Letter

One-loop Bern-Carrasco-Johansson numerators on quadratic propagators from the worldsheet

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We introduce a novel approach for deriving one-loop Bern-Carrasco-Johansson (BCJ) numerators and reveal the world-sheet origin of the one-loop double copy. Our work shows that expanding Cachazo-He-Yuan half-integrands into generalized Parke-Taylor factors intrinsically generates BCJ numerators on quadratic propagators satisfying Jacobi identities. We validate our methodology by successfully reproducing one-loop BCJ numerators for the nonlinear sigma model as well as those of pure Yang-Mills theory in four dimensions with all-plus or single-minus helicities.

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Introduction. Recent advancements in quantum field theory have highlighted the essential role of scattering amplitudes in unraveling the fundamental interactions in nature. At the forefront of these advancements are the Bern-Carrasco-Johansson (BCJ) double-copy relations, rooted in the concept of color-kinematic duality (cf. [1–5]). These relations have unveiled significant structures and simplifications in scattering amplitudes at both tree and loop levels, suggesting a complex interplay within Feynman diagrams that hints at a unified framework underlying gauge and gravitational theories.

Another noteworthy development in this area is the Cachazo-He-Yuan (CHY) formula [6–9], which offers an alternative approach to understanding scattering amplitudes beyond traditional Feynman diagram methods. The CHY formula, known for its intricate geometry and combinatorial intelligence in the realm of world-sheet moduli space, streamlines the derivation of BCJ numerators at the tree level and enhances the efficiency of amplitude calculations. Significant contributions in this field include the identification of relationships between various theories [9–11] and the development of polynomial representations of BCJ numerators for Yang-Mills (YM) theory and many other theories of any multiplicity [12–15].

Moreover, ambitwistor strings [16,17], along with traditional string theories [18–22], have provided deeper insights into the CHY formula from a world-sheet perspective and broadened the applicability of the CHY formula, particularly in the realm of loop amplitudes. Additionally, intersection theories have also proven instrumental to illuminate the mathematical and geometric foundations inherent in the CHY formula [23,24]. It has also been established that manipulating tree-level CHY integrands through their forward limit in higher dimensions generates one-loop CHY integrands [25,26].

Despite considerable advancements, accessing BCJ numerators on quadratic propagators at loop level via world-sheet methods remains a daunting task, with the traditional world-sheet formula introducing loop propagators with linear loop momentum dependence, complicating analyses [25–29]. Although new BCJ double copies have been discovered employing these loop integrands [30,31], the linear aspect hinders simplified integration.

However, significant research by Feng *et al.* [32], among many other works (cf. [33–41]), introduced a method for generating loop integrands with quadratic propagators, pivoting on the expansion of CHY half-integrands via one-loop generalized Parke-Taylor (PT) factors. This approach alleviates the complexities posed by linear dependencies in loop momentum, propelling forward the exploration of loop-level scattering amplitudes.

In this paper, for the first time, we prove that the expansion onto generalized PT factors naturally gives rise

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to BCJ numerators on quadratic propagators that adhere to Jacobi identities using the formulas in [32]. Consequently, this significantly enhances the relevance of the CHY formulas, affirming their utility at the one-loop level. Besides, our method proposes a more flexible double-copy framework building upon the foundations established in previous works [1,2].

As a practical application of our methodology, we demonstrate its efficiency in reconstituting one-loop BCJ numerators for the nonlinear sigma model (NLSM) for arbitrary multiplicities recently proposed in [42] as well as those of pure YM theories in four dimensions with all-plus or single-minus helicity external gluons [43]. Simple double copies among them yield the loop integrands for special Galileon, Born-Infeld, and pure gravity amplitudes [9] with corresponding helicities.

Quadratic propagators from the world sheet.

