Light-cone thermodynamics: Purification of the Minkowski vacuum

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We explicitly express the Minkowski vacuum of a massless scalar field in terms of the particle notion associated with suitable spherical conformal killing fields. These fields are orthogonal to the light wave fronts originating from a sphere with a radius of r_H in flat spacetime: a bifurcate conformal killing horizon that exhibits semiclassical features similar to those of black hole horizons and Cauchy horizons of spherically symmetric black holes. Our result highlights the quantum aspects of this analogy and extends the well-known decomposition of the Minkowski vacuum in terms of Rindler modes, which are associated with the boost Killing field normal to a pair of null planes in Minkowski spacetime (the basis of the Unruh effect). While some features of our result have been established by the theorems of Kay and Wald in the 1990s—on quantum field theory in stationary spacetimes with bifurcate Killing horizons—the added value we provide here lies in the explicit expression of the vacuum.

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I. INTRODUCTION

Light cones emanating from a sphere of radius r_H in Minkowski spacetime satisfy a set of laws that are the analog of the thermodynamic laws satisfied by black holes [1]. When tested with conformally invariantly coupled scalar fields, this is a consequence of the fact that the geometry and (conformal) symmetries of flat spacetime coincide with the geometry and symmetries of certain stationary black hole solutions [2]. In addition to satisfying a version of the zeroth, first, second, and third laws of black hole mechanics, it is shown that suitable accelerated observers following the orbits of spherical conformal Killing vector fields (for whom the light cones are conformal Killing horizons) perceive the Minkowski vacuum as a thermal state with a (conformal) temperature $T = \kappa/(2\pi)$ with κ a natural notion of surface gravity (we review some of the details below). Even though the thermality of the vacuum was derived explicitly in this work, it can be seen as the consequence of a simple adaptation to conformal Killing bifurcate horizon of general theorems by Wald and Kay [3]. While there is a vast literature devoted to the restriction of the Minkowski vacuum to the domain of dependence of a ball [4–8] (and generalizations to maximally symmetric spaces [9]), here we are interested in the description of the Minkowski vacuum in the entire spacetime. In particular, it is the causal complement of the so-called diamond that actually represents (in the sense of [1]) the region accessible to the (analog of the) outside stationary observers of a black hole. We will show that the Minkowski vacuum can be decomposed in terms of modes whose time evolution is adapted to conformal Killing vector fields of flat spacetime that have the previously mentioned light cones as horizons. Our result is the generalization of Unruh's [10] where, instead of uniformly accelerated observers moving away from a plane wave that defines their Rindler horizon, we have a family of suitable radially accelerating observers moving away from a spherical wave front of radius r_H at $t = 0$. Instead of trying to compute explicitly Bogoliubov coefficients via inner products and projections, we will use Unruh's original technique consisting of characterizing positive frequency modes by their analyticity properties in the complex plane of complexified time.

II. THE SPHERICAL CONFORMAL KILLING FIELDS OF INTEREST

We start from the Minkowski metric in spherical coordinates

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\varphi^2, \qquad (1)$$

and then we introduce advanced and retarded time null coordinates

$$\begin{aligned} v &\equiv t + r, \\ u &\equiv t - r. \end{aligned} \qquad (2)$$

In terms of these, the spherical conformal Killing fields of our interest are written as

$$\xi^a = \frac{v^2 - r_H^2}{r_0^2 - r_H^2}\left(\frac{\partial}{\partial v}\right)^a + \frac{u^2 - r_H^2}{r_0^2 - r_H^2}\left(\frac{\partial}{\partial u}\right)^a, \qquad (3)$$

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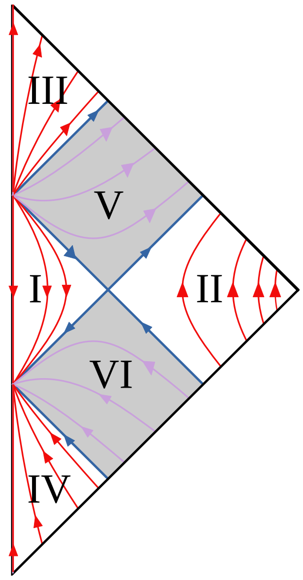

FIG. 1. Integral lines of the spherical conformal Killing field (3). The vector field becomes null on the light wave fronts emanating from a sphere of radius r_H (by choice here at $t = 0$). These light fronts are bifurcate conformal Killing horizons with a bifurcation surface given by the same sphere.

which are completely characterized by two parameters r_0 and r_H. Its norm is given by

$$\xi \cdot \xi = -\frac{(v^2 - r_H^2)(u^2 - r_H^2)}{(r_0^2 - r_H^2)^2}, \tag{4}$$

whose sign divides flat spacetime in six different regions separated by a bifurcate conformal killing horizon where it vanishes (at the light fronts $u = \pm r_H$ and $v = \pm r_H$); see Fig. 1. The interpretation of the free parameters is the following: at $t = 0$ the sphere of radius r_0 is a sphere where $\xi \cdot \xi = -1$, while the sphere of radius r_H is the place where the conformal Killing field vanishes (the bifurcating sphere). We will assume that $r_0 > r_H$, so that the Killing is normalized somewhere in the outside region of the black hole analog in the sense of [1,2]. The vector field (4) is null on the light cones defined by

$$u = u_\pm = \pm r_H, \qquad v = v_\pm = \pm r_H. \tag{5}$$

Hence the conformal Killing vector field (3) divides the spacetime into six separate regions (see Fig. 2) which are the analog of the regions one finds in the Penrose diagram of nonextremal spherical black holes [1,2].

Positive frequency solutions of the massless Klein-Gordon equation, defined with respect to the inertial time t, can be used to construct the one-particle Hilbert space $\mathcal{H}$ of the massless scalar field and then the associated Fock space $\mathcal{F}$ containing all excitations in Minkowski spacetime. Similarly, there is a natural construction of the Fock space associated with any of the four regions where the conformal Killing vector field (3) is timelike. Each arises from the notion the positive frequency solutions with respect to the conformal Killing time in each of these regions. In this paper we will explicitly write the vacuum in $\mathcal{F}$ in terms of several alternative expressions in terms of excitations in the other Fock spaces $\mathcal{F}_{\rm I}$, $\mathcal{F}_{\rm II}$, $\mathcal{F}_{\rm III}$, and $\mathcal{F}_{-\rm III}$.

