Letter

Free fermionic webs of heterotic T folds

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Moduli stabilization is key to obtaining phenomenologically viable string models. Nongeometric compactifications, like T-duality orbifolds (T folds), are capable of freezing many moduli. However, in this paper we emphasize that T folds, admitting free fermionic descriptions, can be associated with a large number of different T folds with varying number of moduli, since the fermion pairings for bosonization are far from unique. Consequently, in one description a fermionic construction might appear to be asymmetric, and hence nongeometric, while in another it admits a symmetric orbifold description. We introduce the notion of intrinsically asymmetric T folds for fermionic constructions that do not admit any symmetric orbifold description after bosonization. Finally, we argue that fermion symmetries induce mappings in the bosonized description that extend the T-duality group.

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Introduction. String theory realizes a unification of gravity, gauge interactions, and their charged matter via the properties of conformal field theories (CFTs) residing on its two-dimensional (2D) worldsheet. Heterotic strings on toroidal orbifolds [1,2] led to some of the most realistic string-derived models to date [3–5]. However, orbifolds and other geometrical backgrounds result in free moduli (such as the metric, B-field, or Wilson lines) on which detailed physics, like gauge and Yukawa couplings, depend.

Strings on tori and their orbifolds admit exact quantization. This was instrumental in the discovery of T dualities [6], like the famous $R \to 1/R$ duality, which sets the effective minimum of the radius R equal to the string scale. Investigations of string backgrounds had a profound impact on mathematics as mirror symmetry showed, which was argued to be a form of T duality [7].

Modding out T-duality symmetries may lead to exotic nongeometric backgrounds [8,9], dubbed T folds. Hence, the landscape of string vacua may be much vaster than suggested by geometrical compactifications alone. Even though nongeometric constructions have been studied far less than their geometric counterparts, they may be vital for phenomenological string explorations, as they are capable of freezing many moduli.

Such T folds may have different actions on their left- and right-moving bosonic coordinate fields, and are thus referred to as asymmetric orbifolds [10,11]. If only order-two symmetries are modded out, an alternative fermionic description may be obtained by bosonization, a CFT equivalence of chiral bosons and fermions in 2D [12]. This led to a detailed dictionary between these two formulations explicated for symmetric $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds [13]. Asymmetric boundary conditions in the fermionic formalism have profound phenomenological consequences, such as the doublet-triplet splitting mechanism [14,15], Yukawa coupling selection [16], and moduli fixing [17].

Although a similar dictionary for asymmetric orbifolds is not this paper's aim, heterotic bosonization ambiguities suggest identifications of seemingly unrelated T folds. This sheds new light on nongeometric moduli stabilization. Fermionic symmetries parametrizing these ambiguities suggest an extension of the T-duality group.

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Order-two bosonic T-fold models. The bosonic formulation of the heterotic string [18] describes d-dimensional Minkowski space by coordinate fields $x = (x^{\mu=2...d-1})$ in the light-cone gauge. The internal coordinate fields $X = (X_R|X_L)$, with right and left chiral parts, $X_R = (X_R^{i=1...D})$ and $X_L = (X_L^{a=1...D+16})$, $D = 10 - d$, are subject to torus periodicities

$$X \sim X + 2\pi N, \qquad N \in \mathbb{Z}^{2D+16}. \tag{1}$$

The worldsheet supersymmetry current,

$$T_F(z) = i\psi_\mu \partial x^\mu + i\chi^i \partial X_R^i, \tag{2}$$

involves the real holomorphic superpartners $\psi = (\psi^\mu)$ and $\chi = (\chi^i)$ of x and X_R, respectively. Here, $(\bar{\partial})$ ∂ denotes the (anti)holomorphic worldsheet derivative, and repeated indices are summed over.

