

Acausality in truncated Müller-Israel-Stewart-type theory

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The causal-stable Müller-Israel-Stewart (MIS) theory is known to have a finite number of out-of-equilibrium derivative-order corrections but requires treating the viscosity tensor as a separate degree of freedom with its own equations of motion, apart from the fundamental fluid degrees of freedom like velocity and temperature. In this work, I will show that it is possible to rewrite the MIS theory only in terms of velocity and temperature, but the resultant constitutive relation for dissipation must include all orders of gradient corrections. In this work, I will argue that an all-order resummation of gradient contributions is equivalent to introducing new “nonfluid” degrees of freedom in the MIS theory. It will also be shown, using the relativistic quantum causality condition, that any finitely truncated order of derivative correction, however high it is, leads to a theory that is acausal, unless the corrections are infinitely summed up to all orders.

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Introduction. Fluid dynamics is an effective theory that describes the dynamics of a near equilibrium system by the evolution of the conserved fields at long wavelength limit [1,2]. The expectation values of these fields at equilibrium define the hydrodynamic state variables which serve as the fundamental fluid degrees of freedom. The out-of-equilibrium dynamics is described in terms of the order-by-order gradient corrections of these fields. For a relativistic system however, the fluid theory needs to qualify some physical criteria such as causal wave propagation and stability against small perturbations. The relativistic version of the long established first-order theories [1,3] turns out to suffer from the pathologies regarding superluminal signal propagation and thermodynamic instability. As a rescue, the higher-order Müller-Israel-Stewart (MIS) theory [4–7] has been suggested to apply as the standard theory of relativistic dissipative fluid dynamics [8–10].

The MIS theory and the recent derivations of other analogous higher-order theories [11–14] can serve as causal as well as stable hydrodynamic formalism with the fundamental fluid variables like velocity and temperature being locally fixed to their equilibrium values by the Landau gauge condition ($T^\mu_\nu u^\nu = -\epsilon u^\mu$, $T^{\mu\nu}$ is the energy-momentum tensor, ϵ and u^μ are the energy density and fluid velocity at their local equilibrium values). The price is paid by the limitation that these fundamental hydrofields are no longer sufficient to describe the fluid dynamics and additional degrees of freedom have to be introduced. Hence, the dissipative fields in the MIS theory are promoted as the

new degrees of freedom and are attributed their individual equations of motion. Although this version of relativistic hydrodynamic theories has been quite popular and phenomenologically successful especially in the context of high energy heavy ion data analysis [15], the physical meaning of these new degrees of freedom still remains somewhat questionable. These quantities do not relate to any conserved fields. In other words, they do not have any equilibrium counterparts. Given the scenario, one might wonder if it is possible to have an equivalent “fluid dynamical” description of MIS theory where the requirement of these nonfluid new degrees of freedom can be eliminated. By fluid dynamical here I mean that (i) the constitutive relation of the stress tensor can be entirely written in the terms of temperature, velocity, and their derivations and the only equations of motion are the stress tensor conservation, and (ii) the fluid variables such as velocity and temperature are fixed by the Landau gauge such that there is no ambiguity in their definition.

Motivated by this idea, in this work, I attempt to rewrite the MIS theory that obeys causality and stability while remaining in the Landau frame (such that the field variables are well defined) but without requiring any additional degrees of freedom. For simplicity, here a conformal system is considered without any conserved charges. We will see that it is indeed possible to generate the identical results of MIS theory only using the velocity and temperature as fluid variables, if the constitutive relation for the dissipation (here the shear tensor) runs up to infinite orders of gradient correction. In this context, we remember that an all-order resummation of gradient contributions is known to be equivalent to introducing non-hydrodynamic modes

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which have been implemented in many ways. In [16,17], the transport coefficients are resummed by making them frequency dependent in order to incorporate all-order gradient corrections. In [18,19], these large orders are dealt with using Borel resummation techniques with Padé approximation which are also leading works of the far from equilibrium attractor theory. MIS theory is one such resummation scheme where the resummation is shown to result in not only non-hydro-modes but new nonfluid degrees of freedom. I will then explicitly prove, using the causality condition $\text{Im}(\omega(k)) \leq |\text{Im}(k)|$ of relativistic quantum theory [20], that any finite-order truncation of the derivative correction series immediately leads to acausality unless the dissipations are promoted as individual degrees of freedom or summed infinitely.

