$$X_8=Z_K+Z_\kappa-Z_8\,, \tag{28}$$
$$(X_0X_8)^{1/2}\sin(\Theta-\Phi)=-(Z_0Z_8)^{1/2}\sin(\theta-\varphi)\,.$$

Therefore, we have a total of five equations in the six unknowns $X_0, X_8, m, m', \Theta, \Phi$. Using the other parameters found in Sec. V, if we choose $m=600$ MeV, then we predict $m'\approx 1080$ MeV. Too little is known experimentally about the masses m and m' to test this result quantitatively. Further, it should be noted that the scalar meson which exists in the ρ region is similar to the κ meson in that they are both very wide. Our technique of using single-particle saturation may not yield very reliable mass or coupling estimates in such cases. Nevertheless, qualitatively the results are not unreasonable.

We have also constructed the $\langle u_3v_3(\alpha v_8+\beta v_0)\rangle$ vertex to see whether it yields a value for $D^{u}_{33}(0)$ and thus, through X_3 found from our extension of Eq. (5), a value for the mass of the isovector scalar meson. We find that the equations evaluated at $(0,0,0)$ do involve $D^{u}_{33}(0)$ but in such a way that they are satisfied independently of its value. Thus, no information about this meson mass can be obtained from this vertex.

VII. DISCUSSION

The results of the previous sections are encouraging. We have developed a simple yet elegant procedure for constructing three-point functions by combining the short-distance expansion and low-energy type expressions. The self-consistency requirements of the procedure are stringent, yet remarkably they lead to physically reasonable results. The extension of these results to other three-point functions should be straightforward. There remain, however, one technical and one theoretical problem to be solved before this can be done.

The technical problem involves how to treat the double-pole contributions in vertices such as $\langle v_3v_3u_8\rangle$ or even $\langle v_0v_8u_8\rangle$ (when mixing is included). We have tried several schemes which lead to consistent parameterizations. However, we have not found a mathematical or physical argument which would single out one as preferable to the others.

The theoretical problem stems from the fact that we have not been able to justify neglecting the $\beta_i(x)I$ term in Eq. (10) in the asymptotic behavior of the two-point functions. If it is included, we find an inconsistency in the asymptotic behavior of the three-point functions as found, first directly and second via a low-energy theorem with a subsequent limit taken on the two-point functions. We must conclude that either the $\beta_i(x)I$ terms vanish or that we have inadvertently dropped the compensating term from our direct expansion. In either case, the otherwise precise meshing of our consistency conditions leads us to conclude that our basic equations are probably correct.

It is unfortunate that even though our solution is so tightly constrained, very few of our results have direct application to measurable effects. Evidently, if one could complete the program for three-point functions, it would be attractive to pursue these ideas further and investigate the possibility of producing approximations to the four-point functions in the low-energy region.

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is easy to see that we expect $\gamma_i(x)=0$ by parity and only β_0 and β_8 to be nonzero from the form of the Lagrangian. However, we do not have a proof that β_0 and $\beta_8=0$, except to remark that the asymptotic forms of Eq. (7) and Eq. (12) would not be consistent if they were nonzero.

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