Holographic description for correlation functions

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We study general correlation functions of various quantum field theories in the holographic setup. Following the holographic proposal, we investigate correlation functions via a geodesic length connecting boundary operators. We show that this holographic description can reproduce the known two- and three-point functions of conformal field theory. Using this holographic method, we further study general two-point functions of a two-dimensional thermal conformal field theory (CFT) and of a scalar field theory living in a de Sitter or anti–de Sitter space. Due to the nontrivial thermal or curvature effect, the two-point functions in an IR limit show different scaling behaviors from those of the UV CFT. We study such nontrivial IR scaling behaviors by applying the holographic method.

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I. INTRODUCTION

After the AdS/CFT correspondence proposal, there were many attempts to account for strongly interacting systems on the holographic dual gravity side. The AdS/CFT conjecture allows us to relate a $(d+1)$-dimensional classical gravity theory to a d-dimensional nongravitational conformal field theory (CFT) [1–4]. Moreover, the AdS/CFT correspondence proposed that the gravity theory can give us information about the nonperturbative features of a quantum field theory (QFT). In this case, the extra dimension in the bulk is identified with the energy scale observing the dual QFT. Therefore, the gravity theory maps to the renormalization group (RG) flow of the dual QFT. From the RG flow point of view, non-Abelian gauge theories are weakly interacting at a UV fixed point and have a conformal symmetry. In the IR limit, they are strongly interacting and reveal various nonperturbative features. In a situation without a well-established mathematical method describing the nonperturbative RG flow, the AdS/CFT correspondence can provide a new chance to look into the nonperturbative IR physics.

In general, quantum correlation plays a crucial role in understanding physics in both the weak and strong coupling limits. Despite its importance, it is still difficult to calculate correlation functions nonperturbatively. On the QFT side, computing nonperturbative correlation functions is a difficult task because all loop quantum corrections must be taken into account. Although the perturbative method is applicable to UV theories, it is no longer valid in the IR region where a coupling constant becomes strong. One way to understand such nontrivial IR physics is to take into account the nonperturbative RG flow. The AdS/CFT correspondence may shed light on constructing the nonperturbative RG flow and understanding the IR physics correctly. Therefore, it would be interesting to investigate how to evaluate the nonperturbative correlation functions and their RG flow in the holographic dual gravity. The main goal of this work is to study correlation functions of various QFTs defined in nontrivial backgrounds, like finite temperatures and curved spacetimes.

One of the interesting features of the AdS/CFT correspondence is that many important physical quantities, like the $q\bar{q}$ potential [5–7] and entanglement entropy [8–18], can be realized as geometrical objects on the dual gravity side. Similarly, it was also conjectured that a two-point function maps to a geodesic curve connecting two boundary operators [19–23]

$$\langle O(\tau_1,\vec{x}_1)O(\tau_2,\vec{x}_2)\rangle = e^{-\Delta L(\tau_1,\vec{x}_1;\tau_2,\vec{x}_2)/R}, \qquad (1.1)$$

where Δ is the conformal dimension of a local operator $O(\tau,\vec{x})$ and $L(\tau_1,\vec{x}_1;\tau_2,\vec{x}_2)$ indicates a geodesic length connecting two local operators. This holographic technique has been widely employed to understand the correlation functions of various dual CFTs [19–40]. When the dual QFT possesses Lorentz symmetry, one can easily derive

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general correlation functions depending on temporal and spatial coordinates simultaneously. For example, if one solves the equation of motion of the Green function on the QFT side, one can determine a QFT's two-point function directly. However, if the QFT is interacting, one has to take into account loop quantum corrections. In the holographic setup, such loop quantum corrections are encoded in the classical gravity. When the Lorentz symmetry of the dual QFT is preserved, a time-independent spatial correlation function is readily generalized to a general correlator by applying the Lorentz transformation. However, when considering QFTs with broken Lorentz symmetry, obtaining general correlation functions is not straightforward even if spatial correlators are known. QFTs with broken Lorentz symmetry often arise in thermal systems or curved spacetimes. In this work, we develop a systematic holographic method to obtain general correlators even when the Lorentz symmetry is broken. After validating the consistency of our method by comparing with the known results of various CFTs [24,25,32,33], we apply our method to curved spaces, like de Sitter (dS) and anti–de Sitter (AdS) spaces, and evaluate general two-point functions in curved spaces. The holographic method developed here can be further applied to QFTs defined in other nontrivial curved spaces.

Computing correlation functions in thermal systems [24–31] is an interesting research area. For thermal systems, although the finite thermal corrections are negligible in the UV region, they can give rise to a significant effect on IR physics. Due to this, IR physics can show a new physics law like a thermodynamic relation, which cannot be explained by the fundamental UV theory. In order to understand such new macroscopic orders, it would be important to know how the correlations of operators depend on the energy scale. By applying the holographic technique, we calculate a general two-point function of a thermal two-dimensional CFT [32–40]. We show that, although thermal CFTs are conformal at the UV fixed point, the screening effect caused by the background thermal fluctuations leads to exponential suppression of the two-point function, which can be reinterpreted as an effective mass proportional to temperature. We also check that the results derived here are consistent with the previously known ones obtained in a different way [24,25].

Since the holographic method developed here is general and systematic, we can apply it to more nontrivial bulk spaces that map to the nontrivial background of the dual QFT. In this work, we study correlation functions in an eternally expanding universe. The dS space describing an eternal inflation is an important background geometry to understand the birth and evolution of our Universe. The correlation functions on that background also provide important information, like the power spectrum and non-Gaussianity, to understand the history of our Universe. Many works studied the correlation function of massive and massless scalar fields in a dS background [41–50]. In this work, we look into the correlation function of a dS space by applying the holographic method developed here. To do so, we take into account an $(d+1)$-dimensional AdS space whose boundary is given by a d-dimensional dS space. After calculating the geodesic length connecting two operators living in the dS boundary, we derive the general two-point function in an eternally expanding universe. We show that this holographic result is coincident with the result of the free scalar field theory defined in a d-dimensional dS space. This work is further generalized to the correlation function in a d-dimensional AdS space.

The rest of this paper is organized as follows. According to the AdS/CFT correspondence, in Sec. II we discuss how to systematically calculate general correlation functions at both zero and finite temperature. After deriving the well-known CFT's two-point functions from the QFT point of view in Sec. II A, we reproduce the same results in the holographic setup without using the Lorentz transformation in Sec. II B. We apply this holographic method to a three-point function in Sec. II C and to a thermal two-point function in Sec. II D. We further study general two-point functions in d-dimensional curved spaces, specifically a dS space in Sec. III A and an AdS space in Sec. III B. Finally, we conclude this work with some remarks in Sec. IV.

