Letter

Darboux-Kadomtsev-Petviashvili system as an integrable Chern-Simons multiform theory in infinite dimensional space

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(Received 25 May 2023; accepted 13 December 2023; published 9 January 2024)

The Darboux-Kadomtsev-Petviashvili system is a universal three-dimensional integrable field theory, which is a generating system for the entire Kadomtsev-Petviashvili hierarchy, and at the same time generalizes the Darboux system, describing orthogonal curvilinear coordinates. In this paper, we establish a hierarchy of Lagrangian multiforms for the Darboux-Kadomtsev-Petviashvili system, derived from a hierarchy of Chern-Simons actions in an infinite-dimensional space of Miwa variables. This provides an integrable variational description of this multidimensionally consistent field theory embedded in infinite dimensional space.

DOI: 10.1103/PhysRevD.109.L021701

Introduction. Conformal field theory on the world sheet and associated aspects of integrability play an essential role in modern physics as in, for example, string theory, AdS/CFT and other dualities, the computation of scattering amplitudes, cf. [1], and in critical phenomena, [2]. In this context, attempts to understand integrable lattice models and field theories in terms of gauge theories in three and four dimension, cf. [3,4], has led to a body of work by Costello, Yamazaki, and Witten [5,6] to consider four-dimensional Chern-Simons (4DCS) theory with two of the real dimensions reserved for a complex spectral parameter, in addition to a $(1+1)$-dimensional space-time, and where in the quantum theory the Yang-Baxter equation appears naturally in a well-defined perturbation expansion. The lattice represents the world lines of particles whose Wilson loops are computed by knot invariants associated with the relevant R-matrix. The 4D Chern-Simons action can thus be viewed as a generating object for integrable field theories that are instrumental in computing correlation functions beyond perturbation theory.

To move to a nonperturbative description of the physics on the world sheet in full, it is important to find Lagrangians for nonlinear integrable field theories in $(2+1)$-dimensions. We propose in this paper that a prime candidate for such a theory is the Darboux-Kadomtsev-Petviashvili system (and its matrix variants), which is a "compounding" of the famous Kadomtsev-Petviashvili (KP) hierarchy, [7], in terms of appropriate variables (Miwa variables, [8]) which depend on parameters which are in fact the lattice parameters of an associated lattice model. The Darboux-KP system is an integrable three-dimensional system containing many integrable two-dimensional field theories as special (dimensional) reductions, [9]. Furthermore, the integrability of the model is here made manifest through a novel variational formalism introduced in [10] called "Lagrangian multiform theory," which incorporates the key integrability characteristics into a variational framework.

The key integrability aspect that Lagrangian multiforms captures is multidimensional consistency (MDC), which means the coexistence of multiple compatible equations on one and the same dependent variable, into a single variational framework, where the Lagrangians are differential or difference d-forms of nontrivial codimension, and where the action is a functional of not only the variational fields, but also of the surfaces over which the d-form is integrated in a "multitime" space of independent variables of arbitrary dimension. For this to work, the Lagrangian components of the d-form (d being related to the dimensionality of the equations in the system) must be very special, in fact "integrable," and they emerge as solutions of a generalized Euler-Lagrange (EL) system of equations, cf. e.g. [11,12]. These EL equations are obtained by varying both the fields and the d-dimensional surface of integration. It was shown in subsequent work that many known integrable systems (both continuous and discrete) admit a Lagrangian multiform structure, (cf. [13] and references therein).

In [14], a Lagrangian 3-form action was presented for a generalized Darboux system (a system [15] originally describing conjugate nets of orthogonal curvilinear

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coordinates [16]) embedded in an infinite-dimensional space of Miwa variables, [8]. It was shown in [19] that the Darboux system encodes the entire KP hierarchy. While in [20] a Lagrangian 3-form structure was presented for the KP hierarchy in terms of the well-known description using pseudodifferential operators, the Lagrangian 3-form structure given in [14] for what we call the Darboux-KP system, and which is given by the system of partial differential equations (PDEs),

$$\frac{\partial B_{p_i,p_j}}{\partial \xi_{p_k}} = B_{p_i,p_k}B_{p_k,p_j}, \quad \text{if } p_i, p_j, p_k \text{ are different,} \tag{1}$$

is remarkably more concise, and covariant. For any choice of three distinct (possibly complex valued) parameters p_i, p_j, p_k, the Darboux-KP system gives rise to a closed-form system of six three-dimensional PDEs, which however is multidimensionally consistent, by virtue of the fact that the PDE system is compatible for any choice of multiple distinct parameters. Furthermore, the Darboux-KP system represents the entire KP hierarchy, where the ξ_p are the actual Miwa variables, which are parameter-dependent (cf. also [8,21]) and compouded in the sense that the vector fields ∂_{ξ_p} can be expanded as an infinite series of derivatives with respect to the KP time variables t_j (see below).

