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Heavy quarks and strong binding: A field theory of hadron structure*

W. A. Bardeen,† M. S. Chanowitz,‡§ S. D. Drell,‡ M. Weinstein,‡ and T.-M. Yan‡||
Stanford University, Stanford, California 94305
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We investigate in canonical field theory the possibility that quarks may exist in isolation as very heavy particles, $M_{\text{quark}} >> 1$ GeV, yet form strongly bound hadronic states, $M_{\text{hadron}} \sim 1$ GeV. In a model with spin-$\frac{1}{2}$ quarks coupled to scalar gluons we find that a mechanism exists for the formation of bound states which are much lighter than the free constituents. Following Nambu, we introduce a color interaction mediated by gauge vector mesons to quarantee that all states with nonvanishing triality have masses much larger than 1 GeV. The possibility of such a solution to a strongly coupled field theory is exhibited by a calculation employing the variational principle in tree approximation. This procedure reduces the field-theoretical problem to a set of coupled differential equations for classical fields which are just the free parameters of the variational state. A striking property of the solution is that the quark wave function is confined to a thin shell at the surface of the hadronic bound state. Though the quantum corrections to this procedure remain to be investigated systematically, we explore some of the phenomenological implications of the trial wave functions so obtained. In particular, we exhibit the low-lying meson and baryon multiplets of SU(6); their magnetic moments, charge radii, and radiative decays, and the axial charge of the baryons. States of nonvanishing momenta are constructed and the softness of the hadron shell to deformations in scattering processes is discussed qualitatively along with the implications for deep-inelastic electron scattering and dual resonance models.

I. INTRODUCTION

The idea of quark constituents has been of very great importance in providing a simple, concrete model for describing and predicting the low-lying quantum states of hadrons and their observed properties.[1] Despite the successes of the quark model, one is puzzled as to why we do not see quarks. Are they nonexistent as isolated observable particles, or, once isolated from the extremely strong forces that bind them as effectively light and nonrelativistic constituents within hadronic matter of zero triality, are they very heavy so that their production thresholds lie beyond present accelerator energies?

Another puzzling feature of the quark model is the question of quark statistics. For example, the successful SU(6) classification of the ground state and low-lying spectrum for baryons is derived on the assumption that the three quarks bind in a totally symmetric state in space, spin, and SU(3) coordinates. To account for this apparent conflict with the requirement of antisymmetry for a state of three spin-$\frac{1}{2}$ fermions, an additional quantum number, labeled "color," is introduced.[2] It is then assumed that physical hadron states are color singlets—i.e., totally antisymmetric in the color quantum number for the three quarks (one red, one white, and one blue) forming the baryon.

The motivation of the present work is to construct a canonical field-theoretic model which accommodates these ideas and successes of the quark model in a consistent, systematic, and *calculable* way. We introduce quarks as the quanta of the fields and assume that there are nine quarks—an SU(3) triplet for each of the three-color states that form an SU(3)′ of color. The nonappearance of quarks will be interpreted in terms of a heavy mass for "bare quarks" ($\gg 1$ GeV). The large quark "bare mass" results from the strong coupling of the quark field with a neutral scalar field. This interaction provides the attraction binding quarks into bound states with masses corresponding to observed hadronic spectra. Formation of the bound state is traced in our approach to the "unconventional" dynamics of the scalar field which is specified so as to produce "spontaneous breakdown" of an underlying symmetry of the Hamiltonian. A strong color interaction mediated by gauge vector bosons pushes the color-nonsinglet states up to very high energies $\gg 1$ GeV while leaving the color-singlet states alone. Hadrons are formed as low-lying bound states of quarks in color-singlet, or zero-triality, states. Our color-singlet selection rule is thus an approximate one, as is the rule for nonappearance of bare quarks since these unwanted states have high mass. In contrast with the standard parton-model[3] approach which conceives of the nucleon as built of effectively free and light constituents in order to explain Bjorken scaling but rationalizes the embarrassment of not observing partons, we first tackle here, using canonical field theory, the puzzle of unseen quarks. Whether our resolution of this problem can explain why scaling is observed remains to be studied.

Evidently the problem of constructing bound states in a canonical field theory with strong couplings is a very difficult one. The progress we are reporting in this paper is based on a variational approach—i.e., we guess a trial form for the ground states and minimize the energy by a variational principle calculation.[4] The coupled classical field equations for the quark-wave functions and the interacting fields so constructed are solved and the states so constructed have much lower energies than do the free bare quarks. In this way, we find unusual bound states in the strong-coupling case that are inaccessible to a straightforward order-by-order perturbation approach. We view this approach as a first approximation to a solution of the strong-coupling problem. Its justification will ultimately rest on the systematic analysis of corrections to our variational "guess" for the form of the nucleon ground state. Such an analysis is not included in the present work. In this paper we report on the construction of low-lying bound hadronic states and the application of our formalism to calculating physical quantities such as magnetic dipole transition amplitudes, the axial charge renormalization, and the charge radius of the hadron.

What emerges from our analysis is a picture of composite hadrons whose lowest mass configurations coincide with the $L=0$, $\underline{35}$ of mesons and $L=0$, $\underline{56}$ of baryons predicted by the quark model. We reproduce the usual SU(6) results for the ratio of proton to neutron magnetic moments $\mu_p/\mu_n = -\frac{3}{2}$ and for the ratio of rates for baryonic electromagnetic ($M1$) transitions such as $\Delta^+ \to p+\gamma$ and mesonic $M1$ transitions such as $\omega \to \pi^0+\gamma$.

We are also led to a prediction for the proton magnetic moment that is in close accord with its experimental value—i.e., we calculate $\mu_p \simeq 3(e/2M)$ where M is the ground-state mass of the baryon $\underline{56}$, there being no breaking of the basic SU(6) or SU(3) symmetry in our model. The experimental proton moment is $(\mu_p)_{\text{expt}} = 2.79(e/2M_p)$. The "radius" of a meson constructed of a $q\bar{q}$ pair is found to be $(\frac{2}{3})^{1/3}$ of the radius of a baryon formed by a qqq color-singlet state. This same factor of $(\frac{2}{3})^{1/3}$ corrects the ratio of their magnetic dipole transition moments relative to the naive quark model. Although one cannot attach any real significance to such a factor while at the same time ignoring major mass splittings, it is difficult to avoid commenting on the fact that a correction factor of $(\frac{2}{3})^{2/3} = 0.76$ to the naive quark model brings the calculated rate for $\omega \to \pi^0+\gamma$ into close agreement with experiment ($\Gamma_{\omega\to\pi^0+\gamma} = 890$ keV). The mean-squared charge radii for the proton and the neutron are 0.7 fm and 0, respectively.[5]

We have also computed the value of the axial charge to be $g_A = \frac{5}{9}$, which is less than $\frac{1}{2}$ the observed value, 1.25. However, we do not know whether this unsatisfactory result is an argument against models of this type because the models being studied do not incorporate partially conserved axial-vector current (PCAC). This is evident from the fact that the π and ρ mesons are degenerate, although the π should be a Goldstone boson associated with chiral symmetry. Whether or not proper inclusion of PCAC will sufficiently modify the axial-vector current in this model is an open question. In Sec. X we discuss this and other sensitivities of our approach. In particular, the use of the variational principle and of the trial bound states for evaluating physical matrix elements as well as the neglect of quantum corrections to the tree approximation have led to considerable simplification of the quantum field theory. The accuracy of this approach in the strong-coupling region remains to be systematically studied.

II. INTUITIVE PICTURE AND SURVEY OF RESULTS

This section of the paper is intended to present the basic idea of our approach with emphasis on the intuitive ideas and away from the formal aspects. It will also serve as a compendium of our results and as a guide to the remaining sections.

A. Intuitive picture of a quark bound state

Before introducing the gauge vector mesons and the "color" interaction along the general lines first presented by Nambu,[6] we want to show how the strong interaction of an elementary quark (fermion) field with a self-coupled scalar field can lead to a low-mass bound state.

The basic idea of our approach is illustrated by the following simple semiclassical model. This model was also discussed by Vinciarelli.[7] Consider a quark described by wave function ψ interacting with a neutral scalar field σ with the Hamiltonian

$$\mathfrak{H} = \int d^3x\,\mathcal{H}(x)\,,$$

$$\mathcal{H}(x) = \psi^\dagger\left(\frac{\vec{\alpha}\cdot\vec{\nabla}}{i} + G\beta\sigma\right)\psi + \tfrac{1}{2}\dot{\sigma}^2 + \tfrac{1}{2}|\vec{\nabla}\sigma|^2 + H(\sigma^2-f^2)^2\,, \tag{2.1}$$

where G, $H \gg 1$ are large dimensionless coupling constants, and f has the dimension of a mass. The form of the quartic self-interaction term exhibits the invariance of the theory under the discrete transformation $\sigma \to -\sigma$. In a quantum field-theory description, Eq. (2.1) describes a spontaneously broken theory and σ has a nonvanishing vacuum expectation value. In the vacuum the field, σ, takes one of two values, $\pm f$. Small vibrations about one of these ground states are usually studied by making the translation $\sigma \to \sigma' = \sigma + f$. One readily finds that the small σ vibrations have the mass $m_\sigma{}^2 = 8Hf^2$ and the small ψ vibrations have mass $M_Q = Gf$. By assumption, the bare quark mass is

$$M_Q = Gf \gg 1 \text{ GeV} . \tag{2.2}$$

Our choice of the specific Hamiltonian (2.1) is arbitrary. We consider it as typical of a class of renormalizable field theories exhibiting spontaneous breakdown. A wider class without spontaneous breakdown is described in the Appendix.

Our key question is, "Do these theories also have quark states with much lower energy than indicated by the bare quark mass?"

For the purpose of developing an intuitive picture of nonperturbative solutions to the field equations, we approach this problem classically, although this is no longer a purely classical question when fermions are present. The point is that in the one-fermion sector when the charge

$$Q = \int \psi^\dagger \psi \, d^3x$$

has unit eigenvalue we are solving a Dirac equation for the quark in the presence of a scalar potential σ. We are faced with the usual question of negative-energy states and must specify that all the negative-energy states in the presence of this potential are filled, and then focus our attention on the lowest positive-energy eigenvalue. Since we are solving for the quark energy in a scalar potential, there is no Klein paradox of the familiar type encountered in the presence of strong, sharp vector potentials and therefore no ambiguity in identifying and interpreting the desired positive-energy "one-particle" solutions.

We proceed classically therefore with $Q = 1$. Classically, we expect that the quark-wave function and the field amplitude σ will avoid one another as indicated in Fig. 1, so as to escape the high-mass energy [Eq. (2.2)].

The importance of this effect increases with the magnitude of $M_Q = Gf$. At the same time, working against the formation of such a hole into which the quark will trap itself are the energies associated with the curvature of the localized quark-wave function, with the curvature of the σ field as it changes its value, and the energy associated with the potential term $H(\sigma^2 - f^2)^2$ extending over the volume where $\sigma \neq \pm f$. As a simple illustrative example of how these contributions balance, consider a potential as in Fig. 1 with $\sigma \to 0$ within a volume of radius R. Denoting by D the thickness of the shell in which the σ-field amplitude falls from $+f$ to 0, we have for the energies contributing to Eq. (2.1)

$$\int \psi^\dagger \frac{\vec{\alpha} \cdot \vec{\nabla}}{i} \psi \, d^3x \sim 1/R , \tag{2.3}$$

$$\int \tfrac{1}{2} |\vec{\nabla}\sigma|^2 d^3x \sim \tfrac{1}{2}(f/D)^2 4\pi R^2 D , \tag{2.4}$$

$$\int H(\sigma^2 - f^2)^2 d^3x \sim Hf^4(\tfrac{4}{3}\pi R^3 + 4\pi k R^2 D) , \tag{2.5}$$

where the estimate [Eq. (2.3)] follows from the uncertainty principle and $k \sim 1$ is a shape-dependent number. The energy of this configuration is given by the sum of (2.3), (2.4), and (2.5)

$$E(R, D) \sim \frac{1}{R} + 2\pi R^2 f^2 / D + Hf^4(\tfrac{4}{3}\pi R^3 + 4\pi k R^2 D) . \tag{2.6}$$

Minimizing with respect to D and R, we find a surface thickness given dimensionally by

$$\frac{\partial E}{\partial D} = 0 \Rightarrow D \sim 1/H^{1/2} f ,$$

and if $H^{1/2} f \gg 1/R$, i.e., if the volume energy dominates the surface energy, then

$$\frac{\partial E}{\partial R} = 0 \Rightarrow R \sim 1/H^{1/4} f .$$

Hence the lowest possible energy is given by

$$E \equiv \min E(R, D) = \frac{4}{3R} \sim f H^{1/4} . \tag{2.7}$$

In this case

$$D/R \sim H^{-1/4} \ll 1 , \tag{2.8}$$

which is consistent with a thin transition-shell region in the strong-coupling limit.

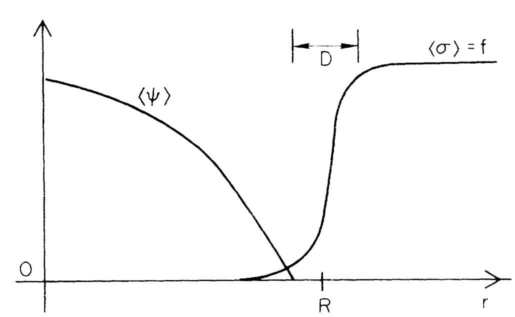

FIG. 1. Classical guess for the solution to the Hamiltonian (2.1) in the one-fermion sector.

Comparing with (2.2), we see that a localized bound state is formed if $G \gg H^{1/4}$. By Eq. (2.8), we see that we are in the strong-coupling domain.

According to Fig. 1, the quark moves as a free massless quantum within the sharp well boundaries, suggesting some of the popular quark-parton model ideas. However, as we shall see in Sec. IV, the treatment as described above is much too naive and crude, although it illustrates the basic idea. We shall learn from a more systematic and careful treatment of Hamiltonian (2.1) in the following sections that what actually emerges for the classical theory is a thin-shell model of the hadron, with the field rapidly changing from $\sigma = +f$ outside to $\sigma = -f$ in a region of thickness $D \sim 1/H^{1/2}f \ll R$, and with the quark confined to a thin shell within a distance $1/Gf$ of R. The energy in this case is $E \sim H^{1/6}f$ rather than the $H^{1/4}f$ found in Eq. (2.7). This solution is illustrated in Fig. 2.

B. Highlights of subsequent developments

In Section III we show how one can reduce the quantum field-theory problem of finding bound states to just the type of classical problem that we have considered above. The method we discuss is to quantize the theory defined by Eq. (2.1) at time $t = 0$ by canonical methods. We then construct a Fock space state, $|s\rangle$, as a trial state with the property

$$Q|s\rangle = |s\rangle \tag{2.9}$$

and show that

$$\langle s|\mathfrak{H}|s\rangle \ll Gf \,. \tag{2.10}$$

Our purpose in this discussion is to show (i) how, for a particular class of variational states, our problem reduces in "tree" approximation to the classical problem, and (ii) to demonstrate in a

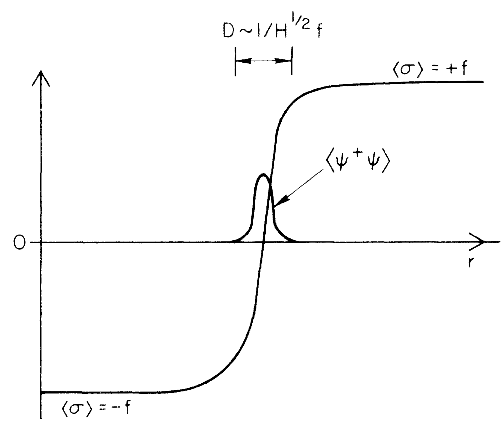

FIG. 2. The solution to (2.1) in the one-fermion sector which is obtained in Sec. IV.

systematic development all the approximations involved in reducing our problem to a classical one, pointing out what we feel are the important unanswered questions. These will involve questions of normal-ordering and corrections to the tree approximation.

Secion IV is devoted to actually solving the classical problem in detail. If we follow the analogy to the polaron problem that was referred to earlier (2.4), the trial state $|s\rangle$ for the variational calculation is formed as a product of a coherent boson state and of a single-quark state constructed in a basis whose coefficients are the localized wave functions in the self-consistent scalar potential; i.e., we write

$$|s\rangle = \exp\left(-i\int g(x)\dot{\sigma}(x)d^3x\right)B_0^\dagger|0_L\rangle \,, \tag{2.11}$$

where the quark field expansion in terms of particle annihilation and antiparticle creation operators B_α and $D_\alpha^\dagger$, respectively, is

$$\psi(x) = \sum_\alpha [B_\alpha U_\alpha(x) + D_\alpha^\dagger V_\alpha(x)] \,, \tag{2.12}$$

and the orthonormality relations

$$\begin{aligned} &\int U_\alpha^\dagger(x)U_\beta(x)d^3x = \delta_{\alpha\beta}\,, \\ &\int U_\alpha^\dagger(x)V_\beta(x)d^3x = 0 \end{aligned} \tag{2.13}$$

are required if the B_α and D_α are to satisfy the usual anticommutation rules: $\{B_\alpha, B_\beta^\dagger\} = \delta_{\alpha\beta}$, etc., $|0_L\rangle$ is a "no particle" state annihilated by the B_α's and D_α's, though not translationally invariant since the localized states are not momentum eigenstates. The classical field $g(x)$ in Eq. (2.11) is the local expectation value of $\sigma(x)$ in the state $|s\rangle$; viz., $\langle s|\sigma(x)|s\rangle = g(x)$. The coupled classical differential equations satisfied by the field $g(x)$ and the quark ground-state function $U_0 \equiv \chi$ derived from (2.1) and (2.11) by requiring that $\langle s|\mathfrak{H}|s\rangle$ be stationary with respect to variations of g and χ are

$$\nabla^2 g - 4Hg(g^2 - f^2) = G\bar{\chi}\chi \tag{2.14}$$

and

$$\left(\frac{\vec{\alpha}\cdot\vec{\nabla}}{i} + G\beta g\right)\chi = \mathcal{E}\chi \,; \tag{2.15}$$

$\mathcal{E}$ appears as a Lagrange multiplier since our trial state is normalized to $\int \chi^\dagger \chi \, d^3x = 1$ by (2.13). The solution of these coupled classical equations gives the lowest ground-state energy consistent with the form of our trial state $|s\rangle$ in Eq. (2.11). As usual in dealing with the Dirac equation, there is no "lowest energy" because of the negative-energy spectrum as commented upon in the Introduction. Here, in referring to the lowest energy state, we make the usual assumption that the negative-energy states are filled and χ is the lowest positive-energy state. In Sec. IV we implement this restriction and exhibit a solution [free of Klein paradoxes since the potential $g(x)$ is scalar].

The solutions of these equations exhibit the properties described in the Introduction. The solution of Eq. (2.14) leading to a bound state is a steplike spherical potential

$$g=f\tanh(\sqrt{2H}f(r-R))[1+O(H^{1/6}/G)] \quad . \tag{2.16}$$

A lower bound on

$$G \gg H^{1/6} \tag{2.17}$$

is required to ensure this bound state to be of lower energy than that of a free quark of mass $M_Q=Gf$. This solution is illustrated in Fig. 2, with the quark confined to a thin shell of thickness $D\sim 1/Gf$ about R. In order to show simply and explicitly how these features of the solution emerge, we present the *exact* solution (discovered by C. K. Lee) to our coupled field equations in 1 space, 1 time dimension.[8] This simple example contains all essential features of the general problem.

In this case we have to solve the coupled equations

$$\frac{d^2}{dx^2}g(x)-4Hg(g^2-f^2)=G\bar{\chi}\chi(x) \tag{2.18}$$

and

$$\left(\frac{1}{i}\alpha\frac{d}{dx}+G\beta g(x)\right)\chi(x)=\mathcal{E}\chi\,. \tag{2.19}$$

Since there is no spin in 1 space dimension, we have the two-component form for $\chi(x)$

$$\chi(x)=\begin{pmatrix}\chi_u(x)\\ \chi_l(x)\end{pmatrix} \tag{2.20}$$

and choose a convenient representation in terms of Pauli matrices

$$\beta=\sigma_3\,;\quad \alpha=\sigma_1\,.$$

First observe that if we set $G\bar{\chi}\chi=0$ in Eq. (2.18), then the resulting equation admits the exact solution

$$g(x)=f\tanh(\sqrt{2H}f(x-x_0))\ . \tag{2.21}$$

If we now consider this as the input potential, we find that Eq. (2.19) then admits the exact solution

$$\chi(x)=N[\cosh(\sqrt{2H}f(x-x_0))]^{-G/\sqrt{2H}}\begin{pmatrix}1\\ i\end{pmatrix}, \tag{2.22}$$

with $\mathcal{E}=0$. Now, using Eq. (2.22) to compute $\bar{\chi}\chi=\chi^\dagger\beta\chi$, we find $\bar{\chi}\chi=0$, and so Eqs. (2.21) and (2.22) provide exact solutions to the coupled equations. The general form of this solution is shown in Fig. 2 and it is obvious that as $G/\sqrt{2H}\to\infty$, the quark is confined to a narrower and narrower region[9]; nevertheless, one sees that the total energy corresponding to the quark part of the Hamiltonian manages to be $\mathcal{E}=0$. To see why this is so, let us examine the two contributions to the energy of the quark. Since $\bar{\chi}\chi=0$, the integral for the mass term vanishes:

$$G\int\bar{\chi}(x)\chi(x)g(x)dx=0\,. \tag{2.23}$$

Secondly, since the upper and lower components of χ have the same slope in x, the kinetic energy term also vanishes:

$$\chi^\dagger(x)\alpha\,\frac{d}{dx}\,\chi(x)=0\,, \tag{2.24}$$

$$\begin{aligned}E&=\int dx\left[\frac{1}{2}\left(\frac{dg}{dx}\right)^2+H(g^2-f^2)^2+\chi^\dagger\left(\frac{\alpha}{i}\,\frac{d}{dx}+G\beta g\right)\chi\right]\\ &=2H\int(g^2-f^2)^2dx\\ &=\tfrac{8}{3}\sqrt{2H}f^3\,.\end{aligned} \tag{2.25}$$

So long as

$$Gf\gg\sqrt{2H}f^3 \tag{2.26}$$

the lowest energy state in the one-quark sector, $Q=\int\chi^\dagger\chi\,dx=1$, is not a free "bare" quark but a localized bound state.

Of primary interest to us here are which features of this 1-plus, 1-dimensional, solution persist in the four-dimensional case. Near $r=R$, Eq. (2.22) becomes (for $G\gtrsim\sqrt{H}$)

$$\chi(\vec{r})=N[\cosh(\sqrt{2H}f(r-R))]^{-G/\sqrt{2H}}\begin{pmatrix}1+O\left(\dfrac{r-R}{R}\right)\\[2ex] \dfrac{i\vec{\sigma}\cdot\vec{r}}{r}\left\{1+O\left(\dfrac{r-R}{R}\right)\right\}\end{pmatrix}\phi^{(+)}_{1/2,m}\ , \tag{2.27}$$

where $\varphi^{(+)}_{1/2,m}$ is the standard two-component angular solution $j=\frac{1}{2}$ and $l=0$, i.e.,

$$\phi^{(+)}_{1/2,\,1/2}=\begin{pmatrix}1\\0\end{pmatrix}$$

and

$$\phi^{(+)}_{1/2,\,-1/2}=\begin{pmatrix}0\\1\end{pmatrix}.$$

$\chi(r)$ is dropping with increasing $r-R$ at the same rate as for $\chi(x)$ in Eq. (2.22). This behavior does not persist all the way to $r=0$ due to the correction terms indicated above, but it does persist until $\chi(r)$ has become negligibly small, as will be shown in Sec. IV. The solution for $g(r)$, Eq. (2.16), is similar to Eq. (2.21). This form is nearly exact at $r=0$ and $r=\infty$ and is modified only slightly near $r=R$.

Introducing Eq. (2.27) into Eq. (2.3) to evaluate the quark kinetic energy, we find readily $1/R$. This difference from the vanishing result in Eq. (2.24) comes from the correction terms in Eq. (2.27), which give the upper and lower components of the wave function slightly different radial dependences. Owing to these terms, it is no longer true that $\bar{\chi}\chi=0$ for all r and, therefore, we also find a correction to Eq. (2.23). What is true, however, is that $\bar{\chi}\chi \ll \chi^\dagger\chi$ in the four-dimensional case as is readily deduced from Eq. (2.27) and, therefore,

$$G\int \bar{\chi}\chi g\, d^3x \ll \frac{1}{R}\ . \tag{2.28}$$

Hence quark confinement in a thin shell leads only to an $\sim 1/R$ contribution to the energy. Finally, we note that the inequality (2.26) becomes

$$H^{1/6}f \ll Gf \tag{2.29}$$

for a tightly bound state.

So far, we have considered localized bound-quark solutions at rest. In Sec. V we extend our solution by constructing variational states with arbitrary nonvanishing average three-momentum. Formally, we do this by guessing a form for the trial state in our Fock space that allows the packet to move in time. Further, we include the constraint that

$$\langle s_v|\vec{\mathrm{P}}^{\mathrm{op}}|s_v\rangle=\vec{\mathrm{p}} \tag{2.30}$$

in performing the variation to minimize the energy. The states so constructed preserve the required relation between energy and momentum,

$$E=M/(1-v^2)^{1/2} \text{ and } \vec{\mathrm{p}}=M\vec{\mathrm{v}}/(1-v^2)^{1/2} \tag{2.31}$$

or

$$\{\langle s_v|\mathfrak{H}|s_v\rangle\}^2-\{\langle s_v|\vec{\mathrm{P}}^{\mathrm{op}}|s_v\rangle\}^2=E^2-\vec{\mathrm{p}}^2 =M^2, \tag{2.32}$$

where v is the velocity of the bound state. The potential exhibits a Lorentz contraction along the direction of motion, viz.,

$$g_v(x_\parallel, x_\perp)=g(\gamma x_\parallel, x_\perp), \tag{2.33}$$

where g_v denotes the potential for a moving state with velocity v, $\gamma=1/(1-v^2)^{1/2}$, and g is the solution for a state at rest. The transformation for the quark state is

$$\chi_v(x_\parallel, x_\perp)=S(\Lambda)\chi(\gamma x_\parallel, x_\perp)e^{i\mathcal{E}v\gamma x_\parallel}, \tag{2.34}$$

where $S(\Lambda)$ is the familiar spinor transformation matrix and $\mathcal{E}$ is the quark energy in the rest state given by Eq. (2.15).

The further problem of constructing actual momentum *eigenstates* along with its attendant complexities are discussed in Sec. IX.