At the horizon, the conformal Killing field (3) satisfies the equation

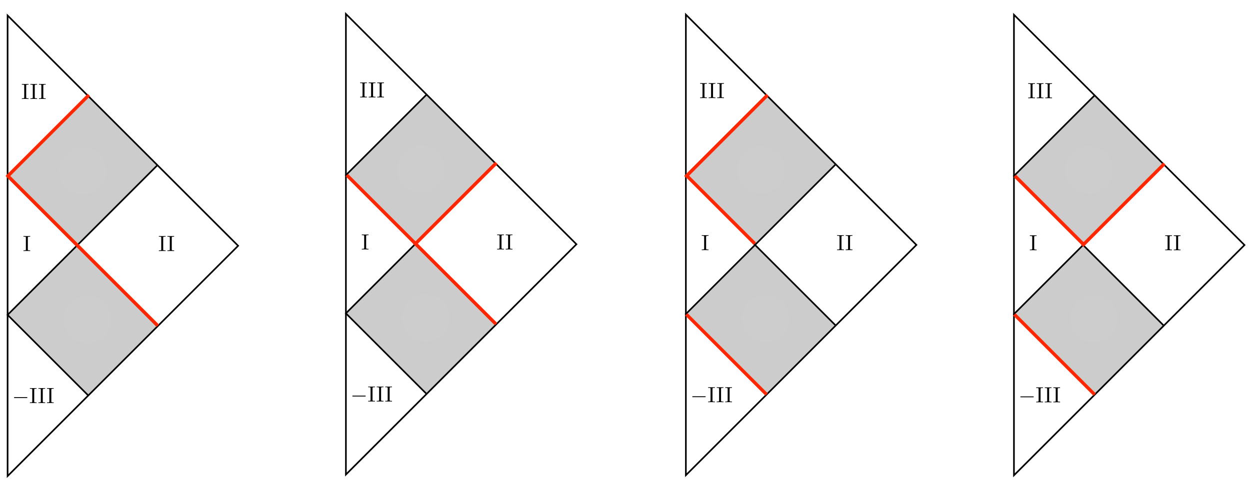

FIG. 2. The causal character of the spherical conformal Minkowski Killing vector field divides flat spacetime into six different regions. The shaded regions correspond to those where the conformal Killing vector field is spacelike. Fock spaces can be constructed according to the positive frequency notion associated with the Killing time in the four white regions where the vector field is timelike. We call such Hilbert spaces $\mathcal{F}_{\rm I}$, $\mathcal{F}_{\rm II}$, $\mathcal{F}_{\rm III}$, and $\mathcal{F}_{-\rm III}$. The Fock space constructed from positive frequency solutions in inertial time will be called $\mathcal{F}$. Solutions of the massless Klein-Gordon equation in Minkowski spacetime can be fully characterized by their value on the portions of null surfaces emphasized in red. This is the key for the different ways one can express the Minkowski vacuum presented in Eqs. (56)–(59).

$$\nabla_a(\xi\cdot\xi) \triangleq -2\kappa\xi_a, \tag{6}$$

where

$$\kappa \equiv \frac{2r_H}{r_0^2 - r_H^2} \tag{7}$$

plays the role of the surface gravity in the analogy with the black hole [1] and corresponds to the temperature notion $T = \kappa/(2\pi)$ appearing in the expression of the Minkowski vacuum that we provide here.

III. MINKOWSKI MODES IN SPHERICAL COORDINATES

In this section we characterize the positive frequency solutions of the massless Klein-Gordon equation with respect to inertial time t when written in spherical coordinates. The choice of spherical coordinates is necessary for the decomposition of these modes in terms of the modes that are positive frequency with respect to the conformal killing time of the conformal spherical Killing vector fields introduced in what follows. In particular, the spherical coordinates are adapted to the geometry of the light cones (the conformal Killing horizons) that play a central role in the decomposition of the Minkowski vacuum that we seek.

We start by writing the Minkowski metric in spherical coordinates

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2(\theta)\mathrm{d}\varphi^2, \tag{8}$$

and the conformally invariant Klein-Gordon equation which, in the previous coordinates and on the flat background, becomes

$$\begin{aligned}0 &= \left(\Box - \frac{1}{6}R\right)\Phi(x)\\ &= \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu)\Phi(x)\\ &= \left(-\frac{\partial^2}{\partial t^2} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial}{\partial\theta}\right)\right.\\ &\quad\left. + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2}{\partial\varphi^2}\right)\Phi(x).\end{aligned} \tag{9}$$

Using the ansatz

$$\Phi_{\omega\ell m}(x) = e^{-i\omega t}Y_{\ell m}(\theta,\varphi)R_\ell(r), \tag{10}$$

the Klein-Gordon equation reads

$$\left(\omega^2 + \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\ell(\ell+1)}{r^2}\right)R_\ell(r) = 0, \tag{11}$$

with two linearly independent solutions given by the spherical Bessel functions $j_\ell(\omega r)$ and $y_\ell(\omega r)$. Regularity at the origin discards the $y_\ell(\omega r)$. Thus, a basis of the solutions of the Klein-Gordon equation in spherical coordinates is given by

$$\Phi_{\omega\ell m}(x) = e^{-i\omega t}Y_{\ell m}(\theta,\varphi)j_\ell(\omega r). \tag{12}$$

Such states are not normalizable in the one-particle Hilbert space $\mathcal{H}$. Nevertheless, it will be convenient to work with them formally. Normalizable states peaked on the relevant quantum numbers can be constructed as superpositions of the previous states.