The Lagrangian 3-form action for the Darboux-KP system takes the form

$$\mathsf{S}[\mathbf{B}(\boldsymbol{\xi});\mathcal{V}] = \int_{\mathcal{V}} \mathsf{L} \tag{2}$$

which is to be considered as a functional not only of the field variables $\mathbf{B}$, but also of the three-dimensional hypersurfaces $\mathcal{V}$ embedded in an infinite-dimensional space of independent variables $\boldsymbol{\xi} = \{\xi_{p_i}\}_{i\in I}$ (where I is some well-chosen index set of cardinality at least 3, but later on we will take $I = \mathbb{Z}$). In the present context the variables ξ_p are labeled by a set of continuous parameters $\{\boldsymbol{p} = (p_j)_{j\in I}\}$, which themselves can be complex valued, and they correspond to the Miwa variables in the KP theory, cf. [8]. The Lagrangian (2) L is here a differential 3-form

$$\mathsf{L} = \sum_{i,j,k\in I} \mathcal{L}_{p_i,p_j,p_k} \mathrm{d}\xi_{p_i} \wedge \mathrm{d}\xi_{p_j} \wedge \mathrm{d}\xi_{p_k}, \tag{3}$$

with Lagrangian components $\mathcal{L}_{p_i,p_j,p_k}$ given by

$$\begin{aligned}\mathcal{L}_{pqr} = {} & \frac{1}{2}(B_{rq}\partial_{\xi_p}B_{qr} - B_{qr}\partial_{\xi_p}B_{rq}) \\ & + \frac{1}{2}(B_{qp}\partial_{\xi_r}B_{pq} - B_{pq}\partial_{\xi_r}B_{qp}) \\ & + \frac{1}{2}(B_{pr}\partial_{\xi_q}B_{rp} - B_{rp}\partial_{\xi_q}B_{pr}) \\ & + B_{rp}B_{pq}B_{qr} - B_{rq}B_{qp}B_{pr}. \end{aligned} \tag{4}$$

The outstanding feature, proven in [14], is that the differential of the Lagrangian 3-form

$$\mathrm{d}\mathsf{L} = \sum_{i,j,k,l\in I} \mathcal{A}_{p_i,p_j,p_k,p_l} \mathrm{d}\xi_{p_i} \wedge \mathrm{d}\xi_{p_j} \wedge \mathrm{d}\xi_{p_k} \wedge \mathrm{d}\xi_{p_l},$$

has a "double zero" (to be explained below) on solutions of the Darboux-KP system (1). As a consequence, the set of generalized EL equations arising from the criticality condition $\delta\mathrm{d}\mathsf{L} = 0$ (in the language of the corresponding variational bicomplex, [22]) produces the whole multidimensionally consistent set of Darboux-KP equations from a single variational principle.

That the set of Darboux-KP equations in terms of Miwa variables generates the entire KP hierarchy, cf. [14,19], can be asserted from the connection between the fields B_{pq}, which can be expressed in terms of the KP τ-function

$$\tau(t_1, t_2, \ldots) = \tau(\{\xi_{p_i}\}_{i\in I}; \{n_{p_i}\}_{i\in I}),$$

in an extended space with discrete variables n_{p_i} associated with the parameters p_i, $i \in I$, as

$$B_{p_ip_j} = \frac{X_{p_ip_j}\tau}{(p_i - p_j)\tau}.$$

Here X_{pq} is the $\mathfrak{gl}(\infty)$ vertex operator, [23], which can be expressed as

$$X_{p_ip_j} = \left[\frac{\prod_{l\in I, l\neq j}(p_l - p_j)^{n_{p_l}}}{\prod_{l\in I, l\neq i}(p_l - p_i)^{n_{p_l}}}\right] T_{p_j}T_{p_i}^{-1},$$

