Letter

Emergent symmetries of relativistic fluid dynamics from local ergodicity

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We show that volume-preserving diffeomorphisms and the chemical shift symmetry defining relativistic Lagrangian ideal fluid dynamics can be derived as an emerging symmetry when ergodicity is assumed to apply locally in a way that is invariant under smooth spacetime foliations. This can be used as a way to derive the ideal hydrodynamic limit in a strongly coupled but strongly fluctuating medium. We comment on the connection with thermalization in small systems, the eigenstate thermalization hypothesis and deviations from the ideal limit.

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Relativistic hydrodynamics has proven to be an extremely useful phenomenological tool [1–3] in the study of heavy ion collisions. This has also reignited the more formal theoretical study of hydrodynamics [4–6], a subject much more complex and subtle on a conceptual level than one initially thinks. The observation of hydrodynamic behavior in "small" systems, both in size and number of degrees of freedom [7] (but also analogous nonrelativistic examples such as ultracold atoms [8] and the "Brazil-nut effect" [9]) demonstrates that conceptual ambiguities can still be relevant for resolving mysteries given to us by experimental data.

One such conceptual issue is that there are several definitions to hydrodynamics: One usually thinks of it as a coarse-grained, long-range effective theory written in terms of conserved charges and their gradients [1,2,4]. In this approach, hydrodynamic evolution is explicitly deterministic, and the connection to local thermal equilibrium is not "built in," leading to concepts such as hydrodynamic attractors far from thermal equilibrium [10,11]. The potential ambiguity comes from the fact that methods from statistical mechanics are necessary to calculate the equation of state and transport coefficients needed to close the hydrodynamic system of equations.

Statistical mechanics however is not a deterministic theory; at best, fluctuations vanish in the thermodynamic limit, but in principle its only assumption is the equal probability of microstates [12,13]. One therefore needs to make sure fluctuations stay small, something usually implicitly assumed in microscopic theories rather than demonstrated [14–24] (Kubo formulas, for instance, are based on taking limits much larger than the fluctuation timescale [3]), though fluctuation-dissipation relations apply also away from the thermodynamic limit [25,26].

As a perhaps related issue, away from perfect equilibrium the definition of "equilibrium" becomes itself ambiguous [27–32]. A full definition of hydrodynamics in this sense, as well as a connection to nonequilibrium quantum field theory that encodes microscopic fluctuations and their hydrodynamic response [33–35], is still missing.

A still different definition of hydrodynamics is based on a Lagrangian picture, where instead of defining currents in the lab frame we follow every cell of the fluid as it propagates and deforms. This definition is particularly elegant in that ideal hydrodynamics is based on local diffeomorphisms [36–38], making it into a "poor people's general relativity" [5,39] (to the extent that perhaps the problem of quantizing gravity can be related, via the entropic hypothesis [40], to the problem, described in the previous paragraph, of defining a fluctuating hydrodynamics where local symmetries still hold [25,41]).

This work makes a link between this last Lagrangian definition of hydrodynamics and the previous two, by showing its elegant local symmetries are in fact emerging if one assumes the ergodic hypothesis [defined via Eq. (1) rather than the usual $\Delta \to \infty$ limit [12]] applies in every cell of the fluid. This link can then be used to define hydrodynamics in a way that might be more appropriate to small systems [7] and as a starting point to examine the fluctuations and redundancies of [32].

Statistical mechanics is based on the emergence of an approximately ergodic system [14] (or rather [12] of the approximation that some observables are described by such a system), whose evolution is governed by an equation of the type

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$$\left.\frac{\Delta x}{\Delta t}\right|_{t-t'\simeq\Delta t} \simeq \int P(q')dq', \tag{1}$$

where x is a microscopic phase space degree of freedom, t is a time sampled across some average interval Δt, and dq' is a time-invariant measure over an ensemble of states.

In other words when sampled under a certain coarse-graining timescale, the system's "evolution" can be approximated via a statistical distribution of microstates. As shown in [12,13], while this is an extremely fragile assumption, dependent on the "indecomposability" of the phase space hypersurface constrained by conservation laws, it can be thought of as an approximate assumption good enough to calculate particular observables (in particular conserved currents) and their "typical" state in strongly coupled systems whose initial conditions are sampled across a broad ensemble. We also note that typically the ergodic hypothesis is formulated for $t - t' \to \infty$, while here this interval defines the size of some cell. This is in line with thermalization going from an asymptotic quantity to a local one in the definition of fluid mechanics.

Thus, for a physical system moving in phase space sufficiently chaotically, with conservation laws of energy and momentum (P^μ and their invariant P) as well as charge Q but not necessarily particle number

$$\left.\frac{\Delta\phi}{\Delta t}\right|_{t-t'=\Delta t} \simeq \frac{1}{d\Omega(Q,P^\mu)}\sum \delta^4_{P^\mu,P^\mu_{\rm macro}(t)}\delta_{Q,Q_{\rm macro}(t)}\delta \times\left(\sum_j^\infty p^\mu_j - P^\mu\right)\delta\left(\sum_j^\infty Q_j - Q\right), \tag{2}$$

where $\phi = \{x_{1,2,\ldots N\to\infty}\}$ is some degree of freedom, p_i is the conjugate momentum and Q_i some conserved charge and $d\Omega(P^\mu, Q)$ is the normalization for that point in spacetime ($\sim e^{dS}$, where dS is the entropy density of the cell). Note that for varying particle numbers the Gibbs factor $N!$ can be taken to be a correction of $d\Omega$, so we do not consider it further. From this the equilibrium state is in the microcanonical ensemble. Other ensembles follow when the number of degrees of freedom and volume are taken to infinity.