Throughout the article, I use natural units ($\hbar = c = k_B = 1$) and flat space-time with mostly positive metric signature $\eta^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$.

MIS theory: An all-order gradient correction theory. To set the chain of arguments, I start with the well-known form of the MIS equations of motion keeping up to the linear terms [21],

$$\partial_\mu T^{\mu\nu} = 0, \quad T^{\mu\nu} = \varepsilon \left(u^\mu u^\nu + \frac{1}{3} \Delta^{\mu\nu} \right) + \pi^{\mu\nu}, \quad (1)$$

$$\pi^{\mu\nu} + \tau_\pi D \pi^{\mu\nu} = -2\eta \sigma^{\mu\nu}. \quad (2)$$

The conformal and uncharged energy-momentum tensor $T^{\mu\nu}$ consists of the equilibrium fields such as energy density ε and hydrodynamic velocity u^μ (along with projection operator $\Delta^{\mu\nu} = \eta^{\mu\nu} + u^\mu u^\nu$) and the viscous correction $\pi^{\mu\nu}$ that depends upon shear viscosity η and relaxation time of shear viscous flow τ_π . The notation $D (= u^\mu \partial_\mu$ in local rest frame) indicates the temporal derivative correction of the hydrodynamic fields and $\sigma^{\mu\nu} = \Delta^{\mu\nu} \partial^\alpha u^\beta$ denotes the traceless, symmetric velocity gradient with $\Delta^{\mu\nu\alpha\beta} = \frac{1}{2} \Delta^{\mu\alpha} \Delta^{\nu\beta} + \frac{1}{2} \Delta^{\mu\beta} \Delta^{\nu\alpha} - \frac{1}{3} \Delta^{\mu\nu} \Delta^{\alpha\beta}$. Here I attempt to derive the combined results of Eqs. (1) and (2) without treating $\pi^{\mu\nu}$ as an independent degree of freedom. Instead of attributing an individual differential equation to $\pi^{\mu\nu}$ like Eq. (2), I express it as a sum of order-by-order gradient corrections in Eq. (1) itself as

$$\begin{aligned} \pi^{\mu\nu} &= \sum_n \pi_n^{\mu\nu}, \\ \pi_1^{\mu\nu} &= -2\eta \sigma^{\mu\nu}, \quad \pi_n^{\mu\nu} = -\tau_\pi D \pi_{n-1}^{\mu\nu}, \quad n \geq 2. \end{aligned} \quad (3)$$

This summation of order-by-order gradient correction in (3) leads to the shear-stress tensor as the following:

$$\pi^{\mu\nu} = -2\eta \left\{ \sum_{n=0}^N (-\tau_\pi D)^n \right\} \sigma^{\mu\nu}, \quad (4)$$

where up to N th order of temporal derivative corrections have been taken. Next, I linearize the conservation equations (1) and (2) for small perturbations of fluid variables around their hydrostatic equilibrium as $\psi(t, x) = \psi_0 + \delta\psi(t, x)$. Here, the subscript 0 indicates global equilibrium and the fluctuations $\delta\psi(t, x)$ are expressed in the plane wave solutions via a Fourier transformation $\delta\psi(t, x) \rightarrow e^{i(kx - \omega t)} \delta\psi(\omega, k)$, with wave 4-vector $k^\mu = (\omega, k, 0, 0)$. Following the linearization, the shear and sound channel dispersion relations from Eq. (4) respectively become

$$(i\omega) + \tilde{\eta}(ik)^2 \left[\sum_{n=0}^N (\tau_\pi i\omega)^n \right] = 0, \quad (5)$$

$$(i\omega)^2 + \frac{4}{3} \tilde{\eta}(i\omega)(ik)^2 \left[\sum_{n=0}^N (\tau_\pi i\omega)^n \right] - \frac{1}{3} (ik)^2 = 0. \quad (6)$$

Here $\tilde{\eta} = \eta/(\varepsilon_0 + P_0) = \eta/(\frac{4}{3}\varepsilon_0)$ for a conformal system.