A. Minkowski-time positivity of frequency as single-ray analyticity

Solutions of the massless Klein-Gordon equation are completely characterized by their value on the union of the future light cone $u = r_H$ and the past light cone $v = r_H$ with $u \leq r_H$. In the surface $u = r_H$,

$$\Phi_{\omega\ell m}(v,\theta,\varphi) = Y_{\ell m}(\theta,\varphi)e^{-i\omega(v+r_H)/2} \times j_\ell\left(\frac{\omega(v-r_H)}{2}\right) \quad \text{for } v \geq r_H, \tag{13}$$

while for $v = r_H$,

$$\Phi_{\omega\ell m}(u,\theta,\varphi) = Y_{\ell m}(\theta,\varphi)e^{-i\omega(r_H+u)/2} \times j_\ell\left(\frac{\omega(r_H-u)}{2}\right) \quad \text{for } u \leq r_H. \tag{14}$$

A generator (light-ray) labeled by θ and φ in the past section of the light cone corresponds to the one labeled by $\theta = \pi - \theta$ and $\varphi = \varphi + \pi$ in a future section. Under such an antipodal map in the sphere one has that

$$Y_{\ell m}(\pi-\theta,\varphi+\pi) = (-1)^\ell Y_{\ell m}(\theta,\varphi), \tag{15}$$

which combined with the property of spherical Bessel functions

$$j_\ell(-x) = (-1)^\ell j_\ell(x) \tag{16}$$

implies that on a single light-cone generator

$$\Phi_{\omega\ell m}(v,\mathrm{ray}) = A_{\mathrm{ray}}e^{-i\omega(v+r_H)/2} \times j_\ell\left(\frac{\omega(v-r_H)}{2}\right) \quad \text{for } v \geq r_H \tag{17}$$

and

$$\Phi_{\omega\ell m}(u,\mathrm{ray}) = A_{\mathrm{ray}}e^{-i\omega(r_H+u)/2} \times j_\ell\left(\frac{\omega(u-r_H)}{2}\right) \quad \text{for } u \leq r_H, \tag{18}$$

where $A_{\rm ray} = Y_{\ell m}(\theta,\varphi) = (-1)^\ell Y_{\ell m}(\pi-\theta,\varphi+\pi)$. It follows that such a solution, when restricted to a single generator, can be written in terms of a single variable $z\in\mathbb{R}$ which will correspond to either u or v depending on the range. We have

$$\Phi_{\omega\ell m}(z,\mathrm{ray}) = A_{\rm ray} e^{-i\omega(z+r_H)/2} j_\ell\left(\frac{\omega(z-r_H)}{2}\right). \quad (19)$$

On a given generator, the previous solutions are given by the product of two entire functions of the variable z [11] (now promoted to a complex variable). Thus, the previous positive frequency solution corresponds to an analytic function when restricted to a single light-cone generator. Since the previous function is analytic, it can only diverge at infinity. Now, the asymptotic behavior of j_ℓ at infinity is given by

$$j_\ell(z) \approx \frac{1}{z}\sin\left(z-\frac{\ell\pi}{2}\right). \quad (20)$$

Hence,

$$\begin{aligned}\Phi_{\omega\ell m}(z,\mathrm{ray}) &\approx \frac{A_{\rm ray}}{\omega(z-r_H)} e^{-i\omega(r_H+z)/2} \\ &\quad\times\left(e^{i\omega(z-r_H)/2+i\ell\pi/2} - e^{-i\omega(z-r_H)/2-i\ell\pi/2}\right) \\ &= \frac{A_{\rm ray}}{\omega(z-r_H)}\left(e^{-i(\omega r_H-\frac{\ell\pi}{2})} - e^{-\frac{i\ell\pi}{2}}e^{-i\omega z}\right). \quad (21)\end{aligned}$$

From the previous equation we conclude that superpositions of modes with $\omega>0$ correspond to analytic functions of z that are bounded in the lower complex plane $[\Im(z)<0]$. Thus, positive frequency solutions of the Klein-Gordon equations are characterized, on the union of the light cones $v=r_H$ and $u=r_H$ when evaluated on a single generator, by analytic functions of z bounded in the lower complex plane.

IV. SPHERICAL CONFORMAL KILLING MODES

In this section we characterize positive frequency solutions with respect to the conformal Killing time in the different regions where the conformal Killing vector field ξ^a defined in (3) is timelike (see Fig. 2). In each of these regions one can define a Fock quantization using standard methods.

A. Regions II, III, and -III

Let us consider the following coordinate transformation [1]:

$$\begin{aligned} t &= \frac{r_H\sinh(\kappa\tau)}{\cosh(\kappa\rho)-\cosh(\kappa\tau)}, \\ r &= -\frac{r_H\sinh(\kappa\rho)}{\cosh(\kappa\rho)-\cosh(\kappa\tau)}, \end{aligned} \quad (22)$$

with κ the surface gravity (7). In terms of null coordinates the previous transformation takes the form

$$\begin{aligned} u &= t-r = -r_H\coth\left(\frac{\kappa\tilde{u}}{2}\right), \\ v &= t+r = -r_H\coth\left(\frac{\kappa\tilde{v}}{2}\right). \end{aligned} \quad (23)$$

The previous coordinate transformation allows one to write the Minkowski metric in Regions II, III, and –III as

$$\mathrm{d}s^2 = \Omega_{\rm II}^2\left(-\mathrm{d}\tau^2 + \mathrm{d}\rho^2 + \kappa^{-2}\sinh^2(\kappa\rho)\mathrm{d}S^2\right), \quad (24)$$

with

$$\Omega_{\rm II} = \frac{r_H\kappa}{\cosh(\kappa\rho)-\cosh(\kappa\tau)}, \quad (25)$$

where each of the regions is characterized by the range of the coordinates that is best defined in terms of $\tilde{u}$ and $\tilde{v}$. Region II is defined by $\tilde{u}\coloneqq\tau-\rho\in\mathbb{R}^+$ and $\tilde{v}\equiv\tau+\rho\in\mathbb{R}^-$, Region III is defined by $\tilde{u}\coloneqq\tau-\rho\in\mathbb{R}^-$ and $\tilde{v}\equiv\tau+\rho\in\mathbb{R}^-$, Region –III is defined by $\tilde{u}\coloneqq\tau-\rho\in\mathbb{R}^+$ and $\tilde{v}\equiv\tau+\rho\in\mathbb{R}^+$. We would like to characterize solutions of the conformally invariant Klein-Gordon equation

$$\left(\Box - \frac{1}{6}R\right)U = 0 \quad (26)$$

when restricted to (suitable portions of) the boundary of Regions II, III, or –III. Under a conformal transformation $g_{ab}\to g'_{ab} = C^2 g_{ab}$ solutions of (26) defined in terms of g_{ab} are mapped into solutions of the same equation in terms of g'_{ab} by the rule $\Phi\to C^{-1}\Phi$ [12,13]. In the new coordinates Eq. (26) reads

