Letter

Stability and causality criteria in linear mode analysis: Stability means causality

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Causality and stability are fundamental requirements for the differential equations describing predictable relativistic many-body systems. In this work, we investigate the stability and causality criteria in linear mode analysis. We discuss the updated stability criterion in $3+1$-dimensional systems and introduce the improved sufficient criterion for causality. Our findings clearly demonstrate that stability implies causality in linear mode analysis. Furthermore, based on the theorems present in this work, we conclude that if updated stability criterion and improved causality criterion are fulfilled in one inertial frame of reference (IFR), they hold for all IFR.

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Introduction. In modern physics, the space-time evolution of predictable relativistic many-body systems is typically described using differential equations. These differential equations must adhere to the principle of causality as required by the theory of relativity [1]. Causality means that signals or information cannot propagate faster than the speed of light. While the physical interpretation of causality is well understood, the establishment of well-defined criteria for causality in various relativistic systems is still rarely discussed. In the context of quantum field theory, it translates into the requirement that the commutators of local operators vanish outside the light cone [2]. For many macroscopic relativistic many-body systems, it is still challenging to derive the general sufficient and necessary criteria for causality, based on our current understanding (also see Refs. [3–5] and the references therein, for recent developments for the causality criteria in relativistic hydrodynamics).

Another essential prerequisite is stability, characterized by minor perturbations in a state gradually diminishing over time. Typically, one can find solutions to the differential equations near equilibrium or eigenstates. Stability ensures that perturbations near equilibrium or eigenstates return to their respective states. Noteworthy examples for stability include circular orbits in the classical central force problem [6], particular solutions in general relativity [7], equilibrium states in thermodynamics [8], and quantum states in quantum mechanics [9,10], among others.

Analyzing stability and causality helps determine the physical viability of a theory and provides additional constraints on its parameters. One famous example is relativistic hydrodynamics. The conventional relativistic hydrodynamics up to the first order in the gradient expansion is found to be acausal and unstable [11]. It has, therefore, been extended to second-order formalisms such as the Müller-Israel-Stewart (MIS) theory [12], Baier-Romatschke-Son-Starinets-Stephanov [13], and Denicol-Niemi-Molnar-Rischke theory [14]. In addition, a stable and causal generalized first-order formalism, known as the Bemfica-Disconzi-Noronha-Kovtun theory has been established [15,16]. Comprehensive discussions of the causality and stability conditions for these theories can be found in Refs. [3,17] and the references therein. Moreover, recent interest has emerged in the stability and causality of effective field theory for hydrodynamics [18].

Physically, causality and stability are intertwined. Causality also implies that all physical observables must reside within the light cone. Assume that there exists a stable mode in one inertial frame of reference (IFR), in which perturbation decays when $t-t_0>0$ with t_0 being initial time. If the perturbation propagates out of the light cone, we will observe that $t'-t'_0<0$ in another IFR due to a Lorentz transformation. This means that the perturbation grows with time in the second IFR. Therefore, it is concluded that acausality leads to an unstable mode [19]. The above argument reveals the connection between acausality and unstable modes; the relationship between causality and stability is worth being discussed deeply. Although the causality and stability conditions overlap in some theories [20], the criteria for causality and stability appear significantly different and seemingly independent.

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A natural question arises as to how we can establish a connection between the stability and causality criteria.

Because of the complexity involved, analyzing the causality and stability of complete differential equations can be challenging. One common approach is to employ linear mode analysis. Despite the significant progress made in recent works [3,19] regarding stability-causality relations, it remains unclear how to deduce these remarkable findings directly from the basic stability and causality criteria in linear mode analysis.

Let us start the story from a general overview of linear mode analysis. Following that, we discuss the updated stability criteria and introduce improved causality criteria. Finally, we uncover the connection between stability and causality in linear mode analysis based on our findings.