Having come this far with a satisfactory single-quark state, in Sec. VI we extend our scheme to the construction of multiquark states and study the spectrum of hadrons seen in nature. The thrust of the argument presented in this section is to answer the question, "If a single quark prefers to dig a hole in the vacuum and trap itself, what happens if one has two or more quarks or quark-antiquark pairs?" Our approach is to construct trial states as in Eq. (2.12) with several quarks present in the self-consistently produced potential $g(x)$. Thus, as in the Hartree-Fock approximation for atoms, the quarks do not interact directly with one another but via their average binding field produced self-consistently. Formally this means that we must do a variational calculation of the sort just discussed, except that now

$$E=n\mathcal{E}+\int d^3x[\tfrac{1}{2}|\vec{\nabla}g|^2+H(g^2-f^2)^2], \tag{2.35}$$

where n stands for the number of (anti) quarks in the ground state of the potential g. If we make the same substitutions as in the one-quark case for the g of Fig. 2, we obtain for the energy

$$E(R,D)\cong\frac{n}{R}+4\pi R^2D(2f/D)^2+Hf^4 4\pi kR^2D, \tag{2.36}$$

where now n denotes the n-quark sector. Minimizing $E(R,D)$ with respect to R and D yields

$$R_n=n^{1/3}R_0 \sim n^{1/3}\,1/fH^{1/6} \tag{2.37}$$

and therefore

$$E=n^{2/3}E_0, \tag{2.38}$$

where R_0 and E_0 denote the results of doing the one-quark calculation. One immediate consequence of Eq. (2.38) is that the ratio of the mean mass of the ground-state meson 0^- $\underline{35}$ to the baryon 0^+ $\underline{56}$ is predicted to be $(\frac{2}{3})^{2/3}$. The experimental significance of the hadron size being $\propto n^{1/3}$ and the ground-state energy $\propto n^{2/3}$ is discussed in Sec. VIII in detail.

In order to proceed beyond this construction of multiquark states to the classification of physical hadron states, we need to introduce "color." In particular, a qqq ground state for baryons will be totally symmetric in space coordinates with each quark in an $l=0$ symmetric s state. It must also be symmetric in spin if we are to achieve an approximate SU(6) symmetry with an $L=0$ $\underline{56}$ baryon ground state. Therefore, antisymmetrization in a "color" quantum number is required.[2] Furthermore, a color interaction must be introduced in order to raise the energies of all color-nonsinglet bound states not yet observed among the low-lying ground states in nature.[6]

Section VII is devoted to a discussion of the way in which the introduction of gauge fields coupled to the "color" of a quark can accomplish this purpose and reproduce the desired classification. The basic idea follows the original observation of Nambu[6] that if colored quarks interact via colored gauge fields, then the interaction will be attractive for color-singlet states and repulsive for color nonsinglets (in the case of states made of particles belonging to a color triplet). What we do is adapt this argument to our self-consistent calculation in order to show how in this scheme only color-singlet states remain with hadronic masses while all nonsinglets are pushed up in energy. Leaving the discussion of true "color"—which corresponds to a non-Abelian theory—aside for the moment, we can here give a good idea of what is going on by highlighting the main ideas for the simpler Abelian case. The detailed presentation both for Abelian and non-Abelian vector gauge interactions is presented in Sec. VII.

The extension which we make of the theory we started with is to introduce a gauge field X_μ and a complex Higgs field ϕ so that our theory is described by a Lagrangian of the form

$$\begin{aligned}\mathcal{L} = &-\tfrac{1}{4}(\partial_\mu X_\nu - \partial_\nu X_\mu)^2 + [(\partial_\mu + i\zeta X_\mu)\phi^*][(\partial^\mu - i\zeta X^\mu)\phi] \\ &- H'(\phi^*\phi - f'^2)^2 + \tfrac{1}{2}(\partial_\mu\sigma)^2 - H(\sigma^2 - f^2)^2 \\ &+ \bar{\psi}(i\gamma^\mu\partial_\mu + \zeta X^\mu\gamma_\mu - G\sigma)\psi. \end{aligned} \tag{2.39}$$

As before, at the classical level (tree approximation), this theory is one in which the vacuum state has $\langle\sigma\rangle = \pm f$ and $\langle\phi\rangle = \pm f'$. Hence, substituting $\sigma \to \sigma + f$ and $\mathrm{mod}\phi \to \mathrm{mod}\phi + f'$, we obtain a theory which describes the following roster of "bare particles": a σ meson of $m_\sigma^2 = 8f^2H$, a fermion of mass $M_Q = Gf$, a ϕ meson of $m_\phi^2 = 4f'^2H'$, and a massive vector of mass $m_B^2 = 2\zeta^2 f'^2$. Choosing G, H, H', and ζ so that all bare quanta are very heavy, we can then, for the same reasons discussed in Sec. III, reduce our problem to that of finding stationary points of the classical energy

$$\begin{aligned} E &= \langle s'|\mathfrak{H}|s'\rangle \\ &= \int d^3x \Big[\tfrac{1}{2}\vec{E}^2 + \tfrac{1}{2}(\vec{\nabla}\times\vec{B})^2 + \tfrac{1}{2}\zeta^2(g' + \sqrt{2}f')^2(\vec{B}^2 + B_0^2) \\ &\qquad + \tfrac{1}{2}(\vec{\nabla}g')^2 + \tfrac{1}{4}H'g'^2(g' + \sqrt{2}f')^2 + \tfrac{1}{2}(\vec{\nabla}g)^2 + H(g^2 - f^2)^2 + \chi^\dagger\Big(\frac{\vec{\alpha}\cdot\vec{\nabla}}{i} + G\beta g + \zeta\vec{\alpha}\cdot\vec{B}\Big)\chi\Big], \end{aligned} \tag{2.40}$$

where $\vec{E}$ is the classical electric-type field associated with the vector potential X_μ, $\vec{B}$ is the associated vector potential, and B_0 is defined as

$$B_0 = \frac{1}{\zeta^2(g' + \sqrt{2}f')^2}(\vec{\nabla}\cdot\vec{E} - \zeta\chi^\dagger\chi). \tag{2.41}$$

We have in this manner reduced the problem to the classical form of the interaction of massive vector "electric" and "magnetic" fields of color in interaction with a color charge density

$$j_0(x) = \chi^\dagger\chi \tag{2.42}$$

and current density

$$\vec{j}(x) = \chi^\dagger\vec{\alpha}\chi. \tag{2.43}$$

In particular, quantum fluctuations are ignored in setting $\langle s'|(\psi^\dagger\psi - \chi^\dagger\chi)^2|s'\rangle = 0$ in writing Eq. (2.40).

As in the classical theory, there is a short-range (Coulomb) repulsion which causes any nonvanishing local charge density to expand. It is this local repulsion which raises the energy of the state by an amount proportional to the coupling constant ζ. Only for color-singlet states does the current density vanish locally so that the energy is not raised by the color interaction. The strength of the color coupling fixes the scale of energy by which states that are not color singlets are raised. Hence the color singlet, or zero-triality selection rule, derived in our theory is approximate and not absolute.[10]

In Sec. VIII we compute physical parameters for the hadronic ground states including $M1$ transition moments, the axial charge, and approximate charge radii using our trial solutions.

Section IX is devoted to constructing momentum eigenstates, a problem we have solved only for charge-$\frac{1}{3}$ states at rest. Remaining difficulties and open problems are discussed.

Section X is devoted to a discussion of what we see to be some of the important questions left totally unanswered to date. One of the most important of these questions on which we can only speculate is whether or not one will ever be able to successfully incorporate PCAC into a scheme of this type.

In Sec. XI we speculate on the structure of excited hadron states. The key observation has to do with the "softness" of our shell solution to deformations of shape—a point which will be discussed heuristically in this section. The basic idea can be illustrated as follows: The potential $g(x)$ is spherical with a contained quark in an s state because this shape gives a surface of smallest area, and hence minimum field energy, while maximizing the volume into which the quark-wave function is squeezed. However, when one excites the quark to a state of higher $l \neq 0$, the hole in the field potential can collapse around the quark-wave function and thereby reduce its surface area and hence its energy without further increasing the curvature of the quark-wave function. Simple models suggest that this mechanism of a soft shell leads to low-lying excitations of the hadron state. The possible connection of this scheme to the dual-string model, scaling, final hadron spectra, etc., is discussed. Our purpose is to show that the potential inherent in this approach, which requires further development, is very broad indeed.

Finally, in Sec. XII we compare our approach to the MIT "bag model" and recent works by Lee and Wick; Chin and Walecka[11]; Creutz and Soh[12]; and Dashen, Hasslacher, and Neveu,[13] who have also studied quark-containment mechanisms in field-theoretic models.

An appendix is devoted to a discussion of a modified version of the simple model discussed in Sec. IV whose purpose is to try to explore how sensitive these results are to the addition of a term which forces the existence of a volume energy in addition to the surface energy in the σ field. In particular, we sketch the arguments of Creutz and Soh,[12] showing how the MIT bag model emerges for a specific choice of parameters.

III. THE VARIATIONAL CALCULATION

We have seen in the last section that a heuristic, semiclassical discussion of the Hamiltonian [Eq. (2.1)] suggests the possible existence of bound states with masses much less than the bare masses of the constituents. In this section we will show how such a semiclassical picture may emerge from a canonical quantum field theory. We have verified that this phenomenon occurs in a strong-coupling theory, where a nonperturbative approach is essential. Our analysis makes use of the variational principle for the expectation value of the Hamiltonian in a trial state.

In carrying out the variational calculation, in addition to making a suitable guess for the trial state, we are forced to make one crucial approximation involving normal-ordering; this is the "tree" approximation. It remains to be shown how good our trial function and use of the tree approximation are in establishing the qualitative character of the strong-coupling solution which we construct. A more complete treatment including renormalization remains for the future and unquestionably requires a more systematic approach whose first step, we would hope, has been established by the work we are reporting here. What we do accomplish in this section is the reduction of the quantum field-theory problem to the classical theory described by solutions of the field equations (2.14) and (2.15).

For simplicity, we will not discuss SU(3) hadrons. Instead, we imagine a world with only a single-quark species and demonstrate in Sec. IV the existence of bound states of mass much less than the bare quark mass. We defer to Sec. VII a discussion of SU(3) hadrons and of the color mechanism which ensures that states of nonzero triality have much larger masses than the hadrons of zero triality.

A. Fock space

We consider the model Lagrangian

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \sigma)^2 - H(\sigma^2 - f^2)^2 + \bar{\psi}(i\gamma^\mu \partial_\mu - G\sigma)\psi, \quad (3.1)$$

where σ and ψ are scalar and fermion fields, respectively, f is a constant parmeter with dimensions of mass, and $G, H > 0$ are dimensionless coupling constants. For our variational approach, we need only consider the system at a single time, which we take to be $t = 0$ (and we usually suppress the time argument in our notation). Only the canonical equal-time commutation relations are needed and at $t = 0$, we may expand the field operators in a normal-mode Fock space basis.

For the scalar field, we choose a plane-wave expansion

$$\sigma(x) = \int \frac{d^3k}{[(2\pi)^3 2\omega_k]^{1/2}} \left(a_k e^{i\vec{k}\cdot\vec{x}} + a_k^\dagger e^{-i\vec{k}\cdot\vec{x}}\right), \quad (3.2)$$

where

$$\omega_k = (\vec{k}^2 + m_\sigma^2)^{1/2}, \quad m_\sigma^2 = 8Hf^2 \quad (3.3)$$

and the operators are quantized by imposing the usual canonical commutation relations. In Eq. (3.3) we have used the mass, $m_\sigma = 2\sqrt{2H}f$, for the small σ oscillations after making the translation $\sigma \to \sigma + f$, as discussed in Sec. II. For the fermion field, we make an expansion in terms of the eigenfunctions of the Dirac equation in an external potential to be specified later when we discuss the variational procedure

$$\psi(x) = \sum_n [B_n U_n(x) + D_n^\dagger V_n(x)]. \tag{3.4}$$

The positive- and negative-energy eigenfunctions U_n and V_n satisfy the orthonormality relations

$$\begin{aligned}\int d^3x\, U_n^*(x) U_m(x) &= \int d^3x\, V_n^*(x) V_m(x)\\ &= \delta_{nm},\\ \int d^3x\, U_n^*(x) V_m(x) &= 0.\end{aligned} \tag{3.5}$$

The nonvanishing equal-time anticommutators are

$$\begin{aligned}\{\psi(x), \psi^\dagger(x')\} &= \delta^3(x - x'),\\ \{B_n, B_m^\dagger\} &= \{D_n, D_m^\dagger\}\\ &= \delta_{nm}.\end{aligned} \tag{3.6}$$

The Hilbert space at $t=0$ is constructed by applying the creation operators $a_k^\dagger$ and $B_n^\dagger, D_m^\dagger$ to the translationally noninvariant no-particle state $|0_L\rangle$ characterized by

$$a_k|0_L\rangle = B_n|0_L\rangle = D_m|0_L\rangle = 0. \tag{3.7}$$

The relation of this expansion to the usual one in terms of plane waves and a translationally invariant trial vacuum will be clarified in terms of the Bogoliubov transformation.[14]

B. Normal-ordering and definition of the Hamiltonian

Our field-theory model with $\mathcal{L}$ given by Eq. (3.1) is a renormalizable theory. Because of the divergences inherent in any renormalizable quantum field theory, the meaning of a product of field operators at the same space-time point is ambiguous and has to be properly defined. In the case of the Hamiltonian, these ambiguities are related to the necessity of a renormalization program designed to remove the ultraviolet divergences in the theory. It is beyond the scope of the present paper to tackle the problem of renormalization in a strong-coupling theory; we define the Hamiltonian by a naive normal-ordering prescription. The prescription depends on the particular expansion chosen for the field operators. Hamiltonians normal-ordered with respect to two different expansions such as Eq. (3.4) and a plane-wave expansion differ by a c-number contribution which is usually a difference of two infinite constants. In order to give such a difference a precise meaning, it would be necessary to regulate and properly renormalize the quantum field theory.[15]

In this paper a very fundamental approximation is to ignore these differences in normal-ordering prescriptions. In other words, the Hamiltonian we are working with is correct only in the so-called "tree" approximation. To the same approximation, the true vacuum state also coincides with the free-field vacuum as defined for small oscillations about $\sigma = f$. Our hope is that when renormalization effects are included, the conclusions will be qualitatively similar although they may be quantitatively different. Specifically, this means we are ignoring the difference in energy between a theory with $\mathfrak{H}$ normal-ordered in the basis (3.7) as constructed for the one-fermion sector and a theory normal-ordered in a translationally invariant trial vacuum.[16]

C. Boson coherent states

The construction of the trial state is guided by our intuitive idea that the boson field develops a localized expectation value in the neighborhood of the fermion source. To describe such a situation, we employ the so-called boson coherent states

$$|g\rangle = U(g)|0_L\rangle, \tag{3.8}$$

where $U(g)$ is a unitary transformation

$$U(g) = \exp\left(-i\int d^3x\, g(x)\dot\sigma(x)\right), \tag{3.9}$$

which displaces the field operator σ

$$\begin{aligned}U^{-1}(g) f(\sigma(x)) U(g) &= f(\sigma(x) + g(x)),\\ U^{-1}(g)\dot\sigma U(g) &= \dot\sigma.\end{aligned} \tag{3.10}$$

Thus, if $f(\sigma)$ is any polynomial function of σ which is normal-ordered term by term, then

$$\begin{aligned}\langle g|f(\sigma)|g\rangle &= \langle 0_L|f(\sigma+g)|0_L\rangle\\ &= f(g).\end{aligned} \tag{3.11}$$

Equation (3.11) shows that the tree approximation rule for taking the expectation value of a function of σ in a coherent state is to replace σ by the c-number amplitude $g(x)$. This procedure gives a concrete realization of the intuitive picture presented in the Introduction.

D. Fermion states and the Bogoliubov transformation

We shall also want to replace the fermion field operator by an arbitrary c-number Dirac spinor wave function when we take the expectation value of $\mathfrak{H}$ in our trial state. For a trial state of fermion number one, we do this by constructing

$$|s\rangle = B_n^\dagger |0_L\rangle, \tag{3.12}$$

where $B_n^\dagger$ is the creation operator for a fermion in an arbitrary state n and $|0_L\rangle$ is the no-particle state in the basis formed as shown in Eqs. (3.4), (3.5), and (3.7). With this procedure, the expectation value of an operator bilinear in the fermion field and normal-ordered in this basis is

$$\langle s|:\psi^\dagger(x)\Gamma\psi(x):|s\rangle = U_n^\dagger(x)\Gamma U_n(x), \tag{3.13}$$

where the arbitrary wave function is to be determined self-consistently by the variational calculation.

If we want to study the relation of the localized no-particle state $|0_L\rangle$ to a translationally invariant trial vacuum state $|0_p\rangle$ or study the relation of the state [Eq. (3.12)] to an expansion in a plane-wave basis, we require a unitary transformation connecting the two representations. This change of basis is called a Bogoliubov transformation. To appreciate the significance of this transformation, let us first construct a trial state of fermion number one of the form

$$|h\rangle = \int d^3p \sum_s h(p,s) b_{p,s}^\dagger |0_p\rangle \tag{3.14}$$

in terms of a plane-wave basis

$$\psi(x) = \int \frac{d^3p}{[(2\pi)^3 2E_p]^{1/2}} \sum_s [b_{ps} u(p,s) e^{i\vec{p}\cdot\vec{x}} + d_{p,s}^\dagger v(p,s) e^{-i\vec{p}\cdot\vec{x}}],$$
$$E_p = (\vec{p}^2 + M^2)^{1/2}, \quad M = Gf. \tag{3.15}$$

Then the expectation value of an operator bilinear in the fermion field and normal-ordered in this basis is

$$\langle h|:\psi^\dagger(x)\Gamma\psi(x):|h\rangle = s^\dagger(x)\Gamma s(x), \tag{3.16}$$

where

$$s(x) = \sum_s \int \frac{d^3p}{[(2\pi)^3 2E_p]^{1/2}} e^{i\vec{p}\cdot\vec{x}} h(p,s) u(p,s). \tag{3.17}$$

However, $s(x)$ is not an arbitrary spinor as required for performing a variational calculation since the positive-energy solutions $u(p,s)$ do not alone form a complete basis. Therefore, Eq. (3.14) is not a suitable trial state. It is apparent that a Bogoliubov transformation must be applied to Eq. (3.14) to mix together the particle and antiparticle plane-wave spinors in order to provide a complete basis for expanding the trial function.

We illustrate how this is accomplished and exhibit the relation of $|0_L\rangle$ to the state $|0_p\rangle$ in what follows. For convenience, we quantize the system in a large but finite volume so that the momentum spectrum becomes discrete. This enables us to treat all expansions on the same footing. The connection between the two bases is

$$B_n = \sum_m (\alpha_{nm} b_m + \beta_{nm} d_m^\dagger),$$
$$D_n^\dagger = \sum_m (\tilde{\alpha}_{nm} b_m + \tilde{\beta}_{nm} d_m^\dagger), \tag{3.18}$$

where

$$\alpha_{nm} = \int d^3x\, U_n^*(x) u_m(x), \quad \beta_{nm} = \int d^3x\, U_n^*(x) v_m(x),$$
$$\tilde{\alpha}_{nm} = \int d^3x\, V_n^*(x) u_m(x), \quad \tilde{\beta}_{nm} = \int d^3x\, V_n^*(x) v_m(x). \tag{3.19}$$

These numbers can be regarded as elements of matrices α, β, $\tilde{\alpha}$, and $\tilde{\beta}$. It follows from the orthonormality of these eigenfunctions that these matrices satisfy the relations

$$\alpha\alpha^\dagger + \beta\beta^\dagger = 1, \quad \alpha^\dagger\alpha + \tilde{\alpha}^\dagger\tilde{\alpha} = 1,$$
$$\tilde{\alpha}\tilde{\alpha}^\dagger + \tilde{\beta}\tilde{\beta}^\dagger = 1, \quad \beta^\dagger\beta + \tilde{\beta}^\dagger\tilde{\beta} = 1, \tag{3.20}$$
$$\alpha\tilde{\alpha}^\dagger + \beta\tilde{\beta}^\dagger = 0, \quad \alpha^\dagger\beta + \tilde{\alpha}^\dagger\tilde{\beta} = 0.$$

In this matrix notation

$$B = \alpha b + \beta d^\dagger,$$
$$D^\dagger = \tilde{\alpha} b + \tilde{\beta} d^\dagger, \tag{3.21}$$

and the inverse is

$$b = \alpha^\dagger B + \tilde{\alpha}^\dagger D^\dagger,$$
$$d^\dagger = \beta^\dagger B + \tilde{\beta}^\dagger D^\dagger. \tag{3.22}$$

It is obvious that the transformation $(b, d^\dagger) \to (B, D^\dagger)$ is unitary. The connection is completed by giving the relation between the two no-particle states (this is a definition of $|0_L\rangle$):

$$|0_L\rangle \equiv \frac{1}{\sqrt{N}} \prod_n B_n D_n |0_p\rangle. \tag{3.23}$$

It can be verified that $|0_L\rangle$ is not a null state. In particular, if the transformation conserves momentum, then α and β are simply numbers

$$B_p = \alpha_p b_p + \beta_p d_{-p}^\dagger,$$
$$D_p^\dagger = \beta_p b_p + \alpha_p d_{-p}^\dagger, \tag{3.24}$$

and the relation reduces to the familiar one,

$$|0_p'\rangle = +\frac{1}{\sqrt{N}} \left[\prod_p \beta_p^*(\alpha_p + \beta_p d_{-p}^\dagger b_p^\dagger)\right] |0_p\rangle. \tag{3.25}$$

If the potential is spherically symmetric, the positive- and negative-energy eigenfunctions do not mix under a spatial rotation; therefore, the B's and D's associated with two coordinates connected by such a rotation are related by

$$B' = RB,$$
$$D' = RD, \tag{3.26}$$

where R is a unitary matrix. Since

$$|\det R| = 1 \tag{3.27}$$

we have

$$\prod_M B_n' D_n' = (\det R)^2 \prod_n B_n D_n\,, \tag{3.28}$$

i.e., the no-particle state $|0_L\rangle$ is rotationally invariant.

Equations (3.12), (3.18), (3.19), and (3.23) give the expansion of a trial state with fermion number one in a momentum basis.

E. Derivation of classical field equations from the variational principle

We now apply the variational principle to the Hamiltonian derived from Eq. (3.1), guessing as the trial state

$$\begin{aligned} |s\rangle &= U(g)\, B_0^\dagger |0_L\rangle \\ &= \exp\left(-i\int d^3x\, g(x)\dot{\sigma}(x)\right) B_0^\dagger |0_L\rangle, \end{aligned} \tag{3.29}$$

where $B_0^\dagger$ is the creation operator associated with the ground-state wave function in Eq. (3.4) and $U(g)$ creates the coherent boson state (3.9). This procedure reduces the quantum field-theory problem to a classical form to which we can apply the heuristic discussion of Sec. III; it can also be solved by mathematical analysis. Specifically, if we assume the Hamiltonian to be normal-ordered term-by-term with respect to $|0_L\rangle$ as discussed earlier, it is straightforward to evaluate the energy of the trial state. The result is

$$\begin{aligned} E &\equiv \langle s|\mathfrak{H}|s\rangle \\ &= \int d^3x \left[\chi^\dagger\left(\frac{\vec{\alpha}}{i}\cdot\vec{\nabla} + G\beta g\right)\chi + \tfrac{1}{2}(\vec{\nabla}g)^2 + H(g^2 - f^2)^2\right]. \end{aligned} \tag{3.30}$$

Zero-point energies associated with the normal-ordering prescription are dropped in writing Eq. (3.30).

Since we have not yet specified what the expansion basis (U_n, V_n) is, except that it forms a complete basis, χ is obviously arbitrary. The idea behind the variational principle is that the best choice of the trial state is such that the corresponding g and χ will *minimize* the energy E. However, this will be true only if the energy operator is positive-definite for *all* g and χ. This is not the case in general, however, since the Dirac part in Eq. (3.30) is not positive-definite. This is, of course, the original difficulty that led Dirac to formulate hole theory which we also must apply here. We proceed as follows: Assume that for any choice of $g(x)$, we solve the Dirac equation exactly

$$\left(\frac{\vec{\alpha}}{i}\cdot\vec{\nabla} + G\beta g\right)\chi = \mathcal{E}(g)\chi \tag{3.31}$$

and take the lowest "positive" eigenvalue $\mathcal{E}(g)$. Since Eq. (3.31) describes the motion of a Dirac particle in a scalar potential, there is no Klein paradox as occurs for sharp localization of a Dirac particle in a strong vector (Coulomb) potential.[17] The solutions for the positive and negative spectra are clearly separated in this case, and so one does not lower the energy of the trial state by including particle-antiparticle pairs flowing into the region of localization as occurs with the Klein paradox. This is seen clearly in the solutions below in Sec. IV.

To ensure that E is now always positive, we require that E be a minimum with respect to arbitrary variation of g. Now

$$E = \int d^3x[\tfrac{1}{2}(\vec{\nabla}g)^2 + H(g^2 - f^2)^2] + \mathcal{E}(g)\,, \tag{3.32}$$

and we have imposed the restriction from Eq. (3.5) that

$$\int d^3x\, \chi^\dagger(x)\chi(x) = 1\ . \tag{3.33}$$

Since

$$\begin{aligned} \frac{\delta\mathcal{E}}{\delta g(x)} &= G\bar{\chi}\chi + \int d^3x' \left[\chi^\dagger\left(\frac{\vec{\alpha}}{i}\cdot\vec{\nabla} + G\beta g\right)\frac{\delta\chi}{\delta g} + \frac{\delta\chi^\dagger}{\delta g}\left(\frac{\vec{\alpha}}{i}\cdot\vec{\nabla} + G\beta g\right)\chi\right] \\ &= G\bar{\chi}\chi + \mathcal{E}(g)\,\frac{\delta}{\delta g(x)}\int d^3x'\chi^\dagger\chi \\ &= G\bar{\chi}\chi \end{aligned} \tag{3.34}$$

the condition $\delta E/\delta g = 0$ leads to the classical field equation

$$\nabla^2 g - 4Hg(g^2 - f^2) = G\bar{\chi}\chi\ . \tag{3.35}$$

Equations (3.31), (3.33), and (3.35) are the same as if we had applied the variational principle to (3.30) with the restriction (3.33). $\mathcal{E}(g)$ then appears as the Lagrangian multiplier enforcing the normalization condition.[18]

IV. SOLUTION OF THE COUPLED EQUATIONS

Unlike the one-dimensional case discussed in the introductory survey, we have not found an exact solution of the coupled differential equations in three dimensions. However, in the strong-coupling limit, we have obtained the leading terms of a solution and the order of magnitude of the small corrections. The solution in this case is very similar to that of the one-dimensional problem.

