New derivation of the amplitude of asymptotic oscillatory tails of weakly delocalized solitons

Gyula Fodor,[1] Péter Forgács,[1,2] and Muneeb Mushtaq[1]

[1]*HUN-REN Wigner Research Centre for Physics, 1525 Budapest 114, P.O. Box 49, Hungary*
[2]*Institut Denis-Poisson CNRS/UMR 7013, Université de Tours, Parc de Grandmont, 37200 Tours, France*

The computation of the amplitude, α, of asymptotic standing wave tails of weakly delocalized, stationary solutions in a fifth-order Korteweg–de Vries equation is revisited. Assuming the coefficient of the fifth order derivative term, $\epsilon^2 \ll 1$, a new derivation of the "beyond all orders in ϵ" amplitude, α, is presented. It is shown by asymptotic matching techniques, extended to higher orders in ϵ, that the value of α can be obtained from the asymmetry at the center of the unique solution exponentially decaying in one direction. This observation, complemented by some fundamental results of Hammersley and Mazzarino [Proc. R. Soc. A **424**, 19 (1989)], not only sheds new light on the computation of α but also greatly facilitates its numerical determination to a remarkable precision for so small values of ϵ, which are beyond the capabilities of standard numerical methods.

I. INTRODUCTION

As it has been observed in numerous systems, spatially localized, "soliton"-type excitations become "weakly" delocalized, by developing asymptotic "shelves" or oscillating tails under the influence of certain, physically important perturbations [1–3]. An approximate picture of a weakly delocalized soliton would consist of a well defined core and of a small amplitude wave. Interestingly, slowly radiating lumps in various field theories, referred to as pulsons/oscillons [4,5] belong to a related class of problems. Even if oscillons are not solitons, they do have numerous physical applications, as many of them exhibit remarkable stability properties and extremely long lifetimes [6–8]. Slowly radiating oscillons have a well-defined core, containing most of their energy, together with small amplitude outgoing waves. The simple idea to approximate slowly radiating oscillons through an assumed adiabatic evolution of appropriate *stationary* configurations, called "quasibreathers"—weakly delocalized lumps with asymptotic standing wave tails [9,10] has proved to be quite fruitful. Quasibreathers are time periodic, and can be thought of as oscillons made stationary by incoming radiation from infinity. Determination of the amplitude of the quasibreather wave tails is necessary to deduce the radiation rate of time-dependent oscillons. It is rather nontrivial to compute, either

numerically or perturbatively, the amplitude of quasibreather standing wave tails, since they are exponentially suppressed, hence beyond all orders in perturbation theory. For some recent reviews on oscillons and quasibreathers see, e.g., [11–13].

As brilliantly shown in the pioneering works of [14–16] the fate of spatially localized solutions in a large class of singularly perturbed equations is conveniently analyzed using methods referred to as "beyond all orders" asymptotics. A number of techniques have been developed, starting with asymptotic matching in the complex plane [14,15], to obtain an exponentially suppressed, but crucial effect of the perturbation.

The Korteweg–de Vries (KdV) equation modified by a fifth order derivative term (fKdV) has been derived in various physical contexts [17–20]. When the fifth order derivative term can be treated as a small perturbation, the fKdV equation admits weakly delocalized solitons with asymptotic wave tails. The amplitude of the asymptotic wave tails are typically "beyond all orders" and it has been computed by several methods [16,21–24]. In a recent paper [25] we have computed higher order perturbative corrections to the leading "beyond all orders" amplitude of the asymptotic wave tails, significantly extending previous result of [21]. In a remarkable, although somewhat lesser known work, Hammersley and Mazzarino [26], have actually produced exact, spatially localized solutions of the fKdV equation. The localized solutions found in Ref. [26], are given in terms of convergent power series, computable to arbitrary precision. Solutions of Ref. [26] are smooth on $\mathbb{R}^+$ with a well-localized core, decreasing monotonically with exponential decay for $x \to \infty$. They are, however, regular

only on the half-line, $\mathbb{R}^+$, and cannot be smoothly continued to regular (decaying or even bounded) solutions to the whole real line, $\mathbb{R}$. This latter fact also follows from the results of Refs. [27,28], where the absence of nontrivial, globally regular decaying solutions of the fKdV equation has been shown.

The main aim of this paper is to connect two, *a priori* quite distinct classes of solutions of the fKdV equation: the decaying and localized solutions of Ref. [26] defined on $\mathbb{R}^+$, on the one hand, with weakly delocalized, globally regular (and even) ones defined on $\mathbb{R}$, which are asymptotically periodic, on the other hand.

To be more precise, we relate the amplitude of asymptotic wave tails of globally regular solutions of Eq. (5) to the "asymmetry" of the localized solution's core. This asymmetry is defined as the value of the third derivative of the decaying solution on $\mathbb{R}^+$ at the point where its first derivative vanishes. From our point of view, one of the main results of Ref. [26] is the obtention of the exact value of the asymmetry as a function of ϵ.

In fact we can connect these two different solution families only for sufficiently small values of the coefficient of the fifth order term, $\epsilon^2 \ll 1$. Even if the validity of the discovered relation is limited for small values of ϵ, it still seems to us not only being of interest in its own right but also rather useful in practice, as it makes possible the computation of the amplitude of standing waves tails for arbitrarily small values of ϵ. This provides a fundamentally new way for the verification of perturbative results for such small values of ϵ that have been numerically inaccessible up to now. It is very much easier to compute numerically the asymmetry of the localized solution to high precision, rather than the amplitude of the wave tail of the weakly delocalized one. Our correspondence starts to noticeably lose both in precision and in applicability for values $\epsilon^2 \gtrsim \frac{1}{8}$. For such "large" values of the fifth order (linear) dispersion term, the fKdV approximation loses, however, its physical applicability due to other non-negligible hydrodynamical corrections, e.g., nonlinear dispersion terms.

Clearly it is of great interest to establish analogous results in other, more complicated and/or higher dimensional cases, in particular for quasibreathers. Perturbative results for small amplitude quasibreathers in various dimensions have been found to be in reasonably good qualitative agreement with numerical simulations [29–31]. Those results are, however, very far from the precision attained by the techniques based on asymmetry-type computations. A major problem for perturbative computations of quasibreather tails is to go beyond the leading order term. Another difficulty is to carry out numerical simulations for sufficiently small values of the perturbative parameter in order to compare the results with the perturbative ones. A suitable generalization of the method presented in this paper is expected be quite a useful first step in that direction.

The plan of the paper is the following: In Sec. II the transformation of the fifth order KdV equation to the form used in Ref. [26] is elucidated, and some of its relevant solutions are briefly discussed. Next, in Sec. III the remarkable construction of the exact (asymmetric) solution of Ref. [26] is reviewed in some detail. In Sec. IV the actual computational method of the exact value of the asymmetry (valid for any value of $\epsilon > 0$) is described in detail. The asymmetry, also computed by a spectral numerical code is shown to agree to the exact results to many significant digits. In Sec. V the perturbative construction of weakly delocalized solutions for $\epsilon \ll 1$, based on Ref. [25], is recapitulated. The perturbative results are exhibited up to order ϵ^{12}. An improved spectral numerical method for the computation of the standing wave tails is also presented. Section VI contains a new derivation of the amplitude of the standing wave tails of the weakly delocalized solutions for $\epsilon \ll 1$. The perturbative results are shown to be in excellent agreement with very high precision numerical calculations. Finally Sec. VII contains our conclusions.

II. STATIONARY FORM OF FIFTH ORDER KDV EQUATION

The fKdV equation is often written as [16,19,25]

$$u_{,t} + \epsilon^2 u_{,XXXXX} + u_{,XXX} + 6uu_{,X} = 0, \quad u = u(t, X), \quad (1)$$

where $\epsilon > 0$ is a (small) parameter, $u_{,t}$ respectively, $u_{,X}$ denote derivatives with respect to t resp. X. Considering stationary, unidirectional traveling waves of speed $c > 0$, we can pass to the comoving frame, $x = X - ct$, and then solutions of Eq. (1) depending only on x, $u = u(x)$, satisfy the following ordinary differential equation (ODE):

$$\epsilon^2 u_{,5} + u_{,3} + (6u - c)u_{,1} = 0, \quad \text{where } u_{,n} = \frac{d^n u}{dx^n}. \quad (2)$$