Linear mode analysis and conventional causality and stability criteria. Linear mode analysis is a widely used method for investigating the stability and causality of differential equations. Here, we take the relativistic hydrodynamics as an illustrative example to introduce the basic idea for linear mode analysis. The main differential equations for relativistic hydrodynamics are the energy-momentum and current conservation equations, expanded in the gradient expansion. In the linear mode analysis, one considers the perturbations of independent macroscopic variables within the system, e.g., energy density δe, number density $\delta\rho$, etc., near the equilibrium. Generally, the hydrodynamic conservation equations for n independent perturbations, $\varphi(t,\vec{x}) = (\delta e, \delta\rho, \ldots)^{\mathrm{T}}$, on top of the irrotational equilibrium state can be formulated as linear partial differential equations [11],

$$\partial_t\varphi(t,\vec{x}) + \mathrm{M}(-i\partial)\varphi(t,\vec{x}) = 0, \tag{1}$$

where matrix $\mathrm{M}(-i\partial)$ is a polynomial of the space derivative ∂_i, $\mathrm{M}(-i\partial) = \sum_{j=0}^{N} \mathrm{M}_{(j)}^{i_1,i_2,\ldots,i_j}\partial_{i_1}\partial_{i_2}\ldots\partial_{i_j}$ with $N \geq 0$ being a finite integer and $\mathrm{M}_{(j)}^{i_1,i_2,\ldots,i_j}$ being a constant $n\times n$ matrix. For simplicity, plane-wave-type perturbations are often adopted in the linear mode analysis,

$$\varphi = \varphi_0 e^{-i\omega t + i\vec{k}\cdot\vec{x}}, \qquad \varphi_0 = \mathrm{const}. \tag{2}$$

Subsequently, the nonzero solutions to the differential equations (1) exist if and only if

$$0 = \mathcal{P}(\omega,\vec{k}) \equiv \det[\omega + i\mathrm{M}(\vec{k})]. \tag{3}$$

The eigenvalues of $-i\mathrm{M}(\vec{k})$ yield the dispersion relations, denoted as $\omega_i = \omega_i(\vec{k})$, $i = 1,2,\ldots,n$. Naturally, the dispersion relations should be constrained by physical requirements:

(i) Stability: All perturbations $|\varphi(0,\vec{x})|$ cannot grow exponentially with time. It gives the conventional stability criterion

$$\mathrm{Im}\,\omega \leq 0, \quad \text{for } \vec{k}\in\mathbb{R}^3. \tag{4}$$

(ii) Causality: The influence of $\varphi(0,\vec{x})$ propagates no faster than the speed of light. A widely accepted asymptotic causality criterion is [21]

$$\lim_{|\vec{k}|\to+\infty}\left\{\frac{|\mathrm{Re}\omega|}{|\vec{k}|} \leq 1, |\omega/\vec{k}| \text{ is bounded}\right\}, \qquad \vec{k}\in\mathbb{R}^3. \tag{5}$$

Here, we define the vector norm $|V| \equiv (\sum_{i=1}^{j} V_i V_i^*)^{1/2}$ for any complex vector $V = (V_1, V_2, \ldots, V_j)$ and adopt notation $\omega = \omega_i(\vec{k})$, $i = 1,2,\ldots,n$ for brevity moving forward. The causality and stability criteria mentioned above are intuitive, but they are not flawless.

(i) A practical challenge arises: the conventional causality and stability criteria (4) and (5) depend on IFR. Commonly, the causality and stability conditions are first derived from the criteria (4) and (5) in the rest frame. Then, the verification of these criteria in other IFR follows. However, this process of examining conditions across different frames is frequently burdensome.

(ii) A concern arises: the conventional causality criterion (5) proves to be inadequate in guaranteeing causality [22].

(iii) A question arises: does stability imply causality? Furthermore, what constitutes the relationship between the stability and the causality criteria?