II. HOLOGRAPHIC DESCRIPTION FOR CORRELATION FUNCTIONS

For CFTs with a Lorentzian signature, the conformal symmetry uniquely determines two- and three-point correlation functions up to overall constants. Denoting the distance between two operators as $|r_1 - r_2| = \sqrt{-|t_1 - t_2|^2 + |\vec{x}_1 - \vec{x}_2|^2}$, two- and three-point functions are given by

$$\langle O(t_1, \vec{r}_1) O(t_2, \vec{r}_2) \rangle = \frac{N}{|r_1 - r_2|^{2\Delta}}, \tag{2.1}$$

$$\langle O(t_1, r_1) O(t_2, r_2) O(t_3, r_3) \rangle = \frac{C_{123}}{|r_1 - r_2|^{\Delta_1+\Delta_2-\Delta_3} |r_2 - r_3|^{\Delta_2+\Delta_3-\Delta_1} |r_3 - r_1|^{\Delta_3+\Delta_1-\Delta_2}}, \tag{2.2}$$

where $\Delta = \Delta_1 = \Delta_2$ and Δ_i means the conformal dimension of the ith operator, $O(t_i, \vec{x}_i)$. Normalizing operators, without loss of generality, allows us to set $N = 1$. The structure constant C_{123} for the three-point function corresponds to the operator product expansion coefficient, which cannot be fixed by the conformal symmetry. According to the AdS/CFT correspondence, a d-dimensional CFT has a one-to-one map to a gravity defined on a $(d+1)$-dimensional AdS space. Therefore, it would be interesting to reproduce the above general correlators in the holographic setup. This holographic study of correlators may be helpful to understand the microscopic and macroscopic correlation for interacting QFTs nonperturbatively.

A. Two-point functions of d-dimensional CFTs

For later comparison with correlators of CFTs at finite temperature or in curved spaces, we first discuss how one can derive the known two-point function in a d-dimensional Euclidean CFT. The metric of a d-dimensional Euclidean space is given by

$$ds^2 = \delta_{\mu\nu} dx^\mu dx^\nu = d\tau^2 + d\vec{x} \cdot d\vec{x}. \tag{2.3}$$

On this flat background geometry, we take into account a massless scalar field theory,

$$S = \frac{1}{2} \int d^d x (\partial\phi)^2. \tag{2.4}$$

The free scalar field theory is conformal and the conformal dimension of ϕ is given by $\Delta_\phi = (d-2)/2$. In this case, the two-point function of ϕ is determined by the following equation:

$$-\partial^\mu \partial_\mu \langle \phi(\tau_1, \vec{x}_1) \phi(\tau_2, \vec{x}_2) \rangle = \delta^{(d)}(|r_1 - r_2|), \tag{2.5}$$

where $|r_1 - r_2| = \sqrt{|\tau_1 - \tau_2|^2 + |\vec{x}_1 - \vec{x}_2|^2}$.

A general solution satisfying this equation is given by

$$\langle \phi(\tau_1, \vec{x}_1) \phi(\tau_2, \vec{x}_2) \rangle \sim \frac{1}{|r_1 - r_2|^{2\Delta_\phi}}. \tag{2.6}$$

If we further consider an operator $O = \phi^n$, the Wick contraction allows the general two-point function to be

$$\langle O(\tau_1, \vec{x}_1) O(\tau_2, \vec{x}_2) \rangle \sim \frac{1}{(|\tau_1 - \tau_2|^2 + |\vec{x}_1 - \vec{x}_2|^2)^\Delta}, \tag{2.7}$$

where the conformal dimension of O is given by $\Delta = n\Delta_\phi$. After the Wick rotation ($\tau = it$), the Euclidean two-point function reduces to a Lorentzian one,

$$\langle O(t_1, \vec{x}_1) O(t_2, \vec{x}_2) \rangle \sim \frac{1}{(-|t_1 - t_2|^2 + |\vec{x}_1 - \vec{x}_2|^2)^\Delta}, \tag{2.8}$$

where $\sqrt{-|t_1 - t_2|^2 + |\vec{x}_1 - \vec{x}_2|^2}$ means a proper distance preserving the $SO(1, d-1)$ Lorentz symmetry. This is the two-point function expected by the conformal symmetry.

In the holographic study, there are two different ways to obtain a two-point function. One is to consider a bulk field that is the dual of a local operator of the boundary theory. After calculating the bulk-to-boundary Green function, we move the bulk field to the boundary to obtain the boundary-to-boundary Green function, which is identified with the two-point function of the dual field theory. It was shown that this boundary-to-boundary Green function leads to the above two-point function in (2.1) [2–4]. Due to the direct relation between the bulk field and its dual operator, this method is useful to understand the internal structure of the two-point function. However, if we consider a non-AdS geometry, it is usually hard to find the bulk-to-boundary Green function. In this case, there is another way to evaluate the two-point function. It was proposed that the two-point function can be described by a geodesic length connecting two boundary operators,

$$\langle O(\tau_1, \vec{x}_1) O(\tau_2, \vec{x}_2) \rangle = e^{-\Delta L(\tau_1, \vec{x}_1; \tau_2, \vec{x}_2)/R}, \tag{2.9}$$

where Δ is the conformal dimension of the boundary operator. Using this proposal, one can reproduce the previous CFT's general two- and three-point functions exactly. In Refs. [22,23], the authors exploited the quotient construction to calculate the geodesic length for a Bañados-Teitelboim-Zanelli (BTZ) black hole. Unfortunately, the quotient construction is only possible for a three-dimensional BTZ black hole. In this section, we discuss another systematic way to evaluate the geodesic length without constructing the quotient space, and then apply this systematic method to general two-point functions of CFTs at finite temperature or in curved spaces.

According to the AdS/CFT correspondence, the known CFT's two-point function in (2.8) must be reproduced in the dual AdS gravity. In previous works [28,29], a spatial (equal-time) correlator at $\tau_1 = \tau_2$ and a temporal (equal-position or autocorrelation) two-point function at $\vec{x}_1 = \vec{x}_2$ were studied. For example, the spatial two-point function is governed by the following geodesic length:

$$L(t, \vec{x}_1; t, \vec{x}_2) = R \int_{x_1}^{x_2} dx \frac{\sqrt{1 + z'^2}}{z}, \tag{2.10}$$

where the prime means a derivative with respect to x. After finding the configuration of the geodesic and applying the holographic formula in (2.9), one finally obtains the spatial two-point function

$$\langle O(t, \vec{x}_1) O(t, \vec{x}_2) \rangle \sim \frac{1}{|\vec{x}_1 - \vec{x}_2|^{2\Delta}}. \tag{2.11}$$

Applying the Lorentz transformation, one can easily generalize this spatial two-point function to the general one in (2.8). This process cannot be applied when the boundary Lorentz symmetry is broken.