Equation (2) can be integrated once, yielding a fourth order ODE, discussed in detail in a number of papers, see, e.g., Refs. [19,25,26]:

$$\epsilon^2 u_{,4} + u_{,2} + 3u^2 - cu = M. \quad (3)$$

By a constant shift of u in Eq. (3) one can always transform the inhomogenous term, M, to zero, while $c \to \pm\sqrt{c^2 + 12M}$, so from now on we assume $M = 0$,

$$\epsilon^2 u_{,4} + u_{,2} + 3u^2 - cu = 0. \quad (4)$$

For a discussion of the physical significance of M, and some related subtleties, see, e.g., Refs. [21,25]. Next, by the following scaling transformations of Eq. (4); $x \to \tilde{x}/s_1$, $u \to s_2 y$, with $s_1^2 = c$, $s_2 = \frac{c}{3}$, one can transform it to the form used in Ref. [26]:

$$\tilde{\epsilon}y_{,4} + y_{,2} + y^2 - y = 0, \qquad y_{,n} = y_{,n}(\tilde{x}) = \frac{d^n y}{d\tilde{x}^n}, \quad (5)$$

where

$$\tilde{\epsilon} = c\epsilon^2, \qquad y = \frac{3}{c}u, \qquad \tilde{x} = xc^{\frac{1}{2}}. \quad (6)$$

We note that from the relation $u(x) = \frac{c}{3}y(xc^{\frac{1}{2}})$, the third derivatives, from which the asymmetry at the origin can be deduced, are related as

$$y_{,3}(\tilde{x}) = 3c^{-\frac{5}{2}}u_{,3}(\tilde{x}c^{-\frac{1}{2}}). \quad (7)$$

The celebrated (single) soliton solution of the KdV equation [$\tilde{\epsilon} = 0$ in Eq. (5)], corresponding to a traveling wave is given as

$$y_s = \frac{3}{2}\text{sech}^2(\tilde{x}/2). \quad (8)$$

Clearly the stationary (single) soliton, y_s, is well localized in space; i.e., $y_s \to 0$ exponentially for $\tilde{x} \to \pm\infty$ and it has a characteristic core, where most of its mass (energy) is concentrated. Also, importantly $y_s(\tilde{x})$ is an even function of $\tilde{x}$.

As already mentioned, the influence of (singular) perturbation on localized solutions often results in the appearance of "shelves" or "weak delocalization." It has been amply discussed in the literature [27,28,32], that the stationary fKdV equation in the comoving frame, Eq. (5) for $\tilde{\epsilon} > 0$, does not admit nontrivial bounded solutions on the real line, $\mathbb{R}$, with decaying boundary conditions for $\tilde{x} \to \pm\infty$. Bounded solutions on $\mathbb{R}$, for sufficiently small values of $\tilde{\epsilon} > 0$, tend to asymptotically periodic functions (oscillating tails) for $\tilde{x} \to \pm\infty$, [15,16,19–21,23,24].

On the other hand, Hammersley and Mazzarino [26], have explicitly constructed smooth solutions, $y(\tilde{x})$, of Eq. (5) for any $\tilde{\epsilon} > 0$ satisfying one-sided decay conditions, say for $\tilde{x} \to \infty$. The solution of Ref. [26] is unique for $\tilde{\epsilon}$ fixed, it decreases monotonically for $\tilde{x} > 0$ with a maximum at the origin, $\tilde{x} = 0$, defined by the vanishing of the first derivative, $y_{,1}(0) = 0$. Importantly these solutions are not even functions of $\tilde{x}$, in that $y_{,3}(0) \neq 0$. As already noted above, in view of the nonexistence theorems of Refs. [27,28], regular solutions decaying for $\tilde{x} \to \infty$ of Eq. (5) on $\mathbb{R}^+$ cannot be continued to globally regular ones on $\mathbb{R}$. We remark here that, according to our numerical experiments asymmetric solutions of Eq. (4), decaying on $\mathbb{R}^+$ run generically into a moving pole-type singularity on $\mathbb{R}^-$. Figure 1 depicts some numerically calculated symmetric and asymmetric solutions of Eq. (4).

The results of Ref. [26] make it possible to obtain a simple, quantitative measure of asymmetry, defined by the value of $y_{,3}(0)$, of regular, decaying solutions as a function of $\tilde{\epsilon} > 0$. In the following we shall connect the value of
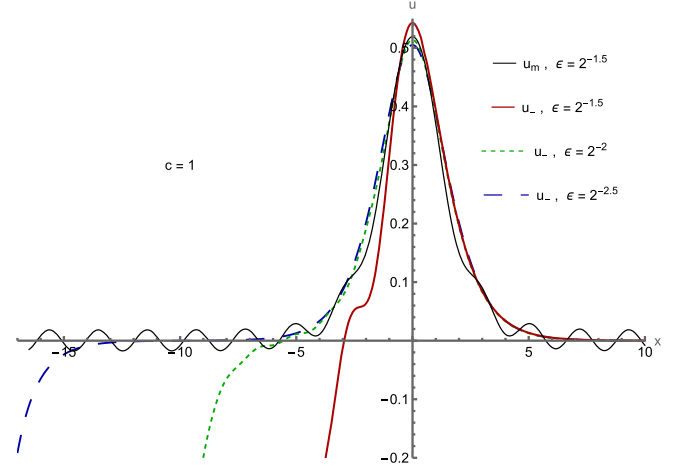

FIG. 1. The black curve shows the minimal tail symmetric solution u_m of Eq. (4) for the parameter values $\epsilon = 2^{-1.5}$, $c = 1$. For the same values of (ϵ, c), the red curve corresponds to the asymmetric solution, u_-, decaying to zero for $x > 0$, and running into a singularity on $\mathbb{R}^-$. The singularity moves farther from the center as ϵ decreases, as illustrated by the green and the blue curves. Since the tail-amplitude decreases very fast with decreasing ϵ, the symmetric solutions are not plotted for the smaller ϵ values.

$y_{,3}(0)$ of decaying solutions on $\mathbb{R}^+$ to the amplitude of the asymptotic wave tails of bounded, globally regular solutions of Eq. (5) which are even functions on $\mathbb{R}$.

III. CONSTRUCTION OF ASYMPTOTICALLY DECAYING, ASYMMETRIC SOLUTIONS

In this section we outline the main steps of the ingenious construction of Hammersley and Mazzarino [26], for further details, the reader should consult the original paper.

Statement of the problem: find regular, monotonically decreasing solutions on $\mathbb{R}^+$ of Eq. (5), subject to the boundary conditions

$$y_{,1}(0) = 0, \quad \lim_{\tilde{x}\to\infty} y(\tilde{x}) = 0, \quad \text{and} \quad y_{,1}(\tilde{x}) < 0 \quad \forall\, \tilde{x} > 0. \quad (9)$$

According to Ref. [26] existence and uniqueness of solutions with bounded derivatives, $y_{,n}(\tilde{x})$, for all $n \geq 1$, satisfying (9), follows from adopting the arguments given in Ref. [33]. Equation (5) admits a familiar first integral:

$$\tilde{\epsilon}\left(y_{,1}y_{,3} - \frac{1}{2}y_{,2}^2\right) + \frac{1}{2}y_{,1}^2 = \frac{1}{2}y^2 - \frac{1}{3}y^3, \quad (10)$$

where the constant of integration has been chosen to be zero, since by assumption y and its derivatives tend to zero for $\tilde{x} \to \infty$. Because Eq. (10) is an autonomous third order ODE, one can reduce it to a second order ODE by considering $y_{,n}$ to be a function of y. As $y(\tilde{x})$ is a monotonically decreasing function for $\tilde{x} \geq 0$ by assumption, both its inverse function, $\tilde{x}(y)$, and $y_{,1}(y)$ are well defined on the

interval $(0, y(0))$. We introduce a new independent variable, $z \propto y$ as

$$y = Yz, \qquad Y = y(0). \tag{11}$$

The new variable, z, varies in the interval $z \in (0, 1)$, because $\tilde{x} \to \infty$ corresponds to $z \to 0$. This tacitly presupposes $Y = y(0) > 0$, which has been shown to hold for all $\tilde{\epsilon} > 0$ in Ref. [26]. Introducing a new unknown, $f(z)$, as

$$y_{,1} = -Y f^{\frac{1}{2}}, \tag{12}$$

one finds

$$y_{,2} = \frac{1}{2} Y f_{,1}, \quad y_{,3} = -\frac{1}{2} Y f^{\frac{1}{2}} f_{,2}, \quad f_{,n} = \frac{d^n f}{dz^n}, \tag{13}$$

so that Eq. (10) is transformed to the second order ODE:

$$\tilde{\epsilon} \left(f f_{,2} - \frac{1}{4} f_{,1}^2 \right) + f = z^2 - \frac{2}{3} Y z^3. \tag{14}$$

The solution, $f(z)$, is conveniently represented by the power series:

$$f(z) = \lambda^2 z^2 \left(1 - \sum_{n=1}^{\infty} c_n z^n \right), \tag{15}$$

where the coefficients, c_n, are unrelated to the speed, denoted by c in the previous section, and λ is the (unique) positive root of the quartic

$$\tilde{\epsilon} \lambda^4 + \lambda^2 - 1 = 0, \qquad \lambda^2 = \frac{\sqrt{1 + 4\tilde{\epsilon}} - 1}{2\tilde{\epsilon}}. \tag{16}$$