It is immediately clear that the ergodic hypothesis is not relativistically covariant, since there is a "t" coordinate on the left-hand side (the right-hand side can be made generally covariant by an on-shell condition). For "global" equilibrium, this is of course not a problem: The system as a whole carries momentum, so there is a preferred frame where its momentum is zero and hence only the energy constraint is needed to maximize entropy.

But this seems in contradiction [31] with the concept of "local" equilibrium, where every cell of a fluid contains "many" degrees of freedom in ergodic equilibrium. To an extent, this is not a deep issue since local equilibrium is not rigorously defined beyond the ideal limit. However, it would be interesting to try to define it from the ergodic hypothesis and approximate deviations from it. To do so, we would like to investigate what kind of symmetries are needed to be imposed on Eq. (2) for a relativistic local ergodic evolution to emerge.

To try to add an element of covariance, we define t via a foliation, Σ_μ.[1] Different Σ_μ's are related via generally noninertial transformations. We would like to concentrate on transformations that respect the causality of the foliation

$$d\Sigma_\mu = \frac{\partial\Sigma_\mu}{\partial\Sigma'_\nu}d\Sigma'_\mu, \qquad \underbrace{(d\Sigma)^2}_{d\tau^2}, \underbrace{(d\Sigma')^2}_{d\tau'^2} > 0, \qquad \left.\frac{d}{dt}\right|_{dt\sim\Delta t} \to \left.\frac{d}{d\Sigma_0}\right|_{d\tau\sim\Delta t}, \qquad d\Sigma_\mu \det\left(\partial_\mu\Sigma_\nu\right)^{-1} \le \Delta. \tag{3}$$

A definition of "ideal" local equilibrium is to ensure that Eq. (2) stays true independently of Σ_μ as long as Σ is causal. In this picture, the averaging is done not on $dt \sim \Delta t$ but on $d\tau \sim \Delta t$, for each cell. The change in foliation must be smooth enough as to not disturb the average Δt (the "microscopic scale," which in this case is the maximum allowed radius of curvature of the foliations).

To see if this is possible, let us remember that p_μ and Q are Noether currents, corresponding to symmetries of the Lagrangian

$$p_\mu = \int d^3\Sigma^\nu T_{\mu\nu}, \qquad T_{\mu\nu} = \frac{\partial L}{\partial\partial^\mu\phi}\Delta_\nu\phi - g_{\mu\nu}L, \qquad \Delta_\nu\phi(x_\mu) = \phi(x_\mu + dx_\nu), \tag{4}$$

$$Q = \int d^3\Sigma^\nu j_\nu, \qquad j_\nu = \frac{\partial L}{\partial\partial^\mu\phi}\Delta_\psi\phi, \qquad \Delta_\psi\phi = \phi(x_\mu) + \delta\phi^*(x_\mu) = |\phi(x)|e^{i(\psi(x)+\delta\psi(x))}, \tag{5}$$

and $\psi = \{\psi_{1,2,\ldots N\to\infty}\}$ is a complex phase.

[1]Here we use the notation first introduced by Cooper and Frye in their treatment of freeze-out [42] and further used in Zubarev hydrodynamics [6], where a "spacelike" volume element can be parametrized by a timelike four-vector denoted by $\Sigma_m u$ whose exterior derivative points to the 4-volume's proper time. This notation is a bit different than the one usually used in general relativity [43], more appropriate for curved spacetimes.

These textbook formulas provide us with an intuition of how to model local equilibrium. All one has to do is to impose ergodicity on every spacetime point and then make this hypothesis foliation independent. Combining Eq. (2) with Eqs. (4) and (5) we get

$$\left.\frac{\Delta\phi}{\Delta\Sigma_0}\right|_{t-t'=\Delta t} = \frac{1}{d\Omega(dP^\mu(\Sigma_0), dQ(\Sigma_0))} \times \sum \delta^4\left(d\Sigma^\nu \underbrace{\frac{\partial L}{\partial\partial^\mu\phi}\partial_\nu\phi - L d\Sigma^\mu}_{\delta L(a_\mu)\sim a_\alpha\partial^\alpha(\delta^\mu_\nu L)} - dP^\mu(\Sigma_0)\right) \times \delta\left(d\Sigma^\mu \underbrace{\frac{\partial L}{\partial\partial^\mu\phi}\frac{\partial\psi}{\partial\Sigma_0}}_{\delta L(\psi)\sim\psi\partial_\mu L} - dQ(\Sigma_0)\right) \tag{6}$$

and enforce the covariance of these equations with any change in foliation that respects causality ($d\Omega$, dP, and dQ refer to the phase space volume, momentum and charge within the cell, respectively). Let us therefore use δa_μ and $\delta\psi$ as coarse-graining degrees of freedom for a generic equation of motion:

$$\left.\frac{\Delta\phi}{\Delta\Sigma_0}\right|_{t-t'=\Delta t} = \frac{1}{d\Omega(P^\mu, Q, \Sigma_0)}\int da_\mu d\psi \delta^4\left(d\Sigma^\nu a_\alpha\partial^\alpha(\delta^\mu_\nu L) - dP^\mu(\Sigma_0)\right)\delta(d\Sigma^\mu\psi\partial_\mu L - dQ(\Sigma_0)). \tag{7}$$