CHY formula for one-loop scattering amplitudes: The one-loop CHY formula yields loop integrands for the scattering of n external, incoming massless particles by integrating over the moduli space of the degenerate tori, specifically, the nodal Riemann sphere localized by the one-loop scattering equations [27,44]:

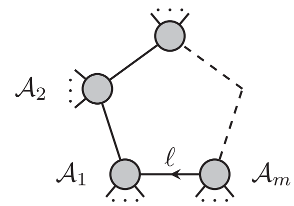
$$\mathcal{M}(\ell) = \frac{1}{\ell^2}\int \underbrace{\prod_{i=2}^{n} \mathrm{d}\sigma_i \delta\left(\frac{2\ell\cdot k_i}{\sigma_i} + \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{s_{ij}}{\sigma_{ij}}\right)}_{\equiv \mathrm{d}\mu_n} \mathcal{I}_L(\ell)\mathcal{I}_R(\ell), \tag{1}$$

where $\sigma_{ij} \equiv \sigma_i - \sigma_j$ and $s_{ij} \equiv 2k_i \cdot k_j$. These loop integrands can also be derived from the $(n+2)$-point massless tree-level CHY formula via the forward limit in higher dimensions [25,26]. Further integration upon the off-shell loop momentum ℓ yields the one-loop amplitudes.

We have fixed the $\mathrm{SL}(2,\mathbb{C})$ gauge redundancy in the nodal Riemann spheres in (1) by setting the two nodal punctures as $\sigma_+ \to 0, \sigma_- \to \infty$, and $\sigma_1 \to 1$. The measure $\mathrm{d}\mu_n$ is universal, while the half-integrands $\mathcal{I}_{L/R}(\ell)$ with equal $\mathrm{SL}(2,\mathbb{C})$ weights encode the dynamics information for a specific theory. The n external momenta satisfy momentum conservation, $\sum_{i=1}^{n} k_i = 0$. For brevity, we introduce multiparticle momenta $k_{12\ldots p} \equiv \sum_{i=1}^{p} k_i$ and the shorthand $\ell_{12\cdots p} \equiv \ell + k_{12\cdots p}$, such that $\ell^2_{12\cdots p}$ signifies the quadratic loop propagator.

One-loop cubic graph and quadratic propagators: An important ingredient in the one-loop double copy construction is the one-loop cubic graph [1,2]. To describe this, we introduce a uniform notation, $g(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m)$, where the sequence $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m$ with $1 \leq m \leq n$ symbolizes all m dangling trees located at the corners of the polygon and the loop momentum ℓ is directed from $\mathcal{A}_m$ to $\mathcal{A}_1$ as illustrated below.

$$g(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m) \equiv \tag{2}$$



The exclusive use of cubic vertices allows for each dangling tree $\mathcal{A}_i$ to be represented by a nested square bracket.

Additionally, we define P_g as the product of all propagators in graph g, encompassing both loop and tree elements. For instance,

$$P_{g([1,2],[[3,4],5],6)} \equiv P_{[1,2],[[3,4],5],6} = \ell^2 \ell^2_{12} \ell^2_{12345} s_{12} s_{34} s_{345}.$$

Henceforth, for simplicity, when referencing a graph $g(\cdots)$ as a subscript, we drop both the g symbol and the parentheses to simplify the notation. Note that P_g depends on the orientation and position of ℓ:

$$P_{\mathcal{A}_m, \mathcal{A}_1, \ldots, \mathcal{A}_{m-1}} = P_{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m}|_{\ell \to \ell_{\mathcal{A}_m}}, \tag{3}$$

$$P_{\mathcal{A}_1, \mathcal{A}_m, \ldots, \mathcal{A}_2} = P_{\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m}|_{\ell \to -\ell_{\mathcal{A}_1}}. \tag{4}$$