It can now be readily checked that, if in Eqs. (4)–(6) the sum over the gradient series is taken up to all orders ($N \rightarrow \infty$), the infinite sum results in a closed form, such as $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ with $x = \tau_\pi i\omega$. The sum exists within the radius of convergence $|\tau_\pi i\omega| < 1$, whose circumference is the location of the first non-hydro-mode of MIS theory ($\omega = -\frac{i}{\tau_\pi}$), beyond which hydrogradient expansion should not be trusted anyway. Applying this technique, Eq. (4) turns out to be

$$\pi^{\mu\nu} = -2\eta(1 + \tau_\pi D)^{-1} \sigma^{\mu\nu}, \quad (7)$$

where this infinite sum appears in the denominator of the expression of $\pi^{\mu\nu}$ in the form of the relaxation operator $(1 + \tau_\pi D)$. We can see Eq. (7) readily takes us to Eq. (2). Hence, we see that the all-order infinite sum in $\pi^{\mu\nu}$ in Eq. (3) is producing the identical results of solving the $\pi^{\mu\nu}$ from Eq. (2) by considering it an independent degree of freedom, without actually doing so. Consequently, from Eqs. (5) and (6) with $N \rightarrow \infty$, we find the well-known dispersion polynomials of MIS theory [22], which at local rest frame for shear and sound channels are given respectively by

$$\tau_\pi \omega^2 + i\omega - \tilde{\eta} k^2 = 0, \quad (8)$$

$$\tau_\pi \omega^3 + i\omega^2 - \left(\frac{4}{3} \tilde{\eta} + \frac{1}{3} \tau_\pi \right) \omega k^2 - \frac{1}{3} i k^2 = 0. \quad (9)$$

So if we choose to integrate out $\pi^{\mu\nu}$ from Eq. (2) by solving it first in terms of the fluid variables using the perturbative technique of derivative expansion, the resultant fluid equations turn out to have an infinite number of derivatives [Eq. (3) with $N \rightarrow \infty$]. This infinite sum over the gradient corrections [Eq. (3)] being “integrated in” [summed in a closed form to generate temporal derivatives in the

denominator such that $\pi^{\mu\nu}$ gets its own differential equation (7)], as the new degrees of freedom in MIS theory, is the key result of the current work.

Based on these results, in this work I argue that the theory well known as the MIS theory [Eqs. (1) and (2) combined], that is known to be free from the acausal signal propagation and thermodynamic instabilities, is not a second-order or any finite higher-order truncated theory. Rather an infinite sum over the derivative-order corrections is required to produce Eqs. (1) and (2). This infinite sum is actually equivalent to attributing a new degree of freedom other than velocity and temperature. Describing the dissipative dynamics of a relativistic system in terms of gradient corrections renders a pathology-free theory only and only if the corrections are summed up to infinite orders. Any truncation at finite order directly leads to the violation of causality and cannot serve as an acceptable hydrodynamic theory.

In the following sections, I will establish that any finitely truncated-order summation of $\pi^{\mu\nu}$ in Eq. (3), however high it is, cannot produce a pathology-free stable-causal theory. In order to resolve the issues of causality and stability, we need to consider all orders of gradient corrections in Eq. (4) ($N \rightarrow \infty$), such that the infinitely summed derivatives generate the relaxation operator $(1 + \tau_\pi D)$ in the denominator of $\pi^{\mu\nu}$. To prove this, I take recourse of the relativistic quantum theory causality condition,

$$\text{Im}(\omega(k)) \leq |\text{Im}(k)|, \quad (10)$$

which indicates the stability invariance of a theory as well [23–25]. In the following, I will show case by case that a truncated theory always violates Eq. (10), while an all-order theory resulting from the infinite sum restores the causality and stability by always satisfying (10) for $\tau_\pi > \tilde{\eta}$.

