Convergence of the polarization tensor in spacetime of three dimensions

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In this paper, we consider the convergence properties of the polarization tensor of graphene obtained in the framework of thermal quantum field theory in three-dimensional spacetime. During the last years, this problem attracted much attention in connection with the calculation of the Casimir force in graphene systems and the investigation of the electrical conductivity and reflectance of graphene sheets. There are contradictory statements in the literature, especially on whether this tensor has an ultraviolet divergence in three dimensions. Here, we analyze this problem using the well-known method of dimensional regularization. It is shown that the thermal correction to the polarization tensor is finite at any D, whereas its zero-temperature part behaves differently for $D = 3$ and 4. For $D = 3$, it is obtained by analytic continuation with no subtracting of infinitely large terms. As for the spacetime of $D = 4$, the finite result for the polarization tensor at zero temperature is found after subtracting the pole term. Our results are in agreement with previous calculations of the polarization tensor at both zero and nonzero temperature. This opens the possibility for a wider application of the quantum-field-theoretical approach in investigations of graphene and other two-dimensional novel materials.

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I. INTRODUCTION

The term polarization tensor has many different meanings and was used for theoretical descriptions of diverse physical phenomena. Here, we reconsider the problem of convergence of the vacuum photon polarization tensor of graphene in quantum electrodynamics (QED) at nonzero temperature in three-dimensional spacetime. Independently of an entirely theoretical interest to calculate the polarization tensor at both zero and nonzero temperature for the case of $D = (2 + 1)$ dimensions [1–5], this problem attracted special attention [6–10] in connection with the advent of the two-dimensional hexagonal structure of carbon atoms called graphene [11].

At energies below a few eV, the electronic properties of graphene are well described by a set of massless or very light quasiparticles with spin $1/2$ obeying the Dirac equation, where the speed of light c is replaced with the Fermi velocity $v_{\mathrm{F}} \approx c/300$ [12–15]. (In the following text, we use the system of units where $\hbar = c = 1$.) This has opened an attractive opportunity of describing the reaction of graphene to an electromagnetic field using the well-established methods of QED in $(2 + 1)$ dimensions, especially the concept of the polarization tensor, i.e., restricting to the one-loop radiative correction in the language of QED. Taking into account that the properties of graphene strongly depend on temperature, this may be done in the framework of thermal quantum field theory.

The polarization tensor derived in $(2 + 1)$-dimensional quantum field theory (QFT) [1,2] was first applied for the theoretical description of the Casimir force between two graphene sheets in Ref. [16]. In Ref. [17], this tensor was generalized for the case of nonzero temperature and calculated at the pure imaginary Matsubara frequencies taking into account the nonzero mass of quasiparticles and chemical potential. The obtained results were used to investigate the Casimir and Casimir-Polder forces in various configurations [18–29].

The analytic continuation of the polarization tensor of graphene to the entire plane of complex frequencies, including the real frequency axis, was performed in Ref. [30]. These results were generalized for graphene sheets possessing a nonzero chemical potential [31]. The obtained polarization tensor of graphene at nonzero temperature was used in the calculation of the Casimir and Casimir-Polder forces in graphene systems [32–41] and the electrical conductivity [42–45] and reflectivity properties of graphene [30,46–48]. Computations of the Casimir force in graphene systems using the polarization tensor have been found to be in excellent agreement with the measurement data of two precision experiments [49–52].

In spite of this progress in the application of thermal QFT to obtaining the polarization tensor of graphene and describing its properties on this basis, the more phenomenological theoretical approach using the Kubo formula is often used in the literature for the same purpose (see, e.g., Refs. [53–72]). However, there are significant conceptual differences between the quantum-field-theoretical and Kubo approaches. For instance, in the framework of the Kubo approach, dissipation is introduced by means of the phenomenological relaxation parameter treated as the imaginary part of the complex frequency. Alternatively, the QFT does not use phenomenological parameters and describes dissipation by means of the imaginary part of the polarization tensor, which arises for the scaled 3-momentum magnitudes exceeding the energy gap in graphene.

In the spatially local approximation, there is agreement between the results obtained using different theoretical approaches [21,30,42–44,46–48]. As for the spatially nonlocal case, the quantum-field-theoretical approach predicts the presence of a double pole at zero frequency in the transverse dielectric permittivity of graphene [41], which is not obtainable in the Kubo approach. It was stated [73] that the presence of a double pole might be connected with an improper regularization of the polarization tensor obtained within thermal QFT.

In this regard, it should be noted that there are contradictory statements in the recent literature concerning the convergence of this tensor. Thus, Ref. [17] found by power counting that the polarization tensor of graphene diverges and made it finite by a Pauli-Villars subtraction, whereas Refs. [27,28,30] concluded that in $2+1$ dimensions it is finite because the ultraviolet divergence is not present due to the gauge invariance. Reference [73] stated that the polarization tensor of graphene obtained by means of the quantum field theory is divergent and suggests an alternative regularization procedure, which brings it into exact coincidence with that obtained by means of the Kubo approach.

In view of the above, we feel that it is necessary to clarify the situation. We demonstrate the calculation of the polarization tensor of graphene using the methods of QED at nonzero temperature in detail and in such a way that the calculation can be followed with a minimum of knowledge of the field-theoretical methods. Thereby, it must be underlined that in standard QED at zero temperature the polarization tensor was calculated long ago in both $(3+1)$ dimensions (see, e.g., [74,75]) and $(2+1)$ dimensions [1,2]. For the latter case, the key moments, gauge invariance, and ultraviolet finiteness were mentioned explicitly in Sec. 2 of Ref. [2].

In the present paper, we reconsider the polarization tensor appearing in the quantum-field-theoretical approach to graphene at nonzero temperature in detail. We use dimensional regularization. First, we demonstrate how the transversality of the polarization tensor can be seen before the momentum integration. Next, we demonstrate that this tensor consists of the zero-temperature part and a thermal correction to it. An immediate analytic calculation shows that the thermal correction to the polarization tensor is finite, so that the ultraviolet divergence, if any, might be contained only in its zero-temperature part. Then, we use the exponential representation for the propagators and carry out the momentum integrations. Finally, after carrying out the next-to-last integration, the transversality also becomes evident in this representation as well as the ultraviolet properties.

Specifically, by considering the polarization tensor in a spacetime of complex D dimensions, we demonstrate that in the case of $D=3$ the finite result is obtained using the regularization by means of analytic continuation from the case $\mathrm{Re}D<2$. In so doing, no pole terms need to be subtracted, i.e., no renormalization is needed. Applying the same procedure to the polarization tensor in a spacetime of $D=4$ dimensions, we show that to obtain the finite result it is necessary to subtract the pole term, i.e., regularization should be followed by the renormalization. Generally speaking, such behavior in the ultraviolet region is well known in QFT as a consequence of the combination of power counting, gauge invariance, and parity. However, an active discussion for graphene, which exists in two spatial dimensions but interacts with an electromagnetic field existing in three-dimensional space, revealed the necessity to demonstrate this behavior in detail. The performed analysis confirms the polarization tensor derived in the literature in the framework of both ordinary and thermal QFT.