in terms of discrete variables, where T_{p_i} and T_{p_j} are the elementary lattice shift operators in the variables n_{p_i} and n_{p_j} respectively. Alternatively, we have

$$\begin{aligned} X_{p_ip_j} = {} & \exp\left(\sum_{l\in I, l\neq i} \frac{\xi_{p_l}}{p_i - p_l} + \sum_{l\in I, l\neq j} \frac{\xi_{p_l}}{p_l - p_j}\right) \\ & \times \exp\left(\sum_{l=1}^{\infty} \frac{1}{l}\left(\frac{1}{p_i^l} - \frac{1}{p_j^l}\right)\frac{\partial}{\partial t_l}\right) \end{aligned} \tag{5}$$

in terms of the continuous variables t_j of the KP hierarchy, where the τ-function obeys the set of Hirota bilinear equations, cf. [7,24]. Here the Miwa variables, ξ_p, are related to the higher time variables, t_j, in the conventional way of writing the KP hierarchy, by

$$\frac{\partial \tau}{\partial \xi_p} = \sum_{j=1}^{\infty} \frac{1}{p^{j+1}}\frac{\partial \tau}{\partial t_j}, \qquad T_{p_i}\tau = \tau\left(\left\{t_l - \frac{1}{l p_i^l}\right\}_{l\in\mathbb{Z}^+}\right).$$

Furthermore, there is the relation

$$\frac{\partial \tau}{\partial \xi_p} = \left(T_p^{-1} \frac{d}{dp} T_p \right) \tau$$

connecting the two realizations.

Higher-dimensional Chern-Simons actions. The aim of this paper is to show that the Lagrangian (4) for the Darboux-KP system arises from an infinite-dimensional matrix-valued Chern-Simons action [25,26]. This would seem to place the notion of integrability in the broad domain of topological field theories, but only in appearance, as this connection with Chern-Simons theory does not make the Darboux-KP system topologically invariant. A connection between the Chern-Simons action and integrable systems, was demonstrated in [27], cf. also [28], where it was shown that for specific choices of (finite-dimensional) gauge groups the CS action leads to integrable models of Davey-Stewartson and Ishimori type in $2+1$ dimensions. However, this connection does not in itself reveal the integrability of those models.

A decade earlier, a connection between $1+1$-dimensional integrable hierarchies and topological field theories (of Wess-Zumino-Witten type) was presented, [21]. Here the coupling constant in front of the topological term in the action contains essentially the spectral parameter. Similar coupling constants reappeared in [6], in the context of the above-mentioned 4D Chern-Simons theory. It would be interesting to determine how these various connections between integrability and topological field theory are interrelated, and moreover how the action incorporates the specifics of certain solution classes.

The conventional Chern-Simons theory over a Lie algebra $\mathfrak{g}$, with associated gauge group G, involves a $\mathfrak{g}$-valued gauge connection 1-form $\boldsymbol{A}$, and the associated curvature 2-form, $\boldsymbol{F} = \mathrm{d}\boldsymbol{A} + \boldsymbol{A} \wedge \boldsymbol{A}$. Here we consider matrix-valued gauge fields only, where the gauge groups of interest are the general linear groups, $GL(n, \mathbb{R})$, in whose Lie algebra, $\mathfrak{gl}(n, \mathbb{R})$, we consider the matrix trace Tr, and where the wedge product $\boldsymbol{A} \wedge \boldsymbol{A}$ is evaluated via the matrix product, and not via the Lie bracket.

The standard CS Lagrangians in dimensions 3 and 5 read

$$CS_3 = \mathrm{Tr}\left(\boldsymbol{A} \wedge \mathrm{d}\boldsymbol{A} + \frac{2}{3} \boldsymbol{A} \wedge \boldsymbol{A} \wedge \boldsymbol{A} \right), \tag{6a}$$

$$CS_5 = \mathrm{Tr}\left(\boldsymbol{A} \wedge \mathrm{d}\boldsymbol{A} \wedge \mathrm{d}\boldsymbol{A} + \frac{3}{2} \boldsymbol{A} \wedge \boldsymbol{A} \wedge \boldsymbol{A} \wedge \mathrm{d}\boldsymbol{A} + \frac{3}{5} \boldsymbol{A} \wedge \boldsymbol{A} \wedge \boldsymbol{A} \wedge \boldsymbol{A} \wedge \boldsymbol{A} \right), \tag{6b}$$

and they are defined through the property that

$$\mathrm{d}CS_3 = \mathrm{Tr}(\boldsymbol{F} \wedge \boldsymbol{F}), \tag{7a}$$

$$\mathrm{d}CS_5 = \mathrm{Tr}(\boldsymbol{F} \wedge \boldsymbol{F} \wedge \boldsymbol{F}). \tag{7b}$$