Choosing the positive root of Eq. (16) ensures that for $\tilde{x} \to \infty$ the decay condition, $y(\tilde{x}) \sim e^{-\lambda \tilde{x}}$, holds. From Eq. (14) one obtains the following recurrence relations for the coefficients c_n:

$$c_1 = \frac{2}{3} Y / (1 + 4\eta^{-1}), \quad q_n c_n = \frac{1}{2} \sum_{r=1}^{n-1} p_{nr} c_r c_{n-r}, \quad n \geq 2, \tag{17}$$

where

$$q_n = \frac{(n+1)^2 + \eta}{(n+1)^2 + 1}, \qquad p_{nr} = 1 - \frac{5}{2} \frac{r(n-r)}{(n+1)^2 + 1}, \tag{18}$$

$$\eta = \frac{1}{\tilde{\epsilon} \lambda^4}.$$

We also note that $\eta = \eta(\tilde{\epsilon})$ can be expressed in terms of $\tilde{\epsilon}$ as

$$\eta(\tilde{\epsilon}) = 1 + \frac{1}{2} \tilde{\epsilon}^{-1} (1 + \sqrt{1 + 4\tilde{\epsilon}}). \tag{19}$$

In order to satisfy the boundary condition $y_{,1}(0) = 0$, the series (15) should converge for $z \to 1$ and $f(1) = 0$. That these conditions can be satisfied is not at all obvious. It has been shown in Ref. [26] that for any $\tilde{\epsilon} > 0$ it is possible to find a $c_1 > 0$ such that

$$\sum_{n=1}^{\infty} c_n = 1. \tag{20}$$

The proof of the validity of Eq. (20) is a crucial point of Ref. [26]. If $c_1 > 0$, then from Eqs. (17) and (18) one obtains that all other $c_n > 0$. Then from Eq. (20) it follows that $0 < c_n < 1$ for all $n \geq 1$, which in turn implies that $f(z)$ is an analytic function of z for $|z| < 1$, and $f(1) = 0$. It is also shown in Ref. [26] that the radius of convergence of the series (15) is exactly 1. Also $f(z) > 0$ for $z \in (0, 1)$, and the boundary conditions (9) are satisfied, indeed. Moreover Y satisfies the inequality $\frac{3}{2} < Y < \frac{3}{2}(1 + 4\eta^{-1})$, which ensures nontriviality and boundedness of the solutions for all $\tilde{\epsilon} > 0$. Clearly, once c_1 has been found, the decaying solution in $\mathbb{R}^+$, $y(\tilde{x})$, is completely determined, albeit implicitly.

It remains to compute the asymmetry $y_{,3}(0)$. The difficulty of this last step is due to the nonanalyticity of $f(z)$, which has a branch point at $z = 1$. We recall that the derivatives of y at $\tilde{x} = 0$, denoted by

$$Y_n = y_{,n}(0), \tag{21}$$

are all finite for $n \geq 1$. Therefore from Eq. (13) one finds for the limit $z \to 1$

$$Y_2 = \frac{1}{2} Y f_{,1}(1) = -\frac{1}{2} \lambda^2 Y \sum_{n=1}^{\infty} n c_n, \quad Y_3 = -\frac{1}{2} Y \lim_{z \to 1} f^{\frac{1}{2}} f_{,2}. \tag{22}$$

Since Y_2 is finite and $Y > 0$, from Eq. (22) one sees that for $z \to 1$, $f(z) \sim (z - 1) f_{,1}(1)$, from where one can deduce

$$f_{,2} \sim -Y_3 \left(-\frac{1}{2} Y Y_2 \right)^{-\frac{1}{2}} (1 - z)^{-\frac{1}{2}}, \quad \text{for } z \to 1, \tag{23}$$

implying also that unless $Y_3 = 0$, the sum, $\sum_{n=1}^{\infty}(n+2)(n+1)c_n z^n$, diverges for $z \to 1$, necessitating more refined methods to express the asymmetry, Y_3, in terms of the coefficients c_n. In Ref. [26] a positive lower bound is established for Y_3:

$$Y_3 \geq 2^{-15} \tilde{\epsilon}^{-\frac{3}{4}} \eta^{\frac{7}{4}} \exp(-\pi \tilde{\epsilon}^{-\frac{1}{2}}), \tag{24}$$

which result captures, remarkably, both the correct $\tilde{\epsilon} \to 0$ limiting behavior of Y_3, $Y_3 \sim \tilde{\epsilon}^{-\frac{5}{2}} \exp(-\pi \tilde{\epsilon}^{-\frac{1}{2}})$, and its large, $\tilde{\epsilon} \to \infty$ limit, $Y_3 \sim \tilde{\epsilon}^{-\frac{3}{4}}$.

Using the analyticity of $f(z)$ inside the unit disk in the complex z plane, Hammersley and Mazzarino find that Y_3 is given as

$$Y_3 = \lambda^2 \left(-\frac{1}{2} \pi Y Y_2 \right)^{\frac{1}{2}} \mathbb{L}, \qquad \mathbb{L} = \lim_{n \to \infty} n^{\frac{5}{2}} c_n, \quad (25)$$

where

$$Y_2 = -\lambda^2 Y \sqrt{c_1(4+\eta) - \eta}, \quad \text{and} \quad Y = \frac{3}{2} c_1(1 + 4\eta^{-1}). \tag{26}$$

We note here a slight disagreement between Eq. (25) and Eq. (4.10) of Ref. [26], due to a missing factor of $1/2$ from Eq. (4.10).

We would like to point out that the result of Eq. (25) can also be deduced from what is often referred to as transfer theorems in analytical combinatorics [34]. Such transfer theorems relate the asymptotic behavior of the coefficients, c_n, for $n \to \infty$ of a generating function, analytic inside the unit disk, to its singularities at $|z| = 1$. In our case we have $f_{,2}(z) \sim (1-z)^{-\frac{1}{2}}$ for $z \to 1$, and a simple application of the transfer theorem of Ref. [34] yields immediately the asymptotic behavior of the coefficients, c_n,

$$\lim_{n \to \infty} c_n \sim n^{-\frac{5}{2}}, \tag{27}$$

and then one easily reproduces Eq. (25).

In summary, all regular, asymptotically decaying solutions on $\mathbb{R}^+$ of Eq. (14) are given by Eq. (15), in terms of convergent series, where all the coefficients, c_n, for $n \geq 2$ are determined, once for some fixed value of $\tilde{\epsilon}$, c_1 has been found. The value of c_1 is determined by the somewhat implicit equation Eq. (20), where all terms c_n, for $n \geq 2$ are to be expressed in terms of c_1. Moreover Eq. (20) contains unbounded powers of the unknown, c_1, therefore it is something of a challenge to obtain precise enough values of c_1. In addition, the actual computation of the asymmetry, $y_3(0) = Y_3$, also necessitates the determination of the quantity $\mathbb{L}$ which is not that straightforward. The difficulty to obtain reliable numerical estimates for the value of $\mathbb{L}$ is due to the slow convergence of its determining sequence, (25), for large values of n. The smaller $\tilde{\epsilon}$ is, the slower is the convergence.

IV. COMPUTATIONAL ISSUES

In this section we outline our computational procedures to determine solutions of Eq. (14) in terms of c_n, which differ somewhat from those employed in Ref. [26]. Instead of using some approximation scheme to solve Eq. (20) for c_1, we obtain approximations to it from the large n asymptotic behavior of the coefficients, c_n, themselves. This might sound paradoxical at first, since it is precisely

the unknown c_1 which determines all the c_n for $n \geq 2$. An important point is to exploit the homogeneity of the recurrence relation, Eq. (17) for the coefficients c_n:

$$c_1 \to c_1(s) := s c_1, \qquad c_n \to c_n(s) := s^n c_n, \quad (28)$$

which permits one to choose $c_1(s) = s$ and view Eq. (20) as an equation for s. We display as an illustration, the first three nontrivial coefficients $c_n(s)$:

$$c_2(s) = \frac{15 s^2}{4(\eta+9)}, \qquad c_3(s) = \frac{45 s^3}{(\eta+9)(\eta+16)},$$

$$c_4(s) = \frac{315(6\eta+59)s^4}{2(\eta+9)^2(\eta+16)(\eta+25)}. \tag{29}$$

As shown in Ref. [26], the sequence of the unique positive roots, s_N, $N = 1, 2 \ldots$ of the correspondingly truncated polynomial equations

$$\sum_{n=1}^{N} c_n(s) = 1, \tag{30}$$

satisfy

$$1 = s_1 > s_2 > s_3 > \ldots > s_N > \ldots > \frac{1}{2}, \tag{31}$$

therefore the decreasing sequence $\{s_N\}_{N=1,2\ldots}$ converges clearly to a positive limit, $s_\infty \geq 1/2$. Furthermore, an important result established in Ref. [26] is that the coefficients defined by $\lim_{N \to \infty} c_n(s_N) = c_n(s_\infty) := c_n$ do satisfy Eq. (20). To compute approximate values of s_∞ for a given $\eta(\tilde{\epsilon})$, Reference [26] used the obvious approximation, i.e., to solve numerically Eq. (30) for $s_N > 0$ with N large enough and use $s_N \approx s_\infty$.