The paper is organized as follows. In Sec. II we consider a general expression for the polarization tensor of graphene at nonzero temperature in a spacetime of D dimensions. Section III is devoted to the zero-temperature part of the polarization tensor and its analytic properties. In Sec. IV the convergence properties of the polarization tensor in both three- and four-dimensional spacetime are considered. In Sec. V we present our conclusions and a discussion.

Recall that we use the system of units where $\hbar=c=1$.

II. REPRESENTATION OF THE POLARIZATION TENSOR AT NONZERO TEMPERATURE IN D DIMENSIONS

In the framework of QFT, the one-loop polarization tensor of graphene was considered in many papers (see, e.g., Refs. [9,16–18,30,31,35,76]). It is represented by the simple diagram shown in Fig. 1, where the solid lines

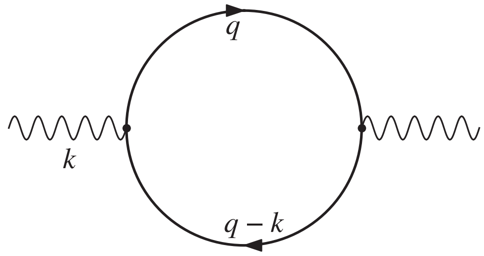

FIG. 1. Feynman diagram representing the one-loop polarization tensor of graphene.

depict the propagators of fermionic quasiparticles which move with the Fermi velocity v_{F} and satisfy the Dirac equation in $2+1$ dimensions,

$$\left[\gamma^0\left(i\frac{\partial}{\partial t}-eA_0\right)+\tilde{\gamma}^1\left(i\frac{\partial}{\partial x^1}-eA_1\right)+\tilde{\gamma}^2\left(i\frac{\partial}{\partial x^2}-eA_2\right)-mv_{\mathrm{F}}{}^2\right]\psi(x)=0. \tag{1}$$

Here, γ^ν are the standard Dirac matrices, $\tilde{\gamma}^{1,2}=v_{\mathrm{F}}\gamma^{1,2}$, $A_\nu=(A_0,A_1,A_2)$ is the vector potential of the electromagnetic field, and m is the mass of quasiparticles bearing the electric charge e.

The important feature of Eq. (1) is that the interaction of charged quasiparticles with the electromagnetic field is introduced by the standard substitution

$$i\frac{\partial}{\partial x^\nu}\longrightarrow i\frac{\partial}{\partial x^\nu}-eA_\nu, \tag{2}$$

where, for a graphene sheet in the plane $x^3=0$, it holds that $x^\nu=(t,x^1,x^2,0)$, $\nu=0,1,2,3$. Note that Eq. (2) contains the speed of light in the factor e/c. (We recall that here $c=1$.) This reflects the fact that the electromagnetic field, although it interacts with the quasiparticles confined in a graphene plane, exists in the $3+1$-dimensional bulk. As a consequence, in the Dirac model of graphene, the electric charge in the system of units with $\hbar=c=1$ is not dimensional (as it holds in the strictly $2+1$-dimensional electrodynamics [1]) but rather dimensionless and results in the standard fine-structure constant $\alpha=e^2\approx 1/137$.

The calculation of the diagram shown in Fig. 1 includes an integration over the internal momentum $q=(q_0,\boldsymbol{q})$ and taking the trace of γ matrices (see Refs. [17,30] for details). Keeping in mind that we are looking for the polarization tensor of graphene at any temperature T, within the Matsubara formalism, an integration over q_0 should be replaced with a summation over the pure imaginary fermionic Matsubara frequencies,

$$q_{0n}\equiv iq_{Dn}=2\pi ik_BT\left(n+\frac{1}{2}\right), \tag{3}$$

where $n=0,\pm1,\pm2,\ldots$ and k_B is the Boltzmann constant. In so doing, the zero component of the external photon wave vector $k=(k_0,\boldsymbol{k})$ is equal to the pure imaginary bosonic Matsubara frequencies,

$$k_{0n}\equiv ik_{Dl}=2\pi ik_BTl. \tag{4}$$

Although here and below we deal with graphene, which is a two-dimensional sheet of carbon atoms, in the following we use the D-dimensional vectors $(q_0,\boldsymbol{q})=(q_0,q^1,\ldots,q^{D-1})$ and $(k_0,\boldsymbol{k})=(k_0,k^1,\ldots,k^{D-1})$, where the dimension of the spatial part is $D-1$, and respective integration measures. The metric tensor is defined as $g_{\mu\nu}=\mathrm{diag}(1,-1,-1,\ldots,-1)$ and the product of two vectors is $qk=q_\nu k^\nu=q_0k^0-\boldsymbol{qk}$. The trace of the metric tensor is $g_\nu^\nu=D$. The point is that, in general, the polarization tensor is ultraviolet divergent, like most radiative corrections in QFT. For instance, simple power counting shows a divergence also in $(2+1)$ dimensions. For this reason, a regularization is necessary. By introducing the D-dimensional spacetime, we take the dimensional regularization, which amounts to formally taking a complex dimension D (see, e.g., Sec. 11.2 in [77]). This allows to find the analytic properties of the polarization tensor as a function of D.

Note that for graphene the Dirac cones are located at the two points at the corners of the Brillouin zone [13]. Then, after taking the trace over the gamma matrices, the resulting polarization tensor in the momentum representation is given by [30]

$$\Pi^{\mu\nu}(ik_{Dl},\boldsymbol{k},T)=-\frac{32\pi\alpha}{v_{\mathrm{F}}^2}k_BT\times\sum_{n=-\infty}^{\infty}\int\frac{d^{D-1}\boldsymbol{q}}{(2\pi)^{D-1}}\frac{Z^{\mu\nu}(ik_{Dl},\boldsymbol{k};iq_{Dn},\boldsymbol{q})}{R(ik_{Dl},\boldsymbol{k};iq_{Dn},\boldsymbol{q})}, \tag{5}$$

where

$$Z^{\mu\nu}(ik_{Dl},\boldsymbol{k};iq_{Dn},\boldsymbol{q})=\eta_{\mu'}^{\mu}\eta_{\nu'}^{\nu}\tilde{Z}^{\mu'\nu'}(ik_{Dl},\boldsymbol{k};iq_{Dn},\boldsymbol{q}) \tag{6}$$

and $\eta_\mu^\nu=\mathrm{diag}(1,v_{\mathrm{F}},v_{\mathrm{F}},\ldots,v_{\mathrm{F}})$.