In hindsight, coming from the variational principle for the multiform theory, the relations (7) are nothing but the double, respectively triple zero conditions for the Euler-Lagrange equations $\boldsymbol{F} = 0$. The general higher form of the CS Lagrangians are given by the formula which is a particular case of [[25], Lemma 3.3],

$$CS_{2n+1} = (n+1) \int_0^1 \mathrm{d}\lambda \mathrm{Tr}(\boldsymbol{A} \wedge \boldsymbol{F}_\lambda^{\wedge n}),$$
$$\text{where } \boldsymbol{F}_\lambda \coloneqq \lambda \mathrm{d}\boldsymbol{A} + \lambda^2 \boldsymbol{A} \wedge \boldsymbol{A}, \tag{8}$$

see also [29]. They obey [30]

$$\mathrm{d}CS_{2n+1} = \mathrm{Tr}(\boldsymbol{F}^{\wedge(n+1)}), \tag{9}$$

where the latter expressions are $2n+2$ forms, whose variational derivative (using the multiform EL equations in the language of the variational bicomplex), are given by

$$\delta \mathrm{d}CS_{2n+1} = (n+1)\mathrm{Tr}(\boldsymbol{F}^{\wedge(n)} \wedge \delta \boldsymbol{F}),$$

which vanishes whenever $\boldsymbol{F} = 0$. The latter would correspond to the usual variational equations in conventional CS theory, cf. e.g. [31], but as we will see later, in our setting $\boldsymbol{F} = 0$ is too stringent a condition. In fact, as we will work later with a restricted set of fields, the corresponding equations of motion are slightly weaker than the standard zero-curvature condition.

In all these CS theories, we fix the dimensionality of the $(2n+1)$-dimensional manifold $\mathcal{M}_{2n+1}$ over which the Lagrangians are integrated through

$$\mathcal{A}_{2n+1}^{CS} = \int_{\mathcal{M}_{2n+1}} CS_{2n+1}. \tag{10}$$

The conventional CS theory, summarized above, is by itself not an integrable theory. In fact, the various CS actions (8) are separate theories living each in a different dimension. We aim to create a universal and integrable CS theory which lives in an *a priori* infinite-dimensional space, and where all higher CS actions coexist. For this we need a gauge connection which is compatible in all odd dimensions, possessing common solutions. This will lead necessarily to a MDC system, and the corresponding CS theory must have a Lagrangian multiform structure.

Higher Lagrangian multiforms from CS Lagrangians. In order to connect the CS theory with Lagrangian multiforms we need to specify the gauge field $\boldsymbol{A}$ in such a way that we recover the Lagrangian 3-form (3) with (4). It turns out that the following special choice achieves this [32]

$$\boldsymbol{B} = \sum_{k,l \in \mathbb{Z}} B_{kl} \mathrm{d}\xi_k E_{k,l}. \tag{11}$$

Here $B_{kl} = B_{kl}((\xi_i)_{i\in\mathbb{Z}})$, and we have simplified the notation by replacing ξ_{p_i} simply by ξ_i, and B_{p_k,p_l} simply by B_{kl}, (we will come back to the role of the parameters p_i later) [33]. Each E_{kl} is an element of the space of all matrices $(M_{ab})_{a,b\in\mathbb{Z}}$, indexed by integers, with $(E_{kl})_{ab} = \delta_{k,a}\delta_{l,b}$, such that $E_{k,l}E_{m,n} = \delta_{l,m}E_{k,n}$ and $\mathrm{Tr}(E_{k,l}) = \delta_{k,l}$. The sum in (11) is infinite; however it can be understood in the "completed graded sense," for the $\mathrm{d}\xi_k E_{k,l}$ are linearly independent, and hence (11) never leads to infinite sums of real numbers.

