Finite-width effects and subtraction corrections for the $A_2 \to 3\pi$ decay

Y. Matsumoto and J. G. Cordes
Department of Physics, Dalhousie University, Halifax, Nova Scotia, Canada
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Effects of interference between ρ-meson poles in the different channels and of subtraction terms for the $A_2 \to 3\pi$ decay are estimated. Goldberg's formula for the interference correction is discussed and tested by numerical calculations. The FDR-FESR (finite dispersion relation–finite-energy sum rules) technique is used to estimate the correction due to the subtraction. It is found that the effect of a finite ρ width and also the subtraction effect lead to small corrections to the decay width.

I. INTRODUCTION

Many two- and three-particle decay processes have been analyzed using amplitudes calculated in a simple narrow resonance (i.e., zero width) approximation. Usually, little attention is paid to the possible effects of (i) the finite width of the resonances involved or (ii) the need for subtractions in some of the amplitudes.

Goldberg[1] has argued that there should be sizable effects due to the interference between resonances in different channels in the $A_1 \to 3\pi$ and $A_2 \to 3\pi$ decays. He finds corrections to the decay widths of ~35% in each process due to overlapping of the ρ bands and ~65% due to overlapping of the σ bands in the $A_1 \to 3\pi$ decay.

Even more drastic are the corrections to the simple pole models for those processes whose amplitudes require subtractions. The use of finite dispersion relations (FDR) was proposed by Aviv and Nussinov[2] as a means of overcoming the problem of unknown subtraction constants. They applied FDR and finite-energy sum rules (FESR) to two-particle scattering amplitudes which are related to the decay amplitudes of interest by crossing. A number of other processes (involving both radiative and purely hadronic decays) have since been treated in this way.[3]

In this paper we reexamine the finite-width effects on the $A_2 \to 3\pi$ decay and also describe an FDR-FESR calculation for the same process.

In Sec. II we give a general formulation of the finite-width problem. Goldberg's results are discussed and compared with our calculations.

In Sec. III, the FDR calculation is described. Final conclusions are given in Sec. IV.

II. EFFECT OF A FINITE ρ WIDTH ON THE DETERMINATION OF $g_{A_2\rho\pi}$

In this section we discuss the effect of the ρ width $\Gamma_{\rho\pi\pi}$ on the two-step decay $A_2 \to \rho\pi \to \pi\pi\pi$.

The simplest possible approximation, of course, is the zero-width approximation in which the ρ is treated as a stable particle. The resulting two-particle phase-space integration may then be done exactly, giving

$$\Gamma^{(0)}(A_2 \to \rho\pi) = \frac{g^2}{20\pi}\,|\vec{q}|^5\,, \tag{1}$$

where $|\vec{q}|$ is the magnitude of the center-of-mass momentum of either the ρ or π. The coupling constant g ($\equiv g_{A_2\rho\pi}$) is defined by the matrix element

$$\langle \rho^a(q)|j_\pi{}^b(0)|A_2{}^c(p)\rangle = g\epsilon_{abc}\epsilon_{\mu\nu\lambda\tau}q^\mu\epsilon^\nu p^\lambda\epsilon^{\tau\sigma}q_\sigma\,, \tag{2}$$

where ϵ^ν is the polarization vector of the ρ and $\epsilon^{\tau\sigma}$ is the polarization tensor of the A_2. We take $m_{A_2} = 1.31$ GeV, $m_\rho = 0.765$ GeV, and $m_\pi = 0.138$ GeV for the particle masses.

The experimental value[4] of the width

$$\Gamma^{\text{expt}}(A_2 \to \rho\pi) \simeq 72\text{ MeV} \tag{3}$$

then implies, on using Eq. (1), the value

$$g = 19.0\text{ GeV}^{-2}\,. \tag{4}$$

If one does not wish to be restricted to this extreme approximation, it is necessary to analyze the four-point function $A_2 \to 3\pi$ in some detail. We begin by defining invariant amplitudes for the process:

$$_{\text{out}}\langle \pi^a(k_1)\pi^b(k_2)\pi^c(k_3)|A_2{}^A(p)\rangle_{\text{in}} = 2(R_1P_\mu + R_2Q_\mu)\epsilon^{\mu\nu}\epsilon_{\nu\tau\lambda\sigma}p^\tau Q^\lambda P^\sigma\,, \tag{5}$$

where the isospin indices on R_1, R_2 have been suppressed, $P = \frac{1}{2}(p + k_2)$ and $Q = \frac{1}{2}(k_3 - k_1)$.