Testing truncated theory. In order to check condition (10), we need to extract the solution $\omega(k)$ from the respective dispersion polynomial. Here, as a test case, I investigate the shear channel frequency solution under the inequality (10). As mentioned in the previous section, for a truncated theory where $\pi^{\mu\nu}$ is taken up to a certain finite order N , the shear channel dispersion polynomial (5) becomes

$$(i\omega) + \tilde{\eta}(ik)^2 \left[\sum_{n=0}^N (\tau_\pi i\omega)^n \right] = 0. \quad (11)$$

I will now show that, for any finite N , the solution of Eq. (11) is not able to satisfy the condition given in (10) for all possible values of k .¹

Truncation at $N = 0$: For $N = 0$, we have the usual Navier-Stokes shear channel which has an exact solution,

$\omega = -i\tilde{\eta}k^2$. For any real k , the condition (10) requires $\tilde{\eta}$ to be positive. But if k is a purely imaginary large number $k = \pm ip$, with $p(\gg 1)$ real and positive, we have $\text{Im}(\omega) = \tilde{\eta}p^2$, which with $\tilde{\eta} > 0$ clearly violates $\text{Im}(\omega) \leq |\text{Im}(k)|$ since $\text{Im}(\omega) \sim p^2$ and $|\text{Im}(k)| \sim p$.

Truncation at $N = 1$: For $N = 1$, we have the exact solution

$$\omega = -i\tilde{\eta}k^2/[1 - \tilde{\eta}\tau_\pi k^2]. \quad (12)$$

For a value of k given by $\tilde{\eta}\tau_\pi k^2 = (1 \pm \epsilon)$ with ϵ to be real, positive, and $\epsilon \ll 1$, $\text{Im}(\omega_\pm) = \pm \frac{1}{\tau_\pi \epsilon}$ which is a large number. Now if τ_π is positive (η is also a positive number [26] considering the constraints of the second law of thermodynamics), then k is a real number with $\text{Im}(k) = 0$ and $\text{Im}(\omega_+) = 1/(\tau_\pi \epsilon)$ is a large positive number. So $\text{Im}(\omega) \leq |\text{Im}(k)|$ is violated. If τ_π is negative, $\text{Im}(\omega_-) = -1/(\tau_\pi \epsilon)$ is again a large positive number with $|\text{Im}(k)| = 1/\sqrt{|\tilde{\eta}\tau_\pi|}$ which is a finite quantity. So $\text{Im}(\omega) \leq |\text{Im}(k)|$ is again violated.

Finite truncation at $N \geq 2$: For $N \geq 2$ but still with finite truncation, Eq. (11) is only possible to solve at limiting values of k . At $k \rightarrow 0$, Eq. (11) gives at least one solution,

$$(i\omega) = \left[\frac{1}{\tilde{\eta}\tau_\pi^N k^2} \right]^{\frac{1}{N-1}}. \quad (13)$$

Here the relaxation time and the wave vector is expressed as

$$\tau_\pi = |\tau_\pi| e^{i\sigma\pi}, \quad k = |k| e^{im\pi}, \quad (14)$$

with $\sigma = 0$ and $\sigma = 1$ corresponding to positive and negative values of relaxation time, respectively, and m is any real number. The imaginary part of frequency and wave number respectively becomes

$$\begin{aligned} \text{Im}(\omega) &= - \left[\frac{1}{\tilde{\eta}|\tau_\pi|^N |k|^2} \right]^{\frac{1}{N-1}} \cos \left\{ \frac{(2m + \sigma N)}{N-1} \pi \right\}, \\ |\text{Im}(k)| &= |k| |\sin(m\pi)|. \end{aligned} \quad (15)$$

We can always choose a domain such as $\frac{1}{2} < (2m + \sigma N)/(N-1) < \frac{3}{2}$, where the cosine function of $\text{Im}(\omega)$ is negative such that $\text{Im}(\omega)$ is a large positive number if $|k|$ is small. But $|\text{Im}(k)|$ is finite at small $|k|$. So clearly condition (10) is violated.

So we see that for any truncated value of N starting from 0, there always exists at least one mode such that the imaginary part of ω becomes greater than the modulus of the imaginary part of k in the complex k -plane and hence has issues with causality and stability. Thus we can safely conclude that any truncated order of gradient correction in Eq. (3) cannot produce a relativistic, dissipative theory that is consistent with the causality-stability assessment.

¹This part of the calculation has been done with the collaboration of Sayantani Bhattacharyya.