The aim of this work is to provide improved causality criteria, simplify the steps for Lorentz transformation, bridge the gap between stability and causality criteria, and reveal the profound stability-causality relations. We propose an improved sufficient criterion for causality. This, combined with new insights into the covariance of the stability and causality criteria, allows for the immediate derivation of significant stability-causality relations in linear mode analysis. Before further discussion, we emphasize that in the current study we concentrate on the systems with well-defined inertial frames. When the system is far from equilibrium or incorporates certain quantum effects [23], the local rest frames may become ill defined and the following criteria may be inapplicable.

Updated stability criterion for a 3 + 1-dimensional relativistic system. The updated stability criterion for a 3 + 1-dimensional relativistic system is

$$\mathrm{Im}\,\omega \leq |\mathrm{Im}\vec{k}|, \quad \text{for } \vec{k} \in \mathbb{C}^3. \tag{6}$$

The inequality (6) is introduced within a $1+1$-dimensional system by imposing the causality on the retarded two-point function in stable systems, as proposed in Ref. [5], and is subsequently proved as the necessary condition for stability across all IFR [24].

The inequality (6) can be proven by employing a contradiction approach. Assume that $\mathrm{Im}\,\omega > |\mathrm{Im}\vec{k}|$ holds true for a specific $\vec{k}$ in one IFR, denoted as K, even while maintaining system stability. We perform a special Lorentz transformation from frame K to another IFR K', characterized by a velocity $\vec{v} = (\mathrm{Im}\,\vec{k})/(\mathrm{Im}\,\omega)$ relative to K. In frame K', as $(\omega', \vec{k}')^{\mathrm{T}} = \Lambda(\vec{v}) \cdot (\omega, \vec{k})^{\mathrm{T}}$ with $\Lambda(\vec{v})$ being the Lorentz transformation matrix, we observe that $\mathrm{Im}\,\vec{k}' = 0$ and $\mathrm{Im}\,\omega' = \gamma^{-1}\mathrm{Im}\,\omega > 0$ with γ being Lorentz factor. This implies the existence of an unstable mode, which violates the stability requirement (4), within the frame K'. This indicates that the assumption $\mathrm{Im}\,\omega > |\mathrm{Im}\,\vec{k}|$, made in any IFR, can render the system unstable.

Furthermore, the subsequent theorem significantly streamlines the intricate calculations associated with transformations between distinct IFR in linear mode analysis.

Theorem 1. The stability criterion (6) holds true across all IFR if it is satisfied in a single IFR.

Proof. It can also be proven by employing a contradiction approach. Assume that $\mathrm{Im}\,\omega' \leq |\mathrm{Im}\,\vec{k}'|$ holds in a IFR K'. Let us suppose that this inequality is violated in another IFR K'', i.e., there exists $\vec{k}'' \in \mathbb{C}^3$ within frame K'' for which $\mathrm{Im}\,\omega'' > |\mathrm{Im}\,\vec{k}''|$. By the Lorentz transformation, $(\omega'', \vec{k}'')^{\mathrm{T}} = \Lambda(\vec{v}) \cdot (\omega', \vec{k}')^{\mathrm{T}}$ where $\vec{v}$ represents the velocity of frame K'' relative to K', we find

$$|\mathrm{Im}\,\omega'|^2 - |\mathrm{Im}\,\vec{k}'|^2 = |\mathrm{Im}\,\omega''|^2 - |\mathrm{Im}\,\vec{k}''|^2 > 0, \tag{7}$$

and $\mathrm{Im}\,\omega' = \gamma(\mathrm{Im}\,\omega'' - \vec{v} \cdot \mathrm{Im}\,\vec{k}'') > 0$. Thus, we arrive at $\mathrm{Im}\,\omega' > |\mathrm{Im}\,\vec{k}'|$, which contradicts the original assumption. It means that frame K'' does not exist. Therefore, the inequality (6) holds across all IFR. ■

Theorem 1 can also be intuitively comprehended through a geometric lens. Specifically, the vector $\mathrm{Im}(\omega, \vec{k})$ does not lay inside the future light cone, as discussed in Ref. [24].