Isospin amplitudes are introduced by the expansion

$$R_i = A_i\delta^{Aa}\delta^{bc} + B_i\delta^{Ab}\delta^{ac} + C_i\delta^{Ac}\delta^{ab}\,, \quad i = 1, 2\,. \tag{6}$$

The amplitudes may now be evaluated in the ρ-dominance approximation (i.e., assuming unsubtracted dispersion relations in the appropriate variables to be valid) to give

$$A_1=2g\gamma\left(\frac{2}{m_\rho^2-t}+\frac{1}{m_\rho^2-u}\right),\quad A_2=\frac{2g\gamma}{m_\rho^2-u}, \tag{7a}$$

$$B_1=2g\gamma\left(\frac{1}{m_\rho^2-s}-\frac{1}{m_\rho^2-u}\right),$$

$$B_2=2g\gamma\left(\frac{1}{m_\rho^2-s}+\frac{1}{m_\rho^2-u}\right), \tag{7b}$$

$$C_1=-2g\gamma\left(\frac{1}{m_\rho^2-s}+\frac{2}{m_\rho^2-t}\right),\quad C_2=-\frac{2g\gamma}{m_\rho^2-s}, \tag{7c}$$

where γ is the $\rho\pi\pi$ coupling constant, normalized so that

$$\Gamma_{\rho\pi\pi}=(\gamma^2/48\pi m_\rho^2)(m_\rho^2-4m_\pi^2)^{3/2}.$$

Finite-width effects due to the instability of the ρ meson can be allowed for in a simple way by making the replacement $m_\rho^2\to m_\rho^2-im_\rho\Gamma_{\rho\pi\pi}$ in Eqs. (7). The width for the decay $A_2^0\to\pi^+\pi^-\pi^0$ is then given by

$$\Gamma(A_2\to\rho\pi\to 3\pi)=\frac{1}{320\pi^3 m_{A_2}^3}\int ds\,dt\,du\,\delta(m_{A_2}^2+3m_\pi^2-s-t-u)\sum_{\text{spin}}|\langle 3\pi|A_2\rangle|^2. \tag{8}$$

Here s, t, u are the usual partial-energy variables for the pion pairs occurring in the final state, and the integration is taken over the whole Dalitz plot for the decay. Ordinarily a numerical integration would be required here, but Goldberg[1] has suggested an approximation scheme which allows an estimate of the finite-width corrections to be made on the basis of a simple integration "by hand." His method is to make the replacements

$$\left|\frac{1}{m^2-s-im\Gamma}\right|^2\to\frac{\pi}{m\Gamma}\,\delta(m^2-s), \tag{9}$$

$$\mathrm{Re}\left(\frac{1}{m^2-s-im\Gamma}\right)^*\left(\frac{1}{m^2-t-im\Gamma}\right)\to\pi^2\delta(m^2-s)\delta(m^2-t). \tag{10}$$

Note that the principal-value terms are omitted on the right-hand side of Eq. (10). We find, on using these δ-function approximations, the result for the corrected A_2 width:

$$\Gamma(A_2\to\rho\pi)=\Gamma^{(0)}(A_2\to\rho\pi)(1+\gamma^2/128). \tag{11}$$

The approximations $m_\pi=0$ and $m_{A_2}=\sqrt{3}m_\rho$ have also been made in obtaining Eq. (11), to facilitate the comparison with Goldberg's results. Instead of the correction factor in Eq. (11), Goldberg gives the correction factor $(1-3\gamma^2/256)$. Numerically, for a ρ width $\Gamma_{\rho\pi\pi}=146$ MeV,[4] Eq. (11) predicts a downward shift in g of ~11% (for a given $A_2\to\rho\pi$ width), while Goldberg's formula predicts an upward shift of ~30%.