B. Holographic description for the general two-point function

Now we discuss how to calculate the general two- and three-point functions without applying the Lorentz symmetry. For convenience, we first calculate Euclidean correlation functions and then obtain Lorentzian ones by applying the Wick rotation. In order to describe a

d-dimensional CFT holographically, we take into account a $(d+1)$-dimensional Euclidean AdS space whose metric is given by

$$ds^2 = \frac{R^2}{z^2}(d\tau^2 + d\vec{x}\cdot d\vec{x} + dz^2), \tag{2.12}$$

where the dual CFT lives at the boundary ($z = 0$).

In order to evaluate general two- and three-point functions, we first investigate a bulk-to-boundary Green function running from $\{\tau, \vec{x}, z\} = \{\tau_J, \vec{x}_J, z_J\}$ to $\{\tau_i, \vec{x}_i, 0\}$, which corresponds to the position of the ith operator $O(\tau_i, \vec{x}_i)$. To find a geodesic connecting these two points, we regard $\vec{x}$ and z as functions of τ. Then, the geodesic length is governed by

$$L(\tau_J, \vec{x}_J, z_J; \tau_1, \vec{x}_1, 0) \equiv \int_{\tau_1}^{\tau_J} d\tau \mathcal{L} = R\int_{\tau_1}^{\tau_J} d\tau \frac{\sqrt{1+\dot{x}^2+\dot{z}^2}}{z}, \tag{2.13}$$

where $x = |\vec{x}|$ and a dot means a derivative with respect to τ. Since the geodesic depends on τ and x implicitly, there exist two conserved charges. The first one is the canonical momentum of $\vec{x}$,

$$\vec{P} = \frac{\partial\mathcal{L}}{\partial\dot{\vec{x}}} = \frac{R\dot{\vec{x}}}{z\sqrt{1+\dot{x}^2+\dot{z}^2}}, \tag{2.14}$$

and the other is the Hamiltonian corresponding to the canonical momentum of τ,

$$H = \frac{\partial\mathcal{L}}{\partial\dot{\vec{x}}}\dot{\vec{x}} + \frac{\partial\mathcal{L}}{\partial\dot{z}}\dot{z} - \mathcal{L} = -\frac{R}{z\sqrt{1+\dot{x}^2+\dot{z}^2}}. \tag{2.15}$$

To fix the conserved charges, we introduce a turning point $\{\tau_t, r_t, z_t\}$, where $\dot{z}(\tau_t) = 0$. Then, the conserved charges at the turning point reduce to

$$\vec{P} = \frac{R\vec{v}}{z_t\sqrt{1+v^2}}, \tag{2.16}$$

$$H = -\frac{R}{z_t\sqrt{1+v^2}}, \tag{2.17}$$

where $v = |\vec{v}|$ denotes the velocity at a turning point, $\vec{v} = \dot{\vec{x}}(\tau_t)$.

Comparing these conserved charges, we find that $\dot{r}$ is given by a constant,

$$\frac{d\vec{x}}{d\tau} = \vec{v}. \tag{2.18}$$

A solution satisfying two boundary conditions, $x_1 = x(\tau_1)$ and $x_J = x(\tau_J)$, reduces to

$$\vec{x}(\tau) = \vec{v}(\tau - \tau_1) + \vec{x}_1. \tag{2.19}$$

where $\vec{v} = \Delta\vec{x}/\Delta\tau$, with $\Delta\vec{x} = \vec{x}_J - \vec{x}_1$ and $\Delta\tau = \tau_J - \tau_1$. Moreover, comparing (2.15) with (2.17) leads to an equation governing the radial motion of a geodesic,

$$\frac{dz}{d\tau} = \pm\frac{\sqrt{1+v^2}\sqrt{z_t^2 - z^2}}{z}. \tag{2.20}$$

A solution satisfying $0 = z(\tau_1)$ and $z_J = z(\tau_J)$ becomes

$$z(\tau) = \sqrt{\frac{(\Delta\tau^2 + \Delta x^2 + z_J^2)(\tau - \tau_1)}{\Delta\tau} - \frac{(\Delta\tau^2 + \Delta x^2)(\tau - \tau_1)^2}{\Delta\tau^2}}, \tag{2.21}$$

where the turning point appears at

$$z_t = \frac{\Delta\tau^2 + \Delta x^2 + z_J^2}{2\sqrt{\Delta\tau^2 + \Delta x^2}}. \tag{2.22}$$

After substituting (2.18) and (2.20) into (2.13), the integration of (2.13) for $\tau_1 < \tau_t < \tau_J$ results in

$$\begin{aligned} L(\tau_J, x_J, z_J; \tau_1, x_1, 0) &= R\left(\int_\epsilon^{z_t} dz - \int_{z_t}^{z_J} dz\right)\frac{z_t}{z\sqrt{z_t^2 - z^2}} \\ &= R\log\frac{\Delta\tau^2 + \Delta x^2 + z_J^2}{z_J} - R\log\epsilon, \end{aligned} \tag{2.23}$$

where ϵ is introduced as a UV cutoff. Following (2.9), the bulk-to-boundary Green function reads

$$\langle O(\tau_J, \vec{x}_J, z_J) O(\tau_1, \vec{x}_1, \epsilon)\rangle = \frac{z_J^\Delta \epsilon^\Delta}{(\Delta\tau^2 + \Delta x^2 + z_J^2)^\Delta}. \tag{2.24}$$

If we take $z_J \to \epsilon$, the bulk-to-boundary Green function further reduces to a boundary-to-boundary Green function or a general two-point function of the dual CFT,

$$\langle \mathcal{O}(\tau_J, \vec{x}_J)\mathcal{O}(\tau_1, \vec{x}_1)\rangle = \frac{1}{(\Delta\tau^2 + \Delta x^2)^\Delta}, \tag{2.25}$$

where the normalized operator is defined as $\mathcal{O} \equiv O/\epsilon^\Delta$. After the Wick rotation, we finally obtain the Lorentzian two-point function in (2.8).