Improved sufficient criterion for causality. Drawing inspiration from the stability criterion (6), we provide a new sufficient criterion for causality based on the theorem below,

Theorem 2. Suppose that the initial data $\varphi(0, \vec{x})$ for differential equations (1) is smooth with respect to $\vec{x}$, and the volume of the support of $\varphi(0, \vec{x})$ is both finite and nonvanishing. If two constants $R > 0$ and $b \in \mathbb{R}$ exist such that

$$\mathrm{Im}\,\omega \leq |\mathrm{Im}\,\vec{k}| + b, \quad \text{for } |\vec{k}| > R, \tag{8}$$

then the influence of the initial data propagates with subluminal speed.

Before proving the aforementioned theorem, we intend to present a simplified equivalent version of criterion (8), denoted as follows: If condition (8) is fulfilled, then there exists an additional real constant $b' \geq b$ such that

$$\mathrm{Im}\,\omega \leq |\mathrm{Im}\,\vec{k}| + b', \quad \text{for } \vec{k} \in \mathbb{C}^3. \tag{9}$$

Let us deduce the inequality (9) from inequality (8). We notice that the $\omega = \omega(\vec{k})$ must be finite for any finite $|\vec{k}|$. This is because the dispersion relation from Eq. (3), represented by $\mathcal{P}(\omega, \vec{k}) = \omega^n + \sum_{m=0}^{n-1} a_m(\vec{k})\omega^m = 0$ with $a_m(\vec{k})$ being a polynomial of $\vec{k}$, will not yield an infinite ω for finite $\vec{k}$. Consequently, there exists a sufficiently large positive constant b' such that $b' \geq b$ and $\mathrm{Im}\,\omega \leq |\mathrm{Im}\,\vec{k}| + b'$ for $|\vec{k}| \leq R$. Here we have implicitly assumed that the perturbed equations can be written as the form (1), which has already ruled out some acausal equations, e.g., the Benjamin-Bona-Mahony equation [22]. Now, let us direct our attention toward proving the causality criterion (8) with the help of its simplified equivalent version (9).

Proof. Suppose that the initial data $\varphi(0, \vec{x})$, possessing finite volume, are enclosed within the closed ball centered at $\vec{x} = 0$ and having a radius of $L > 0$. By employing the general solution of (1), we can express $\varphi(t, \vec{x})$ with $t > 0$ as follows:

$$\varphi(t, \vec{x}) = \int_{\mathbb{R}^3} \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} e^{-\mathrm{M}(\vec{k})t} \tilde{\varphi}(0, \vec{k}), \tag{10}$$

where $\tilde{\varphi}(t, \vec{k}) \equiv \int d^3x e^{-i\vec{k}\cdot\vec{x}} \varphi(t, \vec{x})$ represents the Fourier transformation of $\varphi(t, \vec{x})$. In this case, the causality means the perturbation $\varphi(t, \vec{x})$ cannot persist beyond the region $|\vec{x}| > L + ct$ with any finite $t > 0$, where $c = 1$ is the speed of light [1]. Hence, our task is to demonstrate that $\varphi(t, \vec{x}) = 0$ within the region $|\vec{x}| > L + t$ with any finite $t > 0$, given that the dispersion relations adhere to the inequality (8).