$$\begin{aligned}&\left[-\partial_\tau^2 + \frac{1}{\kappa\sinh^2(\kappa\rho)}\left(\partial_\rho\sinh^2(\kappa\rho)\partial_\rho + \frac{\kappa^2}{\sin(\theta)}\partial_\theta\left(\sin(\theta)\partial_\theta\right)\right.\right. \\ &\quad\left.\left. + \frac{\kappa^2}{\sin^2(\theta)}\partial_\varphi^2\right) + \kappa^2\right]U(x) = 0, \end{aligned} \quad (27)$$

where we used that $R=-6\kappa^2$. Solutions are given by

$$U_{\omega\ell m}(x) = e^{-i\omega\tau}\frac{Q^\ell_{\omega\pm}(\rho)}{\sinh(\kappa\rho)}Y_{\ell m}(\theta,\varphi) \quad (28)$$

with $Q^\ell_{\omega\pm}(\rho)$ satisfying the equation

$$\left(\partial_\rho^2 + \omega^2 - \frac{\ell(\ell+1)\kappa^2}{\sinh^2(\kappa\rho)}\right)Q^\ell_{\omega\pm}(\rho) = 0. \quad (29)$$

A fact that is central in what follows is that the effective potential $-\ell(\ell+1)\kappa^2/\sinh^2(\kappa\rho)$ vanishes exponentially as one approaches any of the internal null boundaries of

Regions II, III, and –III so that solutions of the previous equation are well approximated by free waves

$$U_{\omega\ell m}(x) \approx \frac{e^{-i\omega(\tau\pm\rho)}}{\sinh(\kappa\rho)} Y_{\ell m}(\theta,\varphi), \tag{30}$$

which will lead to a simple plane wave functional dependence when restricted to the null boundaries of the corresponding regions. Explicitly, on the relevant boundaries of the Regions II, III, and –III the solutions are either

$$\Phi_{\omega\ell m}(x) = \Omega_{\rm II}^{-1} U_{\omega\ell m} = \frac{e^{-i\omega\tilde{u}}}{r} Y_{\ell m}(\theta,\varphi) \tag{31}$$

or

$$\Phi_{\omega\ell m}(x) = \Omega_{\rm II}^{-1} U_{\omega\ell m} = \frac{e^{-i\omega\tilde{v}}}{r} Y_{\ell m}(\theta,\varphi), \tag{32}$$

depending on whether we focus on null boundaries of constant v or u, respectively. In the previous equation we have used (22) and (25) to obtain $1/r$ prefactors. Inverting the relationship (23)[1] we can express the solutions with a definite frequency ω—as defined by the accelerated conformal observers—on the past boundary $v = r_H$ of Region II (its past horizon) as

$$\Phi^{\rm II}_{\omega\ell m}(x) = \frac{1}{r_H - u} Y_{\ell m}(\theta,\varphi) e^{-i\frac{\omega}{\kappa}\log\left(\frac{u-r_H}{u+r_H}\right)} \tag{33}$$

for $u \le -r_H$, while for the future horizon $u = -r_H$ of Region II

$$\Phi^{\rm II}_{\omega\ell m}(x) = \frac{1}{v + r_H} Y_{\ell m}(\theta,\varphi) e^{-i\frac{\omega}{\kappa}\log\left(\frac{v-r_H}{v+r_H}\right)} \tag{34}$$

for $v \ge r_H$. Similarly, for the boundary of Region –III, $v = -r_H$, we have

$$\Phi^{-\rm III}_{\omega\ell m}(x) = \frac{1}{u + r_H} Y_{\ell m}(\theta,\varphi) e^{-i\frac{\omega}{\kappa}\log\left(\frac{u-r_H}{u+r_H}\right)}, \tag{35}$$

with $u \le r_H$, while for the boundary of Region III, $u = r_H$, we get

$$\Phi^{\rm III}_{\omega\ell m}(x) = \frac{1}{v - r_H} Y_{\ell m}(\theta,\varphi) e^{-i\frac{\omega}{\kappa}\log\left(\frac{v-r_H}{v+r_H}\right)}, \tag{36}$$

with $v \ge r_H$. Note that the functional form of the modes on the null boundaries of interest is always the same while the range and nature of the variables are different.

B. Region I

Modes in Region I can be described in a way similar to what we have done in the previous section. Instead of (22) one needs to consider the coordinate transformation

$$t = \frac{r_H \sinh(\kappa\tau)}{\cosh(\kappa\rho) + \cosh(\kappa\tau)},$$
$$r = \frac{r_H \sinh(\kappa\rho)}{\cosh(\kappa\rho) + \cosh(\kappa\tau)}. \tag{37}$$

With the new transformation the relation (23) is replaced by

$$v = r_H \tanh\left(\frac{\kappa\tilde{v}}{2}\right),$$
$$u = r_H \tanh\left(\frac{\kappa\tilde{u}}{2}\right). \tag{38}$$

For $\tilde{u} \in \mathbb{R}$ and $\tilde{v} \in \mathbb{R}$ the Minkowski metric in Region I reads

$$ds^2 = \Omega_{\rm I}^2\left(-d\tau^2 + d\rho^2 + \kappa^{-2}\sinh^2(\kappa\rho)dS^2\right), \tag{39}$$

where

$$\Omega_{\rm I} = \frac{r_H \kappa}{\cosh(\kappa\rho) + \cosh(\kappa\tau)}. \tag{40}$$

It follows from the same arguments that the solutions of Eq. (26) on the future null boundary of Region I, $v = r_H$, are given by

$$\Phi^{I}_{\omega\ell m}(x) = \frac{1}{r} Y_{\ell m}(\theta,\varphi) e^{-i\omega\tilde{u}}. \tag{41}$$

Using (38) we can express the modes as a function of Minkowski retarded time[2]

$$\Phi^{\rm I}_{\omega\ell m}(x) = \frac{1}{r_H - u} Y_{\ell m}(\theta,\varphi) e^{-\frac{i\omega}{\kappa}\log\left(\frac{r_H+u}{r_H-u}\right)}. \tag{42}$$

Similarly, on the past null boundary of Region I, $u = -r_H$, the modes are

[1]Explicitly we have

$$\tilde{u} = 2\kappa^{-1}\coth^{-1}\left(-\frac{u}{r_H}\right) = \kappa^{-1}\log\left(\frac{u - r_H}{u + r_H}\right),$$
$$\tilde{v} = 2\kappa^{-1}\coth^{-1}\left(-\frac{v}{r_H}\right) = \kappa^{-1}\log\left(\frac{v - r_H}{v + r_H}\right).$$