Let us now rewrite Eq. (7) under a different foliation and see what kind of constraints will foliation invariance of the ergodic hypothesis bring. Provided both foliations are smooth enough [according to Eq. (3)] the lhs simply becomes

$$\left.\frac{\Delta\phi}{\Delta t}\right|_{t-t'=\Delta t} \to \left.\frac{d\Sigma'_\nu}{d\Sigma_0}\frac{\Delta\phi}{\Delta\Sigma_0}\right|_{\Sigma_0-\Sigma'_0=\Delta t} = \left.\frac{\partial\Sigma'_\nu}{\partial\Sigma_0}\frac{\Delta\phi}{\Delta\Sigma_\nu}\right|_{x_\mu-x'_\mu<\Delta t} \tag{8}$$

and now we eliminate the lhs from the system of equations of Eq. (7) in Σ_μ and Σ'_μ. The resulting equation, a ratio of the rhss, will be

$$\frac{d\Omega(dP'_\mu, dQ', \Sigma'_0)}{d\Omega(dP_\mu, dQ, \Sigma_0)} = \frac{d\Sigma'_0}{d\Sigma_0}\frac{\int da_\mu d\psi\delta^4(d\Sigma^\nu a_\alpha\partial^\alpha(\delta^\mu_\nu L) - dP^\mu(\Sigma_0))\delta(d\Sigma^\mu\psi\partial_\mu L - dQ(\Sigma_0))}{\int da'_\mu d\psi'\delta^4(d\Sigma'_\nu a'_\alpha\partial^\alpha(\delta^\mu_\nu L) - dP'_\mu(\Sigma'_0))\delta(d\Sigma'_\mu\psi'\partial^\mu L - dQ'(\Sigma'_0))}. \tag{9}$$

We also take advantage of the fact that Σ_μ are causal and parametrize $d\Sigma_\mu$ by three Cartesian coordinates $\Phi_{I=1,2,3}$ such that the tangent vector

$$d\Sigma_\mu = \epsilon_{\mu\nu\alpha\beta}\frac{\partial\Sigma^\nu}{\partial\Phi_1}\frac{\partial\Sigma^\alpha}{\partial\Phi_2}\frac{\partial\Sigma^\beta}{\partial\Phi_3}d\Phi_1 d\Phi_2 d\Phi_3, \tag{10}$$

where the Latin letters are 1, 2, and 3 and 0 would be the time coordinate, is a timelike 3-volume element for a certain normalization. Arbitrary changes in foliation can also be decomposed in

$$\frac{\partial\Sigma'_\mu}{\partial\Sigma_\nu} = \Lambda^\nu_\mu \det\frac{d\Phi'_I}{d\Phi_J}, \qquad \det\Lambda^\nu_\mu = 1, \tag{11}$$

where Λ^ν_μ is a Lorentz transformation and $\det\frac{d\Phi_I}{d\Phi_J}$ a rescaling of the volume. Physically, Λ^ν_μ moves between the frame $d\Sigma^\mu_{\rm rest} = d\Phi_1 d\Phi_2 d\Phi_3(1,\vec{0})$.

We can now try define an effective Lagrangian that "automatically" solves Eq. (9) at every time step, thereby guaranteeing Eq. (6) holds in every frame. Good coarse-graining variables are generators of the transformation

$$L(\phi) \simeq L_{\rm eff}(a_\mu(\Sigma), \psi(\Sigma)),$$
$$L_{\rm eff}(a_\mu(\Sigma), \psi(\Sigma)) = L_{\rm eff}(a_\mu(\Sigma'), \psi(\Sigma')) \tag{12}$$

with the latter equality being enforced by solving Eq. (9).

Concentrating on the momentum current conservation, we notice that the dependence on foliation changes of the ratio of $\Omega(\ldots, P_\mu)$ is the same as $\delta(\ldots, P_\mu)$, and the shift in the Lagrangian can be reduced to a local symmetry shift.

The only change possible therefore is that of the foliation inside the δ function $\delta^4(d^3\Sigma_\mu, \ldots)$. One can use the classic relations of δ functions

$$\delta((f(x))) = \sum_i \underbrace{\frac{\delta(x_i - a_i)}{|f'(x_i = a_i)|}}_{f(a_i)=0},$$
$$\phi'_I = \frac{\partial_\alpha\Sigma'_I}{\partial^\alpha\Sigma^J}\Phi_J,$$
$$\delta^4(\Sigma_\mu) = \det_{\mu\nu}\left|\frac{\partial\Sigma^\mu}{\partial\Sigma^\nu}\right|\delta^4(\Sigma'_\mu) \tag{13}$$

to see that the ratio of δ^4 $(d\Sigma^\nu a_\alpha\partial^\alpha(\delta^\mu_\nu L) - dP^\mu(\Sigma_0))$ in Eq. (9) will reduce to a ratio of $\det\frac{\partial\Phi_I}{\partial\Phi_J}$.