Computing the curvature $\boldsymbol{F}$ associated with this gauge field $\boldsymbol{B}$, we get

$$\boldsymbol{F}_B = \sum_{j,k,l\in\mathbb{Z}} (\partial_{\xi_j}B_{kl} - B_{kj}B_{jl})\mathrm{d}\xi_j \wedge \mathrm{d}\xi_k E_{k,l}, \qquad (12)$$

where the coefficients $\partial_{\xi_j}B_{kl} - B_{kj}B_{jl}$ are exactly of the Darboux form of (1). With the choice (11) of gauge field we can now compute the CS action, leading to the Lagrangian 3-form (3) together with (4). The Lagrangian 3-form of the Darboux-KP system arises from the $CS_3(\boldsymbol{B})$ by direct computation, namely

$$\mathsf{L}^{(3)} = CS_3(\boldsymbol{B}) = \sum_{i,j,k\in\mathbb{Z}} \mathcal{L}^{(3)}_{ijk}\mathrm{d}\xi_i \wedge \mathrm{d}\xi_j \wedge \mathrm{d}\xi_k, \qquad (13)$$

in which the coefficients $\mathcal{L}^{(3)}_{ijk} = \frac{2}{3!}\mathcal{L}_{p_ip_jp_k}$ of (4), including a prefactor for convenience.

Calculating the form of the right-hand side in (7a) we find

$$\begin{aligned}\mathrm{Tr}(\boldsymbol{F}_B \wedge \boldsymbol{F}_B) = \sum_{\substack{j,k,l,m\in\mathbb{Z}\\ \text{all indices different}}} &(\partial_{\xi_j}B_{kl} - B_{kj}B_{jl}) \\ &\times (\partial_{\xi_m}B_{lk} - B_{lm}B_{mk})\mathrm{d}\xi_j \\ &\wedge \mathrm{d}\xi_k \wedge \mathrm{d}\xi_m \wedge \mathrm{d}\xi_l. \end{aligned} \qquad (14)$$

In particular, this implies that $\mathrm{Tr}(\boldsymbol{F}_B \wedge \boldsymbol{F}_B)$, has a double zero on the solutions of the generalized Darboux system in (1), which implies that the latter arises as the EL equations of the multiform action (2) with (3), as was asserted in [14].

Remark. While $\mathrm{Tr}(\boldsymbol{F}_B \wedge \boldsymbol{F}_B)$ indeed has a double zero on the solutions of (1), the form $\boldsymbol{F}_B \wedge \boldsymbol{F}_B$ does not necessarily have such a double zero when (1) holds, as this would require that the Darboux system also extends to the case that all three labeled variables are no longer distinct.

Inspired by this Chern-Simons origin of the Darboux-KP multiform action, it is natural also to consider the higher-order CS actions. Analogous to CS_3 providing a system of first order in the derivatives, the higher Chern-Simons actions yield higher order Lagrangian multiforms, which have not yet been considered in the literature. These higher multiforms for the Darboux-KP system are directly obtained from the higher CS actions by using (11). Thus, we obtain the Lagrangian 5-form

$$\begin{aligned}\mathsf{L}^{(5)} &= \mathrm{Tr}\Big(\boldsymbol{B} \wedge \mathrm{d}\boldsymbol{B} \wedge \mathrm{d}\boldsymbol{B} \\ &\quad + \frac{3}{2}\boldsymbol{B} \wedge \boldsymbol{B} \wedge \boldsymbol{B} \wedge \mathrm{d}\boldsymbol{B} + \frac{3}{5}\boldsymbol{B} \wedge \boldsymbol{B} \wedge \boldsymbol{B} \wedge \boldsymbol{B} \wedge \boldsymbol{B}\Big), \\ &= \sum_{j,k,l,m,n\in\mathbb{Z}} \mathcal{L}^{(5)}_{jklmn}\mathrm{d}\xi_j \wedge \mathrm{d}\xi_k \wedge \mathrm{d}\xi_l \wedge \mathrm{d}\xi_m \wedge \mathrm{d}\xi_n, \end{aligned} \qquad (15)$$