[2]The inverse transformation being in this case

$$\tilde{u} = 2\kappa^{-1}\tanh^{-1}\left(\frac{u}{r_H}\right) = \kappa^{-1}\log\left(\frac{u + r_H}{r_H - u}\right),$$
$$\tilde{v} = 2\kappa^{-1}\tanh^{-1}\left(\frac{v}{r_H}\right) = \kappa^{-1}\log\left(\frac{v + r_H}{r_H - v}\right).$$

$$\Phi^{\mathrm{I}}_{\omega\ell m}(x)=\frac{1}{v+r_H}Y_{\ell m}(\theta,\varphi)e^{-\frac{i\omega}{\kappa}\log\left(\frac{r_H+v}{r_H-v}\right)}. \quad (43)$$

All these solutions diverge at $r=0$ due to the vanishing of the conformal factor at that singular point. This is a pathology of the sharp "plane-wave-like" solutions; however, such a divergence cannot survive if we consider suitably normalized wave packets satisfying the usual reflecting boundary conditions at $r=0$.

V. PURIFICATION

The knowledge of the modes on the null boundaries of the different Regions I, II, III, and –III in Fig. 2, as well as their characterization in terms of frequencies with respect to the conformal Killing time τ, is sufficient for writing an explicit expression of the Minkowski vacuum in terms of particle excitations in the Fock quantizations corresponding to these various regions.

A. Vacuum entanglement between Regions I, outgoing II, and III

To find the expression of the Minkowski vacuum in terms of the product of states in $\mathcal{F}_{\mathrm{I}}\otimes(\mathcal{F}^{\mathrm{out}}_{\mathrm{II}}\oplus\mathcal{F}_{\mathrm{III}})$ associated with the Regions I, II (for outgoing modes only), and III, respectively, we focus on the form of the solution of the Klein-Gordon equation on the red null boundaries on the panel on the left of Fig. 2. It is clear that the value of solutions on these null surfaces fully determine the solution everywhere. We can translate this statement in terms of the relevant one-particle Hilbert spaces involved in what follows. The one-particle Hilbert space $\mathcal{H}_{\mathrm{I}}$ is completely characterized by the value of the (normalizable) positive frequency solutions with respect to the conformal killing time in Region I when restricted to the future null boundary of Region I. The one-particle Hilbert space $\mathcal{H}^{\mathrm{out}}_{\mathrm{II}}$ of outgoing modes in Region II is completely characterized by the value of the (normalizable) positive frequency solutions with respect to the conformal killing time in Region II when restricted to the past null boundary of Region II. The one-particle Hilbert space $\mathcal{H}_{\mathrm{III}}$ is completely characterized by the value of the (normalizable) positive frequency solutions with respect to the conformal Killing field in Region III, when restricted to the past null boundary of Region III. The one-particle Hilbert space of positive frequency solutions with respect to inertial time t will be denoted by $\mathcal{H}$.

Now, we will construct a Minkowski inertial time positive frequency solution by combining a definite (conformal time) frequency solution in Regions I, II, and III. According to our previous analysis—recall Eqs. (33), (34), and (42)—these are given by

$$\begin{aligned}
f^{\mathrm{IIout}}_{\omega\ell m}&=\frac{(-1)^\ell Y_{\ell m}(\theta,\varphi)}{r_H-u}\exp\left(-\frac{i\omega}{\kappa}\log\left(\frac{u-r_H}{u+r_H}\right)\right), && u<-r_H \quad \text{(the past horizon of II)},\\
f^{\mathrm{I}}_{\omega\ell m}&=\frac{(-1)^\ell Y_{\ell m}(\theta,\varphi)}{r_H-u}\exp\left(\frac{i\omega}{\kappa}\log\left(\frac{r_H-u}{u+r_H}\right)\right), && -r_H\le u\le r_H \quad \text{(the future horizon of I)},\\
f^{\mathrm{III}}_{\omega\ell m}&=\frac{Y_{\ell m}(\theta,\varphi)}{r_H-v}\exp\left(-\frac{i\omega}{\kappa}\log\left(\frac{v-r_H}{v+r_H}\right)\right), && v>r_H \quad \text{(the past horizon of III)}, \quad (44)
\end{aligned}$$

where the $(-1)^\ell$ has been included to simplify the expressions that follow when writing a positive frequency solution in inertial Minkowski time (recall the need for single-ray analyticity). Consequently, focusing on a single generator of the light cone and using the variable z to represent both u and v, we can write

$$\begin{aligned}
f^{\mathrm{IIout/III}}_{\omega}&=\frac{1}{r_H-z}\exp\left(-\frac{i\omega}{\kappa}\log\left(\frac{z-r_H}{z+r_H}\right)\right), && |z|>r_H,\\
f^{\mathrm{I}}_{\omega}&=\frac{1}{r_H-z}\exp\left(\frac{i\omega}{\kappa}\log\left(\frac{r_H-z}{z+r_H}\right)\right), && |z|<r_H.
\end{aligned} \quad (45)$$

For $\omega>0$, these modes are positive frequency with respect to the time notion associated with the conformal "observers" time τ defined in Regions II/III and I, respectively. Let us promote z to a complex variable and consider the function

$$F_\omega(z)=\frac{1}{r_H-z}\exp\left(-\frac{i\omega}{\kappa}\log\left(\frac{z-r_H}{z+r_H}\right)\right). \quad (46)$$

This function is analytic everywhere with the exception of the real interval $[-r_H,r_H]$ where it has a branch cut and a pole at $z=r_H$ (which corresponds to $r=0$). The branch cut is present due to the infinite blueshift effect approaching the null boundaries for the conformal observers with conformal Killing time. The pole at $z=r_H$ is due to the vanishing of the conformal factor at $r=0$; this pole is present in the wave solutions that we use here for simplicity, and it disappears when considering a suitable basis of normalizable wave packets with the customary reflecting boundary conditions at the origin. Hence, restricting to the lower complex plane, the previous function is analytic and bounded; therefore, according to the analysis below Eq. (21), it is a positive-frequency mode with respect to inertial time t. Indeed, this function can be seen as the restriction of a positive frequency mode on the same components of null boundaries where Eq. (44) is evaluated, and hence fully determining a unique solution of the Klein-Gordon equation.