In an effort to resolve the discrepancy between the two calculations we have carried out the double numerical integration involved in Eq. (8), without making any δ-function approximations, for several values of the ρ width. We find that for $\Gamma_{\rho\pi\pi}\lesssim 50$ MeV the sign of the correction effect is in agreement with Eq. (11), though quantitatively the agreement is only fair. For the experimental value, however, the numerical calculations (still with $m_\pi=0$, $m_{A_2}=\sqrt{3}m_\rho$) predict a 0.5% upward shifting. The overall picture (see Table I) then is of a re-

TABLE I. Values of the $A_2\rho\pi$ coupling constant (g), in units of GeV^{-2}, for several ρ widths and according to various calculation schemes. The percentages in brackets refer to the shifting from the value 16.40 in the $m_\pi=0$, $m_{A_2}=\sqrt{3}m_\rho$ case and from the value 18.98 in the real mass case. The value $\Gamma(A_2\to\rho\pi)=72$ MeV (Ref. 4) has been assumed. The values for $\Gamma_{\rho\pi\pi}$, m_{A_2}, and m_π quoted in the table are in GeV units.

	$m_\pi=0$, $m_{A_2}=\sqrt{3}m_\rho$			$m_\pi=0.138$, $m_{A_2}=1.31$
$\Gamma_{\rho\pi\pi}$	Predicted by Eq. (11)	Predicted by Goldberg	Computed	Computed
0	16.40	16.40	16.40	18.98
0.025	16.03 (−2.3%)	17.02 (+3.8%)	16.18 (−1.3%)	18.89 (−0.5%)
0.050	15.67 (−4.5%)	17.71 (+8.0%)	16.20 (−1.2%)	18.97 (−0.05%)
0.125	14.74 (−10.1%)	20.43 (+24.6%)	16.39 (−0.1%)	19.41 (+2.3%)
0.146	14.51 (−11.5%)	21.46 (+30.9%)	16.49 (+0.5%)	19.57 (+3.1%)

markable insensitivity of g to the value of $\Gamma_{\rho\pi\pi}$. This conclusion is unaffected by using the real masses for m_π and m_{A_2} in the numerical calculations (see Table I). It appears that there must be a nearly exact compensation between (i) the damping of the tail of the ρ peak by the kinematical factor, which vanishes on the boundary of the Dalitz plot, and (ii) the interference effect due to overlapping ρ bands in the decay region.

III. THE FDR CALCULATION

We will apply the FDR technique[2,3] to the decay $A_2^0 \to \pi^+\pi^-\pi^0$, which is related to the scattering process $\pi^+ + A_2^0 \to \pi^+ + \pi^0$ by crossing. Since there is Pomeranchuk exchange allowed in this scattering process, a subtraction is required in the fixed-t dispersion relation for the amplitude $B_1(\nu, t)$ [see Eq. (12) below]. It is the effect of this subtraction that we wish to estimate in this section.

The invariant amplitudes for this process are $B_1(\nu, t)$, which is odd in ν, and $B_2(\nu, t)$, which is even in ν, where $\nu = \frac{1}{2}(s-u)$. Their asymptotic behavior for large ν may be obtained by considering the t-channel center-of-mass frame helicity amplitudes. There is no helicity-zero amplitude, while $B_2(\nu, t)$ contributes only to the helicity-two amplitude. Thus we have for the large-ν behavior

$$B_1(\nu, t) \underset{\nu\to\infty}{\sim} 2g\gamma \sum_j \frac{\pi\beta_1^{(j)}}{\Gamma(\alpha_j)\sin\pi\alpha_j} \times [\nu^{\alpha_j -1} - (-\nu)^{\alpha_j - 1}] , \qquad (12)$$