The quantities $\tilde{Z}^{\mu'\nu'}$ and R are

$$\tilde{Z}^{\mu'\nu'}(ik_{Dl},\boldsymbol{k};iq_{Dn},\boldsymbol{q})=q^{\mu'}(q-\tilde{k})^{\nu'}+(q-\tilde{k})^{\mu'}q^{\nu'}+g^{\mu'\nu'}[-q(q-\tilde{k})+m^2],$$
$$R(ik_{Dl},\boldsymbol{k};iq_{Dn},\boldsymbol{q})=(q^2-m^2+i0)\times[(q-\tilde{k})^2-m^2+i0], \tag{7}$$

where the infinitely small additions $i0$ originate from the fermion propagators, the scaled momentum is

$\tilde{k} = (k_0, v_F\boldsymbol{k})$, $q_0 = q_{0n} = iq_{Dn}$, and $\tilde{k}_0 = k_{0l} = ik_{Dl}$ in accordance with Eqs. (3) and (4). For instance,

$$\tilde{Z}^{00}(ik_{Dl},\boldsymbol{k};iq_{Dn},\boldsymbol{q}) = -q_{Dn}(q_{Dn}-k_{Dl}) + \boldsymbol{q}(\boldsymbol{q}-\tilde{\boldsymbol{k}}) + m^2,$$
$$\tilde{Z}^{11}(ik_{Dl},\boldsymbol{k};iq_{Dn},\boldsymbol{q}) = 2q^1(q^1-\tilde{k}^1) - q_{Dn}(q_{Dn}-k_{Dl}) - \boldsymbol{q}(\boldsymbol{q}-\tilde{\boldsymbol{k}}) - m^2, \tag{8}$$

etc., and

$$R(ik_{Dl},\boldsymbol{k};iq_{Dn},\boldsymbol{q}) = [q_{Dn}^2 + \Gamma^2(\boldsymbol{q}) - i0] \times [(q_{Dn}-k_{Dl})^2 + \tilde{\Gamma}^2(\boldsymbol{q},\boldsymbol{k}) - i0], \tag{9}$$

where

$$\Gamma^2(\boldsymbol{q}) = \boldsymbol{q}^2 + m^2, \qquad \tilde{\Gamma}^2(\boldsymbol{q},\boldsymbol{k}) = (\boldsymbol{q}-\tilde{\boldsymbol{k}})^2 + m^2. \tag{10}$$

It is common knowledge that electrodynamics is a gauge-invariant theory. This means that the Fourier-transformed vacuum current

$$J^\nu(k) = \Pi^{\mu\nu}A_\mu(k) \tag{11}$$

should be invariant under the gauge transformation

$$\delta A_\mu(k) = \tilde{A}_\mu(k) - A_\mu(k) = ik_\mu\chi(k), \tag{12}$$

where $\chi(k)$ is an arbitrary function [78]. As a consequence,

$$\delta J^\nu(k) = ik_\mu\Pi^{\mu\nu}\chi(k) = 0. \tag{13}$$

Thus, for the polarization tensor, the gauge invariance is realized in the form of a transversality condition,

$$k_\mu\Pi^{\mu\nu} = 0. \tag{14}$$

It is easily seen that the polarization tensor of graphene (5) satisfies this condition like that in full QED. Really, using Eqs. (6) and (7), by a simple rewriting, one obtains

$$k_\mu Z^{\mu\nu} = v_F[2q^\nu\tilde{k}q - q^\nu\tilde{k}^2 - \tilde{k}^\nu q^2 + m^2\tilde{k}^\nu] = v_F\{(q^2-m^2)(q-\tilde{k})^\nu - [(q-\tilde{k})^2 - m^2]q^\nu\}, \tag{15}$$

where $\tilde{k}^\nu = \eta^\nu_\beta k^\beta$.

Then, from Eqs. (5) and (15) we find

$$k_\mu\Pi^{\mu\nu} = -\frac{32\pi\alpha}{v_F}k_BT\sum_{n=-\infty}^{\infty}\int\frac{d^{D-1}\boldsymbol{q}}{(2\pi)^{D-1}} \times \left[\frac{(q-\tilde{k})^\nu}{(q-\tilde{k})^2-m^2} - \frac{q^\nu}{q^2-m^2}\right]. \tag{16}$$

Note that $q^0 = q_{0n} = iq_{Dn}$, given by Eq. (3).

The integral in Eq. (16) converges under the condition $\mathrm{Re}D < 2$. Using this condition, the seemingly divergent integral/sum is regularized, allowing the shift of variables $q \to Q + \tilde{k}$, where

$$\mathbf{Q} = \boldsymbol{q} - \tilde{\boldsymbol{k}},$$
$$Q_0 = q_0 - \tilde{k}_0 = q_0 - k_0 = i(q_{Dn} - k_{Dl}), \tag{17}$$

and q_{Dn}, k_{Dl} are defined in Eqs. (3) and (4). As a result, the integrand itself vanishes, i.e., the polarization tensor (5) satisfies the transversality condition (14) even before carrying out the momentum integration.

Now we represent the polarization tensor (5) as the part that is independent of temperature, and the thermal correction to it. For this purpose, the right-hand side of Eq. (5) is rewritten as

$$k_BT\sum_{n=-\infty}^{\infty}f(ik_{Dl},\boldsymbol{k};iq_{Dn}), \tag{18}$$

where

$$f(ik_{Dl},\boldsymbol{k};iq_{Dn}) = -\frac{32\pi\alpha}{v_F^2}\int\frac{d^{D-1}\boldsymbol{q}}{(2\pi)^{D-1}}\frac{Z^{\mu\nu}(ik_{Dl},\boldsymbol{k};iq_{Dn},\boldsymbol{q})}{R(ik_{Dl},\boldsymbol{k};iq_{Dn},\boldsymbol{q})}. \tag{19}$$

(Below we omit the already specified repeated arguments.)

Using the Cauchy residual theorem, the sum (18) can be represented in the form

$$k_BT\sum_{n=-\infty}^{\infty}f(iq_{Dn}) = -\int_{\gamma_1\cup\gamma_2}\frac{dq_D}{2\pi}\frac{f(iq_D)}{e^{i\frac{q_D}{k_BT}}+1}, \tag{20}$$

where the integration contour in the complex q_D plane shown in Fig. 2 consists of the paths γ_1 and γ_2. The validity of Eq. (20) becomes evident when taking into account that the poles of the expression under the integral are at the points $q_{Dn} = 2\pi k_BT(n+1/2)$ (shown as dots in Fig. 2) and calculating the sum of the residues at these poles.

Substituting Eq. (20) into Eq. (5) and interchanging the order of integrations, one obtains

$$\Pi^{\mu\nu}(ik_{Dl},\boldsymbol{k},T) = \frac{32\pi\alpha}{v_F^2}\int\frac{d^{D-1}\boldsymbol{q}}{(2\pi)^{D-1}} \times \left[\int_{\gamma_1}\frac{dq_D}{2\pi}\frac{1}{e^{i\frac{q_D}{k_BT}}+1}\frac{Z^{\mu\nu}}{R} + \int_{\gamma_2}\frac{dq_D}{2\pi}\frac{1}{e^{i\frac{q_D}{k_BT}}+1}\frac{Z^{\mu\nu}}{R}\right]. \tag{21}$$

Here, the integrand in the second term is decreasing in the lower half-plane. To make the integrand in the first term decreasing in the upper half-plane, in the integral along γ_1,