C. Holographic three-point function

Now we take into account the correlation function of three operators located at arbitrary positions. Reference [51] studied how to evaluate a three-point autocorrelation function when three operators are located at the same spatial position. In order to describe a three-point function holographically, we introduce a junction point in the bulk, whose position is denoted by $x_J = \{\tau_J, \vec{x}_J, z_J\}$. The junction point in the bulk represents a three-point vertex [51]

$$-\frac{\lambda}{3!}\int d^{d+1}x\sqrt{-g}\Phi_1(x)\Phi_2(x)\Phi_3(x), \qquad (2.26)$$

where the bulk field Φ is the dual of an operator O. At the leading order of λ, the three-point function of the dual QFT is described by the tree-level Witten diagram in the holographic setup,

$$\begin{aligned} G_3(x_1,x_2,x_3) &= \langle O(x_1)O(x_2)O(x_3)\rangle \\ &= -\lambda\int d^{d+1}x_J\sqrt{-g}G_2(x_1,x_J) \\ &\quad\times G_2(x_2,x_J)G_2(x_3,x_J), \end{aligned} \qquad (2.27)$$

where $x_i = \{\tau_i, \vec{x}_i, 0\}$ indicates the position of the boundary operator and G_n is an n-point function. Using the two-point function studied in the previous section, the three-point function is determined as the sum of minimal geodesic lengths connecting the junction point to three boundary operators [24,51]

$$\begin{aligned} &\langle O(\tau_1,\vec{x}_1)O(\tau_2,\vec{x}_2)O(\tau_3,\vec{x}_3)\rangle \\ &\sim -\lambda\exp\left(-\frac{\sum_{i=1}^{3}\Delta_i L_i(\tau_i,\vec{x}_i,0;\tau_J,\vec{x}_J,z_J)}{R}\right), \end{aligned} \qquad (2.28)$$

where Δ_i indicates the conformal dimension of the ith boundary operator. Using the previous bulk-to-boundary Green function in (2.23), the geodesic length connecting three operators via the junction point is given by

$$\begin{aligned} &\frac{\sum_{i=1}^{3}\Delta_i L_i(\tau_i,\vec{x}_i,0;\tau_J,\vec{x}_J,z_J)}{R} \\ &= \sum_{i=1}^{3}\Delta_i\left(\log\frac{\Delta\tau_i^2+\Delta x_i^2+z_J^2}{z_J} - \log\epsilon\right), \end{aligned} \qquad (2.29)$$

where $\Delta x_i = |\vec{x}_J - \vec{x}_i|$ and $\Delta\tau_i = |\tau_J - \tau_i|$. In this case, the junction point is determined as the position that minimizes the geodesic length. After variation, the junction point must satisfy

$$\begin{aligned} 0 &= \sum_{i=1}^{3}\frac{\Delta_i\Delta r_i}{\Delta r_i^2+z_J^2}, \\ 0 &= \sum_{i=1}^{3}\frac{\Delta_i(\Delta r_i^2 - z_J^2)}{\Delta r_i^2+z_J^2}. \end{aligned} \qquad (2.30)$$

Solving these equations determines the position of the junction point. To do so, we parametrize $\Delta\tau_i$ and Δx_i as follows:

$$\Delta\tau_i = \Delta r_i\cos\Delta\theta_i \quad \text{and} \quad \Delta x_i = \Delta r_i\sin\Delta\theta_i, \qquad (2.31)$$

where $\Delta r_i = |r_J - r_i| = \sqrt{\Delta\tau_i^2+\Delta x_i^2}$, with $\Delta\theta_i = |\theta_J - \theta_i|$. Since the geodesic length is invariant under rotation in the $\tau - x$ plane, we can set $\Delta\tau_i = \Delta r_i$, with $\Delta\theta_i = 0$, without loss of generality. This implies that a general three-point function up to rotation in the τ-x plane is the same as the three-point autocorrelation function studied in Ref [51]. As a result, the general three-point function is given by

$$\begin{aligned} &\langle O(\tau_1,\vec{x}_1)O(\tau_2,\vec{x}_2)O(\tau_3,\vec{x}_3)\rangle \\ &\sim \frac{C_{123}}{|r_1-r_2|^{\Delta_1+\Delta_2-\Delta_3}|r_2-r_3|^{\Delta_2+\Delta_3-\Delta_1}|r_3-r_1|^{\Delta_3+\Delta_1-\Delta_2}}, \end{aligned} \qquad (2.32)$$

where $|r_i - r_j| = \sqrt{|\tau_i-\tau_j|^2+|x_i-x_j|^2}$, and the structure constant C_{123} is given by

$$C_{123} = -\frac{\lambda\gamma^{\Delta}}{2^{\Delta}\Delta_1^{\Delta_1}\Delta_2^{\Delta_2}\Delta_3^{\Delta_3}(\Delta_1-\Delta_2-\Delta_3)^{\Delta_1}(\Delta_2-\Delta_3-\Delta_1)^{\Delta_2}(\Delta_3-\Delta_1-\Delta_2)^{\Delta_3}}, \qquad (2.33)$$

with

$$\begin{aligned} \Delta &= \Delta_1+\Delta_2+\Delta_3, \\ \gamma &= 2(\Delta_2^1\Delta_2^2+\Delta_3^2\Delta_1^2+\Delta_2^2\Delta_3^2) - \Delta_1^4 - \Delta_2^4 - \Delta_3^4. \end{aligned} \qquad (2.34)$$

This is consistent with the three-point function expected in CFT.

D. Holographic general two-point function of two-dimensional thermal CFT

The holographic method studied here is also applicable to other theories. For example, we can study a thermal CFT at finite temperature. Although a finite-temperature effect is negligible in the UV region, in the IR region it crucially modifies the IR correlators. To study this thermal effect, we calculate the two-point function of a two-dimensional thermal CFT whose dual gravity is described by a BTZ black hole,

$$ds^2 = \frac{R^2}{z^2}\left(f(z)d\tau^2 + dx^2 + \frac{1}{f(z)}dz^2\right), \tag{2.35}$$

with a blackening factor

$$f(z) = 1 - \frac{z^2}{z_h^2}. \tag{2.36}$$

For $z_h \to \infty$, the black hole geometry reduces to an AdS space corresponding to the zero-temperature limit. To apply (2.9), we first calculate a geodesic length extending to the BTZ black hole geometry. First, we assume that two local operators are located at the boundary, $\{z,\tau,x\} = \{0,\tau_1,x_1\}$ and $\{0,\tau_2,x_2\}$. Due to the translational symmetry in the τ and x directions, the two-point function is reexpressed in the following form:

$$\langle O(\tau_1,x_1)O(\tau_2,x_2)\rangle = \langle O(-\Delta\tau/2,-\Delta x/2)O(\Delta\tau/2,\Delta x/2)\rangle, \tag{2.37}$$

where $\Delta\tau = |\tau_1 - \tau_2|$ and $\Delta x = |x_1 - x_2|$. Then, the geodesic length in the black hole geometry is governed by

$$L(\tau_1,x_1;\tau_2,x_2) = R\int_{-\Delta\tau/2}^{\Delta\tau/2} d\tau \frac{\sqrt{f^2 + f\dot{x}^2 + \dot{z}^2}}{z\sqrt{f}}. \tag{2.38}$$

The geodesic length, similar to the previous case, relies on τ and x implicitly, so that the canonical momenta of x and τ are conserved. More precisely, the canonical momenta of x and τ are given by

$$P = \frac{R\dot{x}\sqrt{f}}{z\sqrt{f^2 + f\dot{x}^2 + \dot{z}^2}} \quad \text{and} \quad H = \frac{Rf^{3/2}}{z\sqrt{f^2 + f\dot{x}^2 + \dot{z}^2}}. \tag{2.39}$$