with

$$\begin{aligned}\mathcal{L}^{(5)}_{jklmn} = \frac{1}{5!} \sum_{j',k',l',m',n'\in\{j,k,l,m,n\}} &\varepsilon_{j'k'l'm'n'} \\ \times \Big[&B_{p_{j'},p_{l'}}(\partial_{\xi_{p_{k'}}}B_{p_{l'},p_{n'}})(\partial_{\xi_{p_{m'}}}B_{p_{n'},p_{j'}}) \\ &+ \frac{3}{2}B_{p_{j'},p_{k'}}B_{p_{k'},p_{l'}}B_{p_{l'},p_{n'}}(\partial_{\xi_{p_{m'}}}B_{p_{n'},p_{j'}}) \\ &+ \frac{3}{5}B_{p_{j'},p_{k'}}B_{p_{k'},p_{l'}}B_{p_{l'},p_{m'}}B_{p_{m'},p_{n'}}B_{p_{n'},p_{j'}}\Big], \end{aligned} \qquad (16)$$

where $\varepsilon_{\mathrm{jklmn}}$ is the five-dimensional Levi-Civita symbol. As a consequence of the construction, the Lagrangian 5-form has the property that

$$\mathrm{d}\mathsf{L}^{(5)} = \mathrm{Tr}(\boldsymbol{F}_B \wedge \boldsymbol{F}_B \wedge \boldsymbol{F}_B).$$

This again leads to the fact that $\mathrm{d}\mathsf{L}^{(5)}$ has a triple zero on the solutions of the same Darboux system (1), viewed as a multidimensionally consistent system in higher dimensions. [Note again that $\boldsymbol{F}_B \wedge \boldsymbol{F}_B \wedge \boldsymbol{F}_B$ does not necessarily have a triple zero on the solutions of (1).]

Similarly, all higher Lagrangian multiforms $\mathsf{L}^{(2n+1)}$ of odd degree can be constructed in the same way, using the formula (8). In fact, the differential

$$\begin{aligned}\mathrm{d}\mathsf{L}^{(2n+1)} = \mathrm{Tr}(\boldsymbol{F}_B^{\wedge n}) = \sum_{j_1,l_1,j_2,l_2,\ldots,j_n,l_n\in\mathbb{Z}} &(\partial_{\xi_{j_1}}B_{l_nl_1} - B_{l_nj_1}B_{j_1l_1}) \\ &\times (\partial_{\xi_{j_2}}B_{l_1l_2} - B_{l_1j_2}B_{j_2l_2})\ldots \\ &\ldots(\partial_{\xi_{j_n}}B_{l_{(n-1)}l_n} - B_{l_{(n-1)}j_n}B_{j_nl_l}) \\ &\times \mathrm{d}\xi_{j_1} \wedge \mathrm{d}\xi_{l_n} \wedge \mathrm{d}\xi_{j_2} \wedge \mathrm{d}\xi_{l_1} \wedge \ldots \wedge \mathrm{d}\xi_{j_n} \wedge \mathrm{d}\xi_{l_{(n-1)}}, \end{aligned} \qquad (17)$$

has an n-fold zero on the solutions of the generalized Darboux-KP system, by virtue of the fact that none of the indices in the coefficients of the elementary forms on the rhs. coincide (and hence the sum above can be restricted to the case when all indices are different). Essential is that due to the MDC property of the Darboux-KP system, all these higher order CS actions are compatible amongst each other in the sense that the resulting multiform EL equations arising from each degree n are consistent. This allows us to consider all these higher-order CS theories together and create a universal CS action in infinite-dimensional space generating the whole series.

The generating Chern-Simons multiform. Having established a hierarchy of Lagrangian multiforms of odd degree from the higher CS forms for the particular gauge field (11), we note that they are all compatible in the sense that the relevant EL equations all are solved by the same multidimensionally consistent Darboux system (1). In contrast to the usual CS theory (see Sec. 1.3 in Ref. [31]) a critical point of the action can arise even when $\boldsymbol{F}_B \neq 0$. This is because we are not considering gauge fields in the general position, and variations $\delta\boldsymbol{B} = t\boldsymbol{\eta}$ are consequently restricted, i.e. $\boldsymbol{\eta} = \sum_{p,q} \eta_{p,q} \mathrm{d}\xi_p E_{pq}$. Thus, for the $n = 1$ CS action, assuming that integration manifold $\mathcal{V}$ is closed, we obtain by applying Stokes theorem

$$\frac{d}{dt}\int_{\mathcal{V}} CS_3(\boldsymbol{B} + t\boldsymbol{\eta})_{t=0} = 2\int_{\mathcal{V}} \sum_{a,b,c} (\partial_{\xi_a} B_{bc} - B_{ba}B_{ac})\eta_{cb}\mathrm{d}\xi_a \wedge \mathrm{d}\xi_b \wedge \mathrm{d}\xi_c. \tag{18}$$