The key lies in employing the following two inequalities:

$$||e^{-\mathrm{M}(\vec{k})t}|| \leq \frac{a_1}{\epsilon^{n-1}} (1 + |\vec{k}|^{N(n-1)}) e^{[\lambda(\vec{k}) + \epsilon]t}, \tag{11}$$

$$|\tilde{\varphi}(0, \vec{k})| \leq \frac{a_2}{1 + |\vec{k}|^{N(n-1)+4}} e^{L|\mathrm{Im}\vec{k}|}, \tag{12}$$

where $\lambda(\vec{k}) \equiv \max_i \{\mathrm{Im}\omega_i(\vec{k})\}$, $\epsilon \in (0, 1)$, and a_1, a_2 are independent of t, ϵ, and $\vec{k}$. Here, the norm $||\cdot||$ is the spectral norm of matrix [25]. The inequality (12) can be obtained by performing integration by parts $N(n-1) + 4$ times in $\tilde{\varphi}(t, \vec{k}) = \int d^3x e^{-i\vec{k}\cdot\vec{x}} \varphi(t, \vec{x})$ [26]. Proof of inequality (11) utilizes the Cauchy integral formula for matrices [27], and the details can be found in the Supplemental Material [28].

Assisted by these two inequalities, we estimate the integral (10) for $|\vec{x}| > L + t$, employing a method akin to the proof outlined in Theorem 3.1 of Ref. [26]. Upon the fulfillment of the inequality (8) or its simplified equivalent version (9), we have $\lambda(\vec{k}) \leq |\mathrm{Im}\vec{k}| + b'$. Given that the integrand in (10) is an entire analytic function, we change the path $\vec{k} \to \vec{k} + ir\vec{x}$ and apply Cauchy's theorem,

$$|\varphi(t,\vec{x})| = \left| \int_{\mathbb{R}^3} d^3k e^{i(\vec{k}+ir\vec{x})\cdot\vec{x}} e^{-\mathrm{M}(\vec{k}+ir\vec{x})t} \tilde{\varphi}(0,\vec{k}+ir\vec{x}) \right| \propto e^{-r|\vec{x}|(|\vec{x}|-t-L)+(b'+\epsilon)t} \to 0, \tag{13}$$

as $r \to +\infty$, where we have used (11), (12), and the inequality $|e^{-\mathrm{M}(\vec{k})t}\tilde{\varphi}(0,\vec{k})| \leq ||e^{-\mathrm{M}(\vec{k})t}|| \cdot |\tilde{\varphi}(0,\vec{k})|$ [25]. Hence, for any given finite $t > 0$, we deduce $\varphi(t,\vec{x}) = 0$ for $|\vec{x}| > L + t$ in accordance with (13), thereby affirming that the perturbation signal propagates at a speed no greater than that of light. ∎

Analogous to the stability criterion, the subsequent theorem facilitates the extension of the causality criterion (8) or (9) from one IFR to all IFR.

Theorem 3. The causality criterion (8) or (9) holds true across all IFR if it is fulfilled in a single IFR.

Proof. We prove it by employing a contradiction approach again. Let us focus on the inequality (9). Suppose, by contradiction, that (9) holds in a IFR K, but is violated in another IFR K', where K' moves with a velocity $\vec{v}$ relative to K. Thus for any positive real constant $b' > 0$, there exists a $\vec{k}' \in \mathbb{C}^3$ such that $\mathrm{Im}\,\omega' > |\mathrm{Im}\vec{k}'| + b'$ in frame K'. Taking Lorentz transformation, we obtain $\mathrm{Im}\,\omega > |\mathrm{Im}\vec{k}| + \gamma b'(1 - |\vec{v}|)$, which contradicts (9) within frame K. Therefore, the frame K' does not exist. This completes the proof. ∎

In practice, one can verify the causality condition by taking $R \to \infty$ in inequality (8). In large $|\vec{k}|$ limit, the typical dispersion relations exhibit behavior such as $\omega \propto \vec{v}_0 \cdot \vec{k} + \mathcal{O}(|\vec{k}|^0)$ or $\omega \propto c_0(\vec{k}\cdot\vec{k})^{1/2} + \mathcal{O}(|\vec{k}|^0)$, where c_0 and $\vec{v}_0$ are real constants satisfying $|c_0|, |\vec{v}_0| \in [0,1]$. In this scenario, the sufficient criterion (8) holds, ensuring that the system maintains causality.