Testing all-order theory. The all-order infinitely summed theory has the shear channel dispersion polynomial as the following:

$$\tau_\pi(i\omega)^2 - (i\omega) - \tilde{\eta}(ik)^2 = 0. \quad (16)$$

The solution of Eq. (16) is

$$\omega_\pm = \frac{1}{2\tau_\pi} \left[-i \pm \{-1 + 4\tilde{\eta}\tau_\pi k^2\}^{\frac{1}{2}} \right]. \quad (17)$$

Now decomposing ω as $\omega = \omega_R + i\omega_I$ and expressing k as $k = |k|e^{im\pi}$ as before, it is possible to derive

$$2(1 + 2\tau_\pi\omega_I)^2 = \{1 - 4\tilde{\eta}\tau_\pi|k|^2 \cos(2m\pi)\} \pm [1 + 16\tilde{\eta}^2\tau_\pi^2|k|^4 - 8\tilde{\eta}\tau_\pi|k|^2 \cos(2m\pi)]^{\frac{1}{2}}. \quad (18)$$

In the following, it will be proved that for any value of $|k|$ and m , it is not possible to violate condition (10). If possible, violation of (10) demands

$$\omega_I > |k| |\sin(m\pi)|. \quad (19)$$

Equations (18) and (19) simultaneously lead us to the inequality

$$\begin{aligned} & \pm \left[\{1 - 4\tilde{\eta}\tau_\pi|k|^2\}^2 + 16\tilde{\eta}^2\tau_\pi^2|k|^4 \sin^2(m\pi) \right]^{\frac{1}{2}} \\ & > 1 + 4\tilde{\eta}\tau_\pi|k|^2 + 8\tau_\pi|k| |\sin(m\pi)| \\ & + 8\tau_\pi(\tau_\pi - \tilde{\eta})|k|^2 \sin^2(m\pi). \end{aligned} \quad (20)$$

We can see that, if $\tau_\pi > \tilde{\eta}$, the maximum value of the left-hand side and the minimum value of the right-hand side of (20) both are $(1 + 4\tilde{\eta}\tau_\pi|k|^2)$. So the left-hand side of (20) can never exceed the right-hand side and the violation of (10) is never possible once the condition $\tau_\pi > \tilde{\eta}$ is satisfied. Interestingly enough, this is the asymptotic causality condition of MIS shear channel estimated from the large wave number propagating mode that can be obtained from (10) itself.

So we conclude that, once the condition $\tau_\pi > \tilde{\eta}$ is obeyed by the transport coefficients, the causality condition $\text{Im}(\omega) \leq |\text{Im}(k)|$ is satisfied for all possible k values in an all-order theory. Hence, in contrast to the truncated ones, an all-order theory is the only admissible candidate for a reliable hydrodynamic theory for relativistic, dissipative fluids.

Conclusion. In this work, it has been shown that, although in the conventional MIS theory the equations of motion for temperature, velocity, and viscous tensor have finite number of derivatives, integrating out $\pi^{\mu\nu}$ by solving it in terms of the fluid variables perturbatively results in an infinite number of derivatives. For such a theory, the causal signal propagation is not compromised as long as the gradient corrections are summed up to infinite orders. The advantage of such an approach is that, no new degrees of freedom that could not be linked to conserved quantities are needed to be introduced, but still the temperature and velocity can be unambiguously fixed by the Landau gauge. The recently proposed first-order stable and causal Bemfica-Disconzi-Noronha-Kovtun (BDNK) theory [27–32] does not require any additional degrees of freedom, but they are well behaved only away from the Landau frame. Hence, the primary field variables like velocity and temperature have ambiguities in their first principle definition since they do not coincide with those in either Landau’s or Eckart’s frames apart from global equilibrium and consequently lack a definition in terms of the microscopic field theory operator $T^{\mu\nu}$. Of course the theory derived here with infinitely many derivatives has practical limitations for simulation purpose since it is not possible to solve them even numerically for arbitrary initial conditions. But the point of this work is to indicate that if we want to construct a relativistic dissipative hydrodynamic theory purely in terms of fundamental fluid fields like temperature and velocity that have first principle microscopic definition, causality is only maintained if all orders of derivative corrections are taken into consideration. If this all-order sum needs to be averted, it has to be folded in one way or another: either in terms of new degrees of freedom (MIS) or in terms of field redefinition leading to no first principle definition of velocity or temperature (BDNK). In our recent article [33], this tension has been further investigated. It has been shown that in BDNK theory if we want to define the velocity and temperature locally in terms of the stress tensor operator like we do in Landau frame, then the constitutive relation will include all orders of derivative corrections as well.