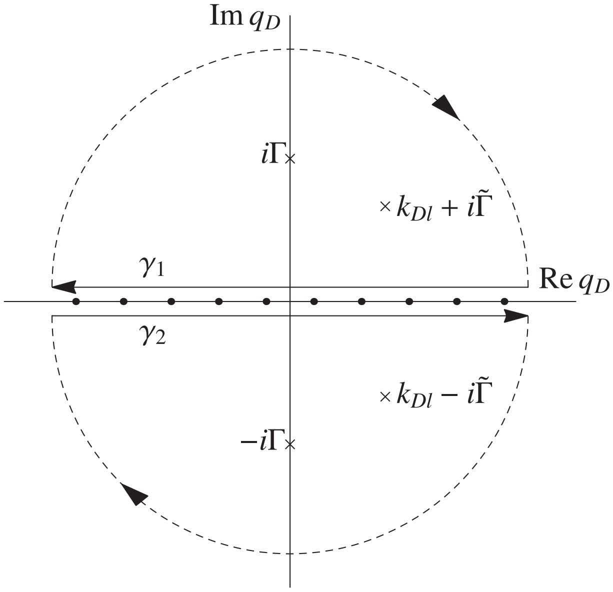

FIG. 2. Complex q_D plane containing the integration paths γ_1 and γ_2. The dots indicate the poles at the fermionic Matsubara frequencies. The four additional poles are shown as crosses (see the text for further discussion).

we use the identity

$$\frac{1}{e^{i\frac{q_D}{k_BT}}+1}=1-\frac{1}{e^{-i\frac{q_D}{k_BT}}+1}. \tag{22}$$

Substituting it into Eq. (21), we bring the polarization tensor to the form

$$\Pi^{\mu\nu}(ik_{Dl},\boldsymbol{k},T)=\Pi_0^{\mu\nu}(ik_{Dl},\boldsymbol{k})+\Delta_T\Pi^{\mu\nu}(ik_{Dl},\boldsymbol{k},T), \tag{23}$$

where

$$\Pi_0^{\mu\nu}(ik_{Dl},\boldsymbol{k})=-\frac{32\pi\alpha}{v_{\rm F}^2}\int_{-\infty}^{\infty}\frac{dq_D}{2\pi}\int\frac{d^{D-1}\boldsymbol{q}}{(2\pi)^{D-1}}\frac{Z^{\mu\nu}}{R} \tag{24}$$

and

$$\Delta_T\Pi^{\mu\nu}(ik_{Dl},\boldsymbol{k},T)=-\frac{32\pi\alpha}{v_{\rm F}^2}\int\frac{d^{D-1}\boldsymbol{q}}{(2\pi)^{D-1}}\times\left[\int_{\gamma_1}\frac{dq_D}{2\pi}\frac{1}{e^{-i\frac{q_D}{k_BT}}+1}\frac{Z^{\mu\nu}}{R}-\int_{\gamma_2}\frac{dq_D}{2\pi}\frac{1}{e^{i\frac{q_D}{k_BT}}+1}\frac{Z^{\mu\nu}}{R}\right]. \tag{25}$$

Note that the minus sign in front of (24) appeared because the direction of the path γ_1 is against the real axis in the complex plane q_D.

The first term on the right-hand side of Eq. (23) given by Eq. (24) has the meaning of the polarization tensor at zero temperature (until the moment it is calculated at the bosonic Matsubara frequencies). As for the second term given by Eq. (25), it explicitly depends on T and has the meaning of the thermal correction.

We begin with calculating the thermal correction. This can be done by closing the integration paths γ_1 and γ_2 with the help of semicircles of infinitely large radii in the upper and lower half-planes, respectively, and applying again the Cauchy residue theorem. In the upper half-plane, there are two poles of the function $Z^{\mu\nu}/R$ at the roots of R. These are $q_D=i\Gamma(\boldsymbol{q})$ and $q_D=i\tilde{\Gamma}(\boldsymbol{q})+k_{Dl}$, where Γ and $\tilde{\Gamma}$ are defined in Eq. (10). In the lower half-plane, the poles of the function $Z^{\mu\nu}/R$ are at $q_D=-i\Gamma(\boldsymbol{q})$ and $q_D=-i\tilde{\Gamma}(\boldsymbol{q})+k_{Dl}$. All of these poles are shown in Fig. 2 as crosses.

Calculating the residues at all four poles and taking into account that the integrals along both semicircles vanish, we rewrite the thermal correction (25) as

$$\Delta_T\Pi^{\mu\nu}(ik_{Dl},\boldsymbol{k},T)=\frac{16\pi\alpha}{v_{\rm F}^2}\int\frac{d^{D-1}\boldsymbol{q}}{(2\pi)^{D-1}}\times\sum_{\lambda=\pm1}\left\{\frac{Z^{\mu\nu}(q_D=i\lambda\Gamma)}{\Gamma(e^{\frac{\Gamma}{k_BT}}+1)[(i\lambda\Gamma-k_{Dl})^2+\tilde{\Gamma}^2]}+\frac{Z^{\mu\nu}(q_D=i\lambda\tilde{\Gamma}+k_{Dl})}{\tilde{\Gamma}(e^{\frac{\tilde{\Gamma}}{k_BT}}+1)[(i\lambda\tilde{\Gamma}+k_{Dl})^2+\Gamma^2]}\right\}. \tag{26}$$

When obtaining this equation, it was used that $\exp[-i\lambda k_{Dl}/(k_BT)]=1$ due to Eq. (4).

Equation (26) can be further simplified because the integrand is symmetric under the substitution $\boldsymbol{q}\to\tilde{\boldsymbol{k}}-\boldsymbol{q}$. Making this substitution and the replacement $\lambda\to-\lambda$ in the second term of this equation, one obtains

$$\Delta_T\Pi^{\mu\nu}(ik_{Dl},\boldsymbol{k},T)=\frac{16\pi\alpha}{v_{\rm F}^2}\int\frac{d^{D-1}\boldsymbol{q}}{(2\pi)^{D-1}}\frac{1}{\Gamma(e^{\frac{\Gamma}{k_BT}}+1)}\times\sum_{\lambda=\pm1}\frac{Z^{\mu\nu}(q_D=i\lambda\Gamma,\boldsymbol{q})+Z^{\mu\nu}(q_D=k_{Dl}-i\lambda\Gamma,\tilde{\boldsymbol{k}}-\boldsymbol{q})}{(k_{Dl}-i\lambda\Gamma)^2+\tilde{\Gamma}^2}. \tag{27}$$

Taking into account that $\Gamma \sim |\boldsymbol{q}|$ when $|\boldsymbol{q}| \to \infty$, it is seen that the integral in Eq. (27) converges exponentially fast for any D. Note that Eq. (27) is easily generalized for the case of graphene with a nonzero chemical potential μ. This is done by making the replacement [79]

$$\frac{1}{e^{\frac{\Gamma}{k_B T}}+1} \longrightarrow \frac{1}{2}\left(\frac{1}{e^{\frac{\Gamma+\mu}{k_B T}}+1}+\frac{1}{e^{\frac{\Gamma-\mu}{k_B T}}+1}\right). \quad (28)$$

Thus, the problem of convergence of the polarization tensor reduces to the question of whether its zero-temperature part (24) converges.