$$B_2(\nu, t) \underset{\nu\to\infty}{\sim} \sum_j \frac{\pi\beta_2^{(j)}}{\Gamma(\alpha_j)\sin\pi\alpha_j} [\nu^{\alpha_j -2} + (-\nu)^{\alpha_j - 2}] , \qquad (13)$$

where $\alpha_j \equiv \alpha_j(t)$ are possible Regge trajectories to be exchanged in the t channel and $\beta_1^{(j)}$, $\beta_2^{(j)}$ are corresponding Regge residue functions (assumed as usual to be constants) to be determined by the FESR. The leading trajectories exchanged are P, P', and ϵ. However, following Aviv and Nussinov,[2] we expect the Pomeron contribution in the decay region to be negligible, and we also omit its contribution to the FESR on the basis of the Harari-Freund conjecture.[5] Therefore, the Regge-pole amplitudes $B_1^R(\nu, t)$ and $B_2^R(\nu, t)$ contributing to our decay process are the right-hand sides of Eqs. (12) and (13), respectively, with j referring only to P' and ϵ. We take $\alpha_{P'}(t) = 0.5 + t$ and $\alpha_\epsilon(t) = -0.8 + t$.[6]

First we determine the residues β_1^j and β_2^j ($j = P', \epsilon$) by FESR. We take the cutoff N of ν to correspond to $s_{\max} = \frac{1}{4}(m_\rho^2 + 3m_g^2) = 4m_\rho^2$, where $m_g = \sqrt{5}\, m_\rho$ is the g-meson mass. We make this choice because the customary choice $\frac{1}{2}(m_\rho^2 + m_g^2) = 3m_\rho^2$ is not large enough compared to the values of s involved in the decay process under consideration (due to the relatively large mass of the A_2). The corresponding cutoff value of ν is

$$N = 4m_\rho^2 - \tfrac{1}{2}(m_{A_2}^2 + 3m_\pi^2) + \tfrac{1}{2}t . \qquad (14)$$

The FESR for the $B_i(\nu, t)$ read

$$\int_0^N d\nu \,\mathrm{Im}\, B_1^P(\nu, t) = \int_0^N d\nu \,\mathrm{Im}\, B_2^R(\nu, t) \qquad (15)$$

and

$$\int_0^N d\nu\, \nu \,\mathrm{Im}\, B_2^P(\nu, t) = \int_0^N d\nu\, \nu \,\mathrm{Im}\, B_2^R(\nu, t) , \qquad (16)$$

where the B_i^P are the pole-dominated low-energy amplitudes given by Eq. (7b). Taking the first two terms in the expansions of these equations about $t=0$ and solving for the β's, we find

$$\beta_1^{P'} = -0.812, \quad \beta_1^\epsilon = 0.644, \qquad \beta_2^{P'} = 0.325, \quad \beta_2^\epsilon = -0.830 \qquad (17)$$

in units of GeV^{-2}.

We assume that our amplitudes satisfy fixed-t finite dispersion relations:

$$B_i(\nu, t) = \frac{1}{2\pi i} \oint d\nu' \frac{B_i(\nu', t)}{\nu' - \nu} = B_i^P(\nu, t) + B_i^H(\nu, t), \quad i = 1, 2 \qquad (18)$$

where

$$B_1^H(\nu, t) = \frac{\nu}{2\pi i} \int_{C_N} d\nu' \frac{B_1^R(\nu', t)}{\nu'^2 - \nu^2} = \sum_j \frac{2g\gamma\beta_1^j N^{\alpha_j - 1}}{\Gamma(\alpha_j)} \sum_{n=0}^{\infty} \left(\frac{\nu}{N}\right)^{2n+1} \frac{1}{\alpha_j - 2n - 2} \qquad (19)$$

and

$$B_2^H(\nu, t) = \frac{1}{2\pi i} \int_{C_N} d\nu' \frac{\nu' B_2^R(\nu', t)}{\nu'^2 - \nu^2} = \sum_j \frac{2g\gamma\beta_2^j N^{\alpha_j - 2}}{\Gamma(\alpha_j)} \sum_{n=0}^{\infty} \left(\frac{\nu}{N}\right)^{2n} \frac{1}{\alpha_j - 2n - 2} , \qquad (20)$$

where C_N is a circle with radius N centered at $\nu = 0$.