Generalized PT factors and their integrals: Inspired by the maximally helicity violating gluon amplitude formula [45], tree-level PT factors are used in the tree-level CHY formula to encode the information of color ordering for theories like YM theory [8]. They only have simple poles and can act as the basis of the tree-level CHY half-integrands (cf. [7,15,46,47]). Building on this, the one-loop variant of PT factors was introduced in the one-loop CHY formula in [44] to similarly convey color-ordering information. In [32], further operators acting on the standard scalar one-loop PT factors were introduced to define the generalized one-loop PT factors,

$$\begin{aligned}\boldsymbol{\ell}_1^{\mu_1,\mu_2,\ldots,\mu_r}\mathrm{PT}(1,2,\ldots,n) &\equiv \left(\prod_{j=1}^{r}\boldsymbol{\ell}_1^{\mu_j}\right)\mathrm{PT}(1,2,\ldots,n)\\ &\equiv \sum_{i=1}^{n}\prod_{j=1}^{r}(\ell^{\mu_j} - k^{\mu_j}_{12\cdots i-1})\\ &\quad\times \mathrm{PT}^{\mathrm{tree}}(+,i,i+1,\ldots,i-1,-),\end{aligned} \tag{5}$$

which have nontrivial dependency on the loop momentum. Leg 1 in (5) plays a special role as we define ℓ as the loop momentum flowing into the subtree that contains leg 1 and we use the subscript in the operator $\boldsymbol{\ell}_1$ to emphasize it. The tree-level PT factor reads $\mathrm{PT}^{\mathrm{tree}}(+,1,2,\ldots,n,-) \equiv \frac{1}{\sigma_1\sigma_{1,2}\sigma_{2,3}\cdots\sigma_{n-1,n}}$.

It was demonstrated in [32] that the CHY integral of two generalized PT factors yields loop integrands with quadratic loop propagators,

$$\frac{1}{\ell^2}\int \mathrm{d}\mu_n \boldsymbol{\ell}_1^{\mu_1,\mu_2,\ldots,\mu_r}\mathrm{PT}(1,\rho(2),\ldots,\rho(n)) \\ \times \boldsymbol{\ell}_1^{\nu_1,\nu_2,\ldots,\nu_t}\mathrm{PT}(1,\sigma(2),\ldots,\sigma(n)) \\ \cong \mathrm{sgn}^{\rho}_{\sigma}\sum_{g\in T(1,\rho)\cap T(1,\sigma)}\frac{\ell^{\mu_1,\mu_2,\ldots,\mu_r}_{A(g,\rho)}\ell^{\nu_1,\nu_2,\ldots,\nu_t}_{A(g,\sigma)}}{P_g}, \tag{6}$$

where ρ and σ denote permutations of the elements $2,3,\ldots,n$ and the symbol $\cong$ signifies that the integrands on both sides yield identical results after loop integration [48]. The summation extends over all graphs that are members of both $T(1,\rho)$ and $T(1,\sigma)$ defined as follows. Considering a one-loop cubic graph $g(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_m)$ defined in (2) with $2\le m\le n$, the loop momentum circulates clockwise and particle 1 is positioned in the initial corner, that is, $\mathcal{A}_1\ni 1$, and the set $T(1,\rho)$ represents all of such cubic graphs with the external legs ordered according to the sequence $(1,\rho)$, in a clockwise arrangement.

$\ell^{\mu_1\mu_2}_{A(g,\rho)}$ implies $\ell^{\mu_1}_{A(g,\rho)}\ell^{\mu_2}_{A(g,\rho)}$ and the shift factor $A(g,\rho)$ in $\ell_{A(g,\rho)}=\ell+k_{A(g,\rho)}$ signifies the subset of particles in the first corner $\mathcal{A}_1$ of graph g that are situated before particle 1 in the cyclic order $(1,\rho)$. Note that $A(g,\rho)$ can be empty. More explicitly, suppose $\rho(b)\rho(b+1)\cdots\rho(n)1\rho(2)\cdots\rho(a)$ belong to $\mathcal{A}_1$; then, $A(g,\rho)=\{\rho(b),\ldots,\rho(n)\}$. The overall sign is delineated as follows:

$$\mathrm{sgn}^{\rho}_{\sigma}\equiv\prod_{i=1}^{|\sigma|-1}\mathrm{sgn}^{\rho}_{\sigma(i),\sigma(i+1)}=\mathrm{sgn}^{\sigma}_{\rho}, \tag{7}$$

where $\mathrm{sgn}^{\rho}_{i,j}$ equals $+1$ if i is ahead of j in ρ, and -1 otherwise.