The new improved sufficient causality criterion (8) or (9) is more stringent and preferable compared to the conventional insufficient inequality (5). We show that the conventional criterion (5) is automatically fulfilled when the dispersion relations obey the inequality (8) in the Supplemental Material [28]. However, the inequality (8) cannot be derived from (5). For example, consider the dispersion relation $\omega = k(1+i)/2$, which indeed satisfies the inequality (5), but does not obey the condition (8). This specific case has been demonstrated to be acausal according to Theorem 2.7 in Ref. [26].

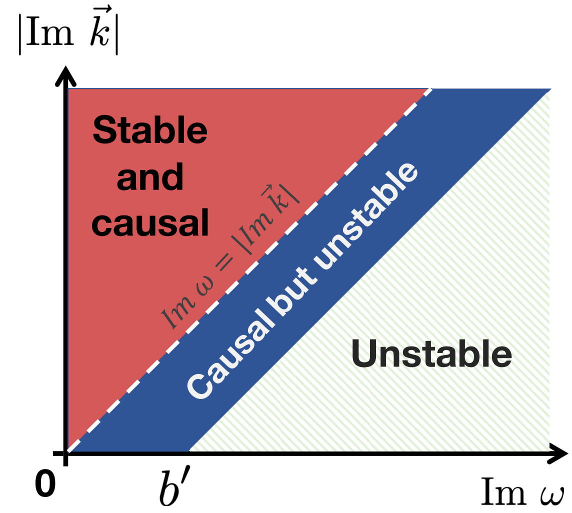

FIG. 1. Illustration for stability and causality criteria.

The updated stability criterion (6) and improved causality criterion (8) or (9) offer a straightforward way to reveal the profound relations between stability and causality in linear mode analysis. Consequently, we derive two corresponding conclusions.

Stability in all IFR means causality. To illustrate the relationship between the stability and causality, we depict the regions that satisfy the stability criterion (6) and causality criterion (9) in Fig. 1. Evidently, the causality criterion (9) is inherently satisfied when the stability criterion (6) holds true in all IFR. However, the reverse does not necessarily hold, e.g., as presented by the blue region in Fig. 1. An example is the tachyon field equation [29], whose dispersion relations fall inside the blue region (also see Supplemental Material [28]). Therefore, stability across all IFR implies causality, while causality does not necessarily entail stability. To prevent any misinterpretation, it is crucial to highlight that if the system has stability solely in a specific IFR while being unstable in others, the presence of stability alone does not guarantee causality. The same conclusion has also been substantiated in Refs. [3,19] through different approaches (also see a preliminary discussion in Ref. [20] written by Pu and co-workers).

This finding is compatible with another significant observation in Ref. [4] referred to as "thermodynamic stability implies causality." The conditions for thermodynamic stability delineated as (i)–(iii) in Ref. [4] maintain Lorentz invariance. Clearly, if the thermodynamic stability holds true in a particular IFR, it holds in any IFR.

Stability and causality in one IFR implies stability and causality across all IFR. The subsequent theorem concerning stability-causality assessment across various frames assists in mitigating the challenge in linear mode analysis, where one is required to examine stability and causality criteria in multiple frames. Similar conclusions have also been reported in Refs. [3,19], relying on the premise of strong hyperbolicity or causality in all IFR.

Theorem 4. If the system described by differential equation (1) exhibits stability and causality in one IFR, then it is stable and causal in all IFR.

Proof. If differential equation (1) is stable and causal in a particular IFR, then it follows that $\mathrm{Im}\,\omega \leq |\mathrm{Im}\vec{k}|$ within this frame (see Theorem 2 in Ref. [24] for the $1+1$-dimensional case and the Supplemental Material [28] for the $3+1$-dimensional case). Consequently, combining Theorems 1–3 presented in this work completes the proof. ■

Theorem 4 provides a practical way for evaluating stability and causality across all IFR. Initially, one selects a suitable IFR, e.g., the rest frame in hydrodynamics. One can derive the stability condition by analyzing conventional stability criterion (4) with the Routh-Hurwitz criterion [16], which is often more straightforward than directly assessing the stability criterion (6). Subsequently, the causality condition can be verified using the causality criterion (8) as $|\vec{k}| \to \infty$. To illustrate this point, we provide an example concerning the stability and causality of MIS theory with bulk viscous pressure only, presented in the Supplemental Material [28]. Interestingly, the calculations become straightforward in isotropic systems by applying the theorem discussed in the coming paragraph.