An order-two generator, defining the orbifold action

$$X \sim e^{2\pi i v} X - 2\pi V, \tag{3}$$

with $v = (v_R|v_L)$, $V = (V_R|V_L) \in \frac{1}{2}\mathbb{Z}^{2D+16})$, is called a shift, a twist, or a rototranslation if $V \not\equiv 0 \equiv v$, $v \not\equiv 0 \equiv V$, or $v, V \not\equiv 0$, respectively. ($\equiv$ means equal up to integral vectors.)

An orbifold is called *symmetric* if there is a basis such that the left- and right-twist parts are equivalent according to

$$v_L \equiv (v_R, 0^{16}) \tag{4}$$

for all its generators simultaneously [19–21], and *asymmetric* if no such basis exists. The addition of 0^{16} is essential, as the vectors v_L and v_R have unequal lengths.

Real free fermionic models. In the free fermionic formulation [22–24], the internal degrees of freedom are described by real holomorphic fermions $f = (y, w)$ with $y = (y^i)$ and $w = (w^i)$ and real antiholomorphic fermions $\bar{f} = (\bar{f}^{u=1...2D+32})$. Worldsheet supersymmetry is realized nonlinearly,

$$T_F(z) = i\psi_\mu \partial x^\mu + i\chi^i y^i w^i. \tag{5}$$

A fermionic model is defined by a set of basis vectors with entries 0 or 1 for real fermions. Each basis vector $\boldsymbol{\beta} = (\beta|\bar{\beta})$ defines boundary conditions

$$f \sim -e^{\pi i \beta} f, \qquad \bar{f} \sim -e^{\pi i \bar{\beta}} \bar{f}. \tag{6}$$

Bosonizations.

Holomorphic bosonization: Assuming that the fermions χ are identical in the supercurrents (2) and (5) and that they generate the same worldsheet supersymmetry in the bosonic and fermionic descriptions, bosonization uniquely relates the currents

$$J^i =: (\lambda^i)^* \lambda^i := :y^i w^i: \;\cong i\partial X_R^i \tag{7a}$$

and complex fermions

$$\lambda^i = \frac{1}{\sqrt{2}}(y^i + i w^i) \cong :e^{iX_R^i}: \tag{7b}$$

to normal ordered exponentials of chiral bosons. Here, $\cong$ emphasizes that these expressions are not identities, but rather that both sides have identical operator product expansions in either formulation.

The bosonization formula relates the boundary conditions in both descriptions. The torus periodicities (1) reflect the 2π ambiguities of X_R in the complex exponentials (7b). Comparing the orbifold conditions (3) of the right-moving bosons X_R with the boundary conditions (6) of the holomorphic fermions y and w in (7) leads to the following identifications:

$$v_R = \frac{1}{2}\beta(w) - \frac{1}{2}\beta(y), \qquad V_R = \frac{1}{2}(1^D) - \frac{1}{2}\beta(y). \tag{8}$$

Antiholomorphic bosonization: Contrary to the holomorphic side, the pairing of the antiholomorphic fermions is arbitrary. Associating odd and even fermion labels to the real and imaginary parts of complex fermions results in an antiholomorphic bosonization procedure given by

$$\bar{J}^a =: (\bar{\lambda}^a)^* \bar{\lambda}^a := :\bar{f}^{2a-1}\bar{f}^{2a}: \;\cong i\bar{\partial} X_L^a, \tag{9a}$$

with

$$\bar{\lambda}^a = \frac{1}{\sqrt{2}}(\bar{f}^{2a-1} + i\bar{f}^{2a}) \cong :e^{iX_L^a}: \tag{9b}$$

for $a = 1, \ldots D + 16$.

Then, by similar arguments as above, the torus periodicities (1) for X_L follow. And splitting $\bar{\beta} = (\bar{\beta}_o, \bar{\beta}_e)$ into two $(D+16)$-dimensional vectors, $\bar{\beta}_o = (\bar{\beta}^{1,3...2D+31})$ and $\bar{\beta}_e = (\bar{\beta}^{2,4...2D+32})$, leads to the identifications

$$v_L = \frac{1}{2}\bar{\beta}_e - \frac{1}{2}\bar{\beta}_o, \qquad V_L = \frac{1}{2}(1^{D+16}) - \frac{1}{2}\bar{\beta}_o. \tag{10}$$

Extension of the T-duality group.