Evaluating the previous solution near the real line from below, namely at $z = u - i\epsilon$, with u real and $\epsilon > 0$, we see that it coincides with $f_\omega^{\mathrm{II/III}}$ in the limit $\epsilon \to 0$ for $|u| > r_H$. Now for $-r_H < u < r_H$ the situation is more subtle due to the presence of the branch cut when $\epsilon = 0$. Indeed, we have

$$\begin{aligned} F_\omega(u - i\epsilon) &= \frac{1}{r_H - u + i\epsilon} \exp\left(-\frac{i\omega}{\kappa}\log\left(\frac{u - r_H - i\epsilon}{u + r_H - i\epsilon}\right)\right) \\ &= \frac{1}{r_H - u + i\epsilon} \exp\left(-\frac{i\omega}{\kappa}\log\left(\frac{r_H - u - i\epsilon}{u + r_H - i\epsilon} e^{-i(\pi - \mathcal{O}(\epsilon))}\right)\right) \\ &= \frac{1}{r_H - u + i\epsilon} e^{-\frac{(\pi - \mathcal{O}(\epsilon))\omega}{\kappa}} \exp\left(-\frac{i\omega}{\kappa}\log\left(\frac{r_H - u - i\epsilon}{u + r_H - i\epsilon}\right)\right) \\ &\underset{\epsilon \to 0}{\longrightarrow} e^{-\frac{\pi\omega}{\kappa}} \bar{f}_\omega^{\mathrm{I}}. \end{aligned} \tag{47}$$

Thus, we have showed that

$$F_\omega(u) = f_\omega^{\mathrm{IIout}} + f_\omega^{\mathrm{III}} + e^{-\frac{\pi\omega}{\kappa}} \bar{f}_\omega^{\mathrm{I}} \tag{48}$$

is a positive frequency in Minkowski time. If instead of working with the previous non-normalizable states we build a basis of wave packets f_ω sharply peaked about the frequency ω, then the following combination is also a positive frequency in Minkowski time:

$$F_\omega(u) = f_\omega^{\mathrm{II-out}} + f_\omega^{\mathrm{III}} + e^{-\frac{\pi\omega}{\kappa}} \bar{f}_\omega^{\mathrm{I}}. \tag{49}$$

We can repeat the procedure considering the function

$$F'_\omega(z) = \frac{1}{r_H - z} \exp\left(\frac{i\omega}{\kappa}\log\left(\frac{r_H - z}{r_H + z}\right)\right). \tag{50}$$

This function is analytic everywhere but in the real intervals $(-\infty, -r_H]$ and $[r_H, +\infty)$ and at the pole $z = r_H$. As before, we evaluate F'_ω at $z = u - i\epsilon$, with u real and $\epsilon > 0$. For $-r_H < u < r_H$ it gives f_ω^{I} in the limit $\epsilon \to 0$. For $u > r_H$ we have

$$\begin{aligned} F'_\omega(u - i\epsilon) &= \frac{1}{r_H - u + i\epsilon} \exp\left(\frac{i\omega}{\kappa}\log\left(\frac{r_H - u + i\epsilon}{r_H + u - i\epsilon}\right)\right) \\ &= \frac{1}{r_H - u + i\epsilon} \exp\left(\frac{i\omega}{\kappa}\log\left(\frac{u - r_H + i\epsilon}{u + r_H - i\epsilon} e^{i(\pi - \mathcal{O}(\epsilon))}\right)\right) \\ &= \frac{1}{r_H - u + i\epsilon} e^{-\frac{(\pi - \mathcal{O}(\epsilon))\omega}{\kappa}} \exp\left(\frac{i\omega}{\kappa}\log\left(\frac{u - r_H + i\epsilon}{u + r_H - i\epsilon}\right)\right) \\ &\underset{\epsilon \to 0}{\longrightarrow} e^{-\frac{\pi\omega}{\kappa}} \bar{f}_\omega^{\mathrm{III}}. \end{aligned} \tag{51}$$

The same result applies for $u < -r_H$ (with $\bar{f}_\omega^{\mathrm{II-out}}$ instead of $\bar{f}_\omega^{\mathrm{III}}$). Thus, also the following linear combination is positive frequency

$$F'_\omega(u) = f_\omega^{\mathrm{I}} + e^{-\frac{\pi\omega}{\kappa}} (\bar{f}_\omega^{\mathrm{IIout}} + \bar{f}_\omega^{\mathrm{III}}), \tag{52}$$

with the wave packet argument still being valid. Hence we can compute the S-matrix S. By inspection

$$\begin{aligned} CF_\omega &= f_\omega^{\mathrm{II-out}} + f_\omega^{\mathrm{III}}, \qquad CF'_\omega = f_\omega^{\mathrm{I}}, \\ DF_\omega &= e^{-\frac{\pi\omega}{\kappa}} \bar{f}_\omega^{\mathrm{I}}, \qquad DF'_\omega = e^{-\frac{\pi\omega}{\kappa}} \left(\bar{f}_\omega^{\mathrm{II-out}} + \bar{f}_\omega^{\mathrm{III}}\right), \end{aligned} \tag{53}$$

where $C: \mathcal{H} \to \mathcal{H}_{\mathrm{I}} \otimes (\mathcal{H}_{\mathrm{IIout}} \oplus \mathcal{H}_{\mathrm{III}})$ is the map that gives the positive conformal frequency part of an inertial time positive frequency solution, and $D: \mathcal{H} \to \bar{\mathcal{H}}_{\mathrm{I}} \otimes (\bar{\mathcal{H}}_{\mathrm{IIout}} \oplus \bar{\mathcal{H}}_{\mathrm{III}})$ is the map that gives the negative conformal frequency part of an inertial time positive frequency solution. It follows from the previous equations that

$$\begin{aligned} DC^{-1}\left(f_\omega^{\mathrm{II-out}} + f_\omega^{\mathrm{III}}\right) &= e^{-\frac{\pi\omega}{\kappa}} \bar{f}_\omega^{\mathrm{I}}, \\ DC^{-1} f_\omega^{\mathrm{I}} &= e^{-\frac{\pi\omega}{\kappa}} \left(\bar{f}_\omega^{\mathrm{IIout}} + \bar{f}_\omega^{\mathrm{III}}\right). \end{aligned} \tag{54}$$