Hence, a critical point of the action exists provided $\partial_{\xi_a} B_{bc} - B_{ba}B_{ac} = 0$ for each triple of different indices a, b, c, but there is no critical-point condition when any of the indices coincide.

Having a restricted set of fields is part of the reason that our model is not topologically invariant. Indeed, the fields of the form (11) are not invariant under changes of coordinates, and neither is the critical-point condition. Instead there is an infinite dimensional group of discrete symmetries arising from the Bäcklund-Moutard-type transformations governing the geometry of the generalized Darboux system, cf. [14] and references therein.

The MDC integrability property of the Darboux-KP system allows us to create a generating action for the entire sequence of higher CS theories, constituting a "graded" Chern-Simons theory in the infinite dimensional space of Miwa variables. Thus, for the field $\boldsymbol{B}$ of (11), we can compute at each dimension $2n + 1$ the action,

$$\int_{\mathcal{V}_{2n+1}} CS_{2n+1}(\boldsymbol{B}),$$

for each $(2n + 1)$-dimensional hypersurface $\mathcal{V}_{2n+1}$ embedded in $\mathbb{R}^{\mathbb{Z}}$, and consider the formal sum, in powers of a dummy parameter $\hbar$,

$$\mathcal{S}_{\hbar}^{(\infty)}[\boldsymbol{B}; \mathcal{V}_\infty] = \sum_{n=1}^{\infty} \frac{\hbar^n}{n+1} \int_{\mathcal{V}_{2n+1}} \mathsf{L}^{(2n+1)},$$

which we could call the *generating Chern-Simons multiform* integrated over the disjoint union $\mathcal{V}_\infty = \coprod_{n=1}^{\infty} \mathcal{V}_{2n+1}$ of submanifolds. This novel object makes sense in view of the MDC property of the Darboux-KP system. We expect it may play a role in the quantization of the KP system, which, from different perspective, can be studied from the vantage point of the representation theory of toroidal algebras along the line of e.g. [34].

Conclusions. For clarity we focused on the scalar KP case, but it is straightforward to extend the results to the matrix KP case by choosing the coefficients of the gauge field (11) to take values in a matrix algebra, $\mathrm{Mat}_N(\mathbb{C})$ and by taking in the Lagrangian (13) not only the trace in the space $\mathrm{Mat}_{\mathbb{Z}}$ of generators E_{ij} but also the trace in $\mathrm{Mat}_N(\mathbb{C})$. We thus recover the matrix Lagrangian of [14,35] up to absorbing the constant matrices J by a point transformation.

Lagrangian multiform theory offers a new perspective on the least-action principle in physics, as it provides as much a way to derive the Lagrangians themselves (which by necessity have to correspond to integrable systems) from first principles as the equations of motion from an extended set of Euler-Lagrange equations. This offers a shift of goal posts in setting up a fundamental theory of physics. Furthermore, the Darboux-KP system offers a way to derive integrable hierarchies of associated $(1 + 1)$-dimensional field theories by expansion of Miwa variables and dimensional reduction. Finally, we expect that the quantization program of the KP system through CS actions, will reveal a perspective on $(2 + 1)$-dimensional quantum field theories and their symmetries on the world sheet.

Acknowledgments. J. F. M. was supported by the EPSRC Programme Grant No. EP/W007509/1 (Combinatorial Representation Theory: Discovering the Interfaces of Algebra with Geometry and Topology), and thanks Ben Lambert for some discussions. F. W. N. was supported by the EPSRC Grant No. EP/W007290/1 (Elliptic discrete integrable systems and bi-elliptic addition formulae), when the work was commenced. D. R. was supported by the Akroyd and Brown Scholarship and the School of Mathematics, University of Leeds.

All authors have contributed in equal measure to the conceptualization, computation and contextualization of the paper. The order of authors is alphabetical.

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