In the next section, we will show how to derive BCJ numerators on quadratic propagators based on (6).

One-loop BCJ numerators from the world sheet. In the study of one-loop CHY formulas for theories such as those with $SU(N)$ or $SO(N)$ color groups, the formulation involves distinctive half-integrands. The first half-integrand, $\mathfrak{C}_n$, represents a color-dressed scalar PT factor, devoid of all kinematic considerations, defined as

$$\mathfrak{C}_n\equiv\sum_{\sigma\in S_{n-1}}C_{1,\sigma}\mathrm{PT}(1,\sigma), \\ \text{where } C_{1,2,\ldots,n}\equiv f^{za_1b}f^{ba_2c}f^{ca_3d}\ldots f^{ya_nz}, \tag{8}$$

with f^{abc} representing the structure constants of the color group.

The second half-integrand, I_n, is in general more complicated. However, as we prove later, for theories that accept a BCJ double copy, their second half-integrand can always be expanded to generalized PT factors (5). This expansion can be organized based on scalar PT factors $\mathrm{PT}(1,\rho)$, with all operators $\boldsymbol{\ell}_1$ and σ-independent variables like k_i, polarizations ϵ_i, etc., consolidated as a single operator $\boldsymbol{N}_{1,\rho}$. Consequently, I_n mirrors the structure of $\mathfrak{C}_n$, described as

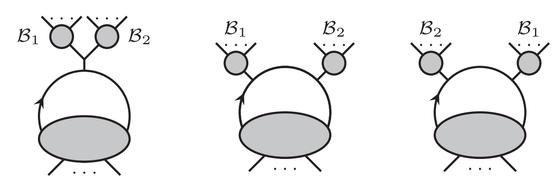
$$I_n=\sum_{\rho\in S_{n-1}}\boldsymbol{N}_{1,\rho}\mathrm{PT}(1,\rho). \tag{9}$$

In the next section, we show the concrete expansions (9) for NLSM and pure YM theory with all-plus or single-minus helicities. More examples of the expansions for low-point super-Yang-Mills amplitudes can be found in [32], suggesting a potential for generalization to higher-point scenarios [50–52]. In this section, our focus is on using the (abstract) expansion (9) as a foundation to illustrate a universal approach for deriving BCJ numerators.

General claim: Our central statement is that the integral $\frac{1}{\ell^2}\int \mathrm{d}\mu_n I_n\mathfrak{C}_n$ inherently generates BCJ numerators for theory $\mathcal{O}$. Specifically, the master BCJ numerator for an n-gon graph can be straightforwardly acquired via the substitution

$$N_{1,\rho(2),\ldots,\rho(n)}\equiv N_{g(1,\rho(2),\ldots,\rho(n))}=\boldsymbol{N}_{1,\rho}|_{\boldsymbol{\ell}_1\to\ell}. \tag{10}$$

Jacobi identities: For any triplet graphs with identical placement and orientation of loop momentum ℓ but differing by a single propagator (as shown below),



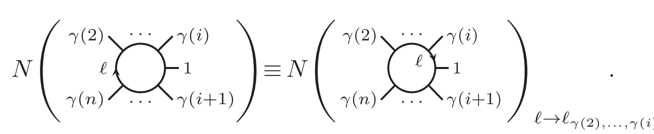
their numerators satisfy Jacobi identities similar to color factors. This denotes an antisymmetrization of the numerators,

$$N_{\ldots,[\mathcal{B}_1,\mathcal{B}_2],\ldots}=N_{\ldots,\mathcal{B}_1,\mathcal{B}_2,\ldots}-N_{\ldots,\mathcal{B}_2,\mathcal{B}_1,\ldots}, \tag{11}$$

where the ellipsis represents consistent dangling tree sequences across the three graphs.