Rather than simply displaying this solution, we show how it comes about by following a more heuristic procedure. We do this in view of its surprising nature of confining the fermion field to a thin shell or bubble surface. First, then, we attempt to construct a solution of the type discussed in Sec. II A, in which the effective mass of the trapped fermion field is zero. Finding that the coupled equations do not allow such a solution, we will be led to a solution in which the effective mass of the trapped fermion field is large and negative. We will find that only these bound states have positive energies that are small compared to the bare masses in the theory.

Since we are seeking the lowest energy state, we expect the classical field g and its source $\bar{\chi}\chi$ to be spherically symmetric. The equations we wish to solve are then

$$\left(\frac{1}{i}\,\vec{\alpha}\cdot\vec{\nabla}+G\beta g\right)\chi=\mathcal{E}\chi\ , \tag{4.1}$$

$$\frac{d^2g}{dr^2}+\frac{2}{r}\,\frac{dg}{dr}-4Hg(g^2-f^2)=G\bar{\chi}\chi\ . \tag{4.2}$$

Our strategy is to make a guess for g, solve the Dirac equation for χ, and then check the Klein-Gordon equation for consistency. According to the heuristic arguments originally presented, we first choose g so that the fermion field has zero effective mass inside a small region of space. That is, we take

$$g(r)=f\,\theta(R-r) \tag{4.3}$$

as illustrated in Fig. 1, where R, the radius of the potential, is to be determined by minimizing E, just as in Eq. (2.6). The solution of the Dirac equation in a spherical square well is well known; we have

$$\chi_l=A_l\begin{bmatrix} ij_l(kr)\phi^{\pm}_{jm} \\ \mp j_{l\pm1}(kr)\vec{\sigma}\cdot\hat{r}\phi^{\pm}_{jm}\end{bmatrix},\quad r<R \tag{4.4}$$

$$=B_l\begin{bmatrix} ik_l(\lambda r)\phi^{\pm}_{jm} \\ -\dfrac{\lambda}{\mathcal{E}_l+Gf}\,k_{l\pm1}(\lambda r)\vec{\sigma}\cdot\hat{r}\phi^{\pm}_{j,m}\end{bmatrix},\quad r>R\ . \tag{4.5}$$

Here the $\pm$ sign corresponds to $j=l\pm\frac{1}{2}$; the j_l and k_l are spherical Bessel functions of half-integral order,

$$k\equiv(\mathcal{E}_l{}^2-m_{\text{eff}}{}^2)^{1/2}\equiv\mathcal{E}_l\ ,$$
$$\lambda\equiv(G^2f^2-\mathcal{E}_l{}^2)^{1/2}\ . \tag{4.6}$$

$\phi^{\pm}_{j,m}$ are two component angular solutions with $j=l\pm\frac{1}{2}$, respectively, and A_l, B_l are normalization constants.[19] For $Gf\gg\mathcal{E}$ and $Gf\gg1/R$, we have $k_l(\lambda R)\cong e^{-\lambda R}/\lambda R$ and continuity at $r=R$ implies the eigenvalue condition

$$j_l(kR)=\pm j_{l\pm1}(kR)\ . \tag{4.7}$$

Equation (4.7) has solutions with $k\sim O(1/R)$. For instance, for $l=0$ we find a ground-state energy at $\mathcal{E}\cong2/R$ with higher energy states spaced at intervals $\sim O(1/R)$. However, this solution is not consistent with the Klein-Gordon equation (4.2). For $r>R$ and far enough from the surface, both sides of Eq. (4.2) are ~0, but for $r<R$ and far enough from the surface, the left-hand side is ~0 but the right-hand side of Eq. (4.2) is $\sim G/R^3$. Thus we are not able to construct such a "heuristic" solution.

The one-dimensional example suggests that we look for a solution with the fermion field confined to the surface $r\sim R$ so that the source term on the right-hand side of Eq. (4.2) will also vanish for $r<R$. This means making the inside potential very deep so that the fermion will have an effective mass $|Gg|\gg\mathcal{E}$ and thereby be restricted to a thin shell near $r\sim R$. In particular, we choose $g=-f$ for $r<R$ and far enough from the surface so that each term in Eq. (4.2) vanishes. Near the surface we expect also as guided by the one-space, one-time dimensional result that $\bar{\chi}\chi\ll\chi^\dagger\chi$ and also in the strong-coupling limit

$$\frac{2}{r}\,\frac{dg}{dr}\sim\frac{f}{RD}\ll\frac{d^2g}{dr^2}\sim\frac{f}{D^2}=\frac{f}{RD}\left(\frac{R}{D}\right)\ .$$

Following Lee and Wick,[11] we solve Eq. (4.2) first by neglecting the $\bar{\chi}\chi$ source term and $(2/r)\,dg/dr$, so

$$\frac{d^2g}{dr^2}-4Hg(g^2-f^2)=0\ . \tag{4.8}$$

This is identical to Eq. (2.18) and we obtain

$$g(x)=f\tanh(\sqrt{2H}\,f(r-R))\ , \tag{4.9}$$

where one of the two integration constants is chosen so that g approaches its vacuum value $g=f$ at large distances. The other constant, the radius R, is adjusted later to minimize the total energy. It is shown that the two neglected terms then cancel on the "average."

The details of the Dirac wave function χ in the transition region depend on the relative magnitudes of G and H. However, the total energy of the state and the optimum choice of R are determined by H alone. To illustrate these points, we consider two extreme cases: (i) $\sqrt{H}\gg G\gg1$, and (ii) $G\gg\sqrt{H}\gg1$.

(i) $\sqrt{H}\gg G\gg1$. In this case, it is a good approximation to replace $g(x)$ by a square-well potential. We are therefore invited to solve the Dirac equation in the potential

$$\begin{aligned} g(x) &= +f\,, \quad r>R \\ &= -f\,, \quad r<R\,. \end{aligned} \tag{4.10}$$

Following the standard procedure for solving the Dirac equation in a central potential, we make the decomposition

$$\chi = \begin{pmatrix} i\,\dfrac{G_l(r)}{r}\,\phi_{jm} \\ \dfrac{F_l(r)}{r}\,\dfrac{\vec\sigma\cdot\vec r}{r}\,\phi_{jm} \end{pmatrix}\,, \tag{4.11}$$

where we have adopted the notations of Ref. 19. It is immediately clear that for a spherically symmetric potential $g(x)$, the only solutions of Eq. (4.1) which are consistent with Eq. (4.2) are those with $j=\frac{1}{2}$ (or $l=0$). Otherwise, the right-hand side of Eq. (4.2) has an angular variation while its left-hand side does not. From now on we will restrict ourselves to the case $l=0$.

The radial wave functions satisfy the equations

$$\begin{aligned} \frac{dG_0}{dr} &= \frac{1}{r}\,G_0 + (\mathcal{E}+Gg)F_0\,, \\ \frac{dF_0}{dr} &= -\frac{1}{r}\,F_0 - (\mathcal{E}-Gg)G_0\,. \end{aligned} \tag{4.12}$$

In the limit $Gf \gg \mathcal{E}$, the solutions are $[\lambda \equiv (G^2f^2-\mathcal{E}^2)^{1/2}]$

$$\begin{aligned} G_0 &= A\sinh(\lambda r)\,, \quad r<R \\ F_0 &= \frac{A\lambda}{\mathcal{E}-Gf}\left(\cosh(\lambda r) - \frac{\sinh(\lambda r)}{\lambda r}\right)\,, \\ G_0 &= Be^{-\lambda R}\,, \quad r>R \\ F_0 &= -\frac{B\lambda}{\mathcal{E}+Gf}\left(1+\frac{1}{\lambda r}\right)e^{-\lambda r}\,, \end{aligned} \tag{4.13}$$

where the eigenvalue $\mathcal{E}$,

$$\mathcal{E} = \frac{1}{R}\,, \tag{4.14}$$

is determined by the continuity of G_0/F_0 at $r=R$. The normalization condition determines

$$A = \frac{Be^{-\lambda R}}{\sinh\lambda R} = \left(\frac{\lambda}{2\pi}\right)^{1/2} e^{-\lambda R}\,. \tag{4.15}$$

The wave function χ is concentrated in the region $r\sim R$. We now compute and compare $\chi^\dagger\chi$ and $\bar\chi\chi$:

$$\begin{aligned} \chi^\dagger\chi &= \frac{\lambda}{4\pi R^2}\,e^{-2\lambda|r-R|}\,, \\ \bar\chi\chi &= \frac{1}{4\pi}\,\frac{1}{R^3}\left(\left|1-\frac{r}{R}\right| + \frac{1}{2\lambda R}\right)e^{-2\lambda|r-R|} \\ &= \frac{1}{\lambda R}\left(\left|1-\frac{r}{R}\right| + \frac{1}{2\lambda R}\right)\chi^\dagger\chi\,, \end{aligned} \tag{4.16}$$

which are appreciably different from zero only in the transition region $r\sim R$. These features of the solution are illustrated in Fig. 3.

We can now take into account the effects of the source term. A condition for R is supplied by requiring the two neglected terms to compensate each other on the average. This condition for R is obtained by multiplying Eq. (4.2) by dg/dr and integrating over r across the transition region. We obtain

$$\int dr\,\frac{2}{r}\left(\frac{dg}{dr}\right)^2 = \int dr\left(\frac{dg}{dr}\right)G\bar\chi\chi\,. \tag{4.17}$$

These integrals are insensitive to the precise upper and lower limits since the integrands decrease rapidly away from $r=R$. According to Eqs. (4.9) and (4.16), $G\bar\chi\chi$ is a much more slowly varying function near $r\sim R$ than dg/dr. Thus we can replace $G\bar\chi\chi$ by its value at $r=R$ in Eq. (4.17). Making use of (4.9) and (4.16), we find that (4.17) gives

$$\frac{1}{R^3} = \tfrac{32}{3}\,\pi\sqrt{2H}f^3\,, \tag{4.18}$$

which is independent of G. The value of R as given by Eq. (4.18) also minimizes the total energy. This can be verified explicitly or by the following formal argument.

The total energy E is given by

$$\begin{aligned} E &= \int d^3x\left[\tfrac{1}{2}(\vec\nabla g)^2 + H(g^2-f^2)^2\right] + \mathcal{E} \\ &= 4\pi R^2\int dr\left(\frac{dg}{dr}\right)^2 + \mathcal{E}\,, \end{aligned} \tag{4.19}$$

which follows from the fact that the two terms in the field energy are equal (4.18) and dg/dr is nonvanishing only when $r\sim R$. If we make the explicit

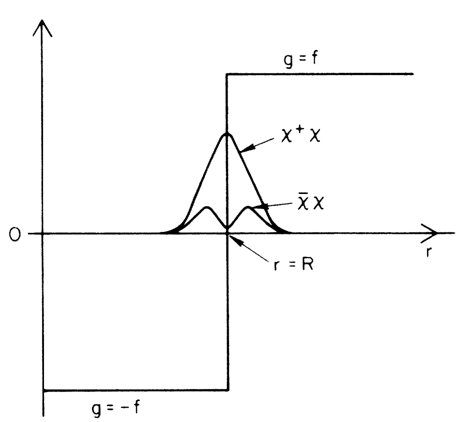

FIG. 3. The solutions to Eqs. (4.1) and (4.2) for the case $\sqrt{H} \gg G \gg 1$.

substitution $g(r) = f\tanh(\sqrt{2H}\,(r-R))$, we find

$$E = \tfrac{16}{3}\pi\sqrt{2H}\,R^2 f^3 + \frac{1}{R}\ , \tag{4.20}$$

which is a minimum at the value of R given in Eq. (4.18). We notice that $\frac{2}{3}$ of the total energy is due to the fermion:

$$E = \frac{3}{2}\ \frac{1}{R}$$
$$= 3(\tfrac{4}{3}\pi)^{1/3}(2H)^{1/6}f\ .$$

A more general way to see that the value of R determined from the integrated meson equation (4.17) minimizes the total energy is to take the derivative of Eq. (4.19) with respect to R. We obtain

$$\frac{\partial E}{\partial R} = 8\pi R\int dr\left(\frac{dg}{dr}\right)^2 + \frac{\partial \mathcal{E}}{\partial R}\ . \tag{4.21}$$

But using Eq. (4.1) we find

$$\frac{\partial \mathcal{E}}{\partial R} = G\int d^3x\left(\frac{dg}{dR}\right)\bar{\chi}\chi$$
$$= -4\pi R^2 G\int dr\left(\frac{dg}{dr}\right)\bar{\chi}\chi\ . \tag{4.22}$$

Consequently $\partial E/\partial R = 0$ implies

$$\frac{2}{R}\int dr\left(\frac{dg}{dr}\right)^2 = G\int dr\left(\frac{dg}{dr}\right)\bar{\chi}\chi\ , \tag{4.23}$$

which coincides with Eq. (4.17). From the inequality (2.29), $H^{1/6} \ll G$, we see that $\bar{\chi}\chi \ll \chi^\dagger\chi$, since it follows from Eq. (4.16) that

$$\bar{\chi}\chi \sim (H^{1/6}/G)^2\chi^\dagger\chi \ll \chi^\dagger\chi\ .$$

(ii) $G \gg \sqrt{H} \gg 1$. In this case the Dirac wave function χ is still given by Eq. (4.13) when $|r-R| \gg 1/\sqrt{2H}f$. However, in the transition region $r\sim R$, a better representation for χ can be found as follows: Introduce the notation

$$u_\pm = G \pm F\ ,$$
$$u_+ = \rho u_-\ .$$

Then Eq. (4.12) becomes

$$\begin{aligned}\frac{du_-}{dr} &= -Ggu_- + \left(\mathcal{E} + \frac{1}{r}\right)\rho u_-\ ,\\ \frac{d\rho}{dr} &= 2Gg\rho + \left(\frac{1}{r} - \mathcal{E}\right) - \rho^2\left(\frac{1}{r} + \mathcal{E}\right)\ .\end{aligned} \tag{4.24}$$

For a solution where g is $+f$ outside the well ($r > R$) and falls to $-f$ inside the well ($r < R$), $\rho = +1$ at $r = 0$ and rapidly decreases away from the origin $r = 0$. At the same time $u_-(r)$ is exponentially increasing toward the surface ($r\sim R$). Hence we only have to solve the equations away from the origin where they reduce to

$$\begin{aligned}\frac{du_-}{dr} &= -Ggu_-\ ,\\ \frac{d\rho}{dr} &= 2Gg\rho + \left(\frac{1}{r} - \mathcal{E}\right)\ .\end{aligned} \tag{4.25}$$

We find immediately that

$$u_- = C[\cosh(\sqrt{2H}f(r-R))]^{-G/\sqrt{2H}}\ . \tag{4.26}$$

The stability of the ρ equation implies that $(1/r) - \mathcal{E} \cong 0$ when $g(x)$ changes sign; hence the eigenvalue for the quark energy $\mathcal{E}$ is

$$\mathcal{E} = \frac{1}{R}\ . \tag{4.27}$$

We notice that $u_\pm$ varies more rapidly than the potential $g(x)$. The half-width of $u_\pm$ is given by $1/G^{1/2}H^{1/4}f$ in contrast with the half-width of $1/Gf$ in Eq. (4.13) of the previous case. Thus we can make the approximation

$$Gg(x) \cong G\sqrt{2H}\,f^2(r-R),\quad r\sim R\ . \tag{4.28}$$

Then Eq. (4.25) gives

$$\rho = +\frac{1}{2G\sqrt{2H}\,f^2R^2} \ll 1\ . \tag{4.29}$$

The normalization condition (3.33) for χ implies

$$\int dr\,u_-^2 = \frac{1}{2\pi}\ . \tag{4.30}$$

Now

$$\bar{\chi}\chi = \frac{1}{R^2}\rho u_-^2$$
$$= \frac{1}{2G\sqrt{2H}f^2R^4}u_-^2\ . \tag{4.31}$$

The solutions are illustrated in Fig. 4.

The condition (4.17) must also be satisfied in the present case. But now it is dg/dr which is slowly

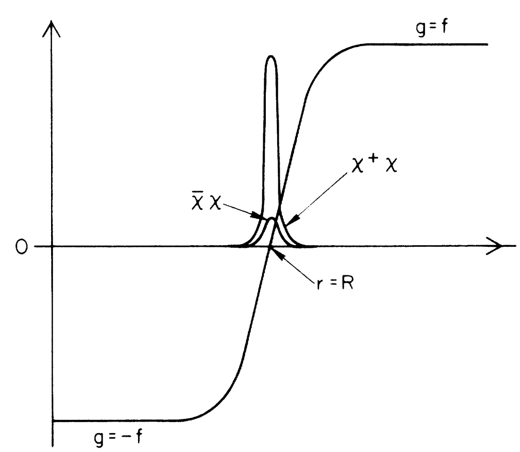

FIG. 4. The solutions to Eqs. (4.1) and (4.2) for the case $G \gg \sqrt{H} \gg 1$.

varying in the transition region, and so it can be replaced by its value at $r=R$. The result gives the same value of R as in Eq. (4.18).

Hence we have shown as claimed that the size and energy of the bound states are determined by H alone, independent of whether $1 \ll G \ll \sqrt{H}$ or $1 \ll \sqrt{H} \ll G$. This conclusion is also valid in the intermediate range of parameters $G \sim \sqrt{H} \gg 1$, although the detailed shape of the wave function is sensitive to G in the transition region as g changes from f to $-f$.

V. STATES WITH NONVANISHING AVERAGE MOMENTUM

In the calculation of the expectation value of the Hamiltonian, we have neglected terms associated with normal-ordering. Since these terms depend on the scalar field $g(x)$ and the wave function χ_n of the fermion, they are different for different states. The question arises whether this is a consistent prescription. In this section we will show that at least this is a Lorentz-covariant approximation. For this purpose we will extend our variational principle to states with nonzero average momentum. As can be verified, the states we constructed above have zero average momentum.

We will be able to establish that the average momentum and energy of such a state are related to the energy of the corresponding state with zero average momentum by the mass shell condition:

$$E_v = M/(1-v^2)^{1/2}\ ,\qquad \vec{P}_v = M\vec{v}/(1-v^2)^{1/2}\ , \tag{5.1}$$

where $\vec{v}=\vec{P}_v/E_v$ is the average velocity of the state and M is E_v at $v=0$.

Again let us first illustrate our procedure in the one-dimensional example without fermions. Then

$$\mathfrak{H} = \int dx\left[\tfrac{1}{2}\dot{\sigma}^2 + \frac{1}{2}\left(\frac{d\sigma}{dx}\right)^2 + H(\sigma^2-f^2)^2\right]\ ,\qquad P_{\rm op} = \int dx\left(-\dot{\sigma}\,\frac{d\sigma}{dx}\right)\ . \tag{5.2}$$

We are interested in minimizing the energy in a state $|v\rangle$

$$E_v = \langle v|\mathfrak{H}|v\rangle \tag{5.3}$$

with the constraint

$$P_v = \langle v|P_{\rm op}|v\rangle\ . \tag{5.4}$$

The coherent states (3.8) automatically give $\langle P_{\rm op}\rangle = 0$. To construct a state with nonvanishing average momentum, let us consider

$$|v\rangle = \exp\left(i\int dx\, g_0(x)\sigma(x)\right) \times \exp\left(-i\int dx\, g_1(x)\dot{\sigma}(x)\right)|0\rangle\ , \tag{5.5}$$

where $g_0(x)$ and $g_1(x)$ are arbitrary real functions to be determined by the variational principle. Now

$$\exp\left(-i\int dx\, g_0(x)\sigma(x)\right) f(\dot{\sigma}) \exp\left(i\int dx\, g_0(x)\sigma(x)\right) = f(\dot{\sigma}+g_0)\ . \tag{5.6}$$

Again using the same normal-ordering prescription, we find

$$E_v - wP_v = \langle v|\mathfrak{H} - w\cdot P_{\rm op}|v\rangle = \int dx\left[\tfrac{1}{2}g_0^2 + \frac{1}{2}\left(\frac{dg_1}{dx}\right)^2 + H(g_1^2-f^2)^2 + g_0 w\,\frac{d}{dx}g_1\right]\ , \tag{5.7}$$

where w is a Lagrange multiplier to take care of the momentum constraint (5.4). The variational principle leads to[20]

$$g_0 + w\,\frac{dg}{dx}g_1 = 0\ ,\qquad -w\,\frac{d}{dx}g_0 - \frac{d^2}{dx^2}g_1 + 4Hg_1(g_1^2-f^2) = 0\ , \tag{5.8}$$

and

$$E_v - wP_v = \int dx\left[-\frac{1}{2}\left(w\,\frac{d}{dx}g_1\right)^2 + \frac{1}{2}\left(\frac{d}{dx}g_1\right)^2 + H(g_1^2-f^2)^2\right]\ . \tag{5.9}$$

Without solving Eq. (5.8) we would intuitively expect that $g_1(x)$ is obtained from $g(x)$ for Lorentz contraction. This is borne out by explicit construction. Let $g(x)$ be the solution for the problem with $P_v=0$, so that $g(x)$ satisfies Eq. (2.18) without the source term. We will show that the choice

$$w = v\ ,\qquad g_1(x) = g(\gamma x),\quad \gamma = \frac{1}{(1-v^2)^{1/2}} \tag{5.10}$$

satisfies the differential equation (5.8). To see this we define

$$x' = \gamma x\ . \tag{5.11}$$

Then

$$w^2 \frac{d^2 g_1(x)}{dx^2} - \frac{d^2 g_1(x)}{dx^2} = -(1-w^2)\left(\frac{dx'}{dx}\right)^2 \frac{d^2 g(x')}{dx'^2}$$
$$= -(1-w^2)\gamma^2 \frac{d^2 g(x')}{dx'^2}$$
$$= -\frac{d^2 g(x')}{dx'^2} \quad . \tag{5.12}$$

The energy also simplifies. Using the definition (5.10) in Eq. (5.7), we have

$$E_v - vP_v = \frac{1}{\gamma}\int dx' \left[\tfrac{1}{2}(1-w^2)\left(\frac{d}{dx}g_1(x)\right)^2 + H[g_1(x)^2 - f^2]^2\right]$$
$$= \frac{1}{\gamma}\int dx' \left[\frac{1}{2}\left(\frac{dg(x')}{dx'}\right)^2 + H[g(x')^2 - f^2]^2\right]$$
$$= \frac{1}{\gamma} M , \tag{5.13}$$

where

$$M = \int dx \left[\frac{1}{2}\left(\frac{dg(x)}{dx}\right)^2 + H[g(x)^2 - f^2]^2\right] \tag{5.14}$$

is the energy in the rest frame.

Another relation between E and P is supplied by the field equation (5.8):

$$M = \int dx \left(\frac{dg}{dx}\right)^2 . \tag{5.15}$$

Also

$$P_v = -\int dx g_0 \frac{d}{dx} g_1$$
$$= v\int dx \left(\frac{dg_1}{dx}\right)^2$$
$$= v\int dx \left(\frac{dg(\gamma x)}{dx}\right)^2$$
$$= v\frac{1}{\gamma}\int dx' \left(\frac{dg(x')}{dx'}\right)^2$$
$$= v\gamma \int dx' \left(\frac{dg(x')}{dx'}\right)^2$$
$$= Mv\gamma . \tag{5.16}$$

Thus E_v and P_v satisfy the mass shell conditions (5.1) and (5.2).

We now proceed to discuss the three-dimensional case. The momentum operator is given by

$$\vec{P}_{op} = \int d^3x \left(-\dot{\sigma}\vec{\nabla}\sigma + \psi^\dagger \frac{1}{i}\vec{\nabla}\psi\right) . \tag{5.17}$$

The trial state in this case is

$$|\vec{v}\rangle = \exp\left(i\int d^3x g_0(x)\sigma(x)\right)$$
$$\times \exp\left(-i\int d^3x g_1(x)\dot{\sigma}(x)\right) B_0^\dagger |0_L\rangle \quad . \tag{5.18}$$

With our usual normal-ordering prescription for $\mathfrak{H}$ and P_{op}, we find

$$E_{\vec{v}} \equiv \langle\vec{v}|\mathfrak{H}|\vec{v}\rangle$$
$$= \int d^3z \left[\tfrac{1}{2}g_0{}^2 + \tfrac{1}{2}(\vec{\nabla}g_1)^2 + H(g_1{}^2 - f^2)^2 + \chi_1^\dagger\left(\frac{\vec{\alpha}\cdot\vec{\nabla}}{i} + \beta G g_1\right)\chi_1\right] ,$$
$$\vec{P}_{\vec{v}} \equiv \langle\vec{v}|\vec{P}_{op}|\vec{v}\rangle$$
$$= \int d^3z \left(-g_0\vec{\nabla}g_1 + \chi_1^\dagger \frac{1}{i}\vec{\nabla}\chi_1\right) . \tag{5.19}$$

The fact that $|\vec{v}\rangle$ is a single-fermion state gives the constraint

$$Q \equiv \int d^3x \chi_1^\dagger \chi_1$$
$$= 1 . \tag{5.20}$$

Introduce the Lagrange multipliers $\vec{w}$ and $\mathcal{E}_1$ to take care of the constraints (5.19) and (5.20). From the variational principle

$$\delta(E_v - \vec{w}\cdot\vec{P}_{\vec{v}} - \mathcal{E}_1 Q) = 0 \tag{5.21}$$

we find the equations

$$g_0 + \vec{w}\cdot\vec{\nabla}g_1 = 0 ,$$
$$-(\vec{w}\cdot\vec{\nabla})g_0 - \nabla^2 g_1 + 4Hg_1(g_1{}^2 - f^2) + G\bar{\chi}\chi_1 = 0 , \tag{5.22}$$
$$\left(\frac{\vec{\alpha}\cdot\vec{\nabla}}{i} + \beta G g_1\right)\chi_1 = \left(\mathcal{E}_1 + \frac{1}{i}\vec{w}\cdot\vec{\nabla}\right)\chi_1 .$$

Again we expect that both $g_1(x)$ and $\chi_1(x)$ are related to $g(x)$ and $\chi(x)$ for $\vec{P}_{\vec{v}} = 0$ states by a Lorentz transformation. Let $g(x,y,z)$, $\chi(x,y,z)$, and $\mathcal{E}$ be the solutions to these equations in the rest frame. Then if we assume the average momentum is along the x axis, we can verify that the boosted functions

$$g_1(x) = g(\gamma x, y, z)$$
$$= g(x', y, z) , \quad x' = \gamma x , \quad \mathcal{E}_1 = \mathcal{E}/\gamma \tag{5.23}$$
$$\chi_1(x) = S(\Lambda)\chi(\gamma x, y, z)e^{i\mathcal{E}v\gamma x} , \quad \vec{w} = \vec{v} , \quad \gamma = 1/(1-v^2)^{1/2}$$

satisfy the field equations. The matrix $S(\Lambda)$ transforms the Dirac spinor properly under a Lorentz boost. The matrices γ^μ transform according to

$$S^{-1}\gamma^0 S = \gamma(\gamma^0 + v\gamma^x) ,$$
$$S^{-1}\gamma^x S = \gamma(\gamma^x + v\gamma^0) . \tag{5.24}$$

Since

$$\bar{\chi}_1\chi_1(x)=\bar{\chi}\chi(x') \tag{5.25}$$

and

$$(w^2-1)\frac{d^2}{dx^2}=(w^2-1)\gamma^2\frac{d^2}{dx'^2}$$
$$=-\frac{d^2}{dx'^2}. \tag{5.26}$$

g_1 obviously **satisfies** the Klein-Gordon equation (5.22). To verify the Dirac equation, let us rewrite Eq. (5.22) as

$$[\gamma^0(\mathcal{E}_1+\vec{v}\cdot\vec{p})-\vec{\gamma}\cdot\vec{p}-Gg_1]\chi_1=0\,, \quad \vec{p}\equiv\frac{1}{i}\vec{\nabla} \tag{5.27}$$

and use

$$S^{-1}(\Lambda)[\gamma^0(\mathcal{E}_1+vp_x)-\gamma_x p_x]S(\Lambda)$$
$$=\gamma[\gamma^0\mathcal{E}_1-\gamma_x p_x(1-v^2)+\gamma_x v\mathcal{E}_1]\,, \tag{5.28}$$
$$P_x[\chi(\gamma x,y,z)e^{i\mathcal{E}v\gamma x}]$$
$$=[p_x\chi(x',y,z)]e^{i\mathcal{E}v\gamma x}+\gamma\mathcal{E}v\chi e^{i\mathcal{E}v\gamma x}\,.$$

We find finally

$$[\gamma^0\mathcal{E}-\gamma_x p_{x'}-\gamma_y p_y-\gamma_z p_z-Gg_1]\chi(x',y,z)=0\,. \tag{5.29}$$

By the same scaling as in Eq. (5.23), we get

$$E_v-w\cdot P_v=\frac{1}{\gamma}M\,, \tag{5.30}$$

where

$$M=\int d^3x[\tfrac{1}{2}(\vec{\nabla}g)^2+H(g^2-f^2)^2]+\mathcal{E} \tag{5.31}$$

is the energy in the rest frame.