In this case, a turning point appears at $\{\tau,x,z\} = \{0,0,z_t\}$ due to the invariance under $\tau \to -\tau$ and $x \to -x$. At the turning point, the conserved quantities read

$$P = \frac{Rv}{z_t\sqrt{f_t + v^2}} \quad \text{and} \quad H = \frac{Rf_t}{z_t\sqrt{f_t + v^2}}, \tag{2.40}$$

where f_t and v are the values of f and $\dot{x}$ at the turning point. Comparing the conserved quantities, we find the relations between x, z, and τ:

$$\frac{dx}{d\tau} = \frac{fv}{f_t}, \tag{2.41}$$

$$\frac{dz}{d\tau} = \frac{(z_h^2 - z^2)\sqrt{z_t^2 - z^2}\sqrt{(v^2+1)z_h^2 - v^2z^2 - z_t^2}}{z_h(z_h^2 - z_t^2)z}. \tag{2.42}$$

Solving (2.42), we can find z_t and v as functions of $\Delta\tau$ and Δx,

$$z_t = z_h \sin\left(\frac{\Delta\tau}{2z_h}\right)\sqrt{1 + \cot\left(\frac{\Delta\tau}{2z_h}\right)^2 \tanh\left(\frac{\Delta x}{2z_h}\right)^2}, \tag{2.43}$$

$$v = \cot\left(\frac{\Delta\tau}{2z_h}\right)\tanh\left(\frac{\Delta x}{2z_h}\right). \tag{2.44}$$

These results finally determine the geodesic length as

$$L(\tau_1,x_1;\tau_2,x_2) = R\left[2\log\frac{1}{\epsilon} + \log\left\{2z_h^2\cosh\left(\frac{\Delta x}{z_h}\right) - 2z_h^2\cos\left(\frac{\Delta\tau}{z_h}\right)\right\}\right]. \tag{2.45}$$

According to the holographic proposal in (2.9), a general Euclidean two-point function becomes, up to normalization,

$$\langle O(\tau_1,x_1)O(\tau_2,x_2)\rangle \sim \frac{1}{|\sin^2(\frac{|\tau_1-\tau_2|}{2z_h}) + \sinh^2(\frac{|x_1-x_2|}{2z_h})|^\Delta}. \tag{2.46}$$

In a UV limit with a short distance and time interval ($\Delta\tau, \Delta x \to 0$), the thermal two-point function reduces to the previous CFT's result in (2.7). This is because finite thermal corrections are negligible in the UV region. Applying the Wick rotation to (2.46), the general Lorentzian two-point function is rewritten as [24]

$$\langle O(t_1,x_1)O(t_2,x_2)\rangle \sim \frac{1}{|-\sinh^2(\frac{|t_1-t_2|}{2z_h}) + \sinh^2(\frac{|x_1-x_2|}{2z_h})|^\Delta}. \tag{2.47}$$

In the IR region with a long distance ($|x_1 - x_2| \gg |t_1 - t_2| \gg z_h$), the spatial two-point function reduces to

$$\langle O(t_1,x_1)O(t_2,x_2)\rangle \sim e^{-|x_1-x_2|/\xi}, \tag{2.48}$$

with the correlation length ξ or the inverse of the effective mass $m_{\rm eff}$,

$$\xi \equiv \frac{1}{m_{\rm eff}} = \frac{1}{2\pi\Delta T_H}, \tag{2.49}$$

where $T_H = 1/(2\pi z_h)$ means the Hawking temperature. Exponential suppression of the thermal correlator in the IR limit is due to the screening effect of thermal background. In another IR limit with a long time interval ($|t_1 - t_2| \gg |x_1 - x_2| \gg z_h$), a similar feature also appears for the temporal two-point function,

$$\langle O(t_1,x_1)O(t_2,x_2)\rangle \sim e^{-|t_1-t_2|/t_{1/2}}, \tag{2.50}$$

where the half-life time $t_{1/2}$ is given by the inverse of the effective mass, $t_{1/2} = 1/m_{\rm eff}$. As expected, the IR correlators in the thermal CFT behave totally different from the UV ones. In the UV region, the correlation suppresses by a power law due to the conformal symmetry, while the IR correlation exponentially suppresses due to the screening effect of the thermal background.

III. CORRELATION FUNCTIONS IN CURVED SPACES

Now we consider a holographic dual of QFTs living in curved spaces like a dS or AdS space, and then investigate correlation functions of such theories. On the QFT side, two-point correlators of a massive field in a global dS space were studied in Refs. [52,53]. In order to describe QFTs living in a d-dimensional dS and AdS space holographically, we have to take into account a $(d+1)$-dimensional AdS space whose boundary is given by a dS or AdS space. To do so, let us first consider a $(d+1)$-dimensional Poincaré AdS space,

$$ds^2 = \frac{R^2}{z^2}(-dt^2 + \delta_{ij}dx^i dx^j + dz^2), \tag{3.1}$$

where $i, j = 1 \cdots d-1$. This AdS metric has a d-dimensional flat boundary at $z = 0$. To describe an AdS space with an AdS boundary, we introduce new coordinates, u and y,

$$z = \frac{uw}{R}, \quad x^a = y^a, \quad \text{and} \quad x^{d-1} = w\sqrt{1 - \frac{u^2}{R^2}}, \tag{3.2}$$

where $a, b = 1, \ldots, d-2$. Then, the AdS metric is rewritten as

$$ds^2 = \frac{R^2}{u^2}\left[\frac{du^2}{1 - u^2/R^2} + \frac{R^2}{w^2}(-dt^2 + \delta_{ab}dy^a dy^b + dw^2)\right]. \tag{3.3}$$

At $u = \epsilon$ in the limit of $\epsilon \to 0$, the boundary reduces to a d-dimensional AdS space,

$$ds^2_{\rm AdS} = \frac{\tilde{R}^2}{w^2}(-dt^2 + \delta_{ab}dy^a dy^b + dw^2), \tag{3.4}$$

where $\tilde{R} = R^2/\epsilon$. Applying the Wick rotation ($\tau = it$), the Euclidean bulk metric becomes

$$ds^2 = \frac{R^2}{u^2}\left[\frac{du^2}{1 - u^2/R^2} + \frac{R^2}{w^2}(d\tau^2 + \delta_{ab}dy^a dy^b + dw^2)\right]. \tag{3.5}$$

This is the metric of a $(d+1)$-dimensional Euclidean AdS space with a d-dimensional AdS boundary.