In the present work we have employed a different method to obtain approximations for s_∞ to rather high precision. Instead of solving Eq. (30) numerically with N sufficiently large for $s_N \approx s_\infty$, we solve (numerically) the recurrence relation Eq. (17) for the quantities $c_n(s = 1) = c_n(1)$, starting with $c_1(1) = 1$. Computing then the ratio $r_n = c_n(1)/c_{n+1}(1)$ for n sufficiently large, we obtain an approximation for s_∞, i.e., $r_n \approx s_\infty$. The reason that the ratio, r_n, can serve as an approximant for s_∞ follows directly from Eqs. (25)–(27) describing the asymptotic behavior of $c_n = c_n(s_\infty)$:

$$c_n \to \mathbb{L} n^{-\frac{5}{2}} \left(1 + \mathcal{O}\left(\frac{1}{n}\right) \right), \quad n \to \infty, \tag{32}$$

recalling that $c_n = c_n(1) s_\infty^n$, it immediately follows that, for $n \to \infty$,

$$\frac{c_n}{c_{n+1}} = \frac{c_n(1)}{c_{n+1}(1)s_\infty} \to 1 + \mathcal{O}\left(\frac{1}{n}\right),$$

$$\text{i.e.,} \quad r_n \to s_\infty\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right). \tag{33}$$

The speed of convergence of r_n is considerably slowed down by a series of finite n corrections of the type $\mathcal{O}(1/n^m)$, $m = 1, 2, \dots$. A closer inspection of the asymptotic, $n \to \infty$, behavior of c_n, together with the application of transfer theorems, reveals that all finite n corrections to the ratio r_n decay as integer powers of $1/n$. Therefore this is a textbook case to increase the rate of convergence of the sequence $\{r_n\}$ considerably, by making use of higher order Richardson extrapolation. Let us recall a compact form of the mth order Richardson extrapolation used in our case [35]:

$$R_m(r_n) = \sum_{k=0}^{m} r_{n+k} \frac{(-1)^{m+k}(n+k)^m}{k!(m-k)!}. \tag{34}$$

For example, for $\tilde{\epsilon} = 1/8$, in standard double precision (~ 16 decimal digits), one obtains already eight decimal digits correctly of $s_\infty = 0.772875581\dots$, from not more than 27 coefficients, $c_n(1)$, by $m =$ seventh order Richardson extrapolation. We note that in Ref. [26] 4000 terms and a first order Richardson extrapolation has been used to obtain six decimal digits of c_1 and of the asymmetry, Y_3, for this value of $\tilde{\epsilon}$. As $\tilde{\epsilon}$ decreases, the number of necessary coefficients, $c_n(s)$, increases. For the values, $2^{-25} \le \tilde{\epsilon} \le 2^{-10}$, considered in Ref. [26], to obtain six significant decimal digits for the asymmetry, 20000 of the coefficients $c_n(s)$, has been computed using quadruple precision, combined with a first order Richardson extrapolation. As it should be clear, the order of Richardson extrapolations is strongly limited by the available numerical precision of r_n, since the number of subtractions induced by the alternating signs in Eq. (34) increases with the order of the extrapolation. Therefore, it is necessary to increase the numerical precision beyond that of the standard one, using suitable software (in this paper ARB [36,37] and Wolfram *Mathematica*), permitting to use relatively high order (~ 20–30) Richardson extrapolations. For the smallest values of $2^{-30} \le \tilde{\epsilon} \le 2^{-25}$ considered in this paper, we have found that performing the computations with the arbitrary precision software ARB to 48 decimal digits, from not more than 200 c_n and Richardson extrapolations of order $m \le 30$, the value of the asymmetry is obtained already to six significant decimal digits. To increase the number of significant digits of the asymmetry to 12 it has been found sufficient to increase the number of decimal digits in ARB to 60.

The optimal order of the Richardson extrapolation can be found by searching for the minimal value of the difference $\Delta R_{m,n} = R_m(r_n) - R_{m-1}(r_n)$, choosing some fixed starting term r_n and increasing the extrapolation order one by one from $m = 0$. We note that the most time consuming task is the computation of the coefficients, $c_n(1)$, for large values ($n \gtrsim 1000$) of n, while computing the Richardson extrapolants, $R_m(r_n)$, goes very fast. Usually, it is a more efficient strategy to increase the order of the Richardson extrapolation together with the number of computational digits, instead of significantly increasing the number of computed coefficients, $c_n(1)$, which permits to decrease the order of Richardson extrapolations. According to our experience, in most cases, the difference $\Delta R_{m,n}$ provides a reasonable error estimate for the result.

The calculation of the limit, $\mathbb{L}$, in Eq. (25) is also greatly improved by the use of Richardson extrapolation due to the same, $1/n^m$ type, corrections encountered when approximating the value of s_∞. As pointed out in Ref. [26], for values of $\tilde{\epsilon} \lesssim 0.01$, it is necessary to further accelerate the convergence of the sequence $\{n^{5/2}c_n\}$, to avoid the computation of an exceedingly large number of terms. The transformation, $c_n \Rightarrow v_n$, put forward in Ref. [26]

$$v_n = n^{\frac{5}{2}}c_n q_0 q_1 \dots q_n, \quad Q := \prod_{n=0}^{\infty} q_n = \frac{\sinh(\pi\sqrt{\eta})}{\sqrt{\eta}\sinh(\pi)}, \tag{35}$$

where q_n is defined in Eq. (18), yields an alternative expression for $\mathbb{L}$:

$$\mathbb{L} = \frac{1}{Q}\lim_{n\to\infty} v_n. \tag{36}$$

This approach has also been used in this paper, together with sufficiently high-order Richardson extrapolations.

We now summarize the algorithm used for the calculation of the asymmetry. We choose some $\tilde{\epsilon}$ value and carry out all calculations to a given number of digits precision. High precision floating point number calculations can be performed either by some algebraic manipulation software (*Mathematica*, Maple, SageMath, etc.) or by using free open source packages in C language (CLN [38], ARB [36]). Starting from $c_1(1) = 1$ we compute the $c_n(1)$ up to some order $n = N$ from the recurrence relation (17). The necessary computational time can be halved exploiting the symmetry $r \to n - r$ of p_{nr}. Then Richardson extrapolation is used on $r_n = c_n(1)/c_{n+1}(1)$ to approximate the value of s_∞ from Eq. (33). The second derivative at the center, Y_2, is found from Eq. (26). The value of $\mathbb{L}$ is approximated by Richardson extrapolation of the series v_n, defined in Eq. (35). Finally, the asymmetry, Y_3, can be obtained from Eq. (25). Applying the above method we have reproduced all the six digits of the results in Table 1 of Ref. [26]. For example, choosing $\tilde{\epsilon} = 2^{-12}$ and computing the coefficients $c_n(1)$ up to $n = 200$ with 56 decimal digits precision, for the computation of Y_3 the optimal order Richardson extrapolation turns out to be 15. This way the asymmetry, $Y_3 \approx 9.20154645 \times 10^{-77}$ is obtained up to 19 decimal

digits of precision. The computational methods presented in this paper allow for the determination of c_1 and Y_3 up to 1000 (or even more) decimal digits of precision.

As Supplemental Material [39] a *Mathematica* notebook and an equivalent C language file that uses the ARB library [36] are provided, to carry out computations of c_1 and Y_3 to high precision for a wide range of $\tilde{\epsilon}$.