On using the amplitudes of Eq. (18) to calculate the A_2 width we get the result [using actual masses, $\Gamma_{\rho\pi\pi} = 146$ MeV, and $\Gamma(A_2 \to \rho\pi) = 72$ MeV]

$$g = 20.11\ \mathrm{GeV}^{-2} , \qquad (21)$$

which is only 2.8% larger than the corresponding simple-pole-model value $g = 19.57\ \mathrm{GeV}^{-2}$, and 6% above the value ($18.98\ \mathrm{GeV}^{-2}$) predicted by the simple zero-width formula Eq. (1).

We have checked the variation in the result (21) due to inclusion of an imaginary part in the ϵ trajectory of the form

$$\alpha_\epsilon(t) = -0.8 + t + i(0.28)(t - 4m_\pi^2)^{1/2}$$

(Ref. 6). By again expanding around $t=0$ we find the change in all four residues to be $\lesssim 30\%$. Since the contribution to the A_2 width from the high-energy part of the amplitudes is, however, already only $\sim 3\%$, this complication does not affect our conclusion. Similarly, varying the ϵ-trajectory intercept from -0.8 to -0.6 is found to have a negligible ($\sim -1.7\%$) effect on the resulting value of g. Finally, we have also carried out the calculation using residues determined by an expansion around $t = 0.8$ GeV^2 instead of $t=0$. The residues do change considerably, but the resulting value for g is 19.4 GeV^{-2}—only 3% below the $t=0$ expansion result.

The relative smallness of the effect of including the $B_i^H(\nu, t)$ in the $B_i(\nu, t)$ can be explained along the following lines. We first note the three main reasons for carrying out an FDR-FESR calculation in the first place: (i) to take into account possible subtraction terms, (ii) to avoid double counting, and (iii) to take into account higher resonances. Now instead of using the s-channel scattering $\pi^+ + A_2^0 \to \pi^+ + \pi^0$, one can relate the decay reaction to the t-channel scattering $\pi^0 + A_2^0 \to \pi^+ + \pi^-$, where the ρ trajectory $\alpha_\rho(s)$ is exchanged in the s channel (no Pomeron exchange). The asymptotic behavior of the invariant amplitude for large $\bar{\nu}$, with fixed s, is $\bar{\nu}^{\alpha_\rho - 1}$, where $\bar{\nu} = \frac{1}{2}(t-u)$. Thus the fixed-$s$ dispersion relations do not require any subtractions. One might expect that avoidance of double counting would have some effect through the s dependence of the residues of the ρ poles which now appear as the first terms of the series for the $B_i^H(\bar{\nu}, s)$, which are defined analogously to Eqs. (18) and (19). However, since the ρ pole is right in the middle of the physical region of the decay, and since this residue is forced to coincide, at the pole position, with that of the simple pole model, this effect is very small.[7] The contribution from higher s-channel poles is relatively small, again because of the location of the ρ poles within the decay region. We conclude, therefore, that the simple pole model should be a very good approximation for the t-channel amplitudes and hence, by crossing, the combined effects of (i)–(iii) should be small also for the s-channel amplitudes. It is evident that this argument can only be expected to be valid within the context of the continuation to the decay region, and the subsequent smoothing resulting from the phase-space integration.

IV. CONCLUSIONS

It has been found that, for the $A_2 \to 3\pi$ decay, Goldberg's estimate of finite ρ-width corrections by calculating the interference effect is unsatisfactory. This is due to the extreme nature of the δ-function approximation which overestimates the contribution of the noninterference terms to the width. We find that the finite ρ-width correction to the coupling constant g is only about $+3\%$, with perhaps an additional 3% correction being suggested by the FDR-FESR calculations. It seems reasonable to suppose that the same kind of compensation may take place in the $A_1 \to \rho\pi$ decay (which has also been analyzed by Goldberg[1]), although we have not yet carried out the detailed calculation required to show this.

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