We can also consider an AdS space with a dS boundary. Introducing another coordinate system,

$$z = \frac{T\bar{u}}{R} \quad \text{and} \quad t = T\sqrt{1 + \frac{\bar{u}^2}{R^2}}, \tag{3.6}$$

the bulk AdS metric (3.1) is rewritten as

$$ds^2 = \frac{R^2}{\bar{u}^2}\left[\frac{d\bar{u}^2}{1 + \bar{u}^2/R^2} + \frac{R^2}{T^2}(-dT^2 + \delta_{ij}dx^i dx^j)\right]. \tag{3.7}$$

When the boundary is located at a fixed $u = \epsilon$, it becomes a d-dimensional dS space,

$$ds^2_{\rm dS} = \frac{1}{H^2 T^2}(-dT^2 + \delta_{ij}dx^i dx^j), \tag{3.8}$$

where the Hubble constant is given by $H = \epsilon/R^2$. If we further introduce $\tau = iT$ and $u = i\bar{u}$, the Lorentzian AdS metric in (3.7) becomes a Euclidean one,

$$ds^2 = \frac{R^2}{u^2}\left[\frac{du^2}{1 - u^2/R^2} + \frac{R^2}{\tau^2}(d\tau^2 + \delta_{ij}dx^i dx^j)\right], \tag{3.9}$$

which has the same form as that of the Euclidean AdS metric with an AdS boundary in (3.5). If we evaluate the two-point function on this Euclidean background, we can easily derive a Lorentzian correlation function on the boundary dS and AdS space.

A. Two-point function in an expanding universe

First, we focus on the dual of a QFT living in a dS space and study its two-point functions holographically. When two local operators are located at $\{u, \tau, \vec{r}\} = \{0, \tau_1, \vec{r}_1\}$ and $\{0, \tau_2, \vec{r}_2\}$, we rearrange the operator's positions to be at $\{u, \tau, \vec{r}\} = \{0, \tau_1, x_1, 0, \ldots, 0\}$ and $\{0, \tau_2, x_2, 0, \ldots, 0\}$, with $|\vec{r}_1 - \vec{r}_2| = |x_1 - x_2|$ due to the $(d-1)$-dimensional rotational symmetry. For the Euclidean AdS space in (3.9), the geodesic length connecting two operators is governed by

$$L(T_1, x_1; T_2, x_2) = \int_{x_1}^{x_2} dx \frac{R}{u}\sqrt{\frac{u'^2}{1 - u^2/R^2} + \frac{R^2}{\tau^2}(1 + \tau'^2)}, \tag{3.10}$$

where τ and u are considered as functions of x and the prime means a derivative with respect to x. It is worth noting that, due to the translation symmetry in the x direction, it is more convenient to take τ and u as functions of x. We first assume that there exists a turning point at $x = x_t$, where $u'(x_t) = 0$. Due to the translation symmetry in the x direction, there is a conserved charge satisfying

$$\frac{\sqrt{R^2-u^2}}{\tau z\sqrt{(1+\tau'^2)(R^2-u^2)+\tau^2 z'^2}}=\frac{1}{\tau_t z_t\sqrt{1+\tau_t'^2}}, \tag{3.11}$$

where $u_t=u(x_t)$, $\tau_t=\tau(x_t)$, and $\tau_t'=\tau'(x_t)$.

Unlike the previous cases, this system has only one conserved charge. Therefore, we have to solve the dynamical equation of τ and u to determine a geodesic. Note that two dynamical equations in this case are not independent because of the conservation law. Combining two dynamical equations of u and τ, we can find the following decoupled equation of τ [54–56]:

$$0=\tau\tau''+\tau'^2+1. \tag{3.12}$$

This allows a general solution,

$$\tau(x)=\sqrt{c_2^2-(c_1-x)^2}, \tag{3.13}$$

where c_1 and c_2 are two integral constants. This solution was also utilized to calculate the holographic entanglement entropy in a three-dimensional AdS space with a dS boundary [55,56]. The dual of the three-dimensional AdS space with a dS boundary corresponds to a two-dimensional inflating universe. Therefore, the holographic entanglement entropy in a three-dimensional AdS space with a dS boundary measures quantum correlations of two macroscopic subsystems in a two-dimensional inflating universe. Since the entanglement entropy is defined at a fixed time, τ must satisfy $\tau(x_1)=\tau(x_2)$ at the dS boundary. In the two-point function calculation we are interested in, two local operators can be located at arbitrary positions and times. Therefore, we do not need to impose $\tau(x_1)=\tau(x_2)$ in the general two-point function calculation. There is another remark. If one calculates the entanglement entropy in higher-dimensional AdS space with a dS boundary, (3.13) is not a solution anymore. For the two-point function calculation in higher-dimensional AdS space with a dS boundary, however, (3.13) still remains as a solution. This is because the two-point function is described by a one-dimensional geodesic curve regardless of the dimension of the bulk space, while the entanglement entropy is governed by a $(d-1)$-dimensional minimal surface in a $(d+1)$-dimensional AdS space.

Plugging (3.13) into (3.11), we obtain

$$\frac{du}{dx}=\pm\frac{c_2\sqrt{u_t^2-u^2}\sqrt{R^2-u^2}}{u(c_2^2-(c_1-x)^2)}. \tag{3.14}$$

A solution of this differential equation is given by

$$u(x)=\frac{\sqrt{R^2[(1+c_3)x+c_2-c_1-(c_1+c_2)c_3]^2-z_t^2[(1-c_3)x+c_2-c_1+(c_1+c_2)c_3]^2}}{2\sqrt{c_3}\sqrt{c_1-c_2-x}\sqrt{c_1+c_2-x}}, \tag{3.15}$$

where c_3 is another integral constant. Here, four undetermined parameters, c_1, c_2, c_3, and z_t, can be fixed by imposing the following four boundary conditions: $\tau_1=\tau(x_1)$, $\tau_2=\tau(x_2)$, $0=u(x_1)$, and $0=u(x_2)$. The first two boundary conditions determine c_1 and c_2 as

$$c_1=\frac{\tau_2^2-\tau_1^2+x_2^2-x_1^2}{2(x_2-x_1)},$$
$$c_2=-\frac{\sqrt{(\tau_1-\tau_2)^2+(x_1-x_2)^2}\sqrt{(\tau_1+\tau_2)^2+(x_1-x_2)^2}}{2(x_1-x_2)}. \tag{3.16}$$

The remaining two boundary conditions determine c_3 as a function of c_1 and c_2,

$$c_3=\frac{\sqrt{(c_1-c_2-x_1)(c_1-c_2-x_2)}}{\sqrt{(c_1+c_2-x_1)(c_1+c_2-x_2)}}. \tag{3.17}$$

Using these results, we finally determine the turning point u_t in terms of the operator's positions,

$$u_t=\frac{R\sqrt{(\tau_1-\tau_2)^2+(x_1-x_2)^2}}{\sqrt{(\tau_1+\tau_2)^2+(x_1-x_2)^2}}. \tag{3.18}$$