To construct the effective Lagrangian invariant under such transformations from Φ_I one therefore

(i) assumes $d\Omega$ is differentiable across changes between Σ and Σ' so $d\Omega(P^\mu, Q, \Sigma'_0)/d\Omega(P^\mu, Q, \Sigma_0) \simeq d\Sigma'_0/d\Sigma_0$, Q transforms as a scalar and P_μ as a four-vector under changes in foliation;

(ii) a local rescaling can add an overall normalization factor to the Lagrangian, proportional to

$$T_0 = b\frac{dL_{\rm eff}}{db}, \qquad b = \sqrt{\det\partial_\mu\phi_I\partial^\mu\phi_J}. \tag{14}$$

To prevent this one must therefore normalize $\Phi_I \to T_0^{-1}\phi_I$ and write the Lagrangian in terms of ϕ_I (so rescalings will cancel out). Note that this renders ϕ_I unitless, as in [37,38].

This procedure gives us a physical intuition for the physical interpretation of T_0 and b in terms of the coordinates ϕ_I. Under the rescaling, two volume cells with the same T_0 will have the same Lagrangian density and hence will be in equilibrium. It is clear that T_0 is therefore the microscopic temperature, possibly weighted by a unitless microscopic degeneracy g (as was done in [5,44]). We will later use this to interpret b and $L_{\rm eff}$.

Once these steps are done all $d\Sigma_0/d\Sigma_0'$ and the effective Lagrangian $L_{\rm eff}$ is invariant under transformations leaving $\det\frac{\partial\phi_I}{\partial\phi_J}$ unchanged:

$$L(a_\mu,\psi) \to L_{\rm eff}(\phi_I,\psi), \qquad L(\phi_I,\psi) = L_{\rm eff}\left(\underbrace{\frac{\partial\phi_I'}{\partial\phi_J}}_{\det\frac{\partial\phi_I'}{\partial\phi_J}=1}\phi_J,\psi\right), \tag{15}$$

ratios of $\delta^4\left(d\Sigma^\nu a_\alpha\partial^\alpha(\delta^\mu_\nu L) - dP^\mu(\Sigma_0)\right)$ cancel out and Eq. (9) reduces automatically to

$$1 = \frac{\delta(d\Sigma^\mu\psi\partial_\mu L - dQ(\Sigma_0))}{\delta(d\Sigma'_\mu\psi'\partial^\mu L - dQ'(\Sigma_0'))} \tag{16}$$

so all coordinate diffeomorphism dependence has been eliminated. To complete our task, one must also construct the symmetries allowing Eq. (16) to be satisfied automatically. We remember that since $d\Sigma^\mu\partial_\mu$ is a scalar, we can evaluate it in the frame where $d\Sigma^\mu = d\Phi_1 d\Phi_2 d\Phi_3(1,\vec{0})$, where it becomes clear that $\partial_\mu L$ will reduce to gradients of $\partial_\mu\Phi_{1,2,3}$ and $\partial_\mu\Psi$. Since one can always choose coordinates where $\partial_\mu\Psi = 0$, imposing Eq. (16) is of course equivalent to imposing invariance of $L(\psi)$ under arbitrary shifts of functions of Φ_I.

Summarizing this main result of our work, for the local ergodicity condition Eq. (7) to apply for a generic foliation Σ, the effective Lagrangian $L_{\rm eff}$, written in terms of generators of local space translations ϕ_I and internal symmetry phases ψ, must be invariant under any $\phi_I \to \phi_I'$ leaving $\det_{IJ}\frac{\partial\phi_I'}{\partial\phi_J}$ and $\psi \to \psi + f(\phi_I)$ invariant. These of course are exactly the symmetries in [36–38].

The identification of conserved currents in terms of $L_{\rm eff}$ was done in [38] by the guaranteed existence, under the above set of diffeomorphisms of the Killing four-vector defined by

$$u_\mu u^\mu = -1, \qquad u_\mu d\Sigma^\mu = u_\mu\partial^\mu\Phi_{I=1,2,3} = 0,$$

which defines the boost from a generic $d\Sigma_\mu$ to one where it is of the form $d\phi_1 d\phi_2 d\phi_3(1,\vec{0})$ as well as two scalars, under diffeomorphism invariance, b introduced in Eq. (14) and $y = u_\mu\partial^\mu\psi$. The latter is related to the Noether current Eq. (5) via [38] the conserved charge current $J^\mu_Q = u_\mu dL/dy$. Thus, the conservation equations defining ideal hydrodynamics, the simplest equations of an infinite family of currents determined by arbitrary $f(\phi_I)$:

$$\partial_\mu s^\mu = \partial_\mu J^\mu_Q = \partial_\mu(f(\phi_I)J^\mu_Q) = 0, \qquad s^\mu = bu^\mu, \tag{17}$$

follow from local ergodicity via the symmetries already developed in [38] (note that Kelvin's theorem is a consequence [37]). This confirms the identification of b with the microscopic entropy and, via basic thermodynamic relations, $L_{\rm eff}$ with the microscopic energy density.