In the one-dimensional case, a second relation is supplied by the field equation. To find another relation between E_v and P_v in this case, we will make use of general properties of the stress-energy tensor. To fully utilize the covariance of a classical field theory, we introduce a set of time-dependent functions from Eq. (5.23):

$$g_1(x,y,z,t)\equiv g(\gamma(x-vt),y,z)\,,$$
$$g_0(x,y,z,t)\equiv-\vec{v}\cdot\vec{\nabla}g_1(x,y,z,t)$$
$$=\frac{\partial}{\partial t}g_1\,, \tag{5.32}$$
$$\chi_1(x,y,z,t)\equiv S(\Lambda)\chi(\gamma(x-vt),y,z)$$
$$\times e^{-i\mathcal{E}\gamma(t-vx)}.$$

We verify that

$$i\frac{\partial}{\partial t}\chi_1=\left(\frac{1}{i}\vec{v}\cdot\vec{\nabla}+\mathcal{E}_1\right)\chi_1\,. \tag{5.33}$$

Thus we can cast the equations for g_1 and χ_1 in the covariant form

$$(i\gamma^\mu\partial_\mu-Gg_1)\chi_1=0\,,$$
$$\partial^2 g_1+4Hg_1(g_1^2-f^2)=-G\bar{\chi}_1\chi_1\,. \tag{5.34}$$

Furthermore,

$$E_v=\int d^3x\,T^{00}\,,$$
$$P^k_{\vec{v}}=\int d^3x\,T^{0k}\,, \tag{5.35}$$

where T^{00} and T^{0k} are components of a symmetric stress-energy tensor

$$T^{\mu\nu}=\tfrac{1}{4}i\bar{\chi}_1(\gamma^\mu\overleftrightarrow{\partial}^\nu+\gamma^\nu\overleftrightarrow{\partial}^\mu)\chi_1+\partial^\mu g_1\partial^\nu g_1$$
$$-g^{\mu\nu}[\tfrac{1}{2}(\partial_\lambda g_1)^2-H(g_1^2-f^2)^2]\,. \tag{5.36}$$

It follows from the covariant field equation (5.34) that $T^{\mu\nu}$ is conserved:

$$\partial_\mu T^{\mu\nu}=\partial_\nu T^{\mu\nu}=0\,. \tag{5.37}$$

But from the explicit construction for g_1 and χ_1 we have

$$\partial_0 T^{00}=-v\frac{\partial}{\partial x}T^{00}\,. \tag{5.38}$$

Then, in terms of $T^{\mu\nu}$, we find the momentum components to be

$$P^1_{\vec{v}}=\int d^3x\,T^{01}$$
$$=-\int d^3x\left(x\frac{\partial}{\partial x}T^{01}\right)$$
$$=\int d^3x\left[x\left(\frac{\partial}{\partial y}T^{02}+\frac{\partial}{\partial z}T^{03}+\frac{\partial}{\partial t}T^{00}\right)\right]$$
$$=-\int d^3x\left(xv\frac{\partial}{\partial x}T^{00}\right)$$
$$=v\int d^3x\,T^{00}$$
$$=E_v v\,,\quad P^2_{\vec{v}}=P^3_{\vec{v}}=0\,. \tag{5.39}$$

Finally we obtain

$$\gamma(E_{\vec{v}}-\vec{v}\cdot\vec{P}_{\vec{v}})=M\,,\quad \vec{P}_{\vec{v}}=E_{\vec{v}}\vec{v}\,. \tag{5.40}$$

Equation (5.40) gives

$$E_{\vec{v}}=\frac{M}{(1-P^2)^{1/2}}\,,$$
$$\vec{P}_{\vec{v}}=\frac{M\vec{v}}{(1-v^2)^{1/2}}\,, \tag{5.41}$$

and

$$E_{\vec{v}}=(P^2_{\vec{v}}+M^2)^{1/2}\,. \tag{5.42}$$

This is a nonlinear relation between the energies

at rest and with average momentum P. Establishment of Eq. (5.42) lends some credence to our seemingly noncovariant normal-ordering prescription.

VI. THE MULTIQUARK STATES

Multiquark bound states may be constructed using the variational method discussed in Sec. III. The variational state consists of a coherent scalar field plus quarks and antiquarks. As in the Hartree-Fock approximation, the quarks and antiquarks are assumed to move in the self-consistent scalar field, the source of which contains contributions from all of the quarks and antiquarks in the state.[21] As for the single quark, the multiquark states are those which minimize the expectation value of the energy. The potential $g(x)$ is similar to the Hartree-Fock field in atomic physics and the (anti) quarks move in the ground states of this self-consistent potential.

To be more explicit, we consider multiquark states of the type

$$|S_N\rangle = U(g)C_1^\dagger \cdots C_N^\dagger |0_L\rangle\,, \tag{6.1}$$

where $C^\dagger$ creates quarks ($B^\dagger$) or antiquarks ($D^\dagger$) in states corresponding to the potential $g(x)$, which defines the coherent state for the scalar field. The energy functional becomes

$$\begin{aligned} E &\equiv \langle S_N|\mathfrak{H}|S_N\rangle \\ &= \sum_{i=1}^{N} \mathcal{E}_i + \int d^3x[\tfrac{1}{2}(\vec{\nabla}g)^2 + H(g^2-f^2)^2]\,, \end{aligned} \tag{6.2}$$

where the quark energies are given by the solution to the Dirac equation

$$\left(\frac{1}{i}\vec{\alpha}\cdot\vec{\nabla} + G\beta g\right)\chi_i = \mathcal{E}_i\chi\,, \tag{6.3}$$

and $g(x)$ is determined by

$$\nabla^2 g - 4Hg(g^2-f^2) = G\sum_{i=1}^{N}\bar{\chi}_i\chi_i\,.$$

The solution we obtain from this system is identical in structure to the solutions found for the single-quark system.

Following our discussion of the single-quark system, we find the energy of a state with N quarks or antiquarks in the ground state to be

$$\begin{aligned} E_N &= \frac{3}{2}\frac{N}{R_N} \\ &= \frac{3}{2}\frac{N^{2/3}}{R_0}\,, \end{aligned} \tag{6.4}$$

where the radius of the system, R_N, is given in terms of the radius for a single quark, R_0, by

$$R_N = N^{1/3}R_0\,,\quad R_0 = (\tfrac{32}{3}\pi\sqrt{2H}\,)^{-1/3}f^{-1}\,.$$

We would like to discuss the hadron spectrum based on this result. First of all, one must bear in mind that any variational calculation can at best give a reasonable approximation to the first few low-lying excited states. The above formula should not be taken seriously for highly excited states. In particular, we have solved the coupled equations (6.3) only for the quarks in $l=0$ states in a spherical potential $g(x)$ as in Sec. IV.

So far, the binding mechanism produces not only the physical hadrons but also nonzero-triality particles of low masses. In Sec. VII we propose a scheme utilizing Nambu's idea of color to promote the physically unobserved states (color nonsinglets) to very high masses. Our scheme, however, leaves the physical hadrons (color singlets) unchanged with a spectrum still given by Eq. (6.3).

We will now discuss the consequence of applying Eq. (6.3) to the color singlets.

A. $q\bar{q}$ system

Both q and $\bar{q}$ are in the $l=0$ states. These states have odd parity since $q\bar{q}$ has an odd intrinsic parity. They consist of the 0^- pseudoscalar and 1^- vector mesons. These are the $\underline{35}$ in SU(6) classification, and are degenerate with the energy

$$E_M = \frac{3}{2}\frac{1}{R_0}(2)^{2/3}\,. \tag{6.5}$$

B. qqq system

All the three quarks are in $l=0$ states. These are the positive-parity (by definition) states with $J=\frac{3}{2}$ and $\frac{1}{2}$, namely the $\underline{56}$ in SU(6) classification. Their common energy is

$$E_B = \frac{3}{2}\frac{1}{R_0}(3)^{2/3}\,. \tag{6.6}$$

Thus E_B/E_M is fixed at $(\frac{3}{2})^{2/3}$.

C. Exotic states

Among color-singlet states there are states with more than one quark-antiquark pair or three quarks. These are the exotic states. So far, there is no experimental evidence for the existence of the exotic states. According to Eqs. (6.5) and (6.6), exotic states appear in our spectrum. For example, a noninteracting two-nucleon system has a mass given by

$$\begin{aligned} E_{2B} &= 2E_B \\ &= \frac{3}{2}\frac{1}{R_0}2(3)^{2/3}\,, \end{aligned} \tag{6.7}$$

while a color-singlet 6-quark state has a mass given by

$$E_{6q} = \frac{3}{2}\frac{1}{R_0}6^{2/3}\,. \tag{6.8}$$

The two masses are related by

$$E_{2B}=2^{1/3}E_{6q}\cong 1.26E_{6q}; \tag{6.9}$$

That is, a $6q$ system has a lower mass than twice the nucleon mass. However, the $6q$ states are highly excited and, as we have emphasized, our variational treatment is more prone to fail for highly excited states. As long as we consider only $l=0$ quark states, we may not form bound states with baryon numbers greater than 2.

From the basis (6.1) we can construct color-singlet states of definite spin and unitary spin. For later applications, we gave a few examples here. Let (u,d) be the nonstrange quarks which form an isospin doublet. We will use arrows to indicate $j_z=+\frac{1}{2}$ or $j_z=-\frac{1}{2}$. For baryon states, the first quarks are red, the second blue, and the third white. For meson states, the quark and antiquark are of the same color and a summation over color is understood. All the following are states with zero average momentum:

$$\sqrt{3}\,|\,\Delta^+,j_z=\tfrac{3}{2}\rangle=|\,d_\uparrow u_\uparrow u_\uparrow\rangle+|\,u_\uparrow d_\uparrow u_\uparrow\rangle+|\,u_\uparrow u_\uparrow d_\uparrow\rangle\,, \tag{6.10}$$

$$\begin{aligned}\sqrt{18}\,|\,P,j_z=\tfrac{1}{2}\rangle=\;&2|\,u_\uparrow u_\uparrow d_\downarrow\rangle-|\,u_\uparrow u_\downarrow d_\uparrow\rangle-|\,u_\downarrow u_\uparrow d_\uparrow\rangle\\&+2|\,u_\uparrow d_\downarrow u_\uparrow\rangle-|\,u_\uparrow d_\uparrow u_\downarrow\rangle-|\,u_\downarrow d_\uparrow u_\uparrow\rangle\\&+2|\,d_\downarrow u_\uparrow u_\uparrow\rangle-|\,d_\uparrow u_\downarrow u_\uparrow\rangle-|\,d_\uparrow u_\uparrow u_\downarrow\rangle\,.\end{aligned}$$

The neutron states are obtained from the proton states by interchange of u and d quarks. For the ω meson and the π^0 we have

$$\begin{aligned}|\,\omega,j_z=0\,\rangle&=\tfrac{1}{2}[\,|\,\bar u_\downarrow u_\uparrow\rangle+|\,\bar u_\uparrow u_\downarrow\rangle+|\,\bar d_\downarrow d_\uparrow\rangle+|\,\bar d_\uparrow d_\downarrow\rangle\,]\,,\\|\,\pi^0\rangle&=\tfrac{1}{2}[-|\,\bar u_\uparrow u_\downarrow\rangle+|\,\bar d_\uparrow d_\downarrow\rangle+|\,\bar u_\downarrow u_\uparrow\rangle-|\,\bar d_\downarrow d_\uparrow\rangle\,]\,.\end{aligned} \tag{6.11}$$

An over-all exponential factor $U(g)=\exp[-i\int d^3x\,g(x)\dot\sigma(x)]$, as in Eq. (6.1), is implicit in the states constructed in Eqs. (6.10) and (6.11).

VII. COLOR SYMMETRY

A. General ideas

As discussed in the previous section, the usual quark-model picture of the ground-state mesons and baryons is obtained so long as the color degree of freedom is added to the quarks.[2] The observed hadronic spectrum is consistent with the existence of only color-singlet bound states of qqq (baryons) and $\bar q q$ (mesons). However, the binding mechanism provided by a singlet scalar field does not distinguish between color-singlet and nonsinglet states giving equivalent binding to all such states including diquark states. It is clear that an additional mechanism must be introduced to unbind the undesired states. Such a mechanism was suggested by Nambu[6] which utilizes a vector interaction coupled to the color degrees of freedom.

As an example, consider the effect of coupling of nucleons via an interaction coupled to the vector isospin current. This interaction leads to a nonrelativistic description of the isospin coupling in terms of two-body potentials in the form

$$V_{ij}=V\vec{t}_i\cdot\vec{t}_j\,,\quad V>0 \tag{7.1}$$

where $\vec{t}_i$ is the isospin of the ith particle and V contains the dependence on the other degrees of freedom. The potential energy of an n-nucleon system may be estimated as

$$\begin{aligned}V(n)&=\tfrac{1}{2}\sum_{i\neq j}V_{ij}\\&=\tfrac{1}{2}V\sum_{i\neq j}\vec{t}_i\cdot\vec{t}_j\\&=\tfrac{1}{2}V[I(I+1)-nt(t+1)]\,,\end{aligned} \tag{7.2}$$

where I is the total isospin of the system and t is the nucleon isospin. This force is seen to be attractive for the deuteron ($I=0$) and repulsive for the dineutron system ($I=1$).

To extend these ideas to quark bound states, the quarks are endowed with the additional internal quantum numbers of color so that there are three triplets of quarks: red, blue, and white. The color interaction is mediated by an octet of non-Abelian gauge bosons coupled to the SU(3) vector currents of the color symmetry.

In analogy with the isospin interaction, the effective potential energy for an n-quark system is

$$V(n)=\tfrac{1}{2}V\sum_{i\neq j}\sum_{a=1}^{8}\lambda_i^a\lambda_j^a\,, \tag{7.3}$$

where $\{\lambda_i^a\}$ are octet coupling matrices to the quarks. The potential energy may be reduced to the form

$$V(n)=\tfrac{1}{2}V(C-nc)\,,\quad V>0 \tag{7.4}$$

where C is the eigenvalue of the Casimir operator for SU(3) of color for the n-particle system

$$C=\sum_a\Big(\sum_i\lambda_i^a\Big)^2\,, \tag{7.5}$$

and $c=\frac{4}{3}$ is the equivalent eigenvalue for the quark. Since C is positive-definite and has zero eigenvalue only for color-singlet states, the strongest attractive interaction occurs for those states which are color singlets.

In this section we will estimate the effects of the color interaction on the quark binding mechanism. We will demonstrate that the color interaction has the effect of increasing the energies of all color nonsinglet states to an arbitrarily high level so as to be consistent with the observed hadron spectrum.[10] However, we should also emphasize that there is an important difference between the original Nambu suggestion and our use of the color interaction. According to Nambu, the color acts as the binding force for color singlets and an unbinding force for color nonsinglets. In our approach, the binding of color singlets is provided by an effective scalar interaction with the quarks. The color interaction couples to the full color current which has zero expectation value for singlet states. Hence, the color interaction serves only to push the nonsinglet states to higher mass, leaving the color-singlet states unaffected.

In the following we will first use an Abelian gauge model to set up the formalism and demonstrate the unbinding mechanism. Then we will discuss the non-Abelian case. For clarity of presentation only the SU(2) case will be discussed in detail. Aside from mathematical complications, the treatment of color SU(3) is analogous to the SU(2) model.

B. The Abelian example

Consider a "quark" field ψ coupled to a scalar σ and a U(1) vector gauge field X_μ. The latter is also coupled to a complex scalar ϕ (Higgs field) which breaks the U(1) gauge symmetry so that the vector field X_μ attains a large mass.[22] The Lagrangian of such a system is

$$\begin{aligned}\mathcal{L} = &-\tfrac{1}{4}(\partial_\mu X'_\nu - \partial_\nu X'_\mu)^2 + [(\partial_\mu + i\zeta X'_\mu)\phi^*][(\partial^\mu - i\zeta X'^\mu)\phi] \\ &+ \tfrac{1}{2}(\partial_\mu\sigma)^2 - H(\sigma^2 - f^2)^2 - H'(\phi^*\phi - f'^2)^2 \\ &+ \bar\psi'(i\not\partial + \zeta \not X')\psi' - G\bar\psi'\psi'\sigma\,,\end{aligned} \tag{7.6}$$

which is invariant under the local gauge transformation

$$\begin{aligned}&X'_\mu(x) \to X'_\mu(x) + \frac{1}{\zeta}\partial_\mu\lambda(x)\,,\\ &\psi'(x) \to e^{i\lambda(x)}\psi'(x)\,,\\ &\phi(x) \to e^{i\lambda(x)}\phi(x)\,.\end{aligned} \tag{7.7}$$

However, the vacuum is unstable and both σ and ϕ acquire a nonvanishing expectation value

$$\langle\phi\rangle_0 = f'\,,\quad \langle\sigma\rangle_0 = f\,. \tag{7.8}$$

To make explicit the nature of the spontaneous symmetry breaking, let us introduce the Kibble transformation

$$\begin{aligned}&\phi(x) = (\tfrac{1}{2})^{1/2}[\sqrt{2}f' + \rho(x)]e^{i\theta(x)/\sqrt{2}f'}\,,\\ &X'_\mu(x) = X_\mu(x) - \frac{1}{\sqrt{2}f'}\partial_\mu\theta(x)\,,\\ &\psi'(x) = e^{-i\theta(x)/\sqrt{2}f'}\psi(x)\,.\end{aligned} \tag{7.9}$$

Then Eq. (7.6) becomes

$$\begin{aligned}\mathcal{L} = &-\tfrac{1}{4}(\partial_\mu X_\nu - \partial_\nu X_\mu)^2 + \tfrac{1}{2}\zeta^2(\rho + \sqrt{2}f')^2 X_\mu X^\mu \\ &+ \tfrac{1}{2}(\partial_\mu\rho)^2 - \tfrac{1}{4}H'\rho^2(\rho + 2\sqrt{2}f')^2 \\ &+ \tfrac{1}{2}(\partial_\mu\sigma)^2 - H(\sigma^2 - f^2)^2 \\ &+ \bar\psi(i\not\partial + \zeta\not X - G\sigma)\psi\,,\end{aligned} \tag{7.10}$$

plus the additional term for Feynman rule calculations in the unitary gauge

$$\Delta\mathcal{L}_{\rm eff} = -i\delta^4(0)\ln\left(1 + \frac{\rho(x)}{\sqrt{2}f'}\right). \tag{7.11}$$

It will be further assumed that ($m_B = \sqrt{2}\,\zeta f'$ is the mass of gauge boson)

$$(m_B, m_\rho, m_\sigma, m_Q) \gg H^{1/6}f\,, \tag{7.12}$$

so the field quanta in the theory are all very heavy and not presently observable. This choice bounds the size of the color charge and, as will be shown below, is responsible for the large upward shift of the energy for a "charged" state—i.e., the analog of a color nonsinglet in this example. The energy density of the theory is given by

$$\begin{aligned}\mathcal{H} = &\tfrac{1}{2}(G^{0k})^2 + \tfrac{1}{2}(\vec\nabla\times\vec{X})^2 + \tfrac{1}{2}\zeta^2(\rho + \sqrt{2}f')^2(\vec{X}^2 + X_0{}^2) \\ &+ \tfrac{1}{2}\dot\rho^2 + \tfrac{1}{2}(\vec\nabla\rho)^2 + \tfrac{1}{4}H'\rho^2(\rho + 2\sqrt{2}f')^2 \\ &+ \tfrac{1}{2}\dot\sigma^2 + \tfrac{1}{2}(\vec\nabla\sigma)^2 + H(\sigma^2 - f^2)^2 \\ &+ \psi^\dagger\left(\frac{1}{i}\vec\alpha\cdot\vec\nabla + G\beta\sigma + \zeta\vec\alpha\cdot\vec{X}\right)\psi\,,\end{aligned} \tag{7.13}$$

where the dependent variable X_0 is given by

$$X_0 = \frac{1}{\zeta^2(\rho + \sqrt{2}f')^2}(\partial_k G^{0k} - \zeta\bar\psi\gamma^0\psi) \tag{7.14}$$

and

$$G^{0k} = \partial^0 X^k - \partial^k X^0\,. \tag{7.15}$$

The canonical variables are X^k and G^{0l} which satisfy the commutation relation

$$[X^k(x), G^{0l}(x')] = i\delta^{kl}\delta^3(x - x')\,. \tag{7.16}$$

According to the general idea of reducing the quantum problem to a classical field theory, we may consider the following trial state of quark number one:

$$|S'\rangle = \exp\left(-i\int d^3x\, W^k(x)G^{0k}(x)\right)\exp\left(i\int d^3x\,\vec{E}(x)\cdot\vec{X}(x)\right)\left(-i\int d^3x\, g(x)\dot\sigma(x)\right)\exp\left(-i\int d^3x\, g'(x)\dot\rho(x)\right)B_0^\dagger|0_L\rangle\,. \tag{7.17}$$

The classical expression for the energy is

$$E' \equiv \langle S'|\mathfrak{H}|S'\rangle$$
$$= \int d^3x\Big[\tfrac{1}{2}\vec{E}^2 + \tfrac{1}{2}(\vec{\nabla}\times\vec{W})^2 + \tfrac{1}{2}\zeta^2(g'+\sqrt{2}f')^2(\vec{W}^2+W_0{}^2) + \tfrac{1}{2}(\vec{\nabla}g')^2 + \tfrac{1}{4}H'g'^2(g'+\sqrt{2}f')^2$$
$$+\tfrac{1}{2}(\vec{\nabla}g)^2 + H(g^2-f^2)^2 + \chi^\dagger\left(\frac{1}{i}\vec{\alpha}\cdot\vec{\nabla} + G\beta g + \zeta\vec{\alpha}\cdot\vec{W}\right)\chi\Big], \tag{7.18}$$

where W_0 is defined by

$$W_0 = \frac{1}{\zeta^2(g'+\sqrt{2}f')^2}(\vec{\nabla}\cdot\vec{E} - \zeta\chi^\dagger\chi). \tag{7.19}$$

To arrive at this expression for E', aside from the normal-ordering problem discussed before, one more approximation has to be introduced. This is associated with how to evaluate the $(\zeta\chi^\dagger\chi)^2$ term in $\mathfrak{H}$. Strictly speaking, $\langle S'|(\zeta\psi^\dagger\psi)^2|S'\rangle = \infty$. This term appears as

$$\langle S'|\zeta^2(\rho+\sqrt{2}f')^2X_0^2|S'\rangle = \left\langle S'\left|\frac{1}{\zeta^2(\rho+\sqrt{2}f')^2}(\partial_k G^{0k} - \zeta\psi^\dagger\psi)^2\right|S'\right\rangle$$
$$= \frac{1}{\zeta^2(g'+\sqrt{2}f')^2}\langle 0_L|B_0(\nabla\cdot E - \zeta\psi^\dagger\psi)^2B_0^\dagger|0_L\rangle$$
$$= \frac{1}{\zeta^2(g'+\sqrt{2}f')^2}[(\vec{\nabla}\cdot\vec{E} - \zeta\chi^\dagger\chi)^2 + \langle 0_L|B_0(\zeta\psi^\dagger\psi - \zeta\chi^\dagger\chi)^2B_0^\dagger|0_L\rangle]. \tag{7.20}$$

The approximation in deriving (7.18) is to ignore the fluctuation term

$$\Delta = \langle 0_L|B_0(\zeta\psi^\dagger\psi - \zeta\chi^\dagger\chi)^2B_0^\dagger|0_L\rangle; \tag{7.21}$$

that is, we set

$$\langle(j^0)^2\rangle \cong \langle j^0\rangle^2 \equiv (J^0)^2. \tag{7.22}$$

In making the approximation (7.22), we hope that the quantum fluctuation effects are small when the theory is properly regulated.

As before, the requirement that the energy E' be a minimum leads to the classical field equations

$$\begin{aligned}
&\vec{E} = \vec{\nabla}W_0,\\
&-\nabla^2W_0 + \zeta^2(g'+\sqrt{2}f')^2W_0 + \zeta\chi^\dagger\chi = 0,\\
&-\nabla^2\vec{W} + \vec{\nabla}(\vec{\nabla}\cdot W) + \zeta^2(g'+\sqrt{2}f')^2\vec{W} + \zeta\chi^\dagger\vec{\alpha}\chi = 0,\\
&-\nabla^2g' + H'(g'+\sqrt{2}f')[(g'+\sqrt{2}f')^2 - 2f'^2]\\
&\qquad + \zeta^2(g'+\sqrt{2}f')(\vec{W}^2 - W_0{}^2) = 0,\\
&-\nabla^2g + 4Hg(g^2-f^2) + G\bar{\chi}\chi = 0,\\
&\left(\frac{1}{i}\vec{\alpha}\cdot\vec{\nabla} + G\beta g - \zeta W_0 + \zeta\vec{\alpha}\cdot\vec{W}\right)\chi = \mathcal{E}\chi.
\end{aligned} \tag{7.23}$$

It is difficult to solve these coupled differential equations self-consistently. In particular, because the vector potential $\vec{W}$ appears, the Dirac equation is not a central field problem so that the total angular momentum j is not a good quantum number and the no-particle state $|0_L\rangle$ is not rotationally invariant. Since we are only interested in the qualitative differences between an electrically charged system and an electrically neutral one, we simplify the problem by first setting

$$\vec{W} = 0 \tag{7.24}$$

in the state $|S'\rangle$. Corrections for nonvanishing $\vec{W}$ will be treated perturbatively. The variational principle now gives Eq. (7.23) with $\vec{W} = 0$.