The obtained solutions determine the geodesic length in the following form:

$$L(\tau_1,x_1;\tau_2,x_2)=\int_\epsilon^{u_t}du\frac{2R^2u_t}{z\sqrt{R^2-z^2}\sqrt{u_t^2-z^2}}$$
$$=R\log\left(\frac{4R^2u_t^2}{(R^2-u_t^2)\epsilon^2}\right), \tag{3.19}$$

where ϵ is introduced as a UV cutoff. In the Euclidean dS space, the general two-point function up to normalization reduces to

$$\langle O(\tau_1,\vec{r}_1)O(\tau_2,\vec{r}_2)\rangle\sim\left(\frac{\tau_1\tau_2}{|\tau_1-\tau_2|^2+|\vec{r}_1-\vec{r}_2|^2}\right)^\Delta, \tag{3.20}$$

where Δ indicates the conformal dimension of O. After the Wick rotation $\tau=iT$, the Lorentzian two-point function becomes

$$\langle O(T_1,\vec{r}_1)O(T_2,\vec{r}_2)\rangle \sim \left(\frac{T_1T_2}{-|T_1-T_2|^2+|\vec{r}_1-\vec{r}_2|^2}\right)^{\Delta}. \quad (3.21)$$

In the above calculation, we exploited the conformal time T for convenience. In order to study the two-point function in the expanding universe, it is more convenient to introduce the cosmological time t,

$$T=\frac{e^{-Ht}}{H}. \quad (3.22)$$

In terms of the cosmological time, the dS metric in (3.8) is rewritten as

$$ds_{\rm dS}^2=-dt^2+e^{2Ht}\delta_{ij}dx^idx^j, \quad (3.23)$$

which describes an eternal inflation. In this eternally expanding universe, the two-point function for $t_2>t_1$ is given by

$$\langle O(t_1,\vec{r}_1)O(t_2,\vec{r}_2)\rangle \sim \frac{T_1^{2\Delta}e^{-\Delta H|t_2-t_1|}}{|-T_1^2(1-e^{-H|t_2-t_1|})^2+(\vec{r}_1-\vec{r}_2)^2|^{\Delta}}, \quad (3.24)$$

where the conformal time is determined as a function of the cosmological time,

$$T_1=\frac{e^{-Ht_1}}{H} \quad \text{and} \quad T_2=T_1e^{-H|t_2-t_1|}. \quad (3.25)$$

A temporal two-point function at early times ($|\vec{r}_1-\vec{r}_2|\ll|t_2-t_1|\ll1/H$) decreases as a power law,

$$\langle O(t_1,\vec{r}_1)O(t_2,\vec{r}_2)\rangle \approx \frac{1}{|t_1-t_2|^{2\Delta}}. \quad (3.26)$$

This is the correlator of a CFT. In the late-time era ($|\vec{r}_1-\vec{r}_2|\ll1/H\ll|t_2-t_1|$), the holographic result shows that the temporal two-point function suppresses exponentially,

$$\langle O(t_1,\vec{r}_1)O(t_2,\vec{r}_2)\rangle \approx e^{-\Delta H|t_2-t_1|}. \quad (3.27)$$

This is a typical feature of the massive operator's correlator. In this case, ΔH plays the role of an effective mass. On the other hand, the spatial two-point function for $|\vec{r}_1-\vec{r}_2|\gg|t_1-t_2|$ leads to the following correlator:

$$\langle O(t_1,\vec{r}_1)O(t_2,\vec{r}_2)\rangle \approx \frac{e^{-\Delta H|t_2-t_1|}}{|\vec{r}_1-\vec{r}_2|^{2\Delta}}. \quad (3.28)$$

This shows that the two-point function always suppresses by a power law in the spatial direction. This implies that the operator behaves as a massless one in the spatial direction, unlike the temporal correlator. When two operators are located at the same time ($t_1=t_2=t$), athe spatial two-point function at the time t is given by

$$\langle O(t,\vec{r}_1)O(t,\vec{r}_2)\rangle \sim \frac{T_0^2e^{-2\Delta H(t-t_0)}}{|\vec{r}_1-\vec{r}_2|^{2\Delta}}, \quad (3.29)$$

where t_0 is an appropriate reference time satisfying $T_0=e^{-Ht_0}/H$. Therefore, the spatial two-point function exponentially suppresses with time due to the expansion of the background spacetime. This is consistent with the results obtained in a different holographic model [55,56].

To understand the obtained holographic result further on the dual QFT side, we take into account a QFT living in a d-dimensional dS space and discuss its two-point function. The Euclidean metric of a d-dimensional dS space can be written as

$$ds_{\rm dS}^2=g_{\mu\nu}dx^\mu dx^\nu=\frac{R^2}{\tau^2}(d\tau^2+\delta_{ij}dy^idy^j), \quad (3.30)$$

where $x^\mu=\{\tau,y^i\}$, with $i,j=1,\ldots,(d-1)$, and τ indicates the Euclidean time. Now, we consider a scalar field on this dS space,

$$S=\frac{1}{2}\int d^dx\sqrt{g}(\partial^\mu\phi\partial_\mu\phi+\xi\mathcal{R}_{\rm dS}^{(d)}\phi^2). \quad (3.31)$$

When the scalar field conformally couples to the dS background, the coefficient ξ is given by

$$\xi=\frac{(d-2)}{4(d-1)}. \quad (3.32)$$

In this theory, the conformal dimension of ϕ is given by $\Delta_\phi=(d-2)/2$. Since the background dS space has a positive-curvature scalar, $\mathcal{R}_{\rm dS}^{(d)}=\frac{d(d-1)}{R^2}$, the scalar field in the dS space has an effective mass,

$$m_{\rm dS}^2=\frac{(d-2)}{4(d-1)}\mathcal{R}_{\rm dS}^{(d)}=\frac{d(d-2)}{4R^2}. \quad (3.33)$$

Therefore, the two-point function of ϕ satisfies

$$\frac{1}{\sqrt{g}}\left[-\partial_\mu\sqrt{g}g^{\mu\nu}\partial_\nu+m_{\rm dS}^2\right]\langle\phi(\tau_1,\vec{y}_1)\phi(\tau_2,\vec{y}_2)\rangle =\frac{\delta^{(d)}(x_1-x_2)}{\sqrt{g}}. \quad (3.34)$$

Solving this equation, we can obtain the following Lorentzian two-point function after the Wick rotation ($\tau=iT$):

$$\langle \phi(\tau_1, \vec{y}_1)\phi(\tau_2, \vec{y}_2)\rangle \sim \left(\frac{T_1 T_2}{|-(T_1-T_2)^2+(\vec{y}_1-\vec{y}_2)^2|}\right)^{\Delta_\phi}. \tag{3.35}$$

If we further consider an operator $O = \phi^n$, its two-point function becomes

$$\langle O(T_1, \vec{y}_1)O(T_2, \vec{y}_2)\rangle \sim \left(\frac{T_1 T_2}{|-(T_1-T_2)^2+(\vec{y}_1-\vec{y}_2)^2|}\right)^{\Delta}, \tag{3.36}$$

where $\Delta = n\Delta_\phi$ at the tree level. This is the two-point function obtained in the previous holographic calculation. If we further take into account interactions and their quantum corrections, they can generate an anomalous dimension, which modifies the conformal dimension of the operator. At the quantum level, as a result, the conformal dimension of O can be different from the tree-level result [29].