Fermionic inversions and permutations: On the antiholomorphic side, the fermionic symmetries contain inversions and permutations: (u) denotes the fermion inversion $\bar{f}^u \to -\bar{f}^u$. The permutation $(u_1 \cdots u_p)$ acts as $\bar{f}^{u_1} \to \bar{f}^{u_2} \cdots \to \bar{f}^{u_p} \to \bar{f}^{u_1}$, leaving the remaining fermions inert. The permutation group contains elements which consist of multiple factors like this, provided their entries are all distinct. It is generated by permutations of two elements (uv). The induced fermionic symmetry actions within the bosonic formulation can be identified using the bosonization (9).

Induced bosonic coordinate transformations: The fermionic symmetries that leave these fermion bosonization pairs intact realize mappings of the bosonic coordinate fields $X_{\rm L}$ to themselves. Their generators and their realizations on the bosonic coordinates are listed in Table I. The bosonic transformations above the middle line of this table are part of the T-duality group, while those below involve translations as well.

Induced mappings of bosonic boundary conditions: Other fermionic symmetries break up fermion bosonization pairs, and hence correspond to mappings between different coordinate fields between which no obvious coordinate transformation exists. However, all fermionic symmetries, generated by inversions and permutations, map the boundary conditions of one orbifold theory to another. The mappings induced by the generators of the fermionic symmetries are collected in Table II. The transformations induced by permutations $(2a\ 2b)$ and $(2a\text{-}1\ 2b\text{-}1)$ combined (in whatever order) lead to the boundary condition mapping associated with $(2a\text{-}1\ 2b\text{-}1)(2a\ 2b)$, as the group property would suggest. Since some actions can be interpreted as T-duality transformations, while others cannot, this hints at an extension of the T-duality group.

The Table II mappings $(2a\text{-}1\ 2a)$, $(2a\text{-}1\ 2b\text{-}1)$, and $(2a\ 2b)$ are of special significance: they mix the twist and shift vector entries. The action of $(2a\text{-}1\ 2a)$ recalls that the shift part of a rototranslation in directions, where the twist acts nontrivially, can be removed via the associated coordinate transformation in Table I. The actions $(2a\text{-}1\ 2b\text{-}1)$ and $(2a\ 2b)$ imply that a pure shift boundary condition can be turned into a rototranslation. By combining these mappings, a web of equivalent (mostly asymmetric) orbifold theories emerges.

Since all these T folds are just different bosonic representations of the same fermionic theory, their physical properties are identical, even though they may not look alike. For example, their modular invariance conditions may seem to disagree, as the number of nonzero entries in the twist vectors under mappings, like $(2a\ 2b)$, changes. However, since only the part of the shift of (3) on which the twist acts trivially takes part in the modular invariance condition [21], their consistency conditions are numerically identical.

A free fermionic T-fold web: The basis vectors $\boldsymbol{\beta}$ for a simple illustrative 6D fermionic model are given in Table III, together with associated twist and shift vectors using the odd-even pairings (9). Within this bosonization, the model is understood as an asymmetric orbifold. The interpretation may change by applying fermionic symmetries.

The permutations $(2\,6)^{p_1}(4\,8)^{p_2}(10\,14)^{p_1}(12\,16)^{p_4}$ with $p_i = 0, 1$, map the twist vector $v_{\rm L}(\mathbf{1}-\mathbf{b}) = \frac{1}{2}(0^{20}) \to$

$$|p_1 p_2 p_3 p_4) = \frac{1}{2}(p_1 p_2 p_1 p_2 p_3 p_4 p_3 p_4\, 0^{12}), \quad (11)$$

while the other twists and shifts remain the same, since $(2a\ 2b)$ leave $V_{\rm L}$ inert (see Table II). When these permutations are successively switched on, the T-fold web, given in Fig. 1, is obtained.