Note that the thermal correction in the form of Eq. (27) admits an immediate analytic continuation to the real frequency axis by putting $ik_{Dl} = k_0 = \omega$. (Compare with similar results obtained for the temperature Green functions in Refs. [80,81] and with Ref. [9].) In a similar way, the polarization tensor at zero temperature along the real frequency axis is obtained from Eq. (24) by putting $ik_{Dl} = k_0 = \omega$ and $q_D = -iq_0$. With this substitution, it takes the form

$$\Pi_0^{\mu\nu}(k) = i\frac{32\pi\alpha}{v_F^2}\int \frac{d^D q}{(2\pi)^D}\frac{Z^{\mu\nu}(k,q)}{R(k,q)}, \quad (29)$$

where $k = (k_0, \boldsymbol{k})$, $q = (q_0, \boldsymbol{q})$, and $d^D q = dq_0 d\boldsymbol{q}$.

According to Eqs. (6) and (7), $Z^{\mu\nu} \sim q^2$ and $R \sim q^4$ in the limit $q^2 \to \infty$. These simple power-counting arguments show that the integral (29) may contain ultraviolet divergences of the order of q^{D-2}, i.e., they diverge linearly and quadratically in three- and four-dimensional spacetime, respectively. Below we show how these expectations are modified by the gauge invariance of the polarization tensor.

III. ZERO-TEMPERATURE PART AND ITS ANALYTIC EXPRESSION IN D DIMENSIONS

In this section, we calculate the zero-temperature polarization tensor (29) in the case of D-dimensional spacetime. For this purpose, we use the following representation for the propagators entering Eq. (29) [75]:

$$\frac{1}{q^2 - m^2 + i0} = \frac{1}{i}\int_0^\infty ds e^{is(q^2-m^2+i0)},$$
$$\frac{1}{(q-\tilde{k})^2 - m^2 + i0} = \frac{1}{i}\int_0^\infty dt e^{it[(q-\tilde{k})^2-m^2+i0]}. \quad (30)$$

For the momenta $q^{\nu'}$ entering $\tilde{Z}^{\mu'\nu'}$ in Eq. (7), we use

$$q^{\nu'} = \frac{1}{i}\frac{\partial}{\partial \xi_{\nu'}} e^{iq^\gamma \xi_\gamma}|_{\xi=0}. \quad (31)$$

This substitution is made for all q entering the function $\tilde{Z}^{\mu'\nu'}$, i.e.,

$$\tilde{Z}^{\mu'\nu'}(k,q) = \tilde{Z}^{\mu'\nu'}\left(k, \frac{1}{i}\frac{\partial}{\partial \xi_{\nu'}}\right) e^{iq^\gamma \xi_\gamma}|_{\xi=0}. \quad (32)$$

Substituting Eqs. (6) and (30) into Eq. (29) with account of the definition of R in Eq. (7) and using Eq. (32), the polarization tensor at zero temperature is presented as

$$\Pi_0^{\mu\nu}(k) = \frac{32\pi\alpha}{iv_F^2}\eta^\mu_{\mu'}\eta^\nu_{\nu'}\int \frac{d^D q}{(2\pi)^D}\int_0^\infty ds \times \int_0^\infty dt \tilde{Z}^{\mu'\nu'}\left(k, \frac{1}{i}\frac{\partial}{\partial \xi_{\nu'}}\right) e^{iM}|_{\xi=0}, \quad (33)$$

where the quantity M is defined as

$$M = s(q^2 - m^2 + i0) + t[(q-\tilde{k})^2 - m^2 + i0] + iq\xi. \quad (34)$$

This expression for M can be identically rewritten in the form

$$M = (s+t)\left[q + \frac{\xi - 2t\tilde{k}}{2(s+t)^2}\right]^2 - \frac{\xi^2}{4(s+t)} + \frac{t}{s+t}\tilde{k}\xi + H, \quad (35)$$

where

$$H = \frac{st}{s+t}\tilde{k}^2 - (s+t)m^2 + i0. \quad (36)$$

It is seen that only the first term in the expression (35) for M depends on q. Then, the integration with respect to q in Eq. (33) can be easily performed. For this purpose, we use the well-known formulas [75]

$$\int_{-\infty}^\infty \frac{dq_0}{2\pi} e^{i(s+t)q_0^2} = \frac{e^{i\frac{\pi}{4}}}{\sqrt{4\pi(s+t)}},$$
$$\int_{-\infty}^\infty \frac{dq_j}{2\pi} e^{-i(s+t)q_j^2} = \frac{e^{-i\frac{\pi}{4}}}{\sqrt{4\pi(s+t)}}, \quad (37)$$

where $s + t > 0$ and $j = 1, 2, \ldots, D-1$.

Combining the necessary number of expressions in Eq. (37), for the D-dimensional spacetime one obtains

$$\int \frac{d^D q}{(2\pi)^D} e^{i(s+t)q^2} = \frac{e^{i\frac{\pi}{4}(2-D)}}{[4\pi(s+t)]^{\frac{D}{2}}}. \quad (38)$$

By applying Eq. (38) with a necessary shift of the integration variable q in Eqs. (33) and (35), one obtains

$$\Pi_0^{\mu\nu}(k) = \frac{32\pi\alpha}{iv_F^2} e^{i\frac{\pi}{4}(2-D)}\eta^\mu_{\mu'}\eta^\nu_{\nu'}\int_0^\infty ds\int_0^\infty \frac{dt}{[4\pi(s+t)]^{D/2}} \times \tilde{Z}_1^{\mu'\nu'} e^{iH}, \quad (39)$$

where

$$\tilde{Z}_1^{\mu'\nu'} = \tilde{Z}^{\mu'\nu'}\left(\tilde{k}, \frac{1}{i}\frac{\partial}{\partial\xi_{\nu'}}\right)\exp\left[\frac{i}{4(s+t)}(4t\tilde{k}\xi - \xi^2)\right]\Bigg|_{\xi=0}. \quad (40)$$

The functional form of the quantity $\tilde{Z}^{\mu'\nu'}$ is presented in the first line of Eq. (7). It is seen that, in order to calculate the quantity (40), one should find how the operators obtained from $q^{\mu'}$, $q^{\mu'}q^{\nu'}$, and q^2 by the replacement of $q^{\mu'}$ with $-i\partial/\partial\xi_{\mu'}$ act on the exponent in Eq. (40). As an example,

$$\frac{1}{i}\frac{\partial}{\partial\xi_{\mu'}}\exp\left[\frac{i}{4(s+t)}(4t\tilde{k}\xi - \xi^2)\right] = \frac{2t\tilde{k}^{\mu'} - \xi^{\mu'}}{2(s+t)}\exp\left[\frac{i}{4(s+t)}(4t\tilde{k}\xi - \xi^2)\right]. \quad (41)$$

By putting here $\xi = 0$, one finds

$$\frac{1}{i}\frac{\partial}{\partial\xi_{\mu'}}\exp\left[\frac{i}{4(s+t)}(4t\tilde{k}\xi - \xi^2)\right]\Bigg|_{\xi=0} = \frac{t}{s+t}\tilde{k}^{\mu'}. \quad (42)$$