The asymptotically decaying asymmetric solution of Eq. (5) can also be calculated directly by using a high precision spectral numerical method. The semi-infinite interval $0 \leq \tilde{x} \leq \infty$ can be transformed into the $\hat{x} \in [-1, 1]$ region by introducing the new independent variable

$$\hat{x} = \frac{\tilde{x} - \xi}{\tilde{x} + \xi}, \tag{37}$$

where $\xi > 0$ is a constant representing a length scale. The monotonically decreasing $y(\hat{x})$ function remains smooth under this transformation, and can be searched for by usual Chebyshev polynomial expansion. It is sufficient to impose the vanishing of $dy/d\hat{x}$ at $\hat{x} = -1$ corresponding to the center, no further boundary conditions are necessary. The reason for this is that the compactification method excludes both oscillatory and unbounded solutions. This phenomenon is referred to as "behavioral boundary conditions" in Ref. [40]. This compactification approach would not work for the calculation of the symmetric solution because of the infinitely many oscillations in the tail region. For $\tilde{\epsilon} \lesssim 2^{-8}$ the central third derivative that we intend to calculate becomes so small that a very large number of collocation points and more than 16 decimal digits precision becomes necessary. Since in the present paper the numerical construction of the decaying solution is just for consistency checking, we do not go here into more details. The method we applied is very similar to the one described in our previous paper [25]. The obtained results for $y_{,3}(0)$ agree to many decimal digits with the computationally much less expensive Hammersley-Mazzarino method described earlier. For illustration we note that our highest resolution numerical computations used 2200 collocation points with 540 decimal digits of precision (applying the ARB library [36]). The running times took about seven hours on a desktop computer, providing an absolute error for $y_{,3}(0)$ as tiny as 10^{-397}, whose magnitude becomes comparable to the value of the asymmetry for $\tilde{\epsilon} \approx 2^{-17}$. This type of spectral numerical calculations would be too slow and memory hungry for $\tilde{\epsilon} \lesssim 2^{-17}$, when the third derivative becomes smaller than this error value. The numerical evaluation of the exact solution does not necessitate significant computational resources, and it is applicable for any value of $\tilde{\epsilon}$. We expect that the asymmetry computations based on perturbative expansions and spectral numerical methods will remain applicable for more complicated analogous problems, such as the radiation of scalar field oscillons, where it is unlikely that analogous results to that of Hammersley and Mazzarino could be found.

V. GLOBALLY REGULAR, NONDECAYING SYMMETRIC SOLUTIONS

In the preceding sections, following Ref. [26], we have presented the exact solution of Eq. (5) decaying exponentially for $\tilde{x} \to \infty$ and being smooth on $\mathbb{R}^+$. As already mentioned, in view of the results of Refs. [27,28] this solution cannot be smoothly extended to the whole real line, $\mathbb{R}$, with decaying boundary conditions for $\tilde{x} \to -\infty$. Our numerical experiments indicate that solutions decaying for $\tilde{x} \to \infty$, become unbounded when continued for negative values of $\tilde{x}$ as $\sim -840\tilde{\epsilon}/(\tilde{x} + x_0)^4$, where x_0 is determined by the value of $\tilde{\epsilon}$, cf. Fig. 1.

As already mentioned, the existence of globally regular solutions of Eq. (4) on $\mathbb{R}$, asymptoting some periodic function for $x \to \pm\infty$ has been established for sufficiently small values of ϵ, [19]. In Refs. [22,41] the existence of a one-parameter family of globally regular solutions of Eq. (4) for sufficiently small values of $\epsilon > 0$, has been proven. Assuming that for $|x| \to \infty$ the solutions are well approximated by those of the linearized version of Eq. (4), one can seek them in the form $\sim e^{ikx/\epsilon}$. One finds that k satisfies

$$k^4 - k^2 - \underbrace{\epsilon^2 c}_{\tilde{\epsilon}} = 0, \quad \text{with real roots} \quad k^2 = \frac{1}{2}(1 + \sqrt{1 + 4\tilde{\epsilon}}). \tag{38}$$

In the following k shall denote the positive root of Eq. (38). The real roots in Eq. (38) determine the standing wave tail of the symmetric (even with respect to reflections $x \to -x$) solution. This tail can be written as $\sim \alpha \sin(k|x|/\epsilon - \delta)$, where $\alpha > 0$, δ are constants, $k/\epsilon > 0$ is the wave number. The tail amplitude, α, turns out to be exponentially small in ϵ, and δ corresponds to the asymptotic phase. Physically, a particular member of this family is singled out, namely the one where the amplitude of the tail is minimal, $\alpha = \alpha_m$ for some value of $\delta = \delta_m$. Both α_m and δ_m are functions of the parameter ϵ.

The two imaginary roots of Eq. (38) correspond to an exponentially growing and a decaying mode, which we write following [21], as $\sim e^{\pm 2\gamma x}$ with $\gamma > 0$. The real and imaginary roots of Eq. (38) are related as

$$4(\epsilon\gamma)^2 + 1 = k^2. \tag{39}$$

The wave speed, c, can be expressed as

$$c = 4\gamma^2(1 + 4\gamma^2\epsilon^2) = 4\gamma^2 k^2. \tag{40}$$

Following previous numerical works [42,43], in Ref. [25] a multiple precision pseudospectral numerical code has been developed to construct globally regular solutions of Eq. (4), aimed to obtain the ϵ dependence of the minimal tail amplitude, α_m, to high precision. The numerically

TABLE I. Numerically calculated values of the minimal tail amplitude α_{m}, the corresponding phase δ_{m}, and the order of optimal truncation N_ϵ for the case $c = 1$. The last column gives the central third derivative of the asymmetric solution u_-.

ϵ	α_{m}	δ_{m}	N_ϵ	$u_{-,3}(0)$
2^{-1}	4.8×10^{-2}	1.2	1	$7.10760589389 \times 10^{-1}$
2^{-2}	1.53×10^{-3}	0.749	4	$8.11499816225 \times 10^{-2}$
2^{-3}	3.2525301×10^{-8}	0.37253582	11	$1.62487857959 \times 10^{-5}$
2^{-4}	$1.94383743304 \times 10^{-18}$	0.187147747984	23	$7.92099610003 \times 10^{-15}$
2^{-5}	$1.27243717968 \times 10^{-39}$	0.0937046561766	49	$4.16436578363 \times 10^{-35}$
2^{-6}	$1.17039547447 \times 10^{-82}$	0.0468692914439	99	$3.06718215011 \times 10^{-77}$
2^{-7}	$2.29646599329 \times 10^{-169}$	0.0234367851656	199	$4.81567051778 \times 10^{-163}$
2^{-8}	$2.12961253593 \times 10^{-343}$	0.0117186606062	400	$3.57282878973 \times 10^{-336}$
2^{-9}	$4.49453680829 \times 10^{-692}$	0.00585936382454	802	$6.03243642172 \times 10^{-684}$

constructed (symmetric) solutions of Eq. (4) have been found to be in excellent agreement with the analytical ones, obtained by perturbation theory in ϵ and by asymptotic matching.

We now briefly recapitulate the perturbative construction of weakly delocalized solutions of Eq. (4) assuming $0 < \epsilon \ll 1$, for details see Ref. [25]. We also follow closely the notations of Ref. [25]. We recall that weakly delocalized solitons have a well-defined, fast decaying core with an exponentially suppressed asymptotic standing wave tail. The core part, u_{c}, is obtained by expanding the solution, u, of Eq. (4) in a power series in ϵ around the KdV soliton, i.e., writing

$$u_{\mathrm{c}}^{(N)} = \sum_{n=0}^{N} u_n \epsilon^{2n}, \qquad u_0 = 2\gamma^2 \operatorname{sech}^2(\gamma x), \quad (41)$$

where N determines the order of the approximation to u_{c}. In Refs. [21,25], the wave speed, c (40), has been considered as a function of ϵ, while γ has been assumed to be an ϵ independent constant. Note that in the present paper another choice, $c = 1$, has been made for all numerical computations. According to Ref. [25] u_n can be expressed as

$$u_n = \gamma^{2n+2} \sum_{j=1}^{n+1} u_{n,j} \operatorname{sech}^{2j}(\gamma x), \quad (42)$$

where $u_{n,j}$ are (rational) numbers. The localized, core part of the solution, u_{c}, is defined as the (formal) expansion (41) for $N \to \infty$, it is to be understood as an asymptotic series in ϵ. This approximation has an optimal order, N_ϵ determined by ϵ, which for small enough values of ϵ behaves as $N_\epsilon \sim 1.56/(\epsilon \sqrt{c})$.