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- [1] L. D. Landau and E. M. Lifshitz, *Course of Theoretical Physics Volume 6: Fluid Mechanics*, 2nd ed. (Elsevier, Amsterdam, 1987).
- [2] L. P. Kadanoff and P. C. Martin, *Ann. Phys. (N.Y.)* **24**, 419 (1963).
- [3] C. Eckart, *Phys. Rev.* **58**, 919 (1940).
- [4] I. Müller, *Z. Phys.* **198**, 329 (1967).
- [5] W. Israel, *Ann. Phys. (N.Y.)* **100**, 310 (1976).
- [6] W. Israel and J. M. Stewart, *Phys. Lett.* **58A**, 213 (1976).
- [7] W. Israel and J. M. Stewart, *Ann. Phys. (N.Y.)* **118**, 341 (1979).
- [8] W. A. Hiscock and L. Lindblom, *Ann. Phys. (N.Y.)* **151**, 466 (1983).
- [9] W. A. Hiscock and L. Lindblom, *Phys. Rev. D* **31**, 725 (1985).
- [10] T. S. Olson, *Ann. Phys. (N.Y.)* **199**, 18 (1990).
- [11] R. Baier, P. Romatschke, D. T. Son, A. O. Starinets, and M. A. Stephanov, *J. High Energy Phys.* 04 (2008) 100.
- [12] G. S. Denicol, H. Niemi, E. Molnar, and D. H. Rischke, *Phys. Rev. D* **85**, 114047 (2012).
- [13] A. Muronga, *Phys. Rev. Lett.* **88**, 062302 (2002).
- [14] A. Jaiswal, *Phys. Rev. C* **88**, 021903 (2013).
- [15] P. Huovinen and P. V. Ruuskanen, *Annu. Rev. Nucl. Part. Sci.* **56**, 163 (2006).
- [16] M. Lublinsky and E. Shuryak, *Phys. Rev. D* **80**, 065026 (2009).
- [17] Y. Bu and M. Lublinsky, *Phys. Rev. D* **90**, 086003 (2014).
- [18] M. P. Heller, R. A. Janik, and P. Witaszczyk, *Phys. Rev. Lett.* **110**, 211602 (2013).
- [19] P. Romatschke, *Phys. Rev. Lett.* **120**, 012301 (2018).
- [20] C. Itzykson and J. B. Zuber, *Quantum Field Theory*, International Series in Pure and Applied Physics (McGraw-Hill, New York, 1980).
- [21] P. Romatschke, *Int. J. Mod. Phys. E* **19**, 1 (2010).
- [22] S. Pu, T. Koide, and D. H. Rischke, *Phys. Rev. D* **81**, 114039 (2010).
- [23] M. P. Heller, A. Serantes, M. Spaliński, and B. Withers, *Phys. Rev. Lett.* **130**, 261601 (2023).
- [24] L. Gavassino, *Phys. Lett. B* **840**, 137854 (2023).
- [25] R. E. Houtl and P. Kovtun, *Phys. Rev. D* **109**, 046018 (2024).
- [26] S. Weinberg, *Astrophys. J.* **168**, 175 (1971).
- [27] F. S. Bemfica, M. M. Disconzi, and J. Noronha, *Phys. Rev. D* **98**, 104064 (2018).
- [28] F. S. Bemfica, M. M. Disconzi, and J. Noronha, *Phys. Rev. D* **100**, 104020 (2019).
- [29] F. S. Bemfica, M. M. Disconzi, and J. Noronha, *Phys. Rev. X* **12**, 021044 (2022).
- [30] P. Kovtun, *J. High Energy Phys.* 10 (2019) 034.
- [31] R. E. Houtl and P. Kovtun, *J. High Energy Phys.* 06 (2020) 067.
- [32] R. E. Houtl and P. Kovtun, *Phys. Rev. D* **106**, 066023 (2022).
- [33] S. Bhattacharyya, S. Mitra, and S. Roy, [arXiv:2312.16407](https://arxiv.org/abs/2312.16407).