Applying these identities recursively enables deriving the numerator N_g of any graph as a linear combination of $(n-1)!$ master BCJ numerators $N_{1,\rho}$ with $\rho\in S_{n-1}$, considering potential loop momentum shifts.

The shift arises because varying ℓ positions within the same n-gon yield different numerator representations,



(12)

That is, $N_{\gamma(2),\ldots,\gamma(i),1,\ldots} \equiv N_{1,\ldots,\gamma(2),\ldots,\gamma(i)}|_{\ell\to\ell_{\gamma(2),\ldots,\gamma(i)}}$, where $\gamma(2)$,$\gamma(3),\ldots,\gamma(n)$ denote a permutation of $2,3,\ldots,n$.

Double copy: Having obtained all BCJ numerators and color factors for the one-loop cubic graphs, we express the loop integrand for theory $\mathcal{O}$ as

$$\frac{1}{\ell^2}\int \mathrm{d}\mu_n I_n \mathfrak{C}_n \cong \mathcal{M}_n^{\mathcal{O}}(\ell) = \sum_{g\in W_1} \frac{N_g C_g}{P_g}, \tag{13}$$

where W_1 denotes all one-loop cubic graphs $g(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_m)$ with $2\leq m\leq n$ and $\mathcal{A}_1\ni 1$.

Consider another theory $\mathcal{O}'$ with half-integrand $I'_n = \sum_{\sigma\in S_{n-1}} N'_{1,\sigma}\mathrm{PT}(1,\sigma)$. The double copy between $\mathcal{O}$ and $\mathcal{O}'$ gives the loop integrand of a third distinct theory,

$$\frac{1}{\ell^2}\int \mathrm{d}\mu_n I_n I'_n \cong \mathcal{M}_n^{\mathrm{DC}}(\ell) = \sum_{g\in W_1} \frac{N_g N'_g}{P_g}, \tag{14}$$

where C_g in (13) is replaced by another set of BCJ numerators $N'_g = N_g|_{\boldsymbol{N}\to\boldsymbol{N}'}$.

Numerators: For clarity, the numerator N_g in (13) and (14) is explicitly formulated as

$$N_g = \mathrm{sgn}_g^{\rho_0} \sum_{(1,\rho)\in T^{-1}(g)} \mathrm{sgn}_\rho^{\rho_0} \boldsymbol{N}_{1,\rho}|_{\boldsymbol{\ell}_1\to\ell_{A(g,\rho)}}. \tag{15}$$

Here, the sum is over all orderings $(1,\rho)$ such that $g\in T(1,\rho)$ and we have selected an arbitrary $(1,\rho_0)$ from them as a reference. The sign $\mathrm{sgn}_\rho^{\rho_0}$ is specified in (7).

In a parallel manner, the color factor is given by

$$C_g = \mathrm{sgn}_g^{\rho_0} \sum_{(1,\sigma)\in T^{-1}(g)} \mathrm{sgn}_\sigma^{\rho_0} C_{1,\sigma}. \tag{16}$$

Notably, the comprehensive overall signs $\mathrm{sgn}_g^{\rho_0}$ in (15) and (16) result in $+1$ upon the multiplication of N_g and C_g in (13), ensuring consistency.