In a similar way, by calculating the remaining derivatives and putting $\xi = 0$ in the obtained results, we arrive at

$$\begin{aligned}
&\frac{1}{i}\frac{\partial}{\partial\xi_{\mu'}}\frac{1}{i}\frac{\partial}{\partial\xi_{\nu'}}\exp\left[\frac{i}{4(s+t)}(4t\tilde{k}\xi - \xi^2)\right]\Bigg|_{\xi=0} \\
&= -\frac{g^{\mu'\nu'}}{2i(s+t)} + \frac{t^2}{(s+t)^2}\tilde{k}^{\mu'}\tilde{k}^{\nu'}, \\
&\frac{1}{i}\frac{\partial}{\partial\xi^{\mu'}}\frac{1}{i}\frac{\partial}{\partial\xi_{\mu'}}\exp\left[\frac{i}{4(s+t)}(4t\tilde{k}\xi - \xi^2)\right]\Bigg|_{\xi=0} \\
&= -\frac{D}{2i(s+t)} + \frac{t^2}{(s+t)^2}\tilde{k}^2,
\end{aligned} \quad (43)$$

where we accounted for $g_{\mu\nu}g^{\mu\nu} = D$.

Using Eqs. (7), (42), and (43), we bring Eq. (40) to the form

$$\tilde{Z}_1^{\mu'\nu'} = g^{\mu'\nu'}\frac{D-2}{2i(s+t)} - \frac{ts}{(s+t)^2}(2\tilde{k}^{\mu'}\tilde{k}^{\nu'} - g^{\mu'\nu'}\tilde{k}^2) + g^{\mu'\nu'}m^2. \quad (44)$$

It is convenient to rewrite the polarization tensor (39) in terms of new integration variables ρ and λ defined as

$$s = \rho\lambda, \qquad t = (1-\rho)\lambda, \quad (45)$$

so that

$$\rho = \frac{s}{s+t}, \qquad \lambda = s + t, \quad (46)$$

where ρ is the so-called Feynman parameter (frequently denoted by x).

It is easily seen that

$$\int_0^\infty ds\int_0^\infty dt g(s,t) = \int_0^1 d\rho\int_0^\infty \lambda d\lambda g(\rho\lambda,(1-\rho)\lambda), \quad (47)$$

where the factor λ in Eq. (47) comes from the Jacobian.

In terms of the variables (45), the quantities $\tilde{Z}_1^{\mu'\nu'}$ from Eq. (44) and H from Eq. (36) take the form

$$\begin{aligned}
\tilde{Z}_1^{\mu'\nu'} &= g^{\mu'\nu'}\frac{D-2}{2i\lambda} - 2\rho(1-\rho)\tilde{k}^{\mu'}\tilde{k}^{\nu'} \\
&\quad + g^{\mu'\nu'}[\rho(1-\rho)\tilde{k}^2 + m^2], \\
H &= \lambda[\rho(1-\rho)\tilde{k}^2 - m^2 + i0] \equiv \lambda H_1(\rho).
\end{aligned} \quad (48)$$

Then, the polarization tensor (39) is given by

$$\Pi_0^{\mu\nu}(k) = \frac{32\pi\alpha}{iv_{\mathrm{F}}^2}e^{i\frac{\pi}{4}(2-D)}\eta^{\mu}_{\mu'}\eta^{\nu}_{\nu'}\int_0^1 d\rho\int_0^\infty\frac{d\lambda}{(4\pi)^{D/2}} \times \lambda^{1-\frac{D}{2}}\tilde{Z}_1^{\mu'\nu'}e^{i\lambda H_1(\rho)}, \quad (49)$$

where $\tilde{Z}_1^{\mu'\nu'}$ and H_1 are defined in Eq. (48). Note that the limit of large momenta corresponds to small λ.

The integral over λ in Eq. (49) can be calculated using the formula [82]

$$\int_0^\infty d\lambda e^{i\lambda H_1(\rho)}\lambda^{w-1} = [-iH_1(\rho)]^{-w}\Gamma(w), \quad (50)$$

where $\Gamma(w)$ is the gamma function. Note that the integral on the left-hand side of Eq. (50) is equal to the gamma function only under the conditions $\mathrm{Re}(-iH_1) > 0$ and $\mathrm{Re}w > 0$. The first of these is satisfied due to the presence of $i0$ in Eq. (48). Below we apply Eq. (50) for the spacetime with $\mathrm{Re}D < 2$, where $\mathrm{Re}w > 0$. The results for the cases $\mathrm{Re}D > 2$ are obtained by standard analytic continuation. (See the next section for the differences between the cases $D = 3$ or $D = 4$.)

Using Eq. (50) in Eq. (49), one finds

$$\begin{aligned}
&\int_0^\infty\frac{d\lambda}{(4\pi)^{D/2}}\lambda^{1-\frac{D}{2}}\tilde{Z}_1^{\mu'\nu'}e^{i\lambda H_1(\rho)} \\
&= g^{\mu'\nu'}\frac{D-2}{2i(4\pi)^{D/2}}\Gamma\left(1-\frac{D}{2}\right)[-iH_1(\rho)]^{\frac{D}{2}-1} \\
&\quad + \frac{1}{(4\pi)^{D/2}}\{-2\rho(1-\rho)\tilde{k}^{\mu'}\tilde{k}^{\nu'} + g^{\mu'\nu'}[\rho(1-\rho)\tilde{k}^2 \\
&\quad + m^2]\}\Gamma\left(2-\frac{D}{2}\right)[-iH_1(\rho)]^{\frac{D}{2}-2}.
\end{aligned} \quad (51)$$

Using the property

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}, \tag{52}$$

the integral (51) can be rewritten in a simpler form,

$$\begin{aligned}\int_0^\infty \frac{d\lambda}{(4\pi)^{D/2}} \lambda^{1-\frac{D}{2}} \tilde{Z}_1^{\mu'\nu'} e^{i\lambda H_1(\rho)} \\ = \frac{2}{(4\pi)^{D/2}} \rho(1-\rho)(g^{\mu'\nu'}\tilde{k}^2 - \tilde{k}^{\mu'}\tilde{k}^{\nu'}) \\ \times \Gamma\left(2-\frac{D}{2}\right)[-iH_1(\rho)]^{\frac{D}{2}-2}.\end{aligned} \tag{53}$$

Inserting Eq. (53) into Eq. (49), we arrive at

$$\begin{aligned}\Pi_0^{\mu\nu}(k) = \frac{64\pi\alpha}{iv_{\rm F}^2} e^{i\frac{\pi}{4}(2-D)} \frac{\eta_{\mu'}^{\mu}\eta_{\nu'}^{\nu}}{(4\pi)^{D/2}} \Gamma\left(2-\frac{D}{2}\right) \\ \times (g^{\mu'\nu'}\tilde{k}^2 - \tilde{k}^{\mu'}\tilde{k}^{\nu'}) \int_0^1 d\rho\rho(1-\rho)[-iH_1(\rho)]^{\frac{D}{2}-2}.\end{aligned} \tag{54}$$

From the tensor structure of Eq. (54) it becomes evident that for $\Pi_0^{\mu\nu}$ the transversality condition (14) is satisfied, as it must be for both the zero-temperature part of the polarization tensor and the thermal correction to it.