Physically the point here is that in the ideal hydrodynamic limit with no fluctuations one can perform a spacetime foliation [6] proportional to the four-vector of Lagrange multipliers under which entropy is maximized, u_μ/T, which must define the direction of every conserved current. Rotations and volume-preserving rescalings represent the residual symmetries after this vector is fixed.

While this ideal hydrodynamics is the same as the usual one [1,2,4], this way of arriving at the ideal hydrodynamics *as a limit* is profoundly different from the usual approach of coarse-graining Boltzmann-type equations around conservation laws, one perhaps more useful to study hydrodynamics in small systems such as droplets of quark-gluon plasma [7], ultracold atoms [8] and perhaps the everyday phenomenon of "the Brazilian-nut effect" [9].

In the usual approach [4] hydrodynamics is a limit of a deterministic transport equation valid in the ensemble average limit. This means a hierarchy

$$L_{\rm micro} \ll \underbrace{L_{\rm diss}}_{\sim\eta/(sT)} \ll \underbrace{L_{\rm macro}}_{\sim(\partial u_\mu)^{-1}}, \tag{18}$$

where $L_{\rm micro}$, the origin of stochasticity, is neglected and the effective theory is obtained by a coarse-graining around $L_{\rm diss}/L_{\rm macro}$ ($L_{\rm diss}$ determines mean free path, or viscosity to entropy ratio η/s).

While the $L_{\rm micro}$ expansion was not studied as extensively as the gradient expansion, it is possible to include it via diagrammatic techniques [21] as well as recently developed approaches based on the Schwinger-Keldysh formalism [22,24]. In these approaches, one can obtain an expansion in $L_{\rm micro}/L_{\rm diss}$ [14–24] as well as a renormalization of dissipative coefficients [21] due to contributions of fluctuations. Unsurprisingly, this approach almost invariably results in $L_{\rm diss}$ being renormalized to $\geq \mathcal{O}(L_{\rm micro})$ even if the "bare" $L_{\rm diss}$ is small. This is a reasonable conclusion, but small systems hydrodynamics [7–9] seems to contradict this conclusion as the goodness of hydrodynamic expansions is remarkably independent on $L_{\rm micro}$, suggesting that perhaps

hydrodynamics is somehow nonanalytic in $L_{\rm micro}/L_{\rm diss}$, as argued in [32].

In the approach taken in this work, we start with the ergodic hypothesis to hold approximately in every cell at a scale Δt, but we do not consider fluctuations to be necessarily small. In fact, the deterministic continuum is coincident with the onset of the regression to the mean by the law of large numbers. Such "coarse graining around ergodicity" in the ideal limit means

$$L_{\rm diss} = 0, \qquad L_{\rm micro} \equiv \Delta \ll L_{\rm macro}. \tag{19}$$

If one could move beyond $L_{\rm diss} = 0$ this could lead to a limit to hydrodynamics that depends very differently on the thermal and mean free path scale. The main point here is that

(i) a strongly coupled system could be "close the ergodic limit" at every cell even if the number of d.o.f.s in each cell is small. In this case the "Khinchin conditions" [12] are not satisfied, so Gibbsian and Boltzmannian equilibria are very different, but the system will be in a local Gibbsian equilibrium at every point;

(ii) since ergodicity and frame independence are different concepts, in such a limit, foliation independence should still be valid. It is in fact far from clear that, provided the system is strongly coupled and chaotic, decreasing the number of d.o.f.s decreases the applicability of the ergodic hypothesis. In this sense, fluctuations should respect foliation independence. Approximate foliation independence means fluctuations are inherently nonlinear and in fact often help the system thermalize. This is particularly appealing if fluctuation-dissipation holds [26].

To expand on point (i), the considerations above are particularly relevant when the system is small enough for quantum effects to be relevant. In fact, one can use the direct connection between the eigenstate thermalization hypothesis [45] (or similarly Berry's conjecture; see [46] and references therein) and the microcanonical density matrix to generalize the derivation here to a quantum system. A generic mixed density matrix commuting with the Hamiltonian of a theory with a conserved charge and energy-momentum is characterized by the density matrix

$$\hat{\rho} = \delta_{P,P'}\delta_{Q,Q'} c_{P,Q} c^*_{P',Q'} |i, P_\mu, Q\rangle\langle j, P_\mu, Q|, \qquad \hat{U}^{-1}\hat{\rho}\,\hat{U} = \hat{\rho}, \tag{20}$$

where

$$\hat{U} = \exp\left[i\hat{P}^\nu \delta x_\nu\right] \exp\left[i\theta\delta\hat{Q}\right]| \tag{21}$$

the phase invariance θ in a first quantized theory just reflects unitarity and $J_{\mu\nu}$ are simply generators of momentum and angular momentum. For a highly mixed thermalized system, ρ becomes a pseudorandom matrix, averaged to $c_{P,Q} c^*_{P',Q'} \simeq 1/d\Sigma(P,Q)$, making it indistinguishable from the microcanonical ensemble examined in this paper.