Example: Here is an example at $n=3$ for (13):

$$\frac{1}{\ell^2}\int \mathrm{d}\mu_3 I_3 \mathfrak{C}_3 \cong \frac{N_{1,2,3}C_{1,2,3}}{P_{1,2,3}} + \frac{N_{1,3,2}C_{1,3,2}}{P_{1,3,2}} + \frac{N_{1,[2,3]}C_{1,[2,3]}}{P_{1,[2,3]}} + \frac{N_{[1,2],3}C_{[1,2],3}}{P_{[1,2],3}} + \frac{N_{[1,3],2}C_{[1,3],2}}{P_{[1,3],2}}, \tag{17}$$

where $N_{[1,2],3} = N_{1,2,3} - N_{2,1,3} = \boldsymbol{N}_{1,2,3}|_{\boldsymbol{\ell}_1\to\ell} - \boldsymbol{N}_{1,3,2}|_{\boldsymbol{\ell}_1\to\ell_2}$. The example of (14) can be easily given by substituting C with N'.

The proof of our general proposal as outlined in (13) and (14) are presented in the Supplemental Material [53].

Refined double copy: In our construction (13) and (14), graph pairs $g(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_m)$ and $g(\mathcal{A}_1,\mathcal{A}_m,\ldots,\mathcal{A}_2)$, typically seen as identical for $3\leq m\leq n$, are distinct in set W_1. We introduce set $\hat{W}_1$ by merging such pairs in W_1 and formulate a new CHY half-integrand tied to I_n as follows:

$$\bar{I}_n = \sum_{\rho\in S_{n-1}} \bar{\boldsymbol{N}}_{1,\rho}\mathrm{PT}(1,\rho),$$
$$\text{with}\quad \bar{\boldsymbol{N}}_{1,\rho} = \frac{1}{2}\left(\boldsymbol{N}_{1,\rho} + (-1)^n \boldsymbol{N}_{1,\rho^T}|_{\boldsymbol{\ell}_1\to-\boldsymbol{\ell}_1-k_1}\right), \tag{18}$$

where ρ^T reverses ρ. Employing $\bar{I}_n$ in place of I_n in the CHY integral (13) reveals new master BCJ numerators $\bar{N}_{1,\rho} = \bar{\boldsymbol{N}}_{1,\rho}|_{\boldsymbol{\ell}_1\to\ell}$ satisfying

$$\bar{N}_{1,\rho^T} = (-1)^n \bar{N}_{1,\rho}|_{\ell\to-\ell-k_1}. \tag{19}$$

This ensures identical contributions from n-gon pairs $g(1,\rho)$ and $g(1,\rho^T)$ to amplitudes, extending to any $g(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_m)$ and $g(\mathcal{A}_1,\mathcal{A}_m,\ldots,\mathcal{A}_2)$, as proved in [53]. Importantly, the new loop integrand matches $M(\ell)$ from (13) after integration, establishing $\bar{I}_n\cong I_n$ as a refined I_n, leading to the refined double copies,

$$\mathcal{M}_n^{\mathcal{O}}(\ell) \cong \sum_{g\in\hat{W}_1} \mathcal{S}_g \frac{\bar{N}_g C_g}{P_g}, \qquad \mathcal{M}_n^{\mathrm{DC}}(\ell) \cong \sum_{g\in\hat{W}_1} \mathcal{S}_g \frac{\bar{N}_g \bar{N}'_g}{P_g}. \tag{20}$$

Here, the symmetry factor $\mathcal{S}_g$ is 1 for bubble graphs but becomes 2 for triangles and larger polygons [60].

In particular, our initial double copy (13) and (14), which naturally arises from world-sheet perspectives, does not mandate (19), allowing unrelated BCJ numerators for identical graphs with differing ℓ orientations. This suggests potential redundancies in loop-level double-copy constructions, urging further exploration into higher-loop BCJ numerators.

In the context of scattering of n identical external bosons, we achieve one-loop crossing-symmetric BCJ numerators [61–64] through averaging permutations of particle labels in the half-integrand $\bar{I}_n$, as delineated in [53].