In Ref. [25] we have used a spectral numerical method to find solutions of Eq. (4) that are symmetric with respect to $x = 0$. For smaller ϵ parameters this procedure can be made significantly more efficient by first calculating the optimally truncated approximation $u_{\mathrm{c}}^{(N_\epsilon)}$, which is correct up to $\mathcal{O}(\epsilon^{2N_\epsilon+2})$ terms, and then numerically solving the equation for the difference $v = u - u_{\mathrm{c}}^{(N_\epsilon)}$,

$$\epsilon^2 v_{,4} + v_{,2} + 3v^2 - cv + 6u_{\mathrm{c}}^{(N_\epsilon)} v = R_\epsilon, \quad (43)$$

$$R_\epsilon = -\epsilon^2 u_{\mathrm{c},4}^{(N_\epsilon)} - u_{\mathrm{c},2}^{(N_\epsilon)} - 3(u_{\mathrm{c}}^{(N_\epsilon)})^2 + cu_{\mathrm{c}}^{(N_\epsilon)}. \quad (44)$$

The approximation $u_{\mathrm{c}}^{(N_\epsilon)}$ and its residual R_ϵ can be calculated very quickly to many digits of precision using the algorithm presented in [25]. The function $u_{\mathrm{c}}^{(N_\epsilon)}$ decays exponentially as $x \to \infty$. The residual R_ϵ is very small for small ϵ, since $u_{\mathrm{c}}^{(N_\epsilon)}$ is an ideal approximation to the real solution. The function R_ϵ has about N_ϵ oscillations with exponentially decreasing amplitude, then the exponential decrease continues monotonically. The same spectral numerical method can be applied as earlier in [25] to solve the differential equation (43) in a finite interval with matching to the linearized tail at the outer boundary. The current approach is more efficient, because the function v remains as small as the oscillating tail even in the core region (see Fig. 2 of [25]). However, because of the numerous oscillations in v, it is still necessary to use large number of collocation points and extended precision (more than 16 decimal digits) computations. Our C language code uses the ARB library for arbitrary-precision ball arithmetic [36]. The obtained numerical results for the minimal tail amplitude are presented in Table I. The result for a given ϵ contains an error of order α_{m}^2, since we match to a linearized tail solution at the outer boundary. For $\epsilon \geq 2^{-3}$ only the reliable digits are presented, while for smaller ϵ we give only the first 12 decimal digits of the results. In order to get more precise amplitudes for $\epsilon \geq 2^{-3}$ we would need to match to higher amplitude nonlinear spatially periodic solutions, which is beyond the scope of the present paper. In the last column of Table I we list the central third derivative of the asymmetric right decaying

solution, $u_{-,3}(0)$, calculated by the analytical methods in Sec. IV. The relation between α_m and $u_{-,3}(0)$ will be the subject of the next section.

The delocalized, standing wave tail part of the symmetric solution has been constructed in the Wentzel–Kramers–Brillouin (WKB) approximation, see, e.g., Ref. [35], in the following way: We look for approximate solutions of Eq. (4) as $u = u_c + u_w$ neglecting the nonlinear term. Then u_w is readily found to obey the equation:

$$\epsilon^2 u_{w,4} + u_{w,2} + 6u_c u_w - c u_w = 0, \quad u_{w,n} = \frac{d^n u_w}{dx^n}. \quad (45)$$

Solutions of Eq. (45) tending to $\alpha \sin(k|x|/\epsilon - \delta)$ for $|x| \to \infty$, are expected to provide $\mathcal{O}(\alpha)$ approximations to those of Eq. (4). In practice one also has to chose some finite order approximation in ϵ for u_c, but since u_w is beyond all orders in ϵ this does not cause any order-mixing problems. In Ref. [25] high order WKB approximations of the solution of Eq. (45), u_w^{WKB}, have been obtained, which can be formally written as

$$u_w^{WKB} = \beta \exp\left[\sum_{n=1}^{\infty} A_{2n}\epsilon^{2n}\right] \sin\left(\frac{kx}{\epsilon} - \delta_w - \delta(x)\right), \quad (46)$$

where β, δ_w are real constants, while the phase function, $\delta(x)$, is written as

$$\delta(x) = \sum_{n=0}^{\infty} \tilde{A}_{2n+1}\epsilon^{2n+1}. \quad (47)$$

The phase function, $\delta(x)$, is to be distinguished from the asymptotic phase, denoted as δ previously and in [25]. The expansion terms, $\tilde{A}_k$, have been determined to rather high ~100 order using algebraic manipulation codes. The even index (amplitude) coefficients, A_{2n}, are even functions in x, they tend to zero exponentially fast for $x \to \pm\infty$, e.g., the first one is given as $A_2 = 15\gamma^2\text{sech}^2(\gamma x)$. The odd index (phase) functions, $\tilde{A}_{2n+1}$ are odd in x, they tend to constants

for $x \to \pm\infty$. For example the first one is given as $\tilde{A}_1 = 6\gamma \tanh(\gamma x)$. The asymptotic behavior of the phase function, $\delta(x)$, has been written in [25] as

$$\lim_{x\to\infty}\delta(x) = \delta_m = \sum_{n=0}^{\infty} \tilde{\delta}_{2n+1}\epsilon^{2n+1}, \quad \text{e.g., } \tilde{\delta}_1 = 6\gamma. \quad (48)$$

The determination of the amplitude of the approximate WKB solution, β, amounts to solving the notoriously difficult connection problem. Following the pioneering works of [14,15], the connection problem has been solved by an asymptotic matching procedure in the complex plane to leading order in ϵ in Ref. [16]. This work has been extended in Refs. [21,22] where it has been pointed out that there are more general solutions depending on a phase parameter corresponding to δ_w. As argued in [25], the amplitude, α, of the asymptotic tail of globally regular (symmetric) solutions of (4) with asymptotic phase δ are not generally minimal, their amplitude being determined as $\alpha = \alpha_m/\cos(\delta - \delta_m) + \mathcal{O}(\alpha_m^2)$.

Within the framework of the perturbative asymptotic matching there is a unique, minimal tail amplitude solution, $u_m(x)$, with $\alpha = \alpha_m(\epsilon)$ corresponding to $\delta_w = 0$, which has the asymptotic phase $\delta = \delta_m$. The result up to $\mathcal{O}(\epsilon^6)$ for the minimal amplitude wavetail has been presented in Ref. [25]. We would like to point out, that we have considerably extended the result in [25] by working out α_m to higher orders in ϵ. The presentation of the details is beyond the scope of this paper, but we show some of the obtained results. The order $2N$ approximation to the minimal amplitude α_m can be given as

$$\alpha_m^{(N)} = \frac{\mathcal{K}}{\epsilon^2} e^{-\frac{\pi k}{2\gamma\epsilon}}\left(1 - \sum_{n=1}^{N} \xi_n(\gamma\epsilon)^{2n}\right),$$
$$\mathcal{K} \approx 19.96894735876096 \cdot \pi, \quad (49)$$

where the wave number is $k = (1 + 4\gamma^2\epsilon^2)^{1/2}$. The first six coefficients, ξ_n, are given as

$$\xi_1 = 5, \quad \xi_2 \approx 6.54406819358, \quad \xi_3 \approx 474.413839489,$$
$$\xi_4 \approx 4233.41235937, \quad \xi_5 \approx 111053.952710, \quad \xi_6 \approx 1782156.51421.$$

For convenience we give the expansion of $k = k(\epsilon)$ up to $\mathcal{O}(\epsilon^{2N})$:

$$\exp\left(-\frac{k\pi}{2\gamma\epsilon}\right) = \exp\left(-\frac{\pi}{2\gamma\epsilon}\right)\left[1 - \pi\gamma\epsilon + \frac{\pi^2}{2}\gamma^2\epsilon^2 - \left(\frac{\pi^2}{6} - 1\right)\pi\gamma^3\epsilon^3 + \ldots\right]. \quad (50)$$

Using (50) it is straightforward to obtain the fully expanded result for $\alpha_m^{(N)}$. To avoid unnecessarily long expressions, in the following we keep k in the exponential term of Eq. (49).

The coefficients $\mathcal{K}$ and ξ_n can be obtained to hundreds of decimal digits of precision relatively easily, using the methods presented in [25], and applying Richardson extrapolation. The high precision results are useful for
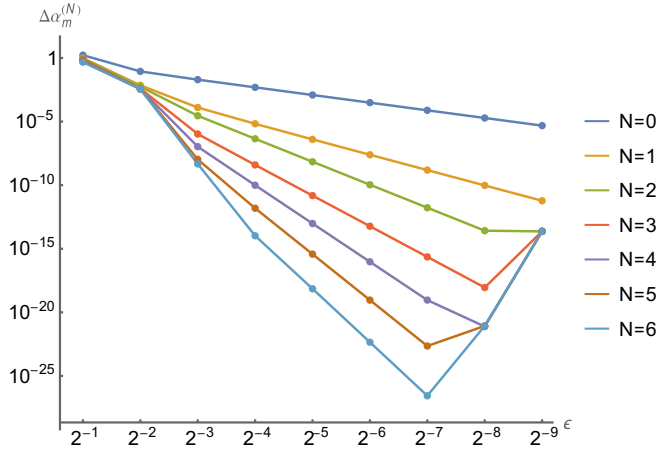
FIG. 2. Relative difference $\Delta\alpha_{\mathrm{m}}^{(N)} = |(\alpha_{\mathrm{m}} - \alpha_{\mathrm{m}}^{(N)})/\alpha_{\mathrm{m}}|$ of the numerical results and the Nth order expansion results as a function of ϵ.

consistency checking when comparing them with the numerical results in Table I, and also with the analytical results in the following sections. It is to be emphasized that beside higher order ϵ^{2M}, $(M > N + 1)$ corrections to $\alpha_{\mathrm{m}}^{(N)}$ in Eq. (49), there are also other corrections of the type $\exp\{-m\pi k/(2\gamma\epsilon)\}$, with $m > 1$ integer, which are, however, exponentially suppressed with respect to $m = 1$. In Fig. 2 we plot the relative difference of the numerical results α_{m} and perturbative results for $\alpha_{\mathrm{m}}^{(N)}$ up to order $N = 6$ for the values of ϵ in Table I. From Fig. 2 it is apparent that for "large" values of $\epsilon > 2^{-2}$ the expansion results for $\alpha_{\mathrm{m}}^{(N)}$ are not particularly meaningful. On the other hand, the numerical results obtained from our spectral code become less and less reliable for $\epsilon < 2^{-7}$, since in those cases we cannot have enough collocation points to describe well the large number of oscillations in u. An important conclusion one can draw from Fig. 2 is that the error of the Nth order approximation, $\alpha_{\mathrm{m}}^{(N)}$, decreases as ϵ^{2N+2}, moreover, its relative error is well estimated by the next order term, $\xi_{N+1}(\gamma\epsilon)^{2N+2}$. It is quite interesting, that the calculated coefficients, ξ_n, do not seem to increase factorially with n, which is generally expected to occur for asymptotic series. The points depicted on Fig. 2 are consistent with convergent behavior for the depicted ϵ range. It is tempting to conjecture that the sum in (49) is convergent for $\epsilon \lesssim 1/2$, but the clarification of this would clearly require further analysis.