Since $\{f_\omega^{\mathrm{I}}\}$ and $\{f_\omega^{\mathrm{II-out}} + f_\omega^{\mathrm{III}}\}$ jointly span $\mathcal{H}_2 = \mathcal{H}_{\mathrm{I}} \otimes (\mathcal{H}_{\mathrm{II-out}} \oplus \mathcal{H}_{\mathrm{III}})$, the previous equation determines the fundamental two-particle state $\mathcal{E} = \overline{DC}^{-1}$ which can be written as

$$\mathcal{E}^{ab} = 2\sum_{\omega} e^{-\frac{\pi\omega}{\kappa}} f_{\omega\mathrm{I}}^{(a}(f_{\omega\mathrm{IIout}} + f_{\omega\mathrm{III}})^{b)}. \tag{55}$$

A standard construction [12] leads to the expression of the state $S|0\rangle$ (where $|0\rangle$ denotes the Minkowski vacuum), namely

$$\boxed{S|0\rangle_M = \prod_{\omega\ell m}\left(\sum_{n=0}^{\infty} e^{-\frac{n\pi\omega}{\kappa}}|n,\omega,\ell,m\rangle_{\mathrm{I}} \otimes \left(|n,\omega,\ell,m\rangle_{\mathrm{II-out}} \oplus |n,\omega,\ell,m\rangle_{\mathrm{III}}\right)\right)}, \tag{56}$$

where $|n,\omega\rangle$ are Fock space "basis" states with a definite number of particles n in the conformal frequency mode ω_i in Region I, the outgoing modes of Region II, or all the modes of Region III, respectively. This is our main result. It clearly shows that the reduced density matrix obtained from tracing $|0\rangle\langle 0|$ over $\mathcal{H}_{\mathrm{I}}$ is thermal with temperature $T = \kappa/(2\pi)$ as in [1].

B. Other equivalent purifications

It is possible to rewrite Eq. (56) in different equivalent forms. For instance, from the form of the conformal invariant Klein-Gordon equation in the region where the conformal Killing field is spacelike (the gray regions in Fig. 2) it is easy to see that an ingoing positive frequency solution in Region II corresponds to a positive frequency solution in Region III (see Appendix). This allows for a trivial identification between the Hilbert spaces $\mathcal{F}_{\mathrm{III}}$ and $\mathcal{F}_{\mathrm{II-in}}$, and the rewriting of Eq. (56) as

$$\boxed{U|0\rangle_M = \prod_{\omega\ell m}\left(\sum_{n=0}^{\infty} e^{-\frac{n\pi\omega}{\kappa}}|n,\omega,\ell,m\rangle_{\mathrm{I}} \otimes |n,\omega,\ell,m\rangle_{\mathrm{II}}\right)}. \tag{57}$$

Similarly, one can write using the trivial isomorphism between $\mathcal{F}_{-\mathrm{III}}$ and $\mathcal{F}_{\mathrm{II-out}}$

$$\boxed{U|0\rangle_M = \prod_{\omega\ell m}\left(\sum_{n=0}^{\infty} e^{-\frac{n\pi\omega}{\kappa}}|n,\omega,\ell,m\rangle_{\mathrm{I}} \otimes \left(|n,\omega,\ell,m\rangle_{-\mathrm{III}} \oplus |n,\omega,\ell,m\rangle_{\mathrm{III}}\right)\right)}. \tag{58}$$

Finally, the time reverse of (56) also holds

$$\boxed{U|0\rangle_M = \prod_{\omega\ell m}\left(\sum_{n=0}^{\infty} e^{-\frac{n\pi\omega}{\kappa}}|n,\omega,\ell,m\rangle_{\mathrm{I}} \otimes \left(|n,\omega,\ell,m\rangle_{-\mathrm{III}} \oplus |n,\omega,\ell,m\rangle_{\mathrm{II-in}}\right)\right)}. \tag{59}$$

In the extremal case $\kappa = 0$, where Region I shrinks to a point, one can show that the Minkowski vacuum coincides with the vacuum of the conformal observers associated with (3). The reason can be traced to the absence of a branch cut in the relationship between the two notions of retarded and advanced times (see appendix in [1]).

VI. CONFORMAL KILLING OBSERVERS ARE UNIFORMLY ACCELERATING

An observer following one integral line of the conformal Killing field has four velocity given by

$$u^a = \frac{\xi^a}{\sqrt{-\xi_b\xi^b}} = \sqrt{\frac{v^2 - r_H^2}{u^2 - r_H^2}}\left(\frac{\partial}{\partial v}\right)^a + \sqrt{\frac{u^2 - r_H^2}{v^2 - r_H^2}}\left(\frac{\partial}{\partial u}\right)^a. \tag{60}$$

This four velocity describes radially accelerating observers with constant acceleration $a^b \coloneqq u^a\nabla_a u^b$, i.e., $u^a\nabla_a(a\cdot a) = 0$. The magnitude of the acceleration is explicitly given by

$$|a| = \sqrt{a_\mu a^\mu} = \kappa\frac{r}{r_H}\frac{1}{\sqrt{\xi_\mu\xi^\mu}}. \tag{61}$$

Standard results in quantum field theory imply that such a uniformly accelerating observer will sense a temperature $T_{\rm obs} = |a|/(2\pi)$, for instance, if sensed by an idealized Unruh-DeWitt detector following such orbits. The temperature measured differs from the conformal invariant notion $T \equiv \kappa/(2\pi)$—appearing in the purification of our formulas—by the constant quantity $r/(r_H\sqrt{\xi_\mu \xi^\mu})$. The mismatch can be understood as follows: the temperature in our purification formulas refers to a conformally invariant property of the Minkowski vacuum of a conformally invariant scalar field. Such a notion coincides with a physical temperature only in the conformal geometry where ξ is an actual Killing field [2]. In all other conformally related geometries, physical thermometers break conformal invariance, and additional geometric considerations are needed in order to relate T and $T_{\rm obs}$. One can be explicit in the case of the Minkowski metric written as in Eq. (39) whose Euclidean continuation obtained via the replacement $\tau \to -i\tau_E$ is

$$\mathrm{d}s^2 = \Omega_{\rm I}^2(-i\tau_E, \rho)\left(\mathrm{d}\tau_E^2 + \mathrm{d}\rho^2 + \kappa^{-2}\sinh^2(\kappa\rho)\mathrm{d}S^2\right), \qquad (62)$$

with

$$\Omega_{\rm I}(-i\tau_E, \rho) = \frac{r_H \kappa}{\cosh(\kappa\rho) + \cos(\kappa\tau_E)}. \qquad (63)$$