To simplify further the Dirac equation, we require that W_0 be spherically symmetric. This is the case for $l = 0$ states, since the source for W_0, $\chi^\dagger\chi$ is then spherically symmetric.

As a first step in finding a self-consistent solution to Eq. (7.23) with $\vec{W} = 0$, we consider the Dirac equation with the magnitude of the color potential $-\zeta W_0$ much smaller than the confining scalar potential $G\beta g$ and with its range corresponding to a length $D' \sim 1/2\zeta f'$ by Eqs. (7.12) and (7.23) of the order of or slightly larger than $D = 1/Gf$. On this scale we can approximately represent $G\beta g - \zeta W_0$ as illustrated in Fig. 5 and solve the simplified Dirac equation.

$$\left(\frac{\alpha\cdot\nabla}{i} + \beta Gg + V\right)\chi = \mathcal{E}\chi, \tag{7.25}$$

with

$$\begin{aligned}
g(r) &= -f, & r &< R\\
&= +f, & r &> R;\\
V(r) &= 0, & r &< R_1 = R - \tfrac{1}{2}D'\\
&= V > 0, & R_1 &< r < R_2 = R + \tfrac{1}{2}D\\
&= 0, & r &> R_2.
\end{aligned} \tag{7.26}$$

Our assumption is that $V \ll Gf$ and the thickness D' in which $V(r) \neq 0$ is of order $D' \gtrsim 1/m_B$. Since g rises to its asymptotic value within this width, we can approximate it as a step function of zero width on this scale. The solution for a $j=\frac{1}{2}$ state is given by

$$\chi = \frac{1}{r}\begin{pmatrix} iG\phi \\ F\sigma\cdot\hat{r}\phi \end{pmatrix}, \tag{7.27}$$

$$\begin{aligned}
G_{\rm I} &= A\lambda r i_0(\lambda r)\,, \\
F_{\rm I} &= -A\frac{\lambda}{Gf-\mathcal{E}}\lambda r i_1(\lambda r)\,, \\
G_{\rm II} &= \lambda' r[C_2 i_0(\lambda' r) + D_2 k_0(\lambda' r)]\,, \\
F_{\rm II} &= -\frac{\lambda'}{Gf-\mathcal{E}+V}(\lambda' r)[C_2 i_1(\lambda' r) + D_2 k_1(\lambda' r)]\,, \\
G_{\rm III} &= \lambda' r[C_3 i_0(\lambda' r) + D_3 k_0(\lambda' r)]\,, \\
F_{\rm III} &= \frac{\lambda'}{Gf+\mathcal{E}-V}\lambda' r[C_3 i_1(\lambda' r) + D_3 k_1(\lambda' r)]\,, \\
G_{\rm IV} &= -\frac{2}{\pi}B\lambda r k_0(\lambda r)\,, \\
F_{\rm IV} &= -\frac{2}{\pi}B\frac{\lambda}{Gf+\mathcal{E}}\lambda r k_1(\lambda r)\,,
\end{aligned} \tag{7.28}$$

$$\begin{aligned}
i_0(z) &= \frac{\sinh z}{z}\,, \\
i_1(z) &= \frac{\cosh z}{z} - \frac{\sinh z}{z^2}\,, \\
k_0(z) &= -\frac{\pi}{2z}e^{-z}\,, \\
k_1(z) &= \frac{\pi}{2z}\left(1+\frac{1}{z}\right)e^{-z}\,,
\end{aligned} \tag{7.29}$$

where

$$\begin{aligned}
\lambda &= [(Gf)^2 - \mathcal{E}^2]^{1/2}\,, \\
\lambda' &= [(Gf)^2 - (\mathcal{E}-V)^2]^{1/2}\,.
\end{aligned} \tag{7.30}$$

We are looking for solutions with $|\mathcal{E}| \ll Gf$. It will be verified later that it is consistent to keep only the leading terms in the asymptotic expansion of i_l and k_l:

$$\begin{aligned}
i_l(z) &\underset{z\to\infty}{\longrightarrow} \frac{1}{2z}e^{z} \quad (l=0,1)\,, \\
k_l(z) &\underset{z\to\infty}{\longrightarrow} (-1)^{l+1}\frac{\pi}{2z}e^{-z}\,.
\end{aligned} \tag{7.31}$$

Then the continuity conditions on the boundaries determine the coefficients

$$\begin{aligned}
C_2 e^{\lambda' R_1} &= A e^{\lambda R_1}\left(1+\frac{V}{2Gf}\right)\,, \\
\pi D_2 e^{-\lambda' R_1} &= A e^{\lambda R_1}\frac{V}{2Gf}\,, \\
C_3 e^{\lambda' R_2} &= A e^{\lambda R}\frac{V}{2Gf}\,, \\
\pi D_3 e^{-\lambda' R_2} &= -A e^{\lambda R}\left(1-\frac{V}{2Gf}\right)\,, \\
B e^{-\lambda R_2} &= \tfrac{1}{2}A e^{\lambda R}\,,
\end{aligned} \tag{7.32}$$

and the eigenvalue

$$\mathcal{E} = V(1-e^{-GfD'})\,. \tag{7.33}$$

The constant A is determined by the normalization condition of χ

$$A^2 = \frac{\lambda}{2\pi}e^{-2\lambda R}\,. \tag{7.34}$$

Now to be consistent with Eq. (7.32), we must impose

$$\frac{1}{R} \ll \mathcal{E} \ll Gf\,. \tag{7.35}$$

We will now check the consistency of other equations in (7.23) for the case $H^2 \gg G$. To maintain the character of the solution for g as used in solving the Dirac equation for χ, we must require that the source term $G\bar{\chi}\chi$ as well as $(2/R)\,dg/dr$ is small, i.e.,

$$|G\bar{\chi}\chi|_R = \frac{1}{4\pi}\frac{G}{R^2}\mathcal{E} \ll Hf^3 \tag{7.36}$$

and

$$\frac{1}{R}\sqrt{H}f^2 \ll Hf^3\,. \tag{7.37}$$

The equation for g' can be approximately satisfied everywhere by

$$g' + \sqrt{2}f' \simeq \sqrt{2}f' + \text{small corrections}$$

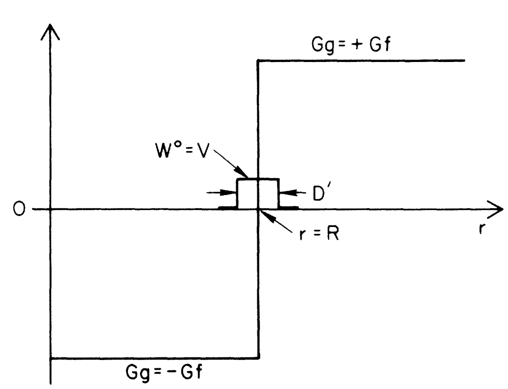

FIG. 5. The scalar potential g and the vector potential W_0 which appear in Eq. (7.23).

if

$$H'f'^2 \gg \zeta^2 W_0^2 \ . \tag{7.38}$$

We will choose the parameters so that Eq. (7.38) is satisfied.

We can now estimate the electrostatic field energy, E_J, the scalar field energies, E_g and $E_{g'}$, and the "fermion energy," E_F, and minimize the total energy to determine R for a hadron in a color-nonsinglet state. Finally we can verify the above inequalities.

The results by straightforward calculation are

$$\begin{aligned} E_J &= \int d^3x \left(\tfrac{1}{2}\vec{E}^2 + \tfrac{1}{2} m_B^2 W_0^2\right) \\ &= \tfrac{1}{2}\int d^3x\, W_0(-\nabla^2 + m_B^2)\, W_0 \\ &= -\tfrac{1}{2}\zeta \int d^3x\, W_0 J_0 \ , \end{aligned} \tag{7.39}$$

where $m_B = \sqrt{2}\,\zeta f'$. Evaluating this in the approximation used in solving the Dirac equation, i.e., setting $-\zeta W_0 = V$ as defined in Eq. (7.26), we find

$$E_J = \tfrac{1}{2} V(1 - e^{-GfD'}) \ . \tag{7.40}$$

The energy associated with the g' field is very small so that

$$E_g + E_{g'} \cong E_g = \tfrac{16}{3}\pi\sqrt{2H}\, R^2 f^3 \ . \tag{7.41}$$

Finally for the fermion energy

$$\begin{aligned} E_F &= \mathcal{E} + \zeta \int d^3x\, \chi^\dagger \chi\, W_0 \\ &= \mathcal{E} - 2E_J \\ &\simeq 0 \ . \end{aligned} \tag{7.42}$$

The total energy is the sum

$$\begin{aligned} E_T &= E_J + E_g + E_{g'} + E_F \\ &= \tfrac{1}{2}|\zeta W_0(R)|(1 - e^{-GfD'}) + \tfrac{16}{3}\pi\sqrt{2H}\, R^2 f^3 \ . \end{aligned} \tag{7.43}$$

The quantity $W_0(R)$ can be computed with the aid of Eq. (7.23), which gives

$$\begin{aligned} W_0(R) &= -\frac{\zeta}{4\pi}\int d^3x\, \frac{e^{-m_B|\vec{x}-\vec{R}|}}{|\vec{x}-\vec{R}|}\chi^\dagger(x)\chi(x) \\ &= -\frac{\zeta}{4\pi}\frac{1}{m_B^2 R^2 D}(1 - e^{-m_B D/2}) \ , \end{aligned} \tag{7.44}$$

where $D = 1/Gf$. Hence we find

$$\begin{aligned} E_T \simeq{}& \frac{\zeta^2}{8\pi m_B^2 R^2 D}(1 - e^{-m_B/2Gf})(1 - e^{-GfD'}) \\ &+ \tfrac{16}{3}\pi\sqrt{2H}\, R^2 f^3 \ . \end{aligned} \tag{7.45}$$

Its minimum, determined by $\partial E_T/\partial R = 0$, gives

$$R^4 = \frac{3}{256\pi^2}\frac{G}{\sqrt{2H}}\frac{1}{f^2 f'^2}(1 - e^{-m_B/2Gf})(1 - e^{-GfD'}) \tag{7.46}$$

and

$$\begin{aligned} E_T ={}& \tfrac{2}{3}(G\sqrt{2H})^{1/2}(1 - e^{-m_B/2Gf})^{1/2} \\ &\times(1 - e^{-GfD'})^{1/2}\left(\frac{f}{f'}\right) f \ , \end{aligned} \tag{7.47}$$

which automatically ensures Eq. (7.37). Now Eq. (7.35) is satisfied if

$$\left(\frac{G}{\sqrt{2H}}\right)^{1/2} \gg \frac{f}{f'} \gg \left(\frac{\sqrt{2H}}{G}\right)^{1/6}\left(\frac{1}{G\sqrt{2H}}\right)^{1/3} \ . \tag{7.48}$$

Equation (7.36) follows simply from the choice $G \ll H^2$. Finally Eq. (7.38) requires

$$\frac{H'^{1/4}}{(G\sqrt{2H})^{1/4}} \gg \frac{f}{f'} \ . \tag{7.49}$$

Both (7.48) and (7.49) can be satisfied without difficulty with proper choice of H' and f'. Now it follows from Eq. (7.48) that

$$Gf \gg E_T \gg H^{1/6} f \ , \tag{7.50}$$

as desired, i.e., the state of nonzero charge, analogous to color nonsinglets in the non-Abelian case of SU(3) of color, is promoted to an energy much higher than in the absence of color interactions, its magnitude depending on the specific choices of parameters ζ, f', and H'. This estimate can be further improved by adding the magnetic interaction energy in a perturbation treatment. The added terms in Eq. (7.18) are

$$\begin{aligned} E_{\vec{W}} &= \tfrac{1}{2}\int d^3x\, \vec{W}\cdot(-\nabla^2 + m_B^2)\vec{W} + \zeta\int d^3x\, \chi^\dagger \vec{\alpha}\chi\cdot\vec{W} \\ &= \tfrac{1}{2}\int d^3x\, \chi^\dagger\vec{\alpha}\chi\cdot\vec{W} \ , \end{aligned} \tag{7.51}$$

which when added to Eq. (7.43) lower the electrostatic energy by

$$\begin{aligned} E_{W_0+\vec{W}} &= \tfrac{1}{2}\zeta\int d^3x\,(-\chi^\dagger\chi W_0 + \chi^\dagger\vec{\alpha}\chi\cdot\vec{W}) \\ &= \tfrac{1}{2}\zeta^2\int d^3x\, d^3x'\, \frac{e^{-m_B|x-x'|}}{4\pi|x-x'|} \\ &\quad\times[J_0(x)J_0(x') - \vec{J}(x)\cdot\vec{J}(x')] \ , \end{aligned} \tag{7.52}$$

where $J^\mu = \bar{\chi}\gamma^\mu\chi$. In the local limit, $m_B \to \infty$, this energy is still positive, although reduced by a factor of $\frac{1}{2}$, and Eq. (7.50) remains valid. To complete this discussion, we next show that the electrically neutral system, the analog of the color singlet, is not shifted in energy by the strong color interaction. As an example of an electrically neutral system, consider two types of fermion with opposite charge called μ^+ and e^-, which have identical coupling to the scalar fields. Then it is clear that if μ^+ and e^- occupy the same state, then

$$J^\mu(x) = 0 \tag{7.53}$$

and, therefore, the energy of such a state is not affected by the electromagnetic coupling and is given by the calculation of the last section without the vector gauge field. Furthermore, if the μ^+ and e^- occupy different states, then one can form the eigenstates of C conjugation by symmetrization or antisymmetrization. For both symmetrized and antisymmetrized states, it can be readily verified that

$$J^\mu(x) = 0 \ ,$$

and their energies are again not affected by the coupling to the gauge field.

C. The non-Abelian case–SU(2)

We now turn to the non-Abelian case to show that the gauge coupling has no effect on the color-singlet states. Our only purpose is to give an order-of-magnitude estimate of the change in energy of the color-nonsinglet states. We will, therefore, simplify the problem as much as possible. For simplicity and clarity of presentation, we consider the group SU(2). The case of SU(3) will be mentioned briefly below. A quark doublet

$$\psi = \begin{pmatrix} \mathcal{P} \\ \mathfrak{N} \end{pmatrix} \tag{7.54}$$

is coupled to an isotopic triplet of vector gauge fields. To completely break the gauge symmetry, we introduce another isotopic spin doublet of complex scalar fields

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \ . \tag{7.55}$$

The Lagrangian of the system is

$$\begin{aligned}\mathfrak{L} = &-\tfrac{1}{4}(\partial_\mu \vec{X}_\nu - \partial_\nu \vec{X}_\mu + \zeta \vec{X}_\mu \times \vec{X}_\nu)^2 \\ &+ [(\partial_\mu + \tfrac{1}{2} i\zeta \vec{\tau}\cdot\vec{X}_\mu)\phi^*][(\partial^\mu - \tfrac{1}{2} i\zeta\vec{\tau}\cdot\vec{X}^\mu)\phi] \\ &- H'(\phi^*\phi - f'^2)^2 \\ &+ \tfrac{1}{2}(\partial_\mu\sigma)^2 - H(\sigma^2 - f^2)^2 \\ &+ \bar{\psi}(i\not\partial - \tfrac{1}{2} i\zeta\vec{\tau}\cdot\vec{\not X} - G\sigma)\psi \ ,\end{aligned} \tag{7.56}$$

where $\times$ denotes the cross product in isotopic spin space. $\mathfrak{L}$ is invariant under the infinitesimal gauge transformation

$$\begin{aligned}&\vec{X}_\mu \to \vec{X}_\mu + \frac{1}{\zeta}\partial_\mu\delta\vec{\omega} - \delta\vec{\omega}\times\vec{X}_\mu \ , \\ &\psi \to (1 + \tfrac{1}{2} i\vec{\tau}\cdot\delta\vec{\omega})\psi \ , \\ &\phi \to (1 + \tfrac{1}{2} i\vec{\tau}\cdot\delta\vec{\omega})\phi \ .\end{aligned} \tag{7.57}$$

Following the standard procedure of eliminating the would-be Goldstone bosons in the unitary gauge, we get

$$\begin{aligned}\mathfrak{L} = &-\tfrac{1}{4}(\partial_\mu \vec{X}_\nu - \partial_\nu \vec{X}_\mu + \zeta \vec{X}_\mu \times \vec{X}_\nu)^2 \\ &+ \tfrac{1}{2}(\partial_\mu\rho)^2 + \tfrac{1}{8}\zeta^2(\rho + \sqrt{2} f')^2 \vec{X}_\mu{}^2 \\ &- \tfrac{1}{4} H'[(\rho + \sqrt{2} f')^2 - 2f'^2]^2 \\ &+ \tfrac{1}{2}(\partial_\mu\sigma)^2 - H(\sigma^2 - f^2)^2 \\ &+ \bar{\psi}(i\not\partial - \tfrac{1}{2} i\zeta\vec{\tau}\cdot\vec{\not X} - G\sigma)\psi \ ,\end{aligned} \tag{7.58}$$

plus the additional term needed for the Feynman rule calculations

$$\Delta\mathfrak{L}_{\text{eff}} = -3i\,\delta^4(0)\ln\left(1 + \frac{\rho(x)}{\sqrt{2} f'}\right) \ . \tag{7.59}$$

The field ρ in the unitary gauge does not respond to an isotopic rotation. The canonical variables are X_a^k and G_a^{0l} which satisfy

$$[X_a^k(x), G_b^{0l}(x')] = i\,\delta^{kl}\delta_{ab}\,\delta^3(x - x') \ .$$

Following the same procedure and approximations as in the Abelian case, we form the trial state

$$|S'\rangle = \exp\left(-i\int d^3x\,\vec{W}^k(x)\cdot\vec{G}^{0k}(x)\right)\exp\left(i\int d^3x\,\vec{E}^k(x)\cdot\vec{X}^k(x)\right)\exp\left(-i\int d^3x\,g(x)\,\dot{\sigma}(x)\right)\exp\left(-i\int d^3x\,g'(x)\,\dot{\rho}(x)\right)B_0^\dagger\,|0_L\rangle \ , \tag{7.60}$$

and calculate the energy

$$\begin{aligned}E' &\equiv \langle S'|\mathfrak{H}|S'\rangle \\ &= \int d^3x \Big\{ \tfrac{1}{2}\vec{E}_k{}^2 + \tfrac{1}{4}(\partial_k\vec{W}_l - \partial_l\vec{W}_k + \zeta\vec{W}_k\times\vec{W}_l)^2 + \tfrac{1}{8}\zeta^2(g' + \sqrt{2} f')^2(\vec{W}_0{}^2 + \vec{W}_k{}^2) \\ &\quad + \tfrac{1}{2}(\vec{\nabla} g')^2 + \tfrac{1}{4} H'[(g' + \sqrt{2} f')^2 - 2f'^2]^2 + \tfrac{1}{2}(\vec{\nabla} g)^2 + H(g^2 - f^2)^2 + \chi^\dagger\left(\frac{1}{i}\vec{\alpha}\cdot\vec{\nabla} + \tfrac{1}{2}\zeta\vec{\tau}\,\alpha^k\cdot\vec{W}^k + G\beta g\right)\chi \Big\} \ ,\end{aligned} \tag{7.61}$$

where

$$\vec{W}_0=\frac{4}{\zeta^2 g'^2}\left[(\partial_k+\zeta\vec{W}_k\times)\vec{E}_k-\zeta\chi^\dagger\tfrac{1}{2}\vec{\tau}\chi\right] . \tag{7.62}$$

The variational principle for E' then gives the classical field equations

$$\begin{aligned}
&\vec{E}_k=(\partial_k+\zeta\vec{W}_k\times)\vec{W}_0 ,\\
&-(\partial_k+\zeta\vec{W}_k\times)(\partial_k+\zeta\vec{W}_k\times)\vec{W}_0+\tfrac{1}{4}\zeta^2(g'+\sqrt{2}f')^2\vec{W}_0+\tfrac{1}{2}\zeta\chi^\dagger\vec{\tau}\chi=0 ,\\
&-(\partial_k+\zeta\vec{W}_k\times)(\partial_k\vec{W}_l-\partial_l\vec{W}_k+\zeta\vec{W}_k\times\vec{W}_l)+\tfrac{1}{4}\zeta^2(g'+\sqrt{2}f')^2\vec{W}_l+\zeta\chi^\dagger\tfrac{1}{2}\vec{\tau}\alpha_l\chi=0 ,\\
&-\nabla^2 g'+H'(g'+\sqrt{2}f')\left[(g'+\sqrt{2}f')^2-2f'^2\right]+\tfrac{1}{4}\zeta^2(g'+\sqrt{2}f')(\vec{W}_k{}^2-\vec{W}_0^2)=0 ,\\
&-\nabla^2 g+4Hg(g^2-f^2)+G\bar{\chi}\chi=0 ,\\
&\left(\frac{1}{i}\vec{\alpha}\cdot\vec{\nabla}-\tfrac{1}{2}\zeta\vec{\tau}\cdot\vec{W}_0+\tfrac{1}{2}\zeta\vec{\tau}\cdot\alpha^k\vec{W}^k+G\beta g\right)\chi=\mathcal{E}\chi .
\end{aligned} \tag{7.63}$$

We have not been able to find a self-consistent solution to these nonlinear coupled differential equations. Among other things, the Dirac differential operator does not commute with the angular momentum operator or with the total isospin operator because of the occurrence of the C-number fields W_0, W^k. Therefore, the eigenfunctions χ are not eigenfunctions of total angular momentum and total isotopic spin. However, the results of the Abelian example suggest that the inclusion of the coherent clouds of the vector gauge fields only lowers the energy of a trial state somewhat but does not alter its value by an order of magnitude. The situation is more complicated in the present non-Abelian case since the vector gauge field is also a source for itself. Nevertheless, we are content to discuss a much simplified trial state of the form

$$|S,N,\bar{N}\rangle=\exp\left(-i\int d^3x\,g\dot{\sigma}\right)\times\left(-i\int d^3x\,g'\dot{\rho}\right)F^{I,I_3}_{N,\bar{N}}(B^\dagger,D^\dagger)\,|0_L\rangle , \tag{7.64}$$

which is a state of N q's and $\bar{N}$ $\bar{q}$'s. The function F is a sum of several terms each of which contains N $B^\dagger$'s and $\bar{N}$ $D^\dagger$'s. It is chosen so that the trial state is a state of definite isospin I and third component I_3. We are furthermore seeking solutions in which the isotopic wave function and spatial wave function factorize.

For a single-quark state we have

$$\begin{aligned}
&F_{\mathcal{P}}=B^\dagger_{\mathcal{P}}\\
&F_{\bar{\mathcal{P}}}=D^\dagger_{\mathcal{P}} ,\ \text{etc.}
\end{aligned} \tag{7.65}$$

For a qq system we have

$$\begin{aligned}
I=0:&\quad F^{0}_{2,0}=\frac{1}{\sqrt{2}}(B^\dagger_{\mathcal{P}_1}B^\dagger_{\mathfrak{N}_2}+B^\dagger_{\mathcal{P}_2}B^\dagger_{\mathfrak{N}_1}),\\
I=1:&\quad F^{1,1}_{2,0}=B^\dagger_{\mathcal{P}_1}B^\dagger_{\mathcal{P}_2} ,\\
&\quad F^{1,0}_{2,0}=\frac{1}{\sqrt{2}}(B^\dagger_{\mathcal{P}_1}B^\dagger_{\mathfrak{N}_2}-B^\dagger_{\mathcal{P}_2}B^\dagger_{\mathcal{P}_1}) ,\\
&\quad F^{1,-1}_{2,0}=B^\dagger_{\mathfrak{N}_1}B^\dagger_{\mathfrak{N}_2} .
\end{aligned} \tag{7.66}$$

For a $q\bar{q}$ system we have

$$\begin{aligned}
I=0:&\quad F^{0,0}_{1,1}=\frac{1}{\sqrt{2}}(D^\dagger_{\mathfrak{N}_1}B^\dagger_{\mathfrak{N}_2}-B^\dagger_{\mathcal{P}_2}D^\dagger_{\mathcal{P}_1}) ,\\
I=1:&\quad F^{1,1}_{1,1}=D^\dagger_{\mathfrak{N}_1}B^\dagger_{\mathcal{P}_2} ,\\
&\quad F^{1,0}_{1,1}=\frac{1}{\sqrt{2}}(D^\dagger_{\mathfrak{N}_1}B^\dagger_{\mathfrak{N}_2}+B^\dagger_{\mathcal{P}_2}D^\dagger_{\mathcal{P}_1}) ,\\
&\quad F^{1,-1}_{1,1}=D^\dagger_{\mathcal{P}_1}B^\dagger_{\mathfrak{N}_2} .
\end{aligned} \tag{7.67}$$

To compute the expectation value of the Hamiltonian, we have to evaluate $\langle S|\vec{j}_0{}^2|S\rangle$. To replace this divergent quantity by a finite expression, we will apply the approximation (7.22) extended to the non-Abelian case. It means that we approximate

$$\langle S|\vec{j}_0{}^2|S\rangle\cong\sum_{I_3}\langle S|\vec{j}_0|S_{I_3}\rangle\langle S_{I_3}|\vec{j}_0|S\rangle , \tag{7.68}$$

where the sum is extended over all states in the same multiplet as S. With this approximation, we can evaluate the matrix element rather easily. From the transformation property of $\vec{j}_0$ under isospin rotation we obtain

$$\langle I=0|\vec{j}_0(x)|I=0\rangle=0$$

for isospin singlet states by the Wigner-Eckart theorem. For these states the gauge coupling has no effect on the energies. For a nonsinglet state we have

$$\langle I, I_3|\vec{j}_0^a(x)|I, I_3\rangle = C_{I_3}^a \sum_{i=1}^{N} \chi_i^\dagger \chi_i \ , \qquad (7.69)$$

so

$$\langle I, I_3|\vec{j}_0(x)\cdot\vec{j}_0(x')|I, I_3\rangle = C\left[\sum_{i=1}^{N} \chi_i^\dagger(x)\chi_i(x)\right] \times\left[\sum_{i=1}^{N}\chi_i^\dagger(x')\chi_i(x')\right] . \qquad (7.70)$$

The constant $+C$ can be easily determined by integrating over x and x':

$$\langle I, I_3|\vec{I}^2|I, I_3\rangle = I(I+1) = N^2 C, \quad C=[I(I+1)]/N^2 \qquad (7.71)$$

so we arrive at

$$\langle S|\vec{j}_0(x)^2|S\rangle = I(I+1)\left[\frac{1}{N}\sum_{i=1}^{N}\chi_i^\dagger(x)\chi_i(x)\right]^2 , \qquad (7.72)$$

where $\chi(x)$ is the normalized wave function for a quark or antiquark. This result neglects some possible exchange terms which will be commented upon later.