VI. RELATING ASYMMETRIC AND SYMMETRIC (EVEN) SOLUTIONS

We shall now relate two important type of solutions of Eq. (4) discussed in Secs. III and V (see Fig. 1). According to Ref. [26] the "asymmetric," decaying solution of Sec. III, u_-, has been proven to be unique. The minimal tail amplitude solution, u_{m}, approximated in Sec. V, is also

expected to be unique among the globally regular, "symmetric" ones, (unfortunately we are not aware of a formal uniqueness proof). For sufficiently small values of ϵ, the amplitude of the asymptotic tail of u_{m} is many orders of magnitude smaller than that of its core. Therefore the heuristic description of u_{m} as the linear superposition of a fast decaying core and of a small amplitude standing wave tail is expected to be a very good approximation.

Consider the difference $\Delta = u_{\mathrm{m}} - u_-$, which tends exponentially fast for $x \gg 1/\epsilon$ to the tail part of u_{m}. Our numerical simulations clearly show that Δ is of order α_{m} for all $x \in \mathbb{R}^+$, i.e., also in the core region. Since both u_{m} and u_- satisfy Eq. (4) it is easily seen that Δ satisfies Eq. (45) with u_w replaced by Δ, when omitting the quadratic term. Therefore we approximate Δ by a suitable solution of (45), and write the minimal tail solution as

$$u_{\mathrm{m}} \approx u_- + u_w^{\mathrm{WKB}} - x_0 u_{\mathrm{c},1}, \quad \text{for } \beta = \alpha_{\mathrm{m}}, \quad \delta_w = 0, \quad (51)$$

where x_0 is a constant to be determined. The term $u_{\mathrm{c},1}$ is the translational zero mode of the (formal) solution, u_{c}, of Eq. (4) hence also that of Eq. (45). The necessity to include the translational mode, $u_{\mathrm{c},1}$, in Eq. (51) can be understood from the fact that the approximation of the minimal amplitude tail, u_w^{WKB} is an odd function with respect to the origin, while u_{m} is even and u_- has been defined to obey the boundary condition, $u_{-,1}(0) = 0$. An alternative viewpoint for Eq. (51) is that one can cancel the tail of u_{m} by subtracting a linear perturbation approximated by u_w^{WKB} and then obtain a shifted version of the asymmetric solution u_-, so that its maximum is at $x = x_0$,

$$u_{\mathrm{m}}(x) \approx u_-(x - x_0) + u_w^{\mathrm{WKB}}(x). \quad (52)$$

In Fig. 3 we plot the difference of the symmetric solution u_{m} and asymmetric solution u_- for the parameter choice $\epsilon = 2^{-3}$, $c = 1$.

Taking derivatives of the approximation (51) at $x = 0$ one obtains the following relations:

$$u_{\mathrm{m}}(0) = u_-(0), \quad (53)$$

$$u_{w,1}^{\mathrm{WKB}}(0) - x_0 u_{\mathrm{c},2}(0) = 0, \quad (54)$$

$$u_{\mathrm{m},2}(0) = u_{-,2}(0), \quad (55)$$

$$u_{-,3}(0) + u_{w,3}^{\mathrm{WKB}}(0) - x_0 u_{\mathrm{c},4}(0) = 0, \quad (56)$$

using that $u_w^{\mathrm{WKB}}(0) = u_{w,2}^{\mathrm{WKB}}(0) = 0$ for $\delta_w = 0$ and $u_{\mathrm{c},1}(0) = u_{\mathrm{c},3}(0) = 0$. Equations (53)–(56) are expected to be valid up to arbitrary orders in ϵ and to first order in $\mathcal{O}(\alpha_{\mathrm{m}})$.

As detailed in [25] the $\mathcal{O}(\alpha_{\mathrm{m}})$ WKB approximant, $u_w^{\mathrm{WKB}}(x)$ has been computed to rather high ($\mathcal{O}(\epsilon^{100})$) orders in ϵ. Using then these results together with those of the core expansion, Eq. (42), from (54) one finds for the parameter x_0 to $\mathcal{O}(\epsilon^{10})$

$$x_0 = -\frac{\alpha_m}{4\gamma^4\epsilon}\left[1 - 9(\gamma\epsilon)^2 - 113(\gamma\epsilon)^4 - 4987(\gamma\epsilon)^6 - \frac{663031}{2}(\gamma\epsilon)^8 - \mathcal{O}(\epsilon^{10})\right]. \tag{57}$$

Using (56) one may obtain the asymmetry in terms of α_m

$$u_{-,3}(0) = \frac{\alpha_m}{\epsilon^3}\left[1 - 5(\gamma\epsilon)^2 - 286(\gamma\epsilon)^4 - 10422(\gamma\epsilon)^6 - \frac{1227721}{2}(\gamma\epsilon)^8 - \mathcal{O}(\epsilon^{10})\right]. \tag{58}$$

Inverting Eq. (58) in the limit $\epsilon \to 0$, we find that α_m is approximated to $\mathcal{O}(\epsilon^{2N+2})$ in terms of the asymmetry as

$$\alpha_{\text{WKB}}^{(N)} = u_{-,3}(0)\epsilon^3\left[1 + \sum_{n=1}^{N} a_n(\gamma\epsilon)^{2n}\right], \tag{59}$$

where a_n are positive rational numbers. The first four coefficients are

$$a_1 = 5, \quad a_2 = 311, \quad a_3 = 13407, \quad a_4 = \frac{1643903}{2}. \tag{60}$$

As further Supplemental Material to this paper we attach a *Mathematica* notebook file [39], which can be used to obtain the above results for x_0, $u_{-,3}(0)$ and $\alpha_{\text{WKB}}^{(N)}$. The code can be used to calculate higher order generalizations up to order $\mathcal{O}(\epsilon^{100})$.

Before quantitatively comparing our results, (58), (59), with available numerical computations a few remarks are in order. As it should be clear from the previous part of this paper, the relations between the asymmetry and the amplitude of the wave tail deduced above, have been obtained in a perturbative framework, valid only for $\epsilon \ll 1$. Furthermore, the wave tail is obtained from a linearized approximation of Eq. (4), i.e., from Eq. (45), in form of a WKB asymptotic expansion in ϵ. Therefore one expects that, for a given value of $\epsilon \ll 1$, Eqs. (58) and (59) can be used up to an ϵ-dependent, "optimal" order of the WKB expansion.