The conformal invariant thermality of the Minkowski vacuum—at the root of our purification formulas—resides in the periodicity of the metric with period $2\pi/\kappa$ in imaginary conformal Killing time τ_E. Regularity of the conformal transformations implies that such periodicity cannot be changed; hence, the conformal invariant character of $T \equiv \kappa/(2\pi)$. Now the physical temperature $T_{\rm obs}$ measured by a local Unruh-DeWitt device is sensitive to the geometric periodicity in imaginary proper time. Such a period depends on the conformal factor and can be computed as follows:

$$\oint_\tau \mathrm{d}s = \int_0^{\frac{2\pi}{\kappa}} \Omega_{\rm I}\mathrm{d}\tau_E = \frac{2\pi}{\kappa}\left(\frac{r_H}{r}\sqrt{\xi_\mu \xi^\mu}\right) = \frac{2\pi}{|a|}. \qquad (64)$$

Thus, even when a single observer following one single orbit of the conformal killing field coincides with an Unruh observer, the form of the Minkowski vacuum state for a conformally invariant scalar field depends (as usual in quantum field theory) on nonlocal features that preclude the naive comparison when using pointlike probes.

VII. DISCUSSION

We have explicitly written the Minkowski vacuum in terms of the particle modes defined by observers moving along spherical conformal killing vector fields. These observers represent accelerated observers moving radially away from a sphere of radius r_H and have causal horizons (conformal Killing horizons), which are given by the light surfaces emanating from that sphere at $t = 0$. The formula we derive is the analog of the one derived by Unruh in terms of Rindler particle states associated with constantly accelerated observers following the boost Killing vector field. We have found the result by exploiting the analyticity properties that define positive frequency solutions in inertial time. A direct derivation using Bogoliubov coefficients computed via the suitably defined Klein-Gordon inner product may be available but does not seem to be the most direct avenue to the final expressions (the use of the characterization of modes in terms of null surfaces is natural and simple using our techniques). A conformal transformation maps Minkowski spacetime to a portion of the Bertotti-Robinson spacetime [2], which describes in a suitable approximation near-horizon physics of a near-extremal black hole. It is potentially interesting to consider using our result to analyze features of quantum field theory on such backgrounds [14]. The massless scalar field quantum theory has been chosen for simplicity. We expect that our results should naturally generalize to any (free) conformal invariant model of quantum fields.

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APPENDIX: SOLUTIONS OF THE KLEIN-GORDON EQUATIONS ADAPTED TO THE CONFORMAL KILLING FIELD IN THE REGIONS WHERE IT IS SPACELIKE

In this section we briefly describe some key features of the solutions of the Klein-Gordon equation in the gray regions of Fig. 2 where the conformal Killing field is spacelike. For concreteness we focus on the region to the future of Region II, which we call Region V (the analysis is basically the same in the other one). The coordinate transformation of interest is given by

$$t = \frac{r_H \cosh(\kappa\tau)}{\sinh(\kappa\tau) + \sinh(\kappa\rho)}, \qquad (\mathrm{A1})$$

$$r = \frac{r_H \cosh(\kappa\rho)}{\sinh(\kappa\tau) + \sinh(\kappa\rho)}, \qquad (\mathrm{A2})$$

with $\tau, \rho \in \mathbb{R}^+$. The double null coordinates are given by

$$v = t + r = r_H \coth\left(\frac{\kappa\tilde{v}}{2}\right), \tag{A3}$$

$$u = t - r = r_H \tanh\left(\frac{\kappa\tilde{u}}{2}\right), \tag{A4}$$

with $u \in \mathbb{R}$ while $v \in \mathbb{R}^+$. In these coordinates, which only cover Region V, the Minkowski metric reads

$$\mathrm{d}s^2 = \Omega_{\mathrm{V}}^2(\mathrm{d}\tau^2 - \mathrm{d}\rho^2 + \kappa^{-2}\cosh^2(\kappa\rho)\mathrm{d}S^2), \tag{A5}$$

with conformal factor given by

$$\Omega_{\mathrm{V}} = \frac{r_H\kappa}{\sinh(\kappa\tau) + \sinh(\kappa\rho)}. \tag{A6}$$

The Klein-Gordon equation, $(\Box - \frac{R}{6})U = 0$, in the conformal metric $ds^2/\Omega_{\mathrm{V}}^2$ reads

$$\left(-\frac{1}{\cosh^2(\kappa\rho)}\partial_\rho(\cosh^2(\kappa\rho)\partial_\rho) + \partial_\tau^2 + \frac{\kappa^2}{\cosh^2(\kappa\rho)} \times\left(\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta) + \frac{1}{\sin^2\theta}\partial_\varphi^2\right) - \kappa^2\right)U = 0. \tag{A7}$$

With the ansatz

$$U_{\omega\ell m} = e^{-i\omega\tau}\frac{Q_{\omega\ell}(\rho)}{\cosh(\kappa\rho)}Y_{\ell m}(\theta,\varphi), \tag{A8}$$

the Klein-Gordon equation reduces to

$$\left(\omega^2 + \frac{\partial^2}{\partial\rho^2} + \frac{\ell(\ell+1)\kappa^2}{\cosh^2(\kappa\rho)}\right)Q_{\omega\ell}(\rho) = 0, \tag{A9}$$

which, on the boundary, reduces to the same wave equation one finds in all the other regions. The consequence of this is that there is a one-to-one correspondence between positive frequency solutions in Region III and ingoing positive frequency solutions in Region II. The same holds true for solutions in Region –III and outgoing solutions in Region II. The reason is that the quantum number ω is conserved across the boundaries as the boundary characteristic data coincide. This implies that one can have a trivial identification between wave packets defining a basis of the one-particle Hilbert spaces $\mathcal{H}_{\mathrm{III}}$ and $\mathcal{H}_{\mathrm{II}}^{\mathrm{in}}$, as well as between elements of a basis of $\mathcal{H}_{-\mathrm{III}}$ and $\mathcal{H}_{\mathrm{II}}^{\mathrm{out}}$.

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