Let us now compute the expectation value of $\mathfrak{H}$. For simplicity we will assume that each of the quarks and antiquarks occupies a different state. But our final result is applicable even if some of the quarks (antiquarks) occupy the same states. We have

$$E = \int d^3x\left[\tfrac{1}{2}(\vec{\nabla}g)^2 + H(g^2-f^2)^2 + \sum_{i=1}^{N}\chi_i^\dagger\left(\frac{\vec{\alpha}\cdot\vec{\nabla}}{i}+\beta G g\right)\chi_i - \sum_{i=1}^{\bar{N}} V_i^\dagger\left(\frac{\vec{\alpha}\cdot\vec{\nabla}}{i}+G\beta G\right)V_i + \frac{\zeta^2}{m_B{}^2}I(I+1)\left(\sum_{i=1}^{N}\chi_i^\dagger\chi_i + \sum_{i=1}^{\bar{N}}V_i^\dagger V_i\right)^2\right] , \qquad (7.73)$$

which makes it clear that if the state is an isosinglet, $I=0$, then E reduces to the case discussed in the previous section.

The energy E is minimized by the following equations:

$$-\nabla^2 g + 4Hg(g^2+f^2) + G\sum \bar{\chi}_i\chi_i - G\sum \bar{V}_i V_i = 0 ,$$
$$\left(\frac{\vec{\alpha}\cdot\vec{\nabla}}{i} + G\beta g + V\right)\chi_i = \mathcal{E}_i\chi_i ,$$
$$\left(\frac{\vec{\alpha}\cdot\vec{\nabla}}{i} + G\beta g + V\right)V_i = -\bar{\mathcal{E}}_i V_i , \qquad (7.74)$$
$$V = \frac{\zeta^2}{m_B{}^2}\frac{1}{N+\bar{N}}I(I+1)\left(\chi_1^\dagger\chi_1 + \cdots + V_{\bar{N}}^\dagger V_{\bar{N}}\right) > 0 .$$

These equations are now exactly of the same structure as in the Abelian case. Thus we conclude as in the Abelian case that

$$Gf \gg E_T \gg H^{1/6} f ,$$

which is much higher than the color-singlet states.

We would like to comment on several possible corrections:

(1) Exchange force. Since the quarks are identical fermions, there exist exchange potentials in addition to the direct-interaction energies we have calculated here. These neglected exchange potentials are off-diagonal terms so they cannot be as big as the direct terms. These exchange terms do not arise in the "color-singlet" states. Since color-singlet states have to be totally antisymmetrical in color, the quarks in such a state are all distinguishable.

(2) Bhabha forces. For a $q\bar{q}$ system there are also additional quantum effects besides the direct potentials computed above. These are annihilation terms. Again, for color-singlet states, these terms do not contribute, since

$$\langle I=0, q\bar{q}|\vec{j}_0|0_L\rangle = 0$$

because the τ_i are traceless.

(3) Self-coupling of the gauge fields. If we accept that the energy of a color-nonsinglet state is of order $Gf \gg E_T \gg H^{1/6}f$, then the gauge field B^0 has a magnitude

$$Gf \gg |\zeta\vec{W}^0| \sim E_T \gg H^{1/6} f .$$

It seems to be self-consistent to assume that this is also true for the spatial components $\zeta\vec{W}_k$. In that case, the self-coupling is smaller by a fractional power of G as compared with the leading terms in Eq. (7.61). If this turns out to be the lowest energy configuration, then our neglect of the coherent states of the vector gauge field will not change our qualitative conclusions.

These discussions for SU(2) also apply to SU(3) color. The only difference is that to completely break the SU(3) gauge symmetry, we need more Higgs scalar fields. One possibility is to introduce two complex triplets. In particular, the quantum annihilation force still vanishes for color singlets of $q\bar{q}$ systems since

$$\langle C=0|j_\mu^a(x)|0_L\rangle = 0$$

because the λ_a are traceless.

In our scheme, we have no explanation for the absence of color-singlet exotic states.

VIII. STATIC PROPERTIES OF THE GROUND-STATE BARYONS AND MESONS

We turn next to the task of calculating the static properties of the color-singlet ground states constructed in Sec. VI. In addition to their masses, as already known from (6.5) and (6.6), these include the magnetic moments of the baryons, the $M1$ transition moments of both baryons and mesons, the axial-vector coupling constant of the nucleon, and the F to D ratio. We also compute the mean-squared charge radii of the baryons and mesons, although these are not strictly static properties since they are probed by finite-momentum-transfer interactions which lead to recoil corrections. The calculations in this section are performed using the states of zero average momentum constructed in Sec. VI. The correct physical amplitudes are defined, however, in terms of zero-momentum eigenstates rather than in terms of localized packets with $\langle \vec{P} \rangle = 0$. We shall construct momentum eigenstates in the following section and find that the corrections to the results obtained here are numerically small. Among the physical parameters being calculated, the $M1$ transition moments for the baryon are related to the magnetic moment by Clebsch-Gordan coefficients for the SU(6) states. However, their numerical relation to the hadron radii and to the meson $M1$ transition moments is determined by the underlying dynamics and wave functions of our theory.

A. Magnetic moment of the proton and neutron

Since we are working with a local Lagrangian field theory, the electromagnetic interaction is introduced via the usual minimal coupling. The magnetic moment of the nucleon is then computed from the energy shift in a weak, constant external magnetic field:

$$\delta E = -\vec{\mu}_p \cdot \vec{B} .$$

In terms of a spin-up trial state for the proton as constructed in (6.10), the magnetic moment is given by

$$\mu_{pz} = \langle p, j_z = \tfrac{1}{2} | \mu_z | p, j_z = \tfrac{1}{2} \rangle , \tag{8.1}$$

where μ_z is the z component of the magnetic moment operator

$$\vec{\mu} = \tfrac{1}{2} \int d^3x\, \vec{r} \times \vec{j}(x) . \tag{8.2}$$

The electromagnetic current operator j in a three-triplet quark model is given by

$$\vec{j} = e \psi^\dagger \vec{\alpha} Q \psi , \tag{8.3}$$

$$\psi = \begin{pmatrix} u \\ d \\ s \end{pmatrix} , \tag{8.4}$$

$$Q = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix} . \tag{8.5}$$

Using the proton state (6.10), we find directly that

$$\vec{\mu}_p = \tfrac{1}{2} e \int d^3x \left[\vec{r} \times (\chi^\dagger \vec{\alpha} \chi) \right]_z , \tag{8.6}$$

where χ is the ground-state wave function of a single quark with $j_z = \frac{1}{2}$. Now it follows from Eq. (4.13) that

$$\vec{r} \times (\chi^\dagger \vec{\alpha} \chi) |_z = \frac{2F_0 G_0}{r^2} \frac{1}{r} (r^2 - z^2) . \tag{8.7}$$

So we get

$$\mu_p = \tfrac{1}{2} e \int d^3x (2F_0 G_0) \frac{1}{r^3} \times \tfrac{2}{3} r^2 , \tag{8.8}$$

where an angular average has been performed. Since to leading order $F_0(r)$ and $G_0(r)$ are equal and are peaked at $r = R$, we have

$$\mu_p = \tfrac{1}{3} e R , \tag{8.9}$$

where we have made use of the normalization condition

$$\int d^3x \frac{1}{r^2} 2F_0 G_0 \cong \int d^3x \frac{1}{r^2} (F_0^2 + G_0^2) = 1 . \tag{8.10}$$

In terms of the mass of $\underline{56}$, $M_{56} = \frac{3}{2}(3/R)$, we obtain finally

$$\mu_p = 3 \left(\frac{e}{2M_{56}} \right) . \tag{8.11}$$

The magnetic moment of other baryons in the ground state $\underline{56}$ can be calculated similarly, where their ratios are given by the Clebsch-Gordan coefficients appropriate to SU(6). For example, for a neutron we find

$$\mu_n = -\tfrac{2}{3} \mu_p . \tag{8.12}$$

B. $M1$ transition moments

The $M1$ transition moments for baryon radiative decay are calculated similarly and their magnitudes are determined in terms of the appropriate SU(6) coefficients.[23] For example, for the radiative decay

$$\Delta \to p + \gamma$$

we have computed

$$\mu_{\Delta}^{*} \equiv \langle p, j_z = \tfrac{1}{2} | \mu_z | \Delta^{+}, j_z = \tfrac{3}{2} \rangle , \qquad (8.13)$$

obtaining

$$\mu_{\Delta}^{*} = \tfrac{2}{3}\sqrt{2}\,\mu_p . \qquad (8.14)$$

As another example, we calculate the $M1$ transition moment for the radiative decay of a meson, viz.,

$$\omega \to \pi^0 + \gamma . \qquad (8.15)$$

The result is

$$\begin{aligned}\mu_{\omega}^{*} &\equiv \langle \pi^0 | \mu_z | \omega, j_z = 0 \rangle \\ &= (\tfrac{2}{3})^{1/3}\mu_p . \end{aligned}\qquad (8.16)$$

The factor $(\frac{2}{3})^{1/3}$ in Eq. (8.16) is the ratio of the radius of a meson state to that of a baryon state and represents a correction to the prediction of the nonrelativistic quark model as discussed in Sec. I which seems to improve the experimental agreement considerably.[24]

C. Charge radii

For computing the mean-squared charge radii of baryons and mesons, we make the approximation using the static definition of the radius. For the proton and neutron, respectively, we find

$$\begin{aligned} e\langle r_p^{\,2}\rangle &\equiv \left\langle p \,\middle|\, \int d^3x [\, r^2 j^0(x)] \,\middle|\, p \right\rangle \\ &= eR^2 , \end{aligned}\qquad (8.17)$$

or

$$\langle r_p^{\,2}\rangle^{1/2} = R , \qquad (8.18)$$

and

$$\langle r_n^{\,2}\rangle = 0 . \qquad (8.19)$$

The same results apply to the charged and neutral meson radii, respectively, with the same factor $(\frac{2}{3})^{1/3}$ appearing as the ratio of radii for the $\bar{q}q$ meson system and the qqq baryon system. Corrections due to mass splittings among the meson $\underline{35}$ and the hadron $\underline{56}$ are not known and may be appreciable, particularly for the relatively light pion.

D. Axial-vector coupling constant g_A

Although our theory as written does not have a conserved (or almost conserved) axial-vector current, we attempt to identify the axial coupling for neutron β decay, g_A, through the matrix elements of the quark current

$$j_{A\mu} = \bar{\psi}\gamma_\mu\gamma_5 \tfrac{1}{2}(\lambda_1 + i\lambda_2)\psi . \qquad (8.20)$$

This is a natural choice for the axial-vector current of the weak interactions since it satisfies the usual commutation rules of current algebra. The axial-vector coupling constant g_A is then given by

$$g_A = \left\langle p \,\middle|\, \int d^3x [j_A^3(x)] \,\middle|\, n \right\rangle , \qquad (8.21)$$

where both the proton and the neutron are in the $j_z = +\frac{1}{2}$ state. Using the explicit representation (6.10) for the proton and neutron states, we have

$$g_A = \tfrac{5}{3} \int d^3x (\chi^{\dagger}\sigma_3\chi) . \qquad (8.22)$$

In the static SU(6), χ is an eigenstate of σ_3 so the integral is unity. However, in our theory, χ is an eigenstate of the total angular momentum but not of the spin. Making the approximation $F_0 = \pm G_0$, we find

$$\chi^{\dagger}\sigma_3\chi = 2G_0^{\,2}\frac{z^2}{r^4} . \qquad (8.23)$$

Therefore,

$$\begin{aligned} g_A &= \tfrac{5}{3}\times\tfrac{2}{3}\int d^3x\, G_0^2/r^2 \\ &= \tfrac{5}{9} , \end{aligned}\qquad (8.24)$$

where we have used the normalization condition (8.10) again. This value of g_A is less than one-half the observed value, 1.25.

We have also computed the ratios for the amplitudes of the weak decay processes, $\Sigma^- \to \Sigma^0$ and $\Sigma^- \to \Lambda$. For the vector part, the ratio is

$$V(\Sigma^- \to \Sigma^0) : V(\Sigma^- \to \Lambda) = 1 : 0 , \qquad (8.25)$$

and that for the axial-vector part it is

$$A(\Sigma^- \to \Sigma^0) : A(\Sigma^- \to \Lambda) = 1 : \tfrac{1}{2}\sqrt{3} . \qquad (8.26)$$

These results agree with the SU(6) prediction and correspond to a so-called F/D ratio of $\frac{2}{3}$.

Comparison of these results with experiment has already been presented in Sec. I. Here we would like to make two additional remarks. These results are not sensitive to the value of the coupling constant G. The corrections to the results given above are smaller by a factor of $G^{-2/3}$. The smallness of our result for g_A is the result of the large ratio of the lower to the upper components of the quark-wave function. Note that with a fixed ratio $G_0 = -aF_0$, the result is

$$g_A = \frac{5}{9}\,\frac{3-a^2}{1+a^2} , \qquad (8.27)$$

which doubles in value to $\sim\frac{10}{9}$ for $a \sim 0.6$. In contrast, the magnetic moment which is given by

$$\mu_p = 3\left(\frac{e}{2M_{56}}\right)\left(\frac{2a}{1+a^2}\right) \qquad (8.28)$$

is maximized in value at $a = 1$ and decreases only by 11% when a decreases to ~ 0.6.

Another factor to be studied is the sensitivity of these results to the use of localized states versus momentum eigenstates for the hadron. As we show in the following section, the corrections to the $M1$ matrix elements are negligible, $\sim O(1/G)$, whereas for the axial charge, they are $\sim O(1/\ln G)$ and perhaps more significant if the color thresholds are found to be not much higher than ~ 10 GeV.

IX. MOMENTUM EIGENSTATES

The states we have constructed so far are described relative to a fixed origin, and the corresponding wave functions are localized in space and concentrated in a rather small region. Especially, the fermion wave function is different from zero only in a very thin spherical shell. This picture of a hadron is surprisingly different from the intuitive one deduced from the empirical information on electromagnetic and purely hadronic reactions. They suggest the view of a hadron as an extended object with almost free pointlike constituents confined inside.

Should we take our "unusual" variational states and wave functions seriously? Variational calculations in nonrelativistic quantum mechanics are known to yield excellent results for energies of ground states even though the trial wave functions are crude. We can test the detailed properties of our trial states by calculating observable matrix elements in terms of them and comparing with experiment. However, in order to go beyond the static properties calculated thus far and confront the theory with experimental data probing the detailed internal structure of the hadron, we must first construct eigenstates of momentum or wave-packet states with a momentum spread comparable to an actual experimental setup. Our average $\langle \vec{P} \rangle$ states do not satisfy this condition, since by the uncertainty principle the wave function contains high momentum components $\approx Gf$ and $\sqrt{2H}f$. Thus we need a definite procedure to construct momentum eigenstates. We must also determine whether the ground-state energies and static hadron properties computed in Sec. VIII remain unchanged, to a good approximation, or are greatly altered if we construct actual eigenstates of momentum for use as our trial functions.

This section is devoted to an attempt to construct momentum eigenstates both for hadrons at rest and with arbitrary momentum. Although this attempt has not been completely successful, we sketch our efforts briefly in order to bring out some of the difficulties we have encountered and to illustrate the corrections introduced by our procedure into calculations of the mass and the other static properties of the ground states calculated in Sec. VIII. In particular, we find that with a particular choice of the scalar meson mass m_σ we are able to construct a single-quark state which is an eigenstate of zero three-momentum and with a mass within 10% of the mass of the local state constructed in Sec. IV. However, when we attempt to generalize this result, we encounter two serious problems:

(1) We are not able to use this method to construct covariant eigenstates of nonvanishing three-momentum. In the $\vec{P} \to \infty$ limit, the results simplify and covariance along the $\vec{P} = \infty$ direction is restored—namely, we have

$$E - P = \frac{M'}{\gamma} \text{ for } \gamma = \frac{1}{(1-v^2)^{1/2}} \to \infty .$$

However, in this case, we have not been able to show by explicit calculation that M' is of the order of the rest mass.

(2) Even for zero three-momentum eigenstates, we cannot generalize the method used in the one-quark sector to multiquark states. The problem is that the explicit Bogoliubov transformation which we use to construct the single-quark state with a translationally invariant $|0_p\rangle$ requires all quarks of the same color to have the same spin, so that, for example, we cannot construct the zero-helicity vector mesons from $|0_p\rangle$. Because of these problems, it is evident that we will eventually need a better prescription for constructing momentum eigenstates than the one we offer in this section.

If the calculations of hadron static properties presented in Sec. VIII are approximately valid, then we should obtain results for three-momentum eigenstates similar to the results obtained for the localized states in Sec. VIII. However, in view of the second problem cited above, we cannot construct multiquark momentum eigenstates for practical calculations. In order to proceed, we introduce an additional assumption: that all the three-momentum of a hadron is carried by its constituent valence quarks and scalar field or, in other words, that the no-particle state $|0_L\rangle$, defined in Sec. III, is an approximate zero eigenstate of the three-momentum operator, $\vec{P}|0_L\rangle = 0$. With this assumption, we are able to verify the results of Sec. VIII. These results and our efforts to solve the problems discussed above are presented in some detail below.

The method which we use for constructing an eigenstate of momentum does not require a knowledge of dynamics, but is simply based on the requirement of translational invariance. The natural procedure in the context of our variational calculation would be to compute the expectation value of $\mathfrak{H}$ with the trial states that are eigenstates of momentum and apply a variational procedure as in Sec. III to minimize the energy and to determine the wave functions. However, such a calculation is extremely difficult in practice. The reason is that the expectation value of the energy is no longer a simple spatial integral of a local energy density. Instead, it becomes a double integral involving overlapping functions due to the superposition of localized states that must be constructed in forming momentum eigenstates. Hence, in this case the variational principle gives rise to integral equations and not to local differential equations.

As a practical attempt, we have tried with partial success an alternative procedure that is both simpler and approximate. Namely, we first construct a state with a specified value of *average* momentum by a variational calculation which minimizes its energy as carried out in Sec. V. We then form an eigenstate of momentum equal in value to the average momentum by applying a momentum projection operator, viz.,

$$|p\rangle = N^{-1/2} \int d^3X e^{i(\vec{p}-\vec{P})\cdot\vec{X}} |\vec{v}\rangle . \tag{9.1}$$

$\vec{P}$ is the momentum operator. $|\vec{v}\rangle$ denotes a state of average momentum

$$\langle \vec{v} | \vec{P} | \vec{v} \rangle = M\gamma\vec{v} = \vec{p} \tag{9.2}$$

and energy

$$\langle \vec{v} | \mathfrak{H} | \vec{v} \rangle = E_v = (M^2 + \vec{p}^2)^{1/2} , \tag{9.3}$$

as in Eqs. (5.41) and (5.42); and the normalization N is given in terms of the overlap integral

$$\langle \vec{p} | \vec{p}' \rangle = (2\pi)^3 \delta(\vec{p} - \vec{p}') N ,$$
$$N = \int d^3\Delta \langle \vec{v} | e^{i(\vec{P}-\vec{p})\cdot\vec{\Delta}} | \vec{v} \rangle . \tag{9.4}$$

A. Construction of a one-quark state with $\vec{p}=0$

The no-particle state $|0_L\rangle$ is not translationally invariant since the $B^\dagger$'s and $D^\dagger$'s create fermions and antifermions localized in space. Thus $|0_L\rangle$ also carries momentum as discussed in Sec. III. To construct a momentum eigenstate in this basis is formally possible, but it is very hard to carry out the explicit calculation. To expose the basic difficulty, we will limit ourselves to the simplest bound state of one quark with zero-momentum eigenvalue. In this case, we can construct the bound state out of a translationally invariant trial vacuum state by a Bogoliubov transformation as described in Sec. III.

We begin with the formal construction of a one-bound-quark state from a translationally invariant vacuum by introducing an operator producing "Cooper pairs":

$$U_F = \exp\left(\sum_s \int d^3p\, \theta(p,s)(b_{ps}^\dagger d_{-ps}^\dagger - d_{-ps} b_{ps}) \right). \tag{9.5}$$

This is a formally unitary operator with the properties

$$\begin{aligned} B_{ps} &= U_F^{-1} b_{ps} U_F \\ &= b_{ps}\cos\theta(p,s) + d_{-ps}^\dagger \sin\theta(p,s) , \\ D_{-ps}^\dagger &= U_F^{-1} d_{-ps}^\dagger U_F \\ &= -b_{ps}\sin\theta(p,s) + d_{-ps}^\dagger \cos\theta(p,s) , \end{aligned} \tag{9.6}$$

or equivalently

$$\begin{aligned} \underline{\psi}(x) &= U_F^{-1}\psi(x) U_F \\ &= \int \frac{d^3p}{[(2\pi)^3 2E_p]^{1/2}} \sum_s [b_{ps} U(p,s) e^{i\vec{p}\cdot\vec{x}} \\ &\qquad + d_{ps}^\dagger V(p,s) e^{-i\vec{p}\cdot\vec{x}}] , \end{aligned} \tag{9.7}$$

where

$$\begin{aligned} U(p,s) &= u(p,s)\cos\theta(p,s) \\ &\qquad - v(-p,-s)\sin\theta(p,s) , \\ V(p,s) &= v(p,s)\sin\theta(-p,-s) \\ &\qquad + u(-p,-s)\sin\theta(-p,-s) . \end{aligned}$$

The state

$$|h\rangle = U_F \exp\left(-i \int d^3x\, g(x)\sigma(x) \right) \times \sum_s \int d^3p\, h(p,s) b_{ps}^\dagger |0_p\rangle \tag{9.8}$$

leads to the expectation value of $\mathfrak{H}$ as before in Eq. (3.30):

$$\begin{aligned} E &= \langle h | \mathfrak{H} | h \rangle \\ &= \int d^3z \left[\tfrac{1}{2}(\vec{\nabla}g)^2 + H(g^2 - f^2)^2 + \chi^\dagger \left(\frac{1}{i} \vec{\alpha}\cdot\vec{\nabla} + G\beta g \right) \chi \right]. \end{aligned} \tag{9.9}$$

Here the ground-state wave function is

$$\begin{aligned} \chi(x) &= \int \frac{d^3p}{[(2\pi)^3 2E_p]^{1/2}} \\ &\quad \times \sum_s h(p,s)[u(p,s)\cos\theta(p,s) \\ &\qquad - v(-p,s)\sin\theta(-p,-s)] e^{i\vec{p}\cdot\vec{x}} , \end{aligned} \tag{9.10}$$

which is an arbitrary spinor since h and θ are arbitrary. Equation (9.8) gives the desired one-quark state and is an explicit construction of the

Bogoliubov transformation described in Sec. III for building our bound states from translationally invariant trial vacuum states.

To obtain Eq. (9.9), we have normal-ordered the boson part as before and we also normal-ordered the transformed fermion part $:U_F^{-1}\mathcal{H}U_F:$ in keeping with the tree approximation neglect of vacuum bubbles. Notice that U_F commutes with P

$$[U_F, \vec{P}] = 0$$

since all the pairs in U_F carry zero momentum by construction.

The procedure of (9.1) and (9.4) projects momentum eigenstates from $|h\rangle$ in (9.8). We now consider $|h\rangle$ to be a state with zero average momentum, so $\vec{p} = 0$.

It follows from

$$e^{-i\vec{P}\cdot\vec{X}}\phi(x)e^{i\vec{P}\cdot\vec{X}} = \phi(x+X)$$

that

$$N = \int d^3\Delta \int d^3z\, \chi^\dagger(z)\chi(z+\Delta) \times\left\langle 0 \left| \exp\left(i\int d^3g\,[g(y-\Delta) - g(y)]\dot{\sigma}(y)\right)\right| 0 \right\rangle. \tag{9.11}$$

In order to further reduce N, we must decompose $\dot{\sigma}(y)$ into its creation and annihilation parts, i.e., its negative- and positive-frequency parts, respectively, at $t=0$:

$$\begin{aligned} &\sigma \equiv \sigma(\vec{y}, 0) \equiv \sigma^{(-)} + \sigma^{(+)}, \quad \sigma^{(+)}|0_p\rangle = 0 \\ &\dot{\sigma} = \dot{\sigma}^{(-)} + \dot{\sigma}^{(+)}, \quad \dot{\sigma}^{(+)}|0_p\rangle = 0\,. \end{aligned} \tag{9.12}$$

The trial vacuum $|0_p\rangle$ for the scalar field is defined in terms of the free Hamiltonian

$$\mathfrak{H}_0 \equiv \tfrac{1}{2}\int d^3x\,[\dot{\sigma}^2 + (\vec{\nabla}\sigma)^2 + 8Hf^2(1-c)\sigma^2|, \tag{9.13}$$

with

$$:\mathfrak{H}_0:|0_p\rangle = 0$$

and the mass of the σ-field normal modes given by

$$m_\sigma^2 \equiv 8Hf^2(1-c)\,. \tag{9.14}$$

The proper choice of c and of the mass of the normal modes will be made so that the expectation value of the energy in the one-quark state will be close to the value already computed in Sec. IV for the localized state at rest with average $\langle\vec{P}\rangle = 0$. It should be made clear that this choice of c and of m_σ in Eq. (9.14) in no way affects the value of E obtained in Sec. III. Different choices of m_σ correspond to neglecting different terms in the normal-ordering of the Hamiltonian, and it will be important and necessary in future work to understand how the different choices feed back into the higher-order corrections to our present approximations.