The analytic continuations of Eq. (54) to the cases of $D=4$ and $D=3$ are considered in the next section.

IV. THREE- AND FOUR-DIMENSIONAL SPACETIMES

We begin with the case of four-dimensional spacetime $D=4$. Keeping in mind the necessity of regularization, let us put $D=4-2\varepsilon$, where ε vanishes when D goes to 4. In this case Eq. (54) takes the form

$$\begin{aligned}\Pi_{0,\varepsilon}^{\mu\nu}(k) = -\frac{4\alpha}{\pi v_{\rm F}^2}\eta_{\mu'}^{\mu}\eta_{\nu'}^{\nu}\Gamma(\varepsilon)(g^{\mu'\nu'}\tilde{k}^2 - \tilde{k}^{\mu'}\tilde{k}^{\nu'}) \\ \times \int_0^1 d\rho\rho(1-\rho)H_1^{-\varepsilon}(\rho).\end{aligned} \tag{55}$$

In fact, the gamma function on the right-hand side of Eq. (55) can be analytically continued to the entire plane of complex ε with the exception of the poles at $\varepsilon = 0, -1, -2, \ldots$ This allows to perform the dimensional regularization of the polarization tensor (55) and subsequent renormalization by subtracting the pole contribution in the form of $1/\varepsilon$.

To do so, we expand the gamma function according to [82]

$$\Gamma(\varepsilon) = \frac{1}{\varepsilon} - \gamma + O(\varepsilon), \tag{56}$$

where γ is the Euler constant. The factor $H_1^{-\varepsilon}$ is represented as

$$H_1^{-\varepsilon} = \exp\left[\ln\left(\frac{H_1(\rho)}{C}\right)^{-\varepsilon}\right] = 1 - \varepsilon\ln\frac{H_1(\rho)}{C} + O(\varepsilon^2), \tag{57}$$

where C is an arbitrary constant with the dimension of H_1.

Substituting Eqs. (56) and (57) into Eq. (55), one obtains

$$\begin{aligned}\Pi_{0,\varepsilon}^{\mu\nu}(k) = -\frac{4\alpha}{\pi v_{\rm F}^2}\eta_{\mu'}^{\mu}\eta_{\nu'}^{\nu}(g^{\mu'\nu'}\tilde{k}^2 - \tilde{k}^{\mu'}\tilde{k}^{\nu'}) \\ \times \int_0^1 d\rho\rho(1-\rho)\left(\frac{1}{\varepsilon} - \ln\frac{H_1(\rho)}{C'}\right),\end{aligned} \tag{58}$$

where $C' = Ce^{-\gamma}$.

It is convenient to rewrite this result in the form

$$\Pi_{0,D\to 4}^{\mu\nu}(k) = \eta_{\mu'}^{\mu}\eta_{\nu'}^{\nu}(g^{\mu'\nu'}\tilde{k}^2 - \tilde{k}^{\mu'}\tilde{k}^{\nu'})\Pi_4(k^2), \tag{59}$$

where

$$\Pi_4(k^2) = \frac{4\alpha}{\pi v_{\rm F}^2}\int_0^1 d\rho\rho(1-\rho)\left(\frac{2}{D-4} + \ln\frac{H_1(\rho)}{C'}\right), \tag{60}$$

and, in accordance with Eq. (48),

$$H_1(\rho) = \rho(1-\rho)\tilde{k}^2 - m^2 + i0. \tag{61}$$

The renormalization in quantum electrodynamics with $D=4$ consists in discarding the pole term in Eq. (60), which corresponds to the logarithmic ultraviolet divergence. This divergence is by two powers less than it follows from a simple power counting for $D=4$ discussed at the end of Sec. II. The decrease in the divergence power is the result of the transversality (gauge invariance) of the polarization tensor ensured by the tensor structure of Eq. (59). By imposing the normalization condition $\Pi_4^{\rm ren}(k^2=0)=0$ (which is justified by the general theory of renormalization in QED), one can fix the arbitrary constant $C' = -m^2$ and arrive at

$$\Pi_4^{\rm ren}(k^2) = \frac{4\alpha}{\pi v_{\rm F}^2}\int_0^1 d\rho\rho(1-\rho)\ln\left[1-\rho(1-\rho)\frac{\tilde{k}^2}{m^2}\right]. \tag{62}$$

This is the well-known result of standard QED [75] if we put $v_{\rm F}=1$ and consider one Dirac point in place of two as for graphene.

Now we pass to the case $D=3$, i.e., to the polarization tensor of graphene at zero temperature. In this case, Eq. (54) takes the form

$$\Pi_{0,D=3}^{\mu\nu}(k) = -\frac{8\alpha}{\sqrt{\pi}v_{\rm F}^2}\eta_{\mu'}^{\mu}\eta_{\nu'}^{\nu}\Gamma\left(\frac{1}{2}\right)(g^{\mu'\nu'}\tilde{k}^2 - \tilde{k}^{\mu'}\tilde{k}^{\nu'}) \times \int_0^1 d\rho \frac{\rho(1-\rho)}{\sqrt{-H_1(\rho)}}. \quad (63)$$

This equation, similar to Eq. (55), is obtained by the analytic continuation of Eq. (54). However, as opposed to Eq. (55), it is finite and does not contain the pole terms. Thus, no subtraction of infinities is needed to obtain the final physical result, i.e., the polarization tensor of graphene behaves like that in the truly three-dimensional QED, which is the super-renormalizable theory (as mentioned in particular in [2]), unlike the standard theory in four dimensions which is “only” renormalizable.

Using the same representation as in Eq. (59),

$$\Pi_{0,D=3}^{\mu\nu}(k) = \eta_{\mu'}^{\mu}\eta_{\nu'}^{\nu}(g^{\mu'\nu'}\tilde{k}^2 - \tilde{k}^{\mu'}\tilde{k}^{\nu'})\Pi_3(k^2), \quad (64)$$

one obtains from Eq. (63)

$$\Pi_3(k^2) = -\frac{8\alpha}{v_{\rm F}^2}\int_0^1 d\rho \frac{\rho(1-\rho)}{\sqrt{m^2 - \rho(1-\rho)\tilde{k}^2}}. \quad (65)$$

The last integral is easily calculated [82]. Thus,

$$\Pi_3(k^2) = -\frac{4\alpha}{v_{\rm F}^2\tilde{k}^2}\left[-m + \frac{4m^2+\tilde{k}^2}{4} \times \int_0^1 d\rho \frac{1}{\sqrt{m^2 - \rho(1-\rho)\tilde{k}^2}}\right]. \quad (66)$$

Using the most convenient expressions for this integral in different regions of parameters, for $\tilde{k}^2 < 0$ we obtain

$$\Pi_3(k^2) = \frac{2\alpha}{v_{\rm F}^2\tilde{k}^2}\left(2m - \frac{4m^2+\tilde{k}^2}{\sqrt{-\tilde{k}^2}}\arctan\frac{\sqrt{-\tilde{k}^2}}{2m}\right). \quad (67)$$