Having established the derivation of master BCJ numerators from the CHY half-integrand expansion, we posit that the inverse is equally valid. When provided with BCJ numerators that satisfy the Jacobi identities (11), we can elevate the master numerators symbolized as $N(1,\rho)$ for n-gons to operators, $N(1,\rho)|_{\ell\to\boldsymbol{\ell}_1}\to\boldsymbol{N}_{1,\rho}$. The CHY half-integrand for theory $\mathcal{O}$ then follows the expansion (9), inherently yielding the same loop integrand for theory $\mathcal{O}$ according to (13). As a corollary, this proves the existence of the expansion (9) for theories that accept BCJ double copies.

Further exploration in this area reveals additional insights. In (1), we have assumed that the CHY integrand for a given theory decomposes into two half-integrands, $\mathcal{I}_L(\ell)$ and $\mathcal{I}_R(\ell)$; however, it could in principle just be a quadratic combination of them, $\sum_i \mathcal{I}_L^{(i)}(\ell)\mathcal{I}_R^{(i)}(\ell)$.

Nevertheless, our investigations confirm that the existence of BCJ numerators for a theory ensures that its CHY integrands can indeed be represented as a product of two half-integrands.

Applications. In this section, we demonstrate our approach through an analysis of one-loop amplitudes for pure gluons in $D = 4$, focusing on all-positive or single-minus helicity configurations. Using straightforward techniques, similar to those employed at the tree level [9–11], we successfully derive the BCJ numerators for NLSM theories as well. This not only demonstrates the straightforwardness of the method in deriving BCJ numerators from the world sheet but also indicates its broad applicability to diverse particle types.

Utilizing spinors, we can express the four-dimensional polarizations ϵ_i and momenta k_i, ensuring $\epsilon_i \cdot \epsilon_j \to 0$ via a specific reference spinor choice [65,66]. For all-plus or single-minus helicity configurations in gluon scattering, as established in [67,68], loop amplitudes involving a gluon, fermion, or scalar in the loop are proportionally equivalent. Thus, we concentrate exclusively on the scalar case.

We derive the corresponding one-loop CHY half-integrand from the tree level in the forward limit [25,29]. The tree-level ones for n external gluons and two scalars, denoted as $+, -$, can be straightforwardly obtained from the well-known reduced Pfaffian for $n + 2$ external gluons [7,8] by extracting its coefficient $\epsilon_+ \cdot \epsilon_-$ [9,10]. When setting all remaining $\epsilon_i \cdot \epsilon_j \to 0$, the Pfaffian simplifies to a determinant. Implementing the forward limit yields the one-loop CHY half-integrand for n external gluons with a scalar propagating in the loop,

$$I_n^{\mathrm{YM}} = -\det \mathtt{C}(\epsilon_i, k_i, \sigma_i), \tag{21}$$

where $\mathtt{C}$ is an $n \times n$ matrix defined as

$$\mathtt{C}_{i,j} \equiv \frac{\epsilon_i \cdot k_j}{\sigma_{i,j}} \quad \text{for } i \neq j \quad \text{and} \quad \mathtt{C}_{i,i} \equiv -\sum_{a=1, a\neq i}^{n} \mathtt{C}_{i,a} - \frac{\epsilon_i \cdot \ell}{\sigma_i}.$$

Applying the matrix tree theorem [69] to expand det $\mathtt{C}$ results in a summation over all labeled trees G with nodes $\{+, 1, 2, \ldots, n\}$ and orientations of the n edges $e(i,j)$ flowing to the root node $+$ [70],

$$I_n^{\mathrm{YM}} = (-1)^{n+1} \sum_G \prod_{e(i,j)} \frac{\epsilon_i \cdot k_j}{\sigma_{i,j}}. \tag{22}$$

By utilizing partial fraction identities and grouping coefficients for each $\mathrm{PT}^{\mathrm{tree}}(+, \pi, -)$ [71], we express (22) as

$$I_n^{\mathrm{YM}} = \sum_{\pi \in S_n} \prod_{i=1}^{n} \epsilon_i \cdot (\ell + Y_i(\pi)) \mathrm{PT}^{\mathrm{tree}}(+, \pi, -), \tag{23}$$

where $Y_i(\pi(1), \cdots, \pi(j), i, \cdots) = k_{\pi(1),\ldots,\pi(j)}$.