B. Two-point correlation in an AdS space

The previous holographic calculation with the dS boundary can be easily generalized to the case with an AdS boundary because the Euclidean bulk AdS metric in (3.5) is invariant under exchanging time and one of the spatial coordinates. When we express the boundary AdS metric as

$$ds^2 = \frac{R^2}{w^2}(dw^2 - dt^2 + \delta_{ab}dx^a dx^b), \tag{3.37}$$

where w ($w \geq 0$) corresponds to the radial coordinate and $a, b = 1, \ldots, (d-2)$, the general two-point function in the holographic setup is given by

$$\langle O(t_1, w_1, \vec{x}_1)O(t_2, w_2, \vec{x}_2)\rangle \sim \left(\frac{w_1 w_2}{|-(t_1-t_2)^2+(w_1-w_2)^2+(\vec{x}_1-\vec{x}_2)^2|}\right)^{\Delta}. \tag{3.38}$$

Similar to the dS case, this result shows that the two-point function suppresses by a power law except for the radial direction w. For $t_1 = t_2 = t$ and $\vec{x}_1 = \vec{x}_2 = \vec{x}$, the two-point function in the radial direction of the AdS space reduces to

$$\langle O(t, w_1, \vec{x})O(t, w_2, \vec{x})\rangle \sim \left(\left|\frac{w_1+w_2}{w_1-w_2}\right|^2 - 1\right)^{\Delta}. \tag{3.39}$$

When the distance of two operators is short, satisfying $|w_1 - w_2| \ll |w_1 + w_2|$, the two-point function behaves like

$$\langle O(t, w_1, \vec{x})O(t, w_2, \vec{x})\rangle \sim \frac{1}{|w_1-w_2|^{2\Delta}}, \tag{3.40}$$

which is equivalent to the two-point function in the other directions. In the large-distance limit satisfying $|w_1 - w_2| \approx |w_1 + w_2|$, however, the two-point function in the w direction shows a different behavior,

$$\langle O(t, w_1, \vec{x})O(t, w_2, \vec{x})\rangle \sim \frac{1}{|w_1-w_2|^{\Delta}}. \tag{3.41}$$

In other words, the scaling dimension of the operator O changes from Δ to $\Delta/2$ as $|w_1 - w_2|$ increases.

From the QFT point of view, similar to the dS case, the previous holographic two-point function can be understood via a scalar field conformally coupled to the background Euclidean AdS space,

$$S = \frac{1}{2}\int d^d x\sqrt{g}(\partial^\mu\phi\partial_\mu\phi + \xi\mathcal{R}^{(d)}_{\mathrm{AdS}}\phi^2), \tag{3.42}$$

where the scalar field has an effective mass due to the curvature of the background AdS space,

$$m^2_{\mathrm{AdS}} = \frac{(d-2)}{4(d-1)}\mathcal{R}^{(d)}_{\mathrm{AdS}} = -\frac{d(d-2)}{4R^2}. \tag{3.43}$$

Therefore, the two-point function of a scalar field must satisfy the following equation in the AdS space:

$$\frac{1}{\sqrt{g}}\left[-\partial_\mu\sqrt{g}g^{\mu\nu}\partial_\nu + m^2_{\mathrm{AdS}}\right]\langle\phi(\tau_1, \vec{y}_1)\phi(\tau_2, \vec{y}_2)\rangle = \frac{\delta^{(d)}(x_1 - x_2)}{\sqrt{g}}. \tag{3.44}$$

The solution of this equation is coincident with the Euclidean version of the holographic result in (3.38).

IV. DISCUSSION

In this paper, we have studied how to calculate general correlation functions in the holographic setup. After regarding the dual gravity of a QFT, we evaluated the geodesic length connecting boundary operators, which is directly related to a boundary-to-boundary Green function in the bulk. According to the AdS/CFT correspondence, this Green function can be regraded as a correlation function of the dual QFT. We showed that this holographic approach reproduces the known correlation functions of CFT and offers a novel method to understand the scale dependence of correlation functions in various situations, like a thermal CFT or QFT in curved spacetimes.

First, we discussed how to calculate general two- and three-point functions of CFT holographically when operators are located at arbitrary positions and times. This is the generalization of the equal-time and equal-position correlation functions studied before [28,29]. For equal-time and equal-position correlators, there is only one conserved charge in the dual description. For the general correlator, however, there are two conserved charges due to the translational symmetries in the temporal and spatial directions. Exploiting these conserved quantities with appropriate boundary conditions, we determined the exact configuration of the geodesic and evaluated its length analytically. Intriguingly, we showed that this holographic calculation reproduces the exact two-and three-point functions expected by the conformal symmetry.

The holographic approach was also applied to the two-point functions of nontrivial QFTs, like a thermal CFT or QFT in an expanding universe. A thermal system or expanding background usually has a parameter—either temperature or the Hubble constant—specifying the system's scale. Since this parameter is finite, its effect is negligible in the UV limit. Therefore, the correlation function of such a system reduces to the CFT result in the UV limit having a short distance and time interval. In the IR limit, however, the finite correction can give rise to a significant effect on the theory that modifies the correlation functions. Although it is important to understand such IR modification, it is generally hard to calculate the IR correlation function exactly because of the nonperturbative feature of IR physics. In the present work, we calculated the exact correlation functions valid over the entire energy scale by applying the holographic method. The holographic result showed that the correlation functions of a thermal CFT or QFT in an expanding universe, as expected, are the same as the CFT results in the UV limit. In other words, the correlation function suppresses by a power law in the short time interval and distance limit. In the IR limit, however, the correlation functions decreases exponentially due to the screening effect for a thermal CFT or the expansion of the background spacetime for the expanding universe.

If we further consider relevant operators deforming the UV CFT, they significantly modify the IR physics by providing a nontrivial RG flow. In this procedure, correlation functions also seriously change. Therefore, it would be interesting to investigate the scale-dependent correlators for a system with a nontrivial RG flow. In this case, a UV CFT can flow to a new IR CFT in which the scaling dimension of an operator changes with the anomalous dimension. We hope to report more interesting results on this issue in future work.

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