TABLE I. Fermionic-symmetry-induced bosonic coordinate field transformations. (Only noninert fields are given.).

Fermionic symmetry	Action on left-moving bosons
$(2a\text{-}1\ 2b\text{-}1)(2a\ 2b)$	$X^a_{\rm L} \leftrightarrow X^b_{\rm L}$
$(2a)$	$X^a_{\rm L} \to -X^a_{\rm L}$
$(2a\text{-}1)$	$X^a_{\rm L} \to \pi - X^a_{\rm L}$
$(2a\text{-}1\ 2a)$	$X^a_{\rm L} \to \frac{1}{2}\pi - X^a_{\rm L}$

TABLE II. Fermionic-symmetry-induced bosonic boundary condition mappings. (Only noninert entries of the vectors $v_{\rm L}$ and $V_{\rm L}$ modulo integral vectors are given.).

Fermionic symmetry	Action on twist and shift entries
$(2a\text{-}1\,2b-1)(2a\ 2b)$	$v^a_{\rm L} \leftrightarrow v^b_{\rm L}, V^a_{\rm L} \leftrightarrow V^b_{\rm L}$
$(2a)$	$v^a_{\rm L} \to -v^a_{\rm L} + 2V^a_{\rm L}, V^a_{\rm L} \to V^a_{\rm L}$
$(2a\text{-}1\ 2a)$	$v^a_{\rm L} \to -v^a_{\rm L}, V^a_{\rm L} \to V^a_{\rm L} - v^a_{\rm L}$
(2a 2b)	$v^a_{\rm L} \to v^b_{\rm L} + V^a_{\rm L} - V^b_{\rm L}, V^a_{\rm L} \to V^a_{\rm L},$ $v^b_{\rm L} \to v^a_{\rm L} + V^b_{\rm L} - V^a_{\rm L}, V^b_{\rm L} \to V^b_{\rm L}$
(2a-1 2b-1)	$v^a_{\rm L} \to v^a_{\rm L} + V^a_{\rm L} - V^b_{\rm L}, V^a_{\rm L} \to V^b_{\rm L},$ $v^b_{\rm L} \to v^b_{\rm L} + V^b_{\rm L} - V^a_{\rm L}, V^b_{\rm L} \to V^a_{\rm L}$
$(2a\text{-}1)$	$v^a_{\rm L} \to v^a_{\rm L} - 2V^a_{\rm L}, V^a_{\rm L} \to -V^a_{\rm L}$

TABLE III. Fermionic basis vectors $\boldsymbol{\beta}$ with the twists v and shifts V corresponding to $\mathbf{1}-\boldsymbol{\beta}$ obtained via (8) and (10).

Fermionic basis vectors	Twist and shift vectors	
$\boldsymbol{\beta}$	$v = (v_{\rm R}\vert v_{\rm L})$	$V = (V_{\rm R}\vert V_{\rm L})$
$\mathbf{1} = \{\psi^{1\ldots4}\chi^{1\ldots4}y^{1\ldots4}w^{1\ldots4}\vert\bar{f}^{1\ldots40}\}$	$(0^4\vert 0^{20})$	$\frac{1}{2}(1^4\vert 1^{20})$
$\mathbf{S} = \{\psi^{1\ldots4}\chi^{1\ldots4}\}$	$(0^4\vert 0^{20})$	$(0^4\vert 0^{20})$
$\boldsymbol{\xi} = \{\bar{f}^{9\ldots40}\}$	$(0^4\vert 0^{20})$	$\frac{1}{2}(0^4\vert 0^4 1^{16})$
$\mathbf{b} = \{\chi^{1\ldots4}w^{1,\ldots4}\vert\bar{f}^{1\ldots4}\bar{f}^{9\ldots12}\}$	$\frac{1}{2}(1^4\vert 0^{20})$	$\frac{1}{2}(0^4\vert 1^2 0^2 1^2 0^{14})$

TABLE IV. The number of $\mathbf{b}$ and $\boldsymbol{\xi}$ twists for the inequivalent T folds associated with the fermionic model in Table III.