As proved in [53], I_n^{YM} can be expanded to the generalized PT factors with ranks ranging from 2 to n,

$$I_n^{\mathrm{YM}} = \sum_{\rho \in S_{n-1}} \prod_{i=1}^{n} \epsilon_i \cdot (\boldsymbol{\ell}_1 + Y_i(1, \rho)) \mathrm{PT}(1, \rho). \tag{24}$$

Utilizing (10), we derive all master BCJ numerators as

$$N_{1,\rho}^{\mathrm{YM}} = \prod_{i=1}^{n} \epsilon_i \cdot \big(\ell + Y_i(1, \rho)\big). \tag{25}$$

One can easily check that $N_{1,\rho^T}^{\mathrm{YM}} = (-1)^n N_{1,\rho}^{\mathrm{YM}}|_{\ell \to -\ell_1}$. Consequently, we directly employ (20) and (15) to compute the loop integrands for all-plus or single-minus YM and GR amplitudes.

Note that we derive (25) directly from $\det \mathtt{C}(\epsilon_i, k_i, \sigma_i)$ (see also [72–75]), ensuring manifest symmetry across all n external gluons. Consequently, the master BCJ numerators, represented as $N_{1,2,\ldots,n}^{\mathrm{YM}} = \prod_{i=1}^{n} \epsilon_i \cdot \ell_{12\cdots i-1}$, exhibit crossing symmetry, as elucidated in [53], in scenarios involving all-plus helicities.

Remarkably, it is straightforward to extend our derivation from gluons to pions. Starting at the one-loop CHY half-integrand (21) for YM theory with a scalar running in the loop, simply replacing $\epsilon_i \to k_i$ in (21), we get the half-integrand for NLSM [9] in the one-loop CHY formula. Crucially, we have not taken the derivative of a Lorentz product of a pair of external polarizations in this procedure, which is different from the one at tree level [9,10,76]. Then, a parallel derivation further demonstrates that $N_{1,2,\ldots,n}^{\mathrm{NLSM}} = \prod_{i=1}^{n} k_i \cdot \ell_{12\cdots i-1}$. The double copies of themselves or together with the gluon ones (25) produce the loop integrands for special Galileon and Born-Infeld theories with corresponding helicities, which will be further studied in [77].

Although the BCJ numerators (25) for gluons have been previously presented in [28,43,68,78–80] and those of pions were recently proposed in [42], our approach reproduces them universally in a streamlined and elegant manner. The success in these specific cases highlights the adaptability of our method, suggesting its potential applicability to a broader array of theories.

Discussion. This research signifies an important advance in quantum field theories, particularly in computing one-loop BCJ numerators. We have pioneered a method for extracting one-loop BCJ numerators, generating one-loop integrands using CHY half-integrands expanded into generalized PT factors. Our strategy, tested on the NLSM as well as pure Yang-Mills theories with all-plus or single-minus helicities, demonstrates robustness and flexibility, showing great potential to enhance computational techniques and uncover previously unknown connections among one-loop amplitudes.

By expressing any one-loop double copy as a CHY integral combining a direct product of two half-integrands, our method potentially alludes to a one-loop variant of the Kawai-Lewellen-Tye (KLT) relations [6,81,82] using quadratic propagators (see Refs. [30,31] using linear propagators), potentially linked to recent studies on the genus-one KLT relations in string amplitudes [83–85].

Although we have concentrated on one-loop BCJ numerators, the foundational principles of our technique hold promise for an extension to higher-loop levels, with various CHY integrands already suggested via ambitwistor strings [86–88], traditional strings [89–91], or double or multiple forward limits [92,93]. Exploring these possibilities is a direction for our upcoming research.

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