From the canonical commutation rules and the definition (9.12) and (9.13) we can calculate the equal-time commutator

$$[\dot{\sigma}^{(-)}(\vec{z}), \dot{\sigma}^{(+)}(\vec{z}\,')] = -\tfrac{1}{2}\int \frac{d^3k}{(2\pi)^3}\,\omega_k e^{-i\vec{k}\cdot(\vec{z}-\vec{z}\,')}, \tag{9.15}$$

with

$$\omega_k \equiv (\vec{k}^2 + m_\sigma^2)^{1/2}\,.$$

This gives for the normalization (9.9)

$$N = \int d^3\Delta\, d^3z\, \chi^\dagger(\vec{z}-\vec{\Delta})\chi(\vec{z})e^{-[Y(0)-Y(\Delta)]}, \tag{9.16}$$

where

$$Y(\Delta) \equiv \tfrac{1}{2}\int \frac{d^3k}{(2\pi)^3}\,\omega_k |g(k)|^2 e^{i\vec{k}\cdot\vec{\Delta}}, \tag{9.17}$$

and where

$$g(k) \equiv \int d^3k\, e^{i\vec{k}\cdot\vec{y}} g(\vec{y})\,.$$

To compute the expectation value of the energy, we also need the equal-time commutation relations

$$[\dot{\sigma}^{(+)}(\vec{z}), \sigma(\vec{z}\,')] = [\dot{\sigma}^{(-)}(\vec{z}), \sigma(\vec{z}\,')] = -\tfrac{1}{2}i\delta^3(\vec{z}-\vec{z}\,')\,.$$

Then a straightforward calculation gives

$$\begin{aligned} E = \frac{1}{N}\int d^3z\, d^3\Delta\, e^{-[Y(0)-Y(\Delta)]} \times\Big\{ &\chi^\dagger(\vec{z}-\vec{\Delta})\left[\frac{1}{i}\,\vec{\alpha}\cdot\vec{\nabla} + G\beta\tfrac{1}{2}[g(\vec{z}-\vec{\Delta}) + g(\vec{z})]\right]\chi(\vec{z}) \\ &+ \chi^\dagger(\vec{z}-\vec{\Delta})\chi(\vec{z})\int d^3y\,\{\tfrac{1}{2}[\vec{\nabla}g(\vec{y})]^2 + H[g(\vec{y})^2 - f^2]^2 \\ &\quad + H[g(\vec{y}-\vec{\Delta}) - g(\vec{y})]^2(cf^2 + \tfrac{3}{2}[g(\vec{y})^2 - f^2] + \tfrac{1}{2}f[g(\vec{y}-\vec{\Delta}) - g(\vec{y})] \\ &\quad + \tfrac{1}{16}[g(\vec{y}-\vec{\Delta}) - g(\vec{y})]^2 + \tfrac{1}{2}[g(\vec{y}-\vec{\Delta}) - g(\vec{y})][g(\vec{y}) - f])\}\Big\}\,. \end{aligned} \tag{9.18}$$

Observe that the Dirac part can be evaluated directly since χ satisfied the Dirac equation according to the procedure described at the beginning of this section. An integration by parts yields

$$E=\mathcal{E}+\frac{1}{N}\int d^3z\,d^3\Delta\,\chi^\dagger(\vec{z}-\vec{\Delta})\chi(\vec{z})\,e^{-[Y(0)-Y(\vec{\Delta})]}$$
$$\times\int d^3y\{\tfrac{1}{2}[\vec{\nabla}g(\vec{y})]^2+H[g(\vec{y})^2-f^2]^2+H[g(\vec{y}-\vec{\Delta})-g(\vec{y})]^2$$
$$\times(cf^2+\tfrac{3}{2}[g(\vec{y})^2-f^2]+\tfrac{1}{2}[g(\vec{y}-\vec{\Delta})-g(\vec{y})]g(\vec{y})+\tfrac{1}{16}[g(\vec{y}-\vec{\Delta})-g(\vec{y})]^2)\}\,. \tag{9.19}$$

The difference between this result and the ground-state energy for a localized $\langle P\rangle=0$ trial state as in Eq. (4.19) arises from three factors in Eq. (9.19):

(1) a correlation or shielding factor

$$e^{-[Y(0)-Y(\vec{\Delta})]}\,; \tag{9.20}$$

(2) a fermion overlap factor

$$\chi^\dagger(\vec{z}-\vec{\Delta})\chi(\vec{z})\,; \tag{9.21}$$

and

(3) terms in the σ-field energy proportional to the difference

$$[g(\vec{y}-\vec{\Delta})-g(\vec{y})]\,. \tag{9.22}$$

Neither of the first two factors differs very much from their value for zero separation $\Delta=0$. It is readily found that $Y(0)<1$ for any ratio of values $G/\sqrt{H}$ in our strong-coupling regime of interest. Furthermore, no sensitive cancellations appear in the fermion density, and the factor (9.21) largely cancels the normalization integral (9.14). However, since the difference (9.22) is nonvanishing and is $O(f)$ for a separation $\Delta\lesssim R$, it can be shown by a straightforward estimate that the last term in Eq. (9.19) contributes an amount of order $H^{1/2}\gg H^{1/6}f$ to the energy for general values of c, viz.,

$$\frac{1}{N}\int d^3z\,d^3\Delta\,e^{-[Y(0)-Y(\vec{\Delta})]}\chi^\dagger(\vec{z}-\vec{\Delta})\chi(\vec{z})Hf^2$$
$$\times\int[g(\vec{y}-\vec{\Delta})-g(\vec{y})]^2\,d^3y$$
$$\approx Hf^4R^3\approx H^{1/2}f\approx H^{1/3}(H^{1/6}f)\approx H^{1/3}M\,.$$

We now show that there is a unique choice of c which will eliminate this large contribution. For this purpose we can replace g^2 (for $G,H\gg1$) by

$$g(\vec{y})^2=g(\vec{y}-\vec{\Delta})^2=f^2\,.$$

In this approximation, the last term of Eq. (9.19) vanishes only if

$$c=\tfrac{3}{4}\,, \tag{9.23}$$

corresponding to the mass in Eq. (9.14),

$$m_\sigma^2=2Hf^2\,. \tag{9.24}$$

With this choice, the energy of the zero-momentum eigenstate stays practically unchanged from the state with average zero momentum. We have checked this explicitly, introducing (9.23) into (9.19) which becomes

$$E=\mathcal{E}+\frac{1}{N}\int d^3z\,d^3\Delta\,\chi^\dagger(\vec{z}-\vec{\Delta})\chi(\vec{z})\,e^{-[Y(0)-Y(\vec{\Delta})]}$$
$$\times\int d^3g\{\tfrac{1}{2}[\vec{\nabla}g(\vec{y})]^2+H[g(\vec{y})^2-f^2]^2$$
$$+\tfrac{3}{4}H(\tfrac{1}{4}[g(\vec{y}-\vec{\Delta})^2-g(\vec{y})^2]^2+[g(\vec{y})^2-f^2][g(\vec{y}-\vec{\Delta})-g(\vec{y})]^2)\}\,. \tag{9.25}$$

The energy computed from Eq. (9.25) is less than 10% lower than the value $3/2R$ obtained in Sec. IV, and the fact that the energy is *reduced* by this small amount indicates that the $\vec{p}=0$ eigenstate is a better approximation to the true state.

We do not understand physically why there is one particular choice of m_σ given by Eq. (9.24) which makes the energy practically unchanged in going from zero average momentum state to the corresponding zero-momentum eigenstate. Undoubtedly, its meaning can be understood only after the true vacuum state is treated properly.

B. Static properties of hadrons with $\vec{p}=\vec{0}$

We would like to generalize the above procedure to construct a hadron trial state which is a three-momentum eigenstate. As is evident from Eq. (9.10), we would need a different Bogoliubov transformation for each quark with different space-spin quantum numbers. This can be achieved for baryons by introducing a separate U_F for each color, but off-diagonal contributions [i.e., $U_F^{-1}(\theta_1)U_F(\theta_2)$ for $\theta_1\neq\theta_2$], then make calculation of transition matrix elements with states such as (6.10) prohibitively difficult. For the mesons the problem is even more serious, since we cannot bind quarks and antiquarks of the same color with different spins and, therefore, cannot construct complete SU(6) multiplets.

An approximation which sidesteps these problems is to return to Eq. (9.1), but leave the Bogoliubov transformation implicit by writing the state in terms of the localized no-particle state $|0_L\rangle$ as in Eq. (6.1) instead of making the transformation explicit as in Eq. (9.8). In this way we avoid having to evaluate off-diagonal matrix elements discussed above. In place of that difficulty, there now appear factors

$$e^{i\vec{P}\cdot\vec{X}}|0_L\rangle \tag{9.26}$$

after the momentum operator is commuted to the right or left in the matrix element. To proceed further, we now make the further approximation of ignoring the momentum carried by the Cooper pairs in the Bogoliubov transformation, i.e., we set

$$e^{i\vec{P}\cdot\vec{X}}|0_L\rangle \cong |0_L\rangle\,. \tag{9.27}$$

We can then repeat the calculations of the $M1$ moments and of the axial charge g_A using Eq. (9.27). The results are found by straightforward calculation which we summarize briefly:

1. The baryon magnet moments and $M1$ transition amplitudes are unaltered up to corrections $\sim O(1/G)$.

2. The axial charge is increased by corrections $\sim 1/\ln G$ and the numerical value of this increase is determined by the magnitude of the bare quark mass $M_Q = Gf$ and of the ratio G^2/H. For a typical calculation with $G^2 \sim H$, this increase is numerically small,

$$g_A = \frac{5}{9}\left(1 + \frac{1}{3\ln(M_Q/M_N)}\right)\,.$$

However, this correction to g_A increases with a decreasing ratio of $H/G^2 \ll 1$. Whether this sensitivity of the numerical result in the solution is real or significant remains to be studied and understood.

C. Construction of eigenstates with $\vec{p}\neq 0$

We now give a brief discussion of our attempt to construct an eigenstate of momentum with $\vec{p}\neq 0$. From the state with average momentum

$$|\vec{v}\rangle = \exp\left(i\int d^3x\, g_0(\vec{x})\sigma(\vec{x})\right)\exp\left(-i\int d^3x\, g_1(\vec{x})\dot{\sigma}(\vec{x})\right) U_F \sum_s \int d^3p_1\, h_1(\vec{p}_1 s)\, b^\dagger_{\vec{p}_1 s}|0_p\rangle \tag{9.28}$$

we construct the corresponding eigenstate

$$|\vec{p}\rangle = \int d^3x\, e^{i(\vec{p}-\vec{P})\cdot\vec{x}}|\vec{v}\rangle\,,$$

where

$$\vec{p} = \langle\vec{P}\rangle = M\vec{v}\gamma$$

and M is the rest mass of the state. The functions g_0 and g_1 are given by (5.22) and (5.23).

The normalization integral N_v in

$$\langle\vec{p}\,|\,\vec{p}\,'\rangle = (2\pi)^3\,\delta^3(\vec{p}-\vec{p}\,')N_v$$

is given by

$$N_{\vec{v}} = \int d^3\Delta\, e^{-i\vec{p}\cdot\vec{\Delta}}\langle\vec{v}\,|e^{i\vec{P}\cdot\vec{\Delta}}|\vec{v}\rangle\,. \tag{9.29}$$

To evaluate $N_{\vec{v}}$, in addition to (9.15) and (9.10), we need the equal-time commutation relations

$$[\dot{\sigma}(\vec{z}), \sigma^{(+)}(\vec{z}\,')] = [\dot{\sigma}(\vec{z}), \sigma^{(-)}(\vec{z}\,')] = -\tfrac{1}{2}i\delta^3(\vec{z}-\vec{z}\,')\,.$$

The result we find is

$$\langle\vec{v}\,|e^{i\vec{P}\cdot\vec{\Delta}}|\vec{v}\rangle = \int d^3z\,\chi_1^*(\vec{z}-\vec{\Delta})\chi_1(\vec{z})K_1(\vec{\Delta})\,, \tag{9.30}$$

where χ_1 given by (5.23) and $K_1(\vec{\Delta})$ is

$$K_1(\vec{\Delta}) = \exp\left[-\tfrac{1}{2}\int\frac{d^3k}{(2\pi)^3}\,\omega_k|g_1(\vec{k})|^2\left(1+\frac{\vec{v}\cdot\vec{k}}{\omega_k}\right)^2(1-e^{i\vec{k}\cdot\vec{\Delta}})\right]\,. \tag{9.31}$$

Incidentally, Eq. (9.30) is a generating functional which yields the expectation values of all moments of $\vec{P}$ in an average momentum state with $\langle\vec{P}\rangle = M\vec{v}\gamma$. The calculation of the energy in the state $|\vec{p}\rangle$ is similar to the calculation of $N_{\vec{v}}$ except it is more complicated and lengthy. We only quote the result

$$
\begin{aligned}
N_{\vec{v}}(E_p-\vec{v}\cdot\vec{p}) = N_{\vec{v}}\, \frac{M}{\gamma}
&+\int d^3\Delta\, d^3z\, e^{-i\vec{p}\cdot\vec{\Delta}} K_1(\vec{\Delta})\chi_1^\dagger(\vec{z}-\vec{\Delta})G\beta\tfrac{1}{2}i\vec{v}\cdot\vec{\nabla}[\bar{g}_1(\vec{z}-\vec{\Delta})-\bar{g}_1(\vec{z})]\chi_1(z)\\
&+\int d^3\Delta\, d^3y\, e^{-i\vec{p}\cdot\vec{\Delta}} K_1(\vec{\Delta})\chi_1^\dagger(\vec{y}-\vec{\Delta})\chi_1(\vec{y})\\
&\quad\times\int d^3z H\{(\tfrac{1}{4}[g_1(\vec{z}-\vec{\Delta})-g_1(\vec{z})+i\vec{v}\cdot\vec{\nabla}(\bar{g}_1(\vec{z}-\vec{\Delta})-\bar{g}_1(\vec{z}))]^2\\
&\qquad+f[g_1(\vec{z}-\vec{\Delta})-g_1(\vec{z})+i\vec{v}\cdot\vec{\nabla}(\bar{g}_1(\vec{z}-\vec{\Delta})-\bar{g}_1(\vec{z}))])^2\\
&\qquad-\tfrac{1}{4}(1+c)f^2[g_1(\vec{z}-\vec{\Delta})-g_1(\vec{z})+i\vec{v}\cdot\vec{\nabla}(\bar{g}_1(\vec{z}-\vec{\Delta})-\bar{g}_1(\vec{z}))]^2\\
&\qquad+\tfrac{3}{2}[g_1^{\,2}(\vec{z})-f^2][g_1(\vec{z}-\vec{\Delta})-g_1(\vec{z})+i\vec{v}\cdot\vec{\nabla}(\bar{g}_1(\vec{z}-\vec{\Delta})-g_1(\vec{z}))]^2\\
&\qquad+\tfrac{1}{2}[g_1(\vec{z})-f][g_1(\vec{z}-\vec{\Delta})-g_1(\vec{z})+i\vec{v}\cdot\vec{\nabla}(\bar{g}_1(\vec{z}-\vec{\Delta})-\bar{g}_1(\vec{z}))]^3\},
\end{aligned}
\tag{9.32}
$$

where the function $\bar{g}_1(\vec{z})$ is defined to be

$$\bar{g}_1(\vec{z})=\int\frac{d^3k}{(2\pi)^3}\frac{1}{\omega_k}g_1(\vec{k})e^{-i\vec{k}\cdot\vec{z}} \tag{9.33}$$

and

$$g_1(\vec{z})=\int\frac{d^3k}{(2\pi)^3}g_1(\vec{k})e^{-i\vec{k}\cdot\vec{z}}\,. \tag{9.34}$$

We make two observations on this complicated expression:

1. There are terms in Eq. (9.32) which are comparable with Gf and $H^{1/2}f$. One of them is the second term associated with the fermion. Hence the energy of the momentum eigenstate is changed by a large amount unless these individually large terms cancel by appropriate choice of G, H, and c or m_σ.

2. The energy is a complicated function of velocity. It does not obviously have the simple velocity dependence required by relative covariance.

Thus our procedure appears neither to be consistent with relativity nor to preserve the energy of the state. However, it may be hoped that by going to the infinite-momentum frame these bad features may disappear. Although the result does greatly simplify for $p\to\infty$, there remain big terms in the nonleading term of the energy, i.e., we find $E_p=p+M'/\gamma$ with $M'\sim Gf$ as

$$\gamma=\frac{1}{(1-v^2)^{1/2}}\to\infty\,.$$

Nevertheless, the covariance along the longitudinal direction is restored.

Notice first from the definition for $\bar{g}_1$ (9.33) and as $\gamma\to\infty$, we get

$$
\begin{aligned}
g_1(\vec{z})+i\vec{v}\cdot\vec{\nabla}\bar{g}_1(\vec{z})=\int\frac{d^3k}{(2\pi)^3}\left(1+\frac{k_\parallel}{|k_\parallel|}\right)g(k)\\
\times\exp(-ik_\parallel\gamma z-i\vec{k}_\perp\cdot\vec{z}_\perp),
\end{aligned}
\tag{9.35}
$$

where $g(\vec{k})$ is the Fourier transform of $g(\vec{z})$. By a change of variable

$$z_\parallel\to\frac{1}{\gamma}z_\parallel$$

we find

$$
\begin{aligned}
&\int d^3z[g_1(\vec{z}-\vec{\Delta})-g_1(\vec{z})+i\vec{v}\cdot\vec{\nabla}(\bar{g}_1(\vec{z}-\vec{\Delta})-\bar{g}_1(\vec{z}))]^n\\
&\underset{\gamma\to\infty}{=}\frac{1}{\gamma}\int\prod_i\frac{d^3k_i}{(2\pi)^3}(2\pi)^3\delta^3(\vec{k}_1+\vec{k}_2+\cdots+\vec{k}_n)\prod_{i=1}^n[2\theta(k_{\parallel i})g(k_i)(e^{i\vec{k}_i\cdot\vec{\Delta}}-1)]\\
&=0\,.
\end{aligned}
\tag{9.36}
$$

This is so since it is impossible to satisfy simultaneously the two conditions

$$
\begin{aligned}
&(1)\ \ k_{\parallel i}\geq 0\,,\\
&(2)\ \ \sum_{i=1}^n k_{\parallel i}=0,\quad i=1,2,\ldots,n
\end{aligned}
\tag{9.37}
$$

unless all $k_{\parallel i}=0$. But the integrand does not have singular support at this point, so the integral vanishes. With the scaling substitution

$$
\begin{aligned}
z_\parallel&\to\frac{1}{\gamma}z_\parallel\,,\\
\Delta_\parallel&\to\frac{1}{\gamma}\Delta_\parallel\,,
\end{aligned}
\tag{9.38}
$$

and if we use the relations (9.36) and (9.38), the energy simplifies to

$$E_p - \vec{v}\cdot\vec{p} = \frac{1}{\gamma}M' , \tag{9.39}$$

where

$$\begin{aligned} M' = M &+ \frac{1}{N\gamma}\int d^3\Delta\, e^{-i\vec{p}\cdot\vec{\Delta}} K_1\left(\frac{1}{\gamma}\Delta_{\parallel}, \vec{\Delta}_{\perp}\right)\int d^3z\, \chi_1^*(\vec{z}-\vec{\Delta})G\beta\tfrac{1}{2}[g(\vec{z}-\vec{\Delta}) - g(\vec{z})]\chi_1(\vec{z}) \\ &+ \frac{1}{N\gamma}\int d^3\Delta\, e^{-i\vec{p}\cdot\vec{\Delta}} K_1\left(\frac{1}{\gamma}\Delta_{\parallel}, \vec{\Delta}_{\perp}\right)\int d^3y\, \chi_1^*(\vec{y}-\vec{\Delta})\chi_1(\vec{y}) \\ &\quad\times\int d^3z\, H\{\tfrac{3}{2}[g(\vec{z})^2 - f^2][g(\vec{z}-\vec{\Delta}) - g(\vec{z}) + g(\vec{z}-\vec{\Delta}) - g(\vec{z})]^2 \\ &\qquad + \tfrac{1}{2}[g(\vec{z}) - f][g(\vec{z}-\vec{\Delta}) - g(\vec{z}) + g(\vec{z}-\vec{\Delta}) - g(\vec{z})]^3\} . \end{aligned} \tag{9.40}$$

Here

$$\vec{p} = \left(\vec{p}_{\perp}, \frac{p_{\parallel}}{\gamma} - \mathcal{E}v\right) = (\vec{p}_{\perp}, (M-\mathcal{E})v) \tag{9.41}$$

and

$$\begin{aligned} N &= (N_v)_{\gamma\to\infty} \\ &= \frac{1}{\gamma}\int d^3\Delta\, e^{-i\vec{p}\cdot\vec{\Delta}} K_1\left(\frac{1}{\gamma}\Delta_{\parallel}, \vec{\Delta}_{\perp}\right) \\ &\quad\times\int d^3z\, \chi^{\dagger}(\vec{z}-\vec{\Delta})(1+\vec{\alpha}\cdot\vec{v})\chi(\vec{z}) \end{aligned} \tag{9.42}$$

We have evaluated the function K_1 and found it to be very insensitive to its arguments, so Eq. (9.39) is a statement of relativistic covariance along the longitudinal axis.

If we approximate $g(\vec{z}) - f$ by

$$g(\vec{z}) - f = -2\theta(R - |\vec{z}|) \tag{9.43}$$

then the function

$$g(\gamma z_{\parallel}, \vec{z}_{\perp})_{\gamma\to\infty} \equiv i\vec{v}\cdot\vec{\nabla}\bar{g}_1(\vec{z}) \tag{9.44}$$

gives

$$g(\vec{z}) \cong \frac{2f}{\pi i}\ln\left|\frac{z_{\parallel} - (R^2 - z_{\perp}^2)^{1/2}}{z_{\parallel} + (R^2 - z_{\perp}^2)^{1/2}}\right| . \tag{9.45}$$

The singularity at $z^2 = R^2$ is only superficial. It is a result of the approximation Eq. (9.43). The argument of the logarithm at $z^2 = R^2$ is of order $H^{\pm 1/3}$.

The second term in Eq. (9.40) which is associated with the fermion has been computed. The result is, to a good approximation, given by

$$\frac{1}{N\gamma}\int d^3\Delta\, e^{-i\vec{p}\cdot\vec{\Delta}} K_1\left(\frac{1}{\gamma}\Delta_{\parallel}, \vec{\Delta}_{\perp}\right)\int d^3z\, \chi_1^*(\vec{z}-\vec{\Delta})G\beta\tfrac{1}{2}g(\vec{z}-\vec{\Delta}) - g(\vec{z})\chi_1(\vec{z}) = -\frac{8}{3\pi}Gf\int_0^{\infty} dx[xj_1^2(\vec{x})] . \tag{9.46}$$

One of the two extra boson terms is small, since $g^2 - f^2$ is nonzero only near the surface of the thin shell. The other term, however, is big and we have only been able to bound its magnitude by

$$\begin{aligned} &\left|\frac{1}{N\gamma}\int d^3\Delta\, e^{-i\vec{p}\cdot\vec{\Delta}} K_1\left(\frac{1}{\gamma}\Delta_{\parallel}, \vec{\Delta}_{\perp}\right)\int d^3y\, \chi_1^*(\vec{y}-\vec{\Delta})\chi_1(\vec{y})\int d^3z\, H\tfrac{1}{2}[g(\vec{z}) - f][g(\vec{z}-\vec{\Delta}) - g(\vec{z}) + g(\vec{z}-\vec{\Delta}) - g(\vec{z})]^3\right| \\ &\qquad \leq \tfrac{16}{3}\pi R^3 H(4f^2)^4\frac{2}{\pi}\ln H^{1/3} \sim \sqrt{H}f\ln H . \end{aligned} \tag{9.47}$$

We have not succeeded in establishing the sign and magnitude of Eq. (9.47) and thereby determining whether there exists a specific condition for remaining on the mass shell M by canceling the contributions of Eqs. (9.46) and (9.47) to leading order as $v/c \to 1$.

X. SOME IMPORTANT PROBLEMS

In previous sections we have seen that a variational approach to a relativistic quantum field theory with spontaneous symmetry breaking and strong coupling reveals several interesting and novel features. In this section we wish to remark briefly on some of the most important problems which remain to be understood.

A. Limitations of the variational principle

We have achieved considerable simplification by using the variational principle. The advantage of this approach is that with it, we can apply our intuition about the classical problem in order to illustrate certain qualitative properties of the theory. On the other hand, this approach has a liability inherent in all variational calculations, that is, to evaluate the validity of our variational guess. Eventually it will be necessary to proceed more systematically in order to verify the existence of the bound states which are suggested by the variational calculation. One possibility is to embark on a systematic study of the quantum fluctuation effects in the field theory beginning with our solution to the classical field equations as the first approximation.

B. Higher-order quantum effects

Our variational calculation has been performed in the tree approximation; therefore, it remains to be seen whether the trial state and the energy and physical matrix elements will be significantly affected by including the quantum fluctuations. This latter point is particularly important in quantum field theory since the Hamiltonian density is an intrinsically singular operator and it requires a careful definition to make it both finite and positive-definite. In conventional perturbation expansion in powers of coupling constants, this definition is provided by the renormalization program. However, the conventional perturbation techniques are inapplicable here since we are interested in the strong-coupling behavior of the quantum field theory. While it may be prohibitively difficult in the strong-coupling quantum theory to derive and verify the validity of our results in quantitative detail, we are hopeful that qualitative answers may be found to such questions as the following:

(a) Does the binding mechanism discussed in the present paper persist when the quantum and renormalization effects are included?

(b) If the binding still occurs, does the character of the solution change qualitatively or quantitatively?

(c) Does there exist a range of parameters such that the solution to the field theory gives a reasonable description of the hadrons?

C. PCAC

PCAC and the role of the pion present a fundamental challenge to all quark models of hadrons. It is very attractive to suppose that the successes of $SU(2)\times SU(2)$ are explained by viewing the pion as a Goldstone boson. On the other hand, in a quark model with SU(9) mass spectra the pion is simply a $q\bar{q}$ bound-state partner of the ρ meson in the $\underline{35}$ and is accorded no special role. How to make these two different viewpoints mutually compatible is at present an unsolved problem.

In our theory we do not have PCAC because the divergence of the axial-vector current $\partial_\nu A_\alpha^\nu = G\sigma\bar{\psi}\gamma_5\lambda_\mu\psi$ is nonvanishing, and, in the strong-coupling limit with $G\gg 1$, is in no sense a "small operator." These difficulties with PCAC may be related to the unsatisfactory result for g_A which we have obtained since Eq. (8.20) may very well define the wrong operator in contrast with the magnetic moment operator which is constructed from the known and conserved electromagnetic current.

If we attempt to restore PCAC by enlarging the σ to a full chiral multiplet, we introduce too many pseudoscalar mesons: the Goldstone bosons themselves as candidates for the π, K, η, as well as the $q\bar{q}$ bound states presumably formed by our mechanism. Alternatively, we may view our model as a semiphenomenological description of the underlying strong dynamics which involves only massless quarks and color vector gluons. By this conjecture, the σ is a bound state as well as the hadrons it binds.

XI. SPECULATIONS

In this section we speculate on possible future applications of our theory. These speculations are based upon crude and naive calculations combined with liberal doses of intuition and wishful thinking. Our main reason for including them is to illustrate the enormously rich structure of a theory of the sort we are studying. The topics we shall touch on include (i) the excited-state spectrum and a possible connection of our model with the dual-string model, (ii) Bjorken scaling in deep-inelastic electron scattering, and (iii) the production mechanism and distribution of final-state hadrons in deep-inelastic electroproduction. As before, our discussion will be based on semiclassical arguments.