	$\boldsymbol{\xi}$					
b	0	2	4	6	8	Sum
0	1	2	3	2	1	9
2	2	11	18	12	3	46
4	3	18	32	19	6	78
6	2	12	19	18	7	58
8	1	3	6	7	5	22
Sum	9	46	78	58	22	213

For the cases with two nonzero p_i's, (11) implies that $v_L = (v_R, 0^{16})$, possibly up to a change of basis. Thus, the resulting bosonic models are interpreted as symmetric orbifolds. In particular, the model obtained after the fermionic permutation (2 6)(4 8) is conventionally considered as the bosonic representation of this fermionic model in which $\boldsymbol{\xi}$ just separates out the $SO(32)$ gauge group, while for all the other cases with two nonzero p_i's, $\boldsymbol{\xi}$ acts as an asymmetric Wilson line.

Table IV provides an overview of all inequivalent T-fold models associated with this fermionic model. It indicates in how many directions $\mathbf{b}$ and $\boldsymbol{\xi}$ act as left-moving twists. Apart from the 16 models depicted in Fig. 1 (of which nine are inequivalent), $\boldsymbol{\xi}$ has an asymmetric twist action, as can inferred from this table. The total number of inequivalent T-fold models associated with the fermionic basis vectors given in Table III is 213. This number rapidly increases for fermionic models defined with more basis vectors. For example, for the fermionic model in which $\boldsymbol{\xi}$ is split into $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$, the number of inequivalent bosonizations becomes 11 273; and for the NAHE set [25–27], it is 85 735.

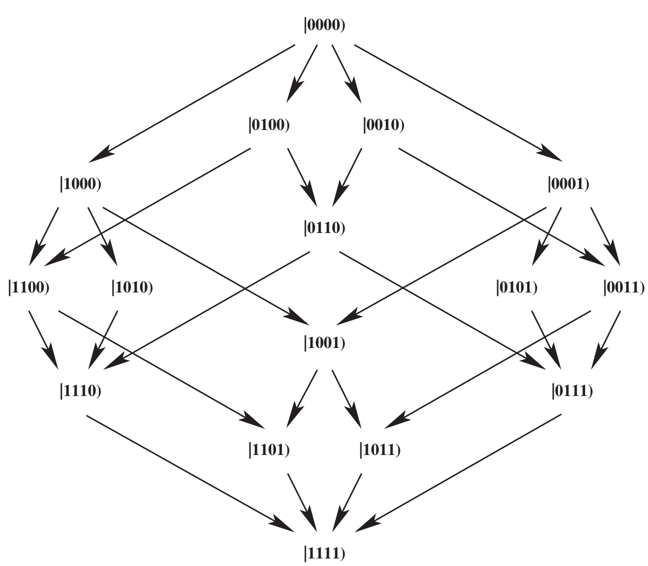

FIG. 1. Web of T folds associated with the fermionic model given in Table III in which only $\mathbf{b}$ acts as a twist (11).

Moduli. The unfixed Narain moduli ($m_{ij} = g_{ij} + b_{ij}$ with metric g_{ij}, B-field b_{ij}, and Wilson lines $m_{ix=1...16}$) of a T fold correspond to the operators

$$m_{ia}\partial X^i_R \bar{\partial} X^a_L, \tag{12}$$

left inert by (3). Symmetric orbifolds always leave at least the diagonal metric moduli $m_{ii} = g_{ii}$ free; asymmetric orbifolds may fix all moduli.