Applying the eigenstate thermalization hypothesis to every cell in every foliation is equivalent to promoting $J_{\mu\nu}$, θ, P, and Q to functions of x_μ and imposing foliation independence on the "pseudorandomness" of $\hat{\rho}$:

$$\left.\frac{d\hat{\rho}}{d\Sigma_0}\right|_{\Sigma_0 - \Sigma_0' \simeq \Delta t} = 0, \qquad \hat{\rho} \simeq \frac{1}{d\Sigma}\hat{\delta}_{E,E'}\hat{\delta}_{Q,Q'} \tag{22}$$

and also

$$\hat{U}^{-1}(x)\hat{\rho}\,\hat{U}(x) \simeq \hat{\rho}, \qquad \hat{U}(x) = \exp[i\hat{T}^{\mu\nu} d^3\Sigma_\mu \delta x_\nu] \exp\left[i\partial_\alpha\theta d^3\Sigma^\alpha \delta\hat{Q}\right] \tag{23}$$

for arbitrary $d^3\Sigma_\mu$. By imposing Σ invariance on the first equation it is not too difficult to see that from Eq. (8) with the density matrix as an "observable" an equation equivalent to Eq. (9) can be derived, and hence quantum dynamics will be equivalent to the classical microcanonical one.

What this reasoning suggests is that if the system can be divided into cells where the eigenstate thermalization hypothesis applies, then a dynamics equivalent to ideal fluid dynamics will naturally emerge even if the number of degrees of freedom is comparatively low and fluctuations are big. In the context of heavy ion physics, therefore, the applicability of the eigenstate thermalization hypothesis suggested in [46] should go hand in hand in the onset of hydrodynamic behavior where spatial anisotropy is also present. Whether the eigenstate thermalization hypothesis can be applied to QCD in the strongly coupled regime of course remains to be understood [47], but perhaps the parton disentanglement mechanism suggested in [48] can generate local ergodicity. To verify this the diffeomorphisms examined here must be examined on the light cone.

Regarding point (ii), to try to relax the $L_{\rm diss} = 0$ limit of Eq. (19), let us now imagine the system is *approximately* locally ergodic, defined by the idea that deviations from ergodicity are within an standard fluctuation. ϕ_I can then be thought of not as coordinates but as probability distributions, each characterized by a Killing vector u_μ. One heuristically arrives at the picture in [32]. To do so one can use escort distributions on top of the ergodic distribution [49]

$$\phi_I \to \hat{\phi}_I \equiv \exp\left[\phi_I - \frac{d\ln\mathcal{Z}}{d\Sigma_I}\right], \qquad Z = \mathrm{Tr}\,\exp\left[d\Sigma_\mu \beta_\nu \hat{T}^{\mu\nu}\right]. \tag{24}$$

One can then link Lagrangian hydrodynamics to Zubarev relativistic hydrodynamics as developed in [6].

In addition let us suppose in accordance to (ii) that the fluctuation-dissipation theorem still applies, so fluctuations around equilibrium give a good estimate of deviations from

ergodicity. Supposing fluctuation-dissipation applies is equivalent to supposing that while b as defined in Eq. (17) is not conserved, the probability of a cell evolving from a starting configuration at $b = b_0$, $\tau = \tau_0$ to $b = b_0 + db$ at $\tau = d\tau = d\Sigma_\mu d\Sigma^\mu$ is

$$\frac{P(\phi_I = \phi_{I0} + d\phi_I, \tau = \tau_0 + d\tau | \phi_I = \phi_{I0}, \tau = \tau_0)}{P(\phi_I = \phi_{I0}, \tau = \tau_0 + d\tau | \phi_I = \phi_{I0} + d\phi_I, \tau = \tau_0)} = \frac{f(b + db)}{f(b)}, \tag{25}$$

i.e., a function of the entropy only. One can easily check that, while this dynamics need not be compatible with Eq. (6), it remains invariant under foliation changes. Keeping the idea that the number of microstates is $\sim \exp(b)$ forces $f(b)$ to be an exponential as well. It is in fact the dynamics of [25] expressed in Lagrangian hydrodynamics. The multiple definitions of ϕ_I should therefore accommodate the gaugelike symmetry of [25] described in [32]. Here, it is worth pointing out that if one uses local ergodicity to describe microscopic fluctuations, it becomes obvious that, as pointed out in [32], only fluctuations of the total $T_{\mu\nu}$ are physical, while fluctuations of u_μ, e, p, and $\Pi_{\mu\nu}$ on their own are essentially equivalent to constrained gauge redundancies, ensembles cannot depend on them individually. In the Lagrangian picture this is evident from the fact that [37,38] all perturbative degrees of freedom are Goldstone modes of the symmetries emergent from ergodicity. Distinguishing a nonhydrodynamic excitation of $\Pi_{\mu\nu}$ from a fluctuation in equilibrium u_μ, e is only possible when the ergodic limit is exact in every cell as well as in the ensemble average, i.e., when the number of microscopic d.o.f.s is infinite in every coarse-grained cell.