Under the conditions $\tilde{k}^2 > 0$, $2m > \sqrt{\tilde{k}^2}$, we have

$$\Pi_3(k^2) = \frac{2\alpha}{v_{\rm F}^2\tilde{k}^2}\left(2m - \frac{4m^2+\tilde{k}^2}{\sqrt{\tilde{k}^2}}\mathrm{arctanh}\frac{\sqrt{\tilde{k}^2}}{2m}\right). \quad (68)$$

Finally, under the conditions $\tilde{k}^2 > 0$, $2m < \sqrt{\tilde{k}^2}$, one obtains

$$\Pi_3(k^2) = \frac{2\alpha}{v_{\rm F}^2\tilde{k}^2}\left[2m - \frac{4m^2+\tilde{k}^2}{\sqrt{\tilde{k}^2}}\left(\mathrm{arctanh}\frac{2m}{\sqrt{\tilde{k}^2}} + i\frac{\pi}{2}\right)\right]. \quad (69)$$

Note that there is a threshold at $\sqrt{\tilde{k}^2} = 2m$.

The two convenient independent quantities characterizing the polarization tensor are Π^{00} and $\mathrm{tr}\Pi^{\mu\nu} = g_{\mu\nu}\Pi^{\mu\nu}$. Using Eq. (64), these are given by

$$\Pi_{0,D=3}^{00}(k) = -v_{\rm F}^2\boldsymbol{k}^2\Pi_3(k^2),$$
$$\mathrm{tr}\Pi_{0,D=3}^{\mu\nu}(k) = v_{\rm F}^2(k^2 + \tilde{k}^2)\Pi_3(k^2), \quad (70)$$

where $\Pi_3(k^2)$ is defined in Eqs. (67)–(69) for different regions of the involved parameters. From Eq. (64) it is seen that if the mass-shell equation $k_0^2 - \boldsymbol{k}^2 = 0$ is satisfied, it holds that $\Pi^{\mu\nu}(k_0 = 0) = 0$.

Equations (64) and (67)–(70) coincide with the results of Refs. [16,30,35] for the polarization tensor of graphene at zero temperature. It should also be mentioned that equivalent results [26] were found in the literature by the method of correlation functions in the random-phase approximation [83–86]. The obtained results are unique and neither Π_0^{00} nor $\mathrm{tr}\Pi_0^{\mu\nu}$ can be modified in any way. As to the thermal correction to the polarization tensor $\Delta_T\Pi^{\mu\nu}$, in Sec. II it was shown that it is finite for any D and uniquely defined. Because of this, it is not the subject of regularization, which refers to only the zero-temperature case.

We emphasize that Eqs. (64) and (67)–(69) for the polarization tensor of graphene at zero temperature, where the Fermi velocity $v_{\rm F}$ is put equal to unity, are in agreement with the well-known results of Refs. [1,2] obtained long ago in the framework of standard $(2+1)$-dimensional QED. (The extra factor of 2 is explained by the presence of two Dirac points for graphene.)

V. CONCLUSIONS AND DISCUSSION

In this paper, we have analyzed the problem of the convergence of the polarization tensor of graphene in the framework of the Dirac model. This is an interesting example regarding the application of methods of low-dimensional thermal QFT to a material of great practical importance. Although in the framework of QFT the polarization tensor of graphene is described by a simple one-loop diagram, which was calculated long ago, there are contradictory statements in the literature (mentioned in Sec. I) concerning its convergence, the necessity of its regularization, and the validity of the obtained results. Taking into account that the quantum-field-theoretical approach to the polarization tensor of graphene suggests the most direct and fundamental way of investigating the electrical conductivity and reflectance of graphene, as well as the Casimir effect in graphene systems, it seems necessary to clarify all of the raised points.

For this purpose, we have performed a detailed calculation of the polarization tensor of graphene and analyzed its analytic properties as a function of the number of spacetime dimensions. We emphasize that this tensor consists of the zero-temperature part plus the thermal correction. In so doing, the thermal correction is represented as an integral that converges in the spacetime of any dimensionality. Thus, the question of regularization is irrelevant to the thermal correction and may be raised only with respect to the zero-temperature part of the polarization tensor.

For experts in QFT, the calculation of the polarization tensor in the framework of $(2+1)$-dimensional QED is a rather simple exercise. Because of this, in the classical papers [1,2] the results of this calculation were presented without derivation. In Refs. [16,30,35], again with no detailed derivation, these results were modified for the case of graphene by taking into account the presence of two fundamental velocities.

As discussed in Sec. I, some of the theoretical predictions made using the quantum-field-theoretical polarization tensor (and especially its trace) are in disagreement with those found with the polarization tensor derived using the Kubo formula. To bring both tensors into agreement, an alternative regularization procedure was suggested [73] by imposing an artificial additional condition irrelevant to the rigorous formalism of quantum field theory.

Our detailed analysis of the convergence of the polarization tensor in $D=(2+1)$-dimensional spacetime shows that, although it is formally represented by a divergent integral, its finite value is obtained by analytic continuation. In so doing, one need not to discard any pole terms, which do not appear in the case $D=3$, i.e., the renormalization is not needed. Just this was meant in Refs. [27,28,30], which stated that for $D=3$ the ultraviolet divergences do not appear. After putting the Fermi velocity equal to the speed of light, our results for the zero-temperature polarization tensor are found to be in agreement with the well-known results of Refs. [1,2]. If the two fundamental velocities are present, our results coincide with those given for graphene in Refs. [16,30,35].

We recall that the situation is different in the case of standard QED with $D=3+1$. In this case, the zero-temperature polarization tensor is also obtained by analytic continuation. However, to obtain the finite result, it is necessary to discard the pole term that arises for $D=4$. This pole corresponds to the ultraviolet divergence deleted by means of the renormalization procedure, which must be performed after a regularization. Therefore, there is a principal difference between the character of the divergences of the polarization tensor for the three- and four-dimensional spacetimes. However, in both cases the final results, obtained by analytic continuation from the case of lower dimensionality and (for $D=4$ only) by discarding the pole term and using the normalization condition, are unique and not subject to any modification.

It is also necessary to stress that the presence of a double pole at zero frequency in the transverse dielectric permittivity of graphene proven [41] by using the polarization tensor plays a decisive role in reaching an agreement between theory and measurements of the Casimir force in graphene systems [49–52]. It is well known that for metallic test bodies the theoretical predictions are in agreement with the results of numerous precise experiments on measuring the Casimir force only if the response of metals to the low-frequency electromagnetic field is described by the dissipationless plasma model possessing a double pole at zero frequency [87,88]. This problem was considered as a failure of the dissipative Drude model, possessing a single pole at zero frequency, in the region of transverse electric evanescent waves [89]. Thus, the prediction of a double pole in the transverse dielectric permittivity of graphene in the framework of quantum field theory, as opposed to the Kubo formula, is in favor of the former.

To conclude, the analysis performed in this paper opens opportunities for the wider use of quantum-field-theoretical methods in the investigation of the properties of graphene and other novel materials.

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