This would suggest that the number of frozen moduli may vary dramatically depending on which bosonic description of a given fermionic model is used. There is no paradox here either: the unfixed scalar deformations of the fermionic model can be identified by the Thirring interactions

$$m_{iuv} y^i w^i \bar{f}^u \bar{f}^v \tag{13}$$

left inert by (6). Thus, the total number of massless untwisted scalars is bosonization independent, and therefore identical in any bosonic realization. Which of them are interpreted as free Narain deformations, however, does depend on the choice of bosonization, as X_L in (12) does.

Intrinsically asymmetric T folds. The previous section showed that whether a real fermionic model should be considered as a symmetric or asymmetric model is very much bosonization dependent. A free fermionic model is called *intrinsically asymmetric* if for any bosonization it corresponds to an asymmetric orbifold. An intrinsically asymmetric T fold is a bosonic model associated with an intrinsically asymmetric fermionic model.

In light of the observation below Eq. (12), a fermionic model that admits a symmetric interpretation has at least inert Thirring interactions (13) with different $u \neq v$ for each i. If not, the fermionic model is intrinsically asymmetric, and hence in any bosonic realization all Narain moduli are frozen. This is, in particular, the case when no Thirring interactions (13) are invariant under (6). An example of such a model is given in Ref. [28].

Simple examples of intrinsically asymmetric free fermionic models can be obtained by taking basis vectors that act as purely holomorphic twists. (For example, consider the twist basis vector $\mathbf{b} = \{\chi^{1...4}, y^{1...4}\}$ in 6D or $\mathbf{b}_1 = \{\chi^{1...4}, y^{1...4}\}$ and $\mathbf{b}_2 = \{\chi^{3...6}, y^{3...6}\}$ in 4D.) As there are no invariant Thirring interactions (13) possible, the corresponding T-fold models are necessarily intrinsically asymmetric.

Discussion. This letter focused on heterotic T folds that admit fermionic descriptions. Even though the key observation that bosonization in a fermionic CFT is not unique is not new—its striking consequences seem not to have been appreciated so far: a single free fermionic model can be associated with a large number of seemingly unrelated bosonic theories. Some may admit a symmetric orbifold interpretation, while most others are asymmetric, but in many different ways.

In light of this, studies of nongeometric constructions, and T folds in particular, may need to be revised, since seemingly different nongeometries may, in fact, be equivalent. In particular, in the bosonic orbifold literature, it would be inconceivable that symmetric and asymmetric orbifolds can be identified. Moreover, the number of frozen moduli turns out to be a bosonization-dependent quantity; only the total number of massless untwisted scalars is identical in any description. Only for an intrinsically asymmetric T fold are all Narain moduli fixed in any bosonic description. In addition, the induced bosonic actions of fermionic symmetries hint at an extension of the T-duality group of toroidal and $\mathbb{Z}_2$ orbifold compactifications.

The findings presented here were derived at free fermionic points. However, the induced transformations of the bosonic boundary conditions by the fermionic symmetries may be considered without referring to the fermionic description. Hence, it is an interesting question whether the suggested extension of the T-duality group discussed above is a general duality symmetry of string theory or if it exists at free fermionic points only.

Our analysis is partially motivated by quasirealistic model building using the free fermionic formulation (see, e.g., Ref. [28]) to give rise to some central features of the Standard Model and its supersymmetric extensions, such as the existence of three generations charged under the Standard Model gauge group with potentially viable Yukawa couplings to Higgs doublets. While this paper focused on the moduli of the internal manifolds, there exist free fermionic models in which the moduli space is further restricted [29], which shows the need for a deeper understanding of the moduli space in these quasirealistic examples, which our analysis may provide. Moreover, the methods adopted in the supersymmetric cases considered here can also be utilized in nonsupersymmetric string constructions, as well as in tachyon-free models that are obtained from compactifications of tachyonic ten-dimensional string vacua [30]. In this respect, the understanding of the correspondence between the fermionic and bosonic representations of string vacua is essential in order to obtain a more profound understanding of the string dynamics at the Planck scale.

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