It is also worth pointing out that any long-range potential between degrees of freedom will, by the fluctuation-dissipation theorem applied to the microcanonical ensemble [26], be indistinguishable from an anisotropic collective flow in its sampling dynamics. This leads exactly to the picture advocated around Eq. (36) [32], where a large class of deformations of the phase space function has the potential to make Vlasov-type and Boltzmann-type terms in kinetic theory cancel out, leading to an ensemble of equivalent local equilibrium states. Superficially this looks like the plasma instability scenario [11], but the "equivalent equilibrium" states are really indistinguishable from bona fide equilibrium. One can see this by considering that if local ergodicity is approximately respected in each cell as well as across ensemble averages, hydrodynamic signatures should be expected to translate across fluctuations, if their fluctuation should reflect initial state fluctuations, as indeed seems to be the case [1,50]. If hydrodynamical observables show thermalization driven by plasma instabilities, one would expect parametrically larger fluctuations [51].

The above reasoning assumes that the potentials are local and symmetric enough to not generate long-range correlations. The alternatives of course are "solids" and "jellies" [37,52–54], where long-range correlations break local deformation, isotropy and translation symmetries. Looking at the arguments governing the applicability of the ergodic hypothesis [12], it is clear that while such systems have an equilibrium state characterized by the local maximum of entropy, their accessible phase space volume is generally far away from the indecomposable limit [12] because of long-range correlations imposing constraints local to phase space. Thus, local ergodicity in the sense of this work does not apply to these systems. Moreover, the arguments presented here make apparent why such systems are generally *fragile*: The lack of phase space indecomposability [12] ensures that small perturbations will lead to large deviations from the ergodic hypothesis in every cell, and hence it is expected that it becomes relatively easy to "break" such systems, i.e., bring them far away from local equilibrium. The preparation of such states will require a careful cancellation of dynamics at different scales. Thus, relativistic fluids and the symmetries defining them are unique in that local ergodicity ensures both the existence of continuous deformation symmetries and the fact that adjacent microstates to equilibrium will be sampled, by local perturbations, in a way that guarantees the continuation of near-local equilibrium.

Making a link of this work with hydrodynamics with spin [5,55] is a nontrivial endeavor because of the fact that the interplay of spin with angular momentum lead to nonlocal phase space hypersurfaces with a high deviation from indecomposability [12] (the presence of subregions of phase space with limited connection can be interpreted as the appearance of an intermediate scale [56]). Nevertheless, it is possible that hydrodynamics with spin can *only* be defined this way, given the fact that spin-vorticity coupling alters microstate distributions at the same order as hydrodynamic fluctuations [5], something arising as nonlocality from transport models [57].

Finally, as a wild speculation taking inspiration from [40], the addition of a horizon term to the entropy, $d\Omega$, and a generalization of the ergodicity hypothesis from time averages to worldline or proper time averages might open the way to an entropic gravity scenario where the equivalence principle is respected exactly, as suggested in [25,41].

In conclusion we have shown that the local symmetries associated with ideal Lagrangian hydrodynamics can be thought of as emerging from a hypothesis of local ergodicity strong enough to be invariant under smooth spacetime foliations. This opens a way of thinking about hydrodynamics which is explicitly "nonperturbative" with respect to fluctuations and more appropriate to study the onset of hydrodynamic behavior in small systems.