A. The excited-state spectrum and the dual-string model

The key question in the treatment of excited states is how rigidly the "classical" potential $g(x)$ (the σ expectation value) resists changing when a quark is excited. If $g(x)$ remains very nearly spherically symmetric, then a quark with nonzero orbital angular momentum l will have an energy $M_l=(l+1)^{2/3}M_0$ which is the spectrum for the Dirac equation in the potential $Gg(x)=Gf\tanh(\sqrt{2H}f(r-R))$. However, it is evident from Eq. (4.2) that $g(x)$ will not remain exactly spherical when the quark is in an $l\neq 0$ state.

In fact, if angular momentum is imparted to a quark along, say, the z direction, its wave function will develop nodes along this direction and extend primarily in orthogonal directions. We, therefore, expect the scalar potential to collapse in shape around the quark since it can thereby reduce the surface area of the confining bubble and thereby the field energy carried by the scalar field $g(x)$. At the same time, this deformation will not further squeeze the quark-wave function which, when $l>0$, is not using all the space available to it and so it will not increase its energy. Thus, we intuitively expect that the shape of the self-consistent scalar field will be distorted when the confined quarks carry angular momentum.

For a very crude estimate of the excitation energy associated with a deformed potential or confining field bubble, we consider a torus as illustrated in Fig. 6 with inner radius a and major radius b. The same heuristic argument used in the intuitive discussion of Sec. II gives the field energy associated with $g(x)$ as

$$E_g = k(4\pi^2 ab)4f^3H^{1/2} \tag{11.1}$$

after minimizing with respect to the thickness $D\sim 1/H^{1/2}f$ of the transition region for $g(x)$ to change from $+f$ to $-f$; $k\sim 1$ as in the spherical case. The fermion energy in analogy with our previous result might be expected to take the form[25]

$$\mathcal{E}\sim\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^{1/2}. \tag{11.2}$$

If we minimize the energy (11.1) plus (11.2) with respect to a and b, we find $a=b\equiv\frac{1}{2}R$, and if we set $k=1$,

$$\begin{aligned} E_{\text{torus}} &= \frac{3}{2}\,\frac{2\sqrt{2}}{R} \\ &= 3\sqrt{2}f\,(2\sqrt{2}\pi^2H^{1/2})^{1/3}. \end{aligned} \tag{11.3}$$

In terms of the energy E_s^j of a quark with angular momentum $j=l+\frac{1}{2}$ in a spherical shell, this energy can be written

$$\frac{E_t}{E_s^j}=\frac{(2\pi)^{1/3}}{(j+\frac{1}{2})^{2/3}}\ , \tag{11.4}$$

which is $\sim 1.2>1$ for $j=\frac{3}{2}$ and $\sim 0.9<1$ for $j=\frac{5}{2}$. Equation (11.4) shows that the energy of the toroid is not very different from the second and third excited states of the sphere even though $g(x)$ is very different in structure. It should also be emphasized that such a nonspherical solution for $g(x)$ must describe a superposition of many eigenstates of different total angular momenta, as in the case of a rigid rotor. Therefore, the true energy of the lowest state in this sum is smaller than what we have calculated.

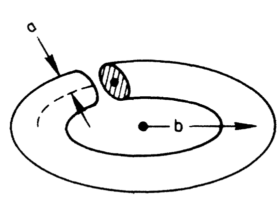

FIG. 6. Potential with torus shape.

Because of the softness of $g(x)$ discussed above, it is evident that we will not know anything about the details of the excited states of our theory until we learn to solve the general problem for deformed, excited states. Nevertheless, the above discussion of the low excitation energies of the toroidal type of configuration suggests a possible connection between these ideas and the general scheme envisioned in the dual-string model.

According to the preceding discussion, there will be a large number of nearby states corresponding to rotational and vibrational excitations of the toroid ("string"). Since the energy of the toroid is proportional to its surface area, and for $a/b\ll 1$ the quark energy will be like $1/a$, clearly the nearby excitations will be those which do not change the length or cross-sectional area of the "string." Presumably, the energy associated with these time-dependent motions will be approximately described by ascribing an effective mass density per unit length to the torus. This leads to a correspondence between the spectrum of the excited states in our model and the picture in a Virasoro-type dual-string model. From this point of view, the dual-string picture may emerge as a phenomenological description of the large density of states (collective stringlike excitations) available in a canonical relativistic field theory of the type being considered.

B. Scaling in deep-inelastic electron scattering

The fact that the quark mass is effectively small only in a thin shell makes any simple explanation of scaling in electroproduction hard to come by. One possibility, however, is that the softness of the shell to quark excitation and the small quark effective mass in the shell itself where $|g(x)|\ll f$ provides the dense set of excited states required so that scaling can occur.

Accepting for the moment the conjecture that the softness of the bag can provide an explanation for observed scaling, one sees what may be an important difference between $e\bar{e}$ annihilation and deep-inelastic lepton scattering processes. For the deep-inelastic processes, the virtual photon scatters from the proton bound state, and the onset of scaling is controlled, as suggested by the preceding discussion, by an energy scale of $\lesssim 1$ GeV associated with the excitations of the deformable shell. But in electron-positron annihilation into hadrons, there is no pre-prepared bound state and the important scale may be the bare quark and scalar σ-gluon production thresholds which are much larger than 1 GeV. The point is that in order to have scaling behavior, the time scale for production of the quark $\tau_{\rm prod} \sim 1/\sqrt{s}$ should be brief relative to the interaction time as controlled by the bare quark mass, $1/Gf$, and the range of the scalar interaction, $\sim 1/H^{1/2}f$. As discussed in Sec. VII, these, as well as the color thresholds, are energies $\gg 1$ GeV. This speculation suggests striking changes in the energy dependence of the total cross section as we first cross color thresholds and then the bare quark and scalar-gluon production thresholds. Thus scaling might appear at present machine energies for deep-inelastic scattering, but might require much larger energies to appear in electron-positron annihilation to hadrons. We emphasize, however, that this is all speculation and it remains an open question whether a simple scaling mechanism exists in our model.

We also comment that the observed rapid falloff of the nucleon elastic form factors may arise from the fact that in the presence of many low-lying deformed "bubble" states the probability for a nucleon which is excited by a highly virtual photon to remain in the ground state is rather small. Another question related to elastic form factors is whether they have nodes because of the thin-shell nature of the wave functions of the quark constituents. To answer this question, we have to understand the Lorentz contraction effect and the overlap factor for the "Cooper pairs" of a nucleon at rest and a moving nucleon.

C. Production of hadronic final states

A simple heuristic picture which seems to possess most of the general features of the inside-outside cascade postulated by Bjorken[26] and discussed by Casher, Kogut, and Susskind,[27] and others can be easily imagined.

Basically, the idea is that a photon comes in and hits one of the three quarks in a proton. This quark recoils from the other two quarks destroying local color charge neutrality and unshielding large color current densities. In analogy to a superconductor, an effect like the Meissner effect will probably take place to confine the resulting large "magnetic" fields to a finite region. This is accomplished by having a region surrounding the quarks become normal (i.e., $\langle \phi^*\phi\rangle = 0$) with large color supercurrents flowing on the boundary. As quarks separate the "normal region" grows into a long tube [since the term $(\phi^*\phi - f'^2)^2$ tends to keep the volume of the normal region as small as possible], and one obtains a restoring force between the quarks that does not fall off like $1/r^2$ (where r is the distance between the quarks).

As the surface of the shell increases in area, the threshold for producing quark-antiquark pairs decreases since they have more space in which to move, and so there will be a critical distance at which the energy stored in the confined color field will exceed the $q\bar{q}$ production threshold. At this moment, a $q\bar{q}$ pair will be produced and the color field will break and join separating sets of quarks. The shell will then break in two, corresponding to two states having the quantum numbers of a baryon and a meson. The process will repeat itself until the resulting fragments no longer have enough energy to separate. These regions would then oscillate and decay into hadrons via a different mechanism. As a consequence of the existence of these two different mechanisms, one would expect to have a set of excited clusters formed possibly spaced by a fixed distance in rapidity, which would decay into ordinary hadrons. Hence the general picture of an inside-outside cascade producing a plateau with short-range correlations in rapidity would seem natural from this point of view.

XII. COMPARISON WITH RELATED WORKS

In this section we compare our approach to the MIT bag model[28] and recent works by Lee and Wick,[11] Chin and Walecka,[11] Creutz,[12] and Dashen, Hasslacher, and Neveu.[13]

A. MIT bag model and Creutz's work

In the MIT model a hadron is a finite region of space to which almost free quanta of the hadronic fields (quarks or partons) are confined. It is obtained from free-field theory with two modifications:

(1) adding to the stress tensor $T^{\mu\nu}$ a term $g^{\mu\nu}B$, called the volume tension, which acts to compress the bag against the outward pressure of the quark gas;

(2) imposing boundary conditions such that the hadronic fields be confined in a finite region of space: the interior of a hadron or the bag.

Because of the boundary conditions, the MIT bag model is not a local field theory. Our model, on the other hand, is based on a conventional local field theory. A possible connection between the two models is discussed by Creutz and by Creutz and Soh.[12] At the classical level, Creutz has demonstrated that the MIT bag model of a scalar field with Dirichlet boundary conditions can be obtained from a local field theory with two scalar fields in a strong-coupling limit. One of the scalar fields produces the bag to confine the other scalar field. Recently, Creutz and Soh have also shown that the MIT bag model for fermions can be obtained from a local field theory. In both cases, the scalar field which produces the bag has a quartic self-coupling of the general type discussed in the Appendix where a brief account of the arguments of Creutz and Soh is presented.

Instead of a volume tension as in the MIT model, the scalar field energy in our model provides a surface tension. Furthermore, the quarks inside the potential do not appear to be free nor are they massless. One consequence is that radial excitations are absent in our model, but are present in the MIT bag model.[29] Presumably, this qualitative difference will also have important consequences in the behavior of form factors and structure functions when momentum transfers are large.

Finally the mechanism for quark confinement in the two models is different. Ours is only an approximate scheme in which the isolated quarks, as well as color nonsinglets, have high but finite threshold.[30] On the other hand, in the MIT model, if the quarks are coupled to a non-Abelian gauge field associated with color, then only color-singlet states can exist. This is an exact selection rule which follows from the boundary conditions for the color gauge fields and Gauss's law. Since this selection rule exists for any nonvanishing color gauge couplings, it is interesting to study what happens as the color gauge coupling is turned off smoothly.

B. Abnormal nuclear states and normal nuclear matter at high density

In a very interesting paper Lee and Wick[11] have discussed ideas very similar to these presented in our work, namely, they have also investigated the theoretical possibility that in a limited domain in space, the expectation value of a neutral spin-0 field may be quite different from its normal vacuum expectation value. Lee and Wick are mainly concerned with the formation of very heavy nuclei, while our primary interest is the possibility of constructing low-mass hadrons from heavy quarks. In the former case, since the atomic number is large, Lee and Wick assume that nucleons are approximately described by a degenerate Fermi distribution, characterized by a maximum Fermi momentum. In the tree approximation, Lee and Wick then find that when the coupling is sufficiently strong and density is high, the classical field $g(x)$ (in our notation) is favored to take the value 0 inside the nucleus. Thus the nucleons are effectively massless inside a heavy nucleus. In our case, however, the number of quarks in a hadron is so few that statistical mechanics does not apply. Instead, we have to actually solve the Dirac equation as well as the coupled equation for the scalar field. The quarks are found to have a large and negative mass inside a hadron instead of being massless.

Similar techniques have been used by Walecka[11] and Chin and Walecka[11] to study nuclear matter at high density. However, in their model, the scalar field does not have σ-model self-interactions, so that there is no spontaneous symmetry breakdown.

C. The work of Dashen, Hasslacher, and Neveu

Recently, Dashen, Hasslacher, and Neveu[13] have developed a technique for finding approximately the spectrum of bound states in a field theory without knowing the bound-state wave functions. Their starting point is a Feynman path-integral representation for the resolvent operator. It is the analog of WKB approximation in nonrelativistic quantum mechanics. This method of finding the bound-state spectrum reduces the problem to solving the same classical field equations as in our work. In our case, these classical field equations arise from the minimization of the energy in a particular class of trial states. Assuming that the trial states resemble the true states, we may compute, in addition to the bound-state energy, other (static) properties of the state as illustrated in Sec. VIII. Dashen, Hasslacher, and Neveu have applied their technique to the $(1+1)$-dimensional version of our model and find the exact classical solutions. They have also calculated the first quantum correction to these classical solutions in the weak-coupling case.

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APPENDIX

In this appendix we consider a more general class of models, in which the meson-meson interactions are not restricted to the σ-model form considered in Sec. II. We again find that the "quark" is confined to a thin spherical shell but, unlike the σ-model solution, we find that the meson field energy from the enclosed spherical volume may be much larger than the meson field energy from the shell. As a result, we recover the result suggested by the heuristic argument of Sec. II, that $E \propto fH^{1/4}$ (assuming that $G \gg H^{1/4}$). More precisely, for a many-quark system, in place of Eq. (6.4), we have

$$E = \frac{4}{3}\frac{n^{3/4}}{R_0}, \tag{A1}$$

where now

$$R_0 \propto \frac{1}{fH^{1/4}}\,. \tag{A2}$$

The local Hamiltonian density we consider is

$$\mathcal{H}(\vec{z}) = \tfrac{1}{2}[\dot{\phi}(\vec{z})]^2 + \tfrac{1}{2}[\nabla\phi(\vec{z})]^2 + U(\phi(\vec{z})) + \psi^\dagger(\vec{z})\left(\frac{\vec{\alpha}\cdot\vec{\nabla}}{i} + \beta G[\phi(\vec{z}) + f]\right)\psi(\vec{z})\,, \tag{A3}$$

in which the meson-meson interactions are given by

$$U(\phi) = \frac{c}{4!}\phi^4 + \frac{b}{3!}\phi^3 + \frac{a}{2}\phi^2\,. \tag{A4}$$

A convenient parameterization is

$$U(\phi) = H[\phi^4 + \tfrac{4}{3}(f_+ + f_-)\phi^3 + 2f_+f_-\phi^2]\,. \tag{A5}$$

We require $H > 0$ so that $U(\phi)$ has an absolute minimum and $f_+f_- > 0$ so that $\phi = 0$ is a minimum. We also make the choice

$$f_- \geq f_+ \geq \tfrac{1}{2}f_- > 0\,, \tag{A6}$$

so that U can be depicted as in Fig. 7, with a local minimum at $\phi = -f_-$ and a local maximum at $\phi = -f_+$. With the choice $f_+ = \frac{1}{2}f_- = f$, we recover the (displaced) σ-model Hamiltonian considered above.

We now proceed with the variational calculation. Forming a trial state as in Eq. (2.11) and varying the energy, we recover coupled differential equations

$$\left[\frac{\vec{\alpha}\cdot\vec{\nabla}}{i} + \beta G(g + f)\right]\chi = \mathcal{E}\chi\,, \tag{A7}$$

$$\frac{d^2g}{dr^2} + \frac{2}{r}\frac{dg}{dr} - \frac{\partial U}{\partial g} = G\bar{\chi}\chi\,. \tag{A8}$$

In first approximation, we let g be a square well

$$g(r) = -f_-\theta(R - r)\,, \tag{A9}$$

and provided that

$$G(f_- - f) \gg \frac{1}{R} \tag{A10}$$

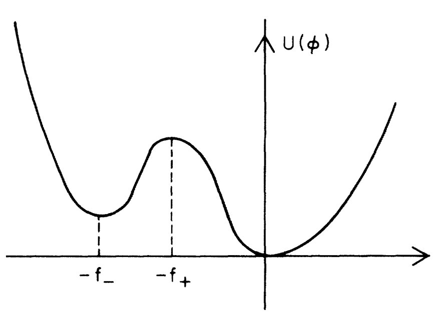

FIG. 7. The potential of Eq. (A5).

the Dirac equation (A7) has the familiar "shell" solution encountered in Sec. IV. Since

$$\left.\frac{\partial U}{\partial g}\right|_{-f_-} = \left.\frac{\partial U}{\partial g}\right|_0 = 0 \tag{A11}$$

the meson equation (A8) is also satisfied everywhere except near the shell.

For the energy of an n-quark state with all quarks in $l = 0$ angular momentum states, we have

$$E \cong \tfrac{4}{3}\pi R^3 U(-f_-) + 4\pi R^2 D U(-f_-) + 4\pi k R^2 \frac{f_-^2}{D} + \frac{n}{R}\,, \tag{A12}$$

where, as in Eq. (2.6), we have introduced a surface region of width D and a shape-dependent number k of order 1. Requiring $\partial E/\partial D = 0$, we have

$$D \cong \frac{k^{1/2}f_-}{[U(-f_-)]^{1/2}}\,, \tag{A13}$$

and provided that $R \gg D$, we may neglect the surface terms in Eq. (A12) so that $\partial E/\partial R = 0$ implies

$$R^4 = \frac{n}{4\pi U(-f_-)}\,, \tag{A14}$$

with E given by Eq. (A1). Using Eqs. (A13) and (A14) together with Eq. (A6), we see that the assumption $R \gg D$ is justified provided that $H \gg 1$.

Although we have not been able to give an explicit solution for g which specified g more completely near $r = R$, we will show, as in Sec. IV, that the Klein-Gordon equation (A8) "averaged" across the surface is automatically satisfied provided $\partial E/\partial R = 0$ and the Dirac equation (A7) is satisfied. That is, we shall verify

$$\int_{r_1}^{r_2} d^3z \frac{\partial g}{\partial r}\left(\frac{\partial^2 g}{\partial r^2} + \frac{2}{r}\frac{\partial g}{\partial r} - \frac{\partial U}{\partial g} - G\bar{\chi}\chi\right) = 0\,, \tag{A15}$$

where r_2 (r_1) is sufficiently greater (less) than R so that $\partial g/\partial r$ is negligible. Writing

$$d^3z = d\Omega\, dr\, r^2$$

and

$$\frac{\partial g}{\partial r}\frac{\partial^2 g}{\partial r^2} = \frac{1}{2}\frac{\partial}{\partial r}\left(\frac{\partial g}{\partial r}\right)^2,$$

we use an integration by parts to rewrite the first term in Eq. (A15) so that Eq. (A15) becomes

$$\int_{r_1}^{r_2} d^3z \frac{\partial g}{\partial r}\left(\frac{1}{r}\frac{\partial g}{\partial r} - \frac{\partial U}{\partial g} - G\bar{\chi}\chi\right) = 0\,. \tag{A16}$$

Now write

$$E = \int d^3z \left[\frac{1}{2}\left(\frac{\partial g}{\partial r}\right)^2 + U(g)\right] + \mathcal{E}\,, \tag{A17}$$

where $\mathcal{E}$ is the fermion energy

$$\mathcal{E} = \int d^3z\, \chi^\dagger\left(\frac{\alpha \cdot \nabla}{i} + \beta G g\right)\chi\,.$$

From the Dirac equation, precisely as in Eq. (4.21), we find

$$\frac{\partial \mathcal{E}}{\partial R} = -\int d^3x \frac{\partial g}{\partial r} G\bar{\chi}\chi\,. \tag{A18}$$

Differentiating the meson contribution to the energy and using the fact that g is a function of $r-R$, so that $\partial g/\partial r = -\partial g/\partial R$, we find after an integration by parts

$$\frac{\partial}{\partial R}\int d^3z\left[\frac{1}{2}\left(\frac{\partial g}{\partial r}\right)^2 + U(g)\right] = \int d^3z\, \frac{\partial g}{\partial r}\left(\frac{1}{r}\frac{\partial g}{\partial r} + \frac{\partial U}{\partial g}\right). \tag{A19}$$

Combining Eqs. (A18) and (A19), we see that $\partial E/\partial R = 0$ implies the averaged Klein-Gordon equation (A16).

For completeness, we now sketch briefly the arguments of Creutz and Soh that with a proper choice of parameters, it is possible to have a solution to (A7) and (A8) with all the characteristics of the MIT bag model with fermions.

We are looking for a solution with the following properties:

1. The classical field $g(x)$ is approximately a constant inside a sphere of radius R, and quickly reaches its vacuum value 0 outside.

2. Inside the sphere, the fermion mass is effectively zero.

According to (A8), we have for $g(x)$ inside the potential well, i.e., for $|x|<R$ and for small deviations from the minimum of $U(g) = -f_-$

$$\begin{aligned} g(x) &= -f_- + \frac{-1}{4\pi}\int d^3y \frac{e^{-m'|x-y|}}{|x-y|}\, G\bar{\chi}\chi(y) \\ &\cong -f_- - \frac{1}{m'^2} G\bar{\chi}\chi(x)\,, \end{aligned} \tag{A20}$$

where $m'^2 = 4Hf_-^2(1-f_+/f_-)$. For a massless fermion moving in a square well, we specify $f = f_-$ in (A7), and so χ is given by (4.4) and (4.5). Thus

$$\bar{\chi}\chi \sim \frac{1}{R^3}\,, \quad r<R\,. \tag{A21}$$

We require that the spatially varying part of $g(x)$ be small, that is, by (A20)

$$|g(x) + f_-| = \frac{1}{m'^2} G\bar{\chi}\chi \ll f_- \tag{A22}$$

or

$$\frac{G}{m'^2}\frac{1}{R^3} \ll f_-\,. \tag{A23}$$

The volume tension constant B of the MIT bag is identified as the energy at the secondary minimum

$$\begin{aligned} B &= U(-f_-) \\ &= \tfrac{2}{3} H f_-^4\left(\frac{f_+}{f_-} - \frac{1}{2}\right). \end{aligned} \tag{A24}$$

The surface energy associated with the transition region of $g(x)$ has been estimated by Creutz,

$$E_s \sim \text{const} \times \sqrt{2M}R^2 f_-^3\,, \tag{A25}$$

and is stored in a thickness of $1/\sqrt{2M}f_-$. In order for the volume energy to dominate as in the MIT model, we require

$$BR^3 \gg \sqrt{2M}R^2 f_-^3\,. \tag{A26}$$

Under these conditions, the total energy of the system is

$$E_n = \frac{an}{R} + \tfrac{4}{3}\pi R^3 B\,, \tag{A27}$$

where $a \simeq 2.04$ and n is the number of fermions and antifermions in the bag. Equation (A27) has a minimum at

$$\frac{(an)^{1/4}}{R} = (r\pi B)^{1/4}, \tag{A28}$$

with the value

$$E_n(\min) = \frac{4}{3}\frac{an}{R}.$$

For strong binding to occur, we must have

$$\frac{1}{R} \ll Gf\,. \tag{A29}$$

The requirement that the fermion is effectively massless inside is the statement

$$G(g+f) \ll \frac{1}{R}\,, \tag{A30}$$

which can be satisfied if

$$\begin{aligned} &f = f_-\,, \\ &\frac{H}{G^4} \gg \frac{f_+}{f_-} - \frac{1}{2}\,. \end{aligned} \tag{A31}$$

Equation (A29) implies

$$\frac{G^4}{H} \gg \frac{f_+}{f_-} - \frac{1}{2}\,. \tag{A32}$$

Now Eqs. (A23) and (A25) require

$$\left(\frac{H}{G^2}\right)^{2/3} H^{-1/3} \gg \frac{f_+}{f_-} - \frac{1}{2} \gg H^{-1/3}\,. \tag{A33}$$

All these conditions can be fulfilled, for example, by the choice

$$\begin{aligned} &H \sim G^4\,, \\ &1 \gg \frac{f_+}{f_-} - \frac{1}{2} \gg H^{-1/3}\,. \end{aligned} \tag{A34}$$

Finally, (A8) implies a condition similar to (4.17).

It is

$$B = \int dr\, G \frac{dg}{dr} \bar{\chi}\chi .$$

Since $\bar{\chi}\chi$ is slowly varying compared with dg/dr, we obtain

$$B = Gf_{-}\bar{\chi}\chi(R), \tag{A35}$$

which can be verified in the limit $Gf_{-} \to \infty$ to reduce to the boundary condition in the MIT model for a spherically symmetric solution:

$$2B = -\frac{\partial}{\partial r}(\bar{\chi}\chi), \quad r = R . \tag{A36}$$

It can be readily shown that conditions (A33) and (A34) ensure that the bag solution has a lower energy than a shell solution with the field $g(r)$ rising back to the value 0 as $r \to 0$.

The bag solution is not realized when $B = 0$ for a potential with a symmetry leading to a spontaneous breakdown which was the model used in the discussion in this paper since then the field will remain at the value $g = -f_{-}$ for all space, there being in that case no volume energy. However, for $B > 0$ but so small that

$$0 < \frac{f_{+}}{f_{-}} - \frac{1}{2} \ll H^{-1/3}$$

instead of (A34), we also find baglike fermion wave functions but with energies given by $H^{1/6}f$ as in the shell solution.

Thus we see that by making different choices of the parameters in the Hamiltonian, we obtain solutions with very different phenomenological implications.

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†Institute of Theoretical Physics, Physics Dept., Stanford University, Stanford, Calif. 94305.

‡Stanford Linear Accelerator Center, Stanford, Calif. 94305.

§Present address: Lawrence Berkeley Laboratory, Univ. of California, Berkeley, Calif. 94720.

‖Permanent address: Laboratory of Nuclear Studies, Cornell University, Ithaca, N.Y. 14850. Work supported in part by the National Science Foundation.

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[9]For $G \lesssim \sqrt{H}$ the thickness of the quark distribution is $\sim 1/Gf$; for $G \gtrsim \sqrt{H}$ it is $\sim 1/G^{1/2}H^{1/4}f$.

[10]It is, of course, possible that an absolute selection rule can be obtained by considering a limit in which the energy of colored states is infinite.

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[15]The c number in the local expansion depends on the external potential.

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[17]See, for instance, J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964), p. 40.

[18]For antifermions, there is an extra minus sign in the

energy as a result of the anticommutating nature of the D's and normal-ordering:

$$\overline{E} = \int d^3x \left[\tfrac{1}{2}(\nabla g)^2 + H(g^2 - f^2)^2 - \overline{\chi}\left(\frac{\vec{\alpha}}{i}\cdot\vec{\nabla} + G\beta g\right)\chi\right].$$

We then require that χ be the eigenfunction with the lowest negative energy:

$$\overline{\mathcal{E}}(g) = -\mathcal{E}(g)$$

and

$$\overline{E}(g) = \int d^3x \, [\tfrac{1}{2}(\nabla g)^2 + H(g^2 - f^2)^2] + \mathcal{E}(g)$$
$$= E(g).$$

This shows explicitly the symmetry between fermion and antifermion states. In the tree approximation, minimization of the energy in our trial state leads us to a classical field theory defined by the classical Hamiltonian (3.30) and classical field equations (3.31) and (3.38). The essential assumption is that the classical solution provides a useful zeroth-order approximation to a quantum field theory.

[19]Cf. J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964), p. 40.

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[30]See, however, Ref. 10.