Application: Asymptotic criterion for an isotropic system. Given that numerous discussions revolve around causality and stability within isotropic systems, such as the case of conventional relativistic hydrodynamics in the rest frame, it is fitting for us to examine the stability and causality conditions tailored for isotropic systems, serving as a practical application. In an isotropic system, a simple asymptotic criterion, as presented in the following theorem, becomes a necessary condition for stability and a sufficient condition for causality across all IFR.

Theorem 5. Considering $\vec{k} = k\hat{\mathrm{n}}$ where $k \in \mathbb{C}$, $\hat{\mathrm{n}} \in \mathbb{C}^3$ and $\hat{\mathrm{n}} \cdot \hat{\mathrm{n}} = 1$. If the nonzero dispersion relations obtained from Eq. (3) are $\hat{\mathrm{n}}$ independent and satisfy inequality (6), then there exist only three asymptotic behaviors at $k \to \infty$,

$$\omega = c_1 k + d_1 + \mathcal{O}(|k|^a), \tag{14}$$

$$\omega = c_2 k^{-2m-1} + d_2 k^{-2m-2} + \mathcal{O}(|k|^{a-2m-2}), \tag{15}$$

$$\omega = c_3 k^{-2m} + \mathcal{O}(|k|^{a-2m}), \tag{16}$$

where $a < 0$, $m = 0, 1, 2, \ldots$, and c_i, d_i are constants obeying $c_1 \in [-1, 1]$, $\mathrm{Im}\,c_2 = 0$, $\mathrm{Im}\,c_3 \leq 0$, $\mathrm{Im}\,d_{1,2} \leq 0$.

The proof of the above theorem follows the asymptotic analysis in Ref. [30] and in the Supplemental Material [28]. We emphasize that the three dispersion relations (14)–(16) mentioned above are necessary conditions for stability. If the asymptotic behaviors of an isotropic system do not adhere to these conditions, it might be causal but must be unstable. Interestingly, the three dispersion relations (14)–(16) satisfy the conventional causality criterion (5). This observation helps explain why the conventional causality criterion (5) has been considered a necessary condition for a covariantly stable and causal isotropic system for a long time.

Summary. In this work we have investigated the updated stability criterion and improved causality criterion for the $3+1$-dimensional relativistic system across all IFR. Notably, our findings indicate that the previously widely used causality criterion (5) needs to be substituted with the improved asymptotic criterion (8) or (9). Based on Theorems 1–3, we reveal the underlying connection between stability and causality in linear mode analysis. Stability in all IFR implies causality, while causality alone does not necessarily require stability. Furthermore, if a system is stable and causal in one IFR, stability and causality holds in all IFR. The findings alleviate the challenge of linear mode analysis, which involves verifying the stability and causality conditions across various frames. As an application, we also study the linear stability and causality of the $3+1$-dimensional isotropic systems and derive the new criterion (14)–(16) that are necessary for stability and sufficient for causality in all IFR. Finally, it is important to emphasize that our theorems are model-independent and can be applied to other relativistic systems beyond relativistic hydrodynamics.

Note added.—Recently, we were informed of Ref. [31] which works on a similar topic and appeared on arXiv on the same day.

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[28] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevD.109.L031504 for more details. In part I, we discuss the causality, stability and related topics studied in other methods. In part II, we provide a detailed derivation of the inequality (11). In part III, we present three examples illustrating the application of our newly established theorems.

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