As we have seen at the end of Sec. IV, for each specific choice of ϵ the central third derivative $u_{-,3}(0)$ can be calculated relatively easily and quickly up to hundreds of digits of precision. Hence it is a natural choice to use (59) to approximate the minimal tail amplitude α_m. The summation in the equation represents an asymptotic expansion, which has an optimal order of truncation. For the choice $\epsilon = 2^{-3}$ in Fig. 4 we plot the relative difference

$$\Delta\alpha_{\text{WKB}}^{(N)} = |(\alpha_m - \alpha_{\text{WKB}}^{(N)})/\alpha_m| \tag{61}$$

of the numerically calculated α_m and the approximation $\alpha_{\text{WKB}}^{(N)}$ as a function of the number of terms N. Since the numerical value is much more precise in this case, $\Delta\alpha_{\text{WKB}}^{(N)}$ represents the error of the Nth order approximation $\alpha_{\text{WKB}}^{(N)}$. For all the ϵ values that we tested we have found that the best value of N in $\alpha_{\text{WKB}}^{(N)}$ agrees with N_ϵ or $N_\epsilon - 1$,
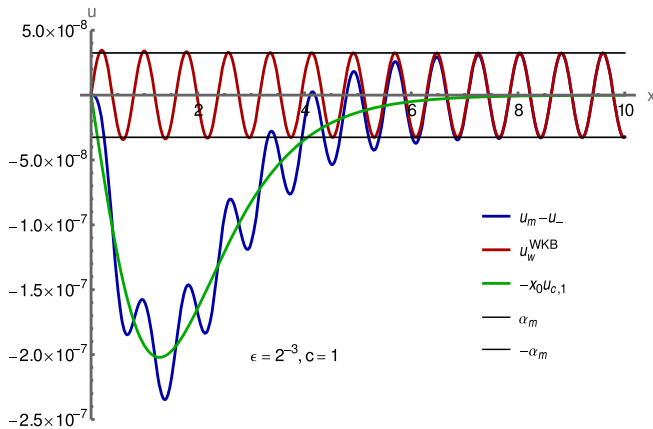

FIG. 3. The blue curve shows the difference $u_m - u_-$ for $\epsilon = 2^{-3}$ and $c = 1$. The green curve, $-x_0 u_{c,1}$, is a translational mode of the linearized equation (45), with $x_0 = -1.03371 \times 10^{-6}$. The difference of these two yields the red curve, $u_w^{\text{WKB}} \approx u_m - u_- + x_0 u_{c,1} \approx u_m(x) - u_-(x - x_0)$, where the equalities hold up to error of 4×10^{-13} for the selected ϵ value.
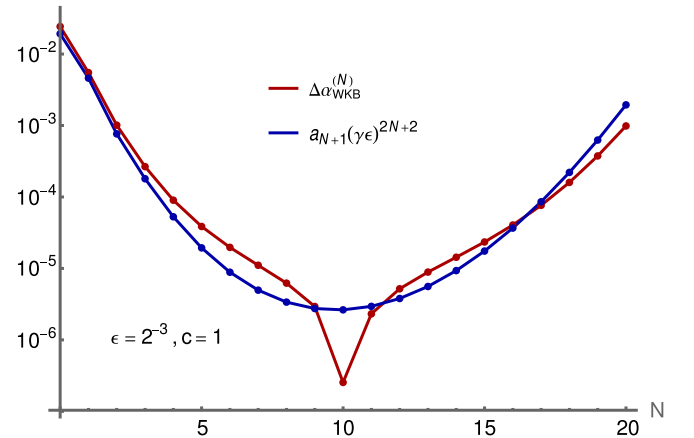

FIG. 4. The red points show the relative difference $\Delta\alpha_{\text{WKB}}^{(N)}$ for the parameter choice $\epsilon = 2^{-3}$, $c = 1$. The blue points give the relative contribution of the next term in (59), $a_{N+1}(\gamma\epsilon)^{2N+2}$, which can be used to estimate the error of the approximation. Both take the smallest value for $N = 10$, which is just one less than the optimal truncation order for the core expansion, $N_\epsilon = 11$ given in Table I.
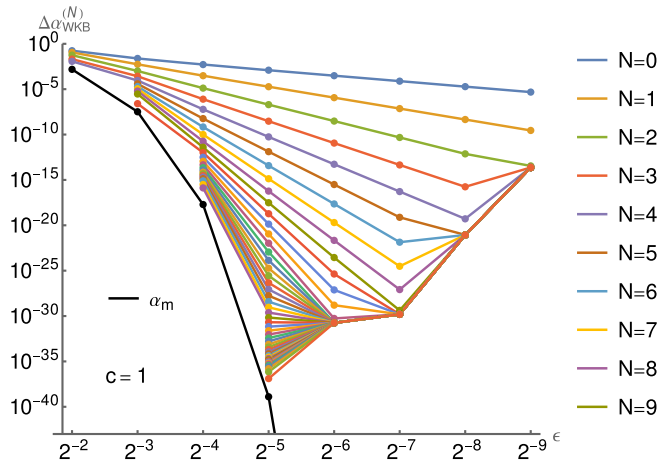
FIG. 5. For different values of ϵ we show how the relative difference $\Delta\alpha_{\text{WKB}}^{(N)}$ changes as we increase the order of the approximation N. To make the plot more transparent we drop those points which give worse approximation than the lower order ones. For $\epsilon = 2^{-2}, 2^{-3}, 2^{-4}, 2^{-5}$ we stop at the optimal WKB truncation, $N = 4, 10, 22, 48$, respectively. For $\epsilon < 2^{-5}$ our best spectral numerical values for α_m are less precise than the higher order $\alpha_{\text{WKB}}^{(N)}$ values, hence the points saturate at the numerical error. The relative error of the optimally truncated result approaches α_m, shown by the black points, consistently with the expectation that the error of both the numerical and WKB tail amplitude is of order α_m^2.

where N_ϵ is the optimal truncation order for the core expansion, listed in Table I. Since in the small ϵ limit $N_\epsilon \sim 1.56/(\epsilon\sqrt{c})$, the error of the optimal order approximation by (59) decreases exponentially with decreasing ϵ. In Fig. 5 we show how the relative difference $\Delta\alpha_{\text{WKB}}^{(N)}$ decreases for several values of ϵ.

The above results clearly show that for small ϵ parameter values, compared to the spectral numerical code, it is more precise and efficient to use $u_{-,3}(0)$ and the WKB method to calculate the tail amplitude α_m of the symmetric solution. The central derivative of the asymmetric

TABLE II. For the ϵ values in this table the central asymmetry $u_{-,3}(0)$ of the solution u_- is used to calculate the minimal tail amplitude α_m of the symmetric solution.

ϵ	α_m	$u_{-,3}(0)$
2^{-10}	$4.95890409131 \times 10^{-1390}$	$5.32457637656 \times 10^{-1381}$
2^{-11}	$1.50219375669 \times 10^{-2786}$	$1.29037422688 \times 10^{-2776}$
2^{-12}	$3.43832141771 \times 10^{-5580}$	$2.36279631071 \times 10^{-5569}$
2^{-13}	$4.49808545327 \times 10^{-11168}$	$2.47284858324 \times 10^{-11156}$
2^{-14}	$1.92344734444 \times 10^{-22344}$	$8.45941084312 \times 10^{-22332}$
2^{-15}	$8.79023705549 \times 10^{-44698}$	$3.09278970949 \times 10^{-44684}$

solution $u_{-,3}(0)$ can be relatively easily calculated by the Hammersley-Mazzarino method presented in Sec. IV. for any ϵ that is larger than 2^{-15}. In Table II we list the values of α_m and $u_{-,3}(0)$ for smaller ϵ values than in Table I. In these cases α_m has been calculated from $u_{-,3}(0)$ using appropriately many terms in (59). In Table II we give "only" 12 decimal digits, but we have calculated these results to several hundred digits.

It is instructive to compare the tail amplitude, $\alpha_{\text{WKB}}^{(N_\epsilon)}$, calculated by the optimal order WKB method to the ϵ expansion results of $\alpha_m^{(N)}$ in Eq. (49). In Fig. 6 we depict the relative difference $\Delta\alpha_\epsilon^{(N)} = |(\alpha_{\text{WKB}}^{(N_\epsilon)} - \alpha_m^{(N)})/\alpha_{\text{WKB}}^{(N_\epsilon)}|$ for a large range of ϵ. The compact formula of the $\mathcal{O}(\epsilon^{12})$ result for $\alpha_m^{(N)}$ given by Eq. (49) is simple to evaluate. The WKB expansion is known to much higher orders ($\gtrsim\mathcal{O}(\epsilon^{200})$), implying it being much more precise. The WKB result is, however, prohibitively complicated to be displayed to such high orders. We note that, for $\epsilon < 2^{-4}$, $\alpha_m^{(6)}$ yields more than 16 significant decimal digits, which is good enough for most cases.

The ϵ expansion result (49) for the symmetric solution can also be used together with the WKB transformation (58) to calculate the central third derivative of the asymmetric solution. Since the coefficients ξ_n are known numerically, we give only numerical coefficients here,

$$u_{-,3}(0) = \frac{\mathcal{K}}{\epsilon^5} e^{-\frac{\pi k}{2\gamma\epsilon}} [1 - 10(\gamma\epsilon)^2 - 267.544068194(\gamma\epsilon)^4 - 9433.69349852(\gamma\epsilon)^6$$
$$- 561740.239659(\gamma\epsilon)^8 - 45224879.2541(\gamma\epsilon)^{10} - 4611707502.31(\gamma\epsilon)^{12} - \ldots], \tag{62}$$

where $\mathcal{K} \approx 19.96894735876096 \cdot \pi$. It is now easy to obtain the $\tilde{\epsilon}$ expansion of the asymmetry, $Y_3 = y_{,3}(0)$, to leading order in α_m using Eqs. (7), (38), (39) alternatively (40):

$$Y_3 = \frac{3\mathcal{