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[1] R. Derradi de Souza, T. Koide, and T. Kodama, Prog. Part. Nucl. Phys. **86**, 35 (2016).
[2] P. Romatschke and U. Romatschke, *Relativistic Fluid Dynamics In and Out of Equilibrium* (Cambridge University Press, Cambridge, England, 2019), ISBN 978-1-108-48368-1, 978-1-108-75002-8.
[3] S. Jeon and U. Heinz, Int. J. Mod. Phys. E **24**, 1530010 (2015).
[4] G. Denicol and D. Rischke, *Microscopic Foundations of Relativistic Fluid Dynamics* (Springer, New York, 2021).
[5] D. Montenegro, R. Ryblewski, and G. Torrieri, Acta Phys. Pol. B **50**, 1275 (2019).
[6] F. Becattini, M. Buzzegoli, and E. Grossi, Particles **2**, 197 (2019).
[7] J. L. Nagle and W. A. Zajc, Annu. Rev. Nucl. Part. Sci. **68**, 211 (2018).
[8] A. Adams, L. D. Carr, T. Schäfer, P. Steinberg, and J. E. Thomas, New J. Phys. **14**, 115009 (2012).
[9] A. P. J. Breu, H.-M. Ensner, C. A. Kruelle, and I. Rehberg, Phys. Rev. Lett. **90**, 014302-1 (2003).
[10] G. Giacalone, A. Mazeliauskas, and S. Schlichting, Phys. Rev. Lett. **123**, 262301 (2019).
[11] S. Mrowczynski, B. Schenke, and M. Strickland, Phys. Rep. **682**, 1 (2017).
[12] A. Y. Khinchin, *Mathematical Foundations of Statistical Mechanics* (Martino Fine Books, 2014), https://archive.org/details/khinchin-mathematical-foundations-of-statistical-mechanics.
[13] E. T. Jaynes, Am. J. Phys. **33**, 391 (1965).
[14] Lifshitz and Landau, *Statistical Mechanics II* (Butterworth-Heinemann, 1980), https://www.amazon.com/Statistical-Physics-Theory-Condensed-State/dp/0750626364.
[15] L. P. Csernai, S. Jeon, and J. I. Kapusta, Phys. Rev. A **56**, 6668 (1997).
[16] M. Singh, C. Shen, S. McDonald, S. Jeon, and C. Gale, Nucl. Phys. **A982**, 319 (2019).
[17] J. I. Kapusta, B. Muller, and M. Stephanov, Phys. Rev. C **85**, 054906 (2012).
[18] Y. Akamatsu, A. Mazeliauskas, and D. Teaney, Phys. Rev. C **95**, 014909 (2017).
[19] P. Kovtun, J. Phys. A **45**, 473001 (2012).
[20] N. Mullins, M. Hippert, and J. Noronha, Phys. Rev. D **108**, 076013 (2023).
[21] P. Kovtun, G. D. Moore, and P. Romatschke, Phys. Rev. D **84**, 025006 (2011).
[22] H. Liu and P. Glorioso, Proc. Sci. TASI2017 (**2018**) 008.
[23] A. Jain and P. Kovtun, J. High Energy Phys. 01 (2024) 162.
[24] X. An, G. Basar, M. Stephanov, and H. U. Yee, Phys. Rev. C **100**, 024910 (2019).
[25] G. Torrieri, J. High Energy Phys. 02 (2021) 175.
[26] M. Bonanca, Phys. Rev. E **78**, 031107 (2008).
[27] R. E. Hoult and P. Kovtun, J. High Energy Phys. 06 (2020) 067.
[28] F. S. Bemfica, M. M. Disconzi, and J. Noronha, Phys. Rev. X **12**, 021044 (2022).
[29] J. Noronha, M. Spaliński, and E. Speranza, Phys. Rev. Lett. **128**, 252302 (2022).
[30] L. Gavassino, M. Antonelli, and B. Haskell, Phys. Rev. Lett. **128**, 010606 (2022).
[31] L. Gavassino, Found. Phys. **50**, 1554 (2020).
[32] T. Dore, L. Gavassino, D. Montenegro, M. Shokri, and G. Torrieri, Ann. Phys. (Amsterdam) **442**, 168902 (2022).
[33] P. B. Arnold, Int. J. Mod. Phys. E **16**, 2555 (2007).
[34] F. Gelis, *Quantum Field Theory* (Cambridge University Press, Cambridge, England, 2019), ISBN-13: 978-1108480901.
[35] E. A. Calzetta and B. L. B. Hu, *Nonequilibrium Quantum Field Theory* (Oxford University Press, New York, 2009), ISBN 978-1-00-929003-6, 978-1-00-928998-6, 978-1-00-929002-9, 978-0-511-42147-1, 978-0-521-64168-5.
[36] R. Jackiw, V. P. Nair, S. Y. Pi, and A. P. Polychronakos, J. Phys. A **37**, R327 (2004).
[37] S. Dubovsky, T. Gregoire, A. Nicolis, and R. Rattazzi, J. High Energy Phys. 03 (2006) 025.
[38] S. Dubovsky, L. Hui, A. Nicolis, and D. T. Son, Phys. Rev. D **85**, 085029 (2012).
[39] C. Barcelo, S. Liberati, and M. Visser, Living Rev. Relativity **8**, 12 (2005).
[40] T. Jacobson, Phys. Rev. Lett. **75**, 1260 (1995).
[41] G. Torrieri, Eur. Phys. J. Special Topics **232**, 3505 (2023).
[42] F. Cooper and G. Frye, Phys. Rev. D **10**, 186 (1974).
[43] E. Gourgoulhon, arXiv:gr-qc/0703035.
[44] G. Torrieri, Phys. Rev. D **85**, 065006 (2012).
[45] L. D'Alessio, Y. Kafri, A. Polkovnikov, and M. Rigol, Adv. Phys. **65**, 239 (2016).
[46] F. Becattini, arXiv:0901.3643.
[47] M. Srdinsek, T. Prosen, and S. Sotiriadis, Phys. Rev. Lett. **132**, 021601 (2024).
[48] D. E. Kharzeev, Phil. Trans. A. Math. Phys. Eng. Sci. **380**, 20210063 (2021).
[49] C. Beck and F. Schlögl, *Thermodynamics of Chaotic Systems* (Cambridge University Press, Cambridge, England, 1993).

[50] S. Vogel, G. Torrieri, and M. Bleicher, Phys. Rev. C **82**, 024908 (2010).
[51] S. Mrowczynski and E. V. Shuryak, Acta Phys. Pol. B **34**, 4241 (2003), https://ui.adsabs.harvard.edu/abs/2003AcPPB..34.4241M.
[52] M. Baggioli and S. Grieninger, J. High Energy Phys. 10 (2019) 235.
[53] L. Gavassino, M. M. Disconzi, and J. Noronha, arXiv:2302.05332.
[54] A. Nicolis, R. Penco, F. Piazza, and R. Rattazzi, J. High Energy Phys. 06 (2015) 155.
[55] F. Becattini and M. A. Lisa, Annu. Rev. Nucl. Part. Sci. **70**, 395 (2020).
[56] M. Hongo, X. G. Huang, M. Kaminski, M. Stephanov, and H. U. Yee, J. High Energy Phys. 11 (2021) 150.
[57] N. Weickgenannt, E. Speranza, X. l. Sheng, Q. Wang, and D. H. Rischke, Phys. Rev. Lett. **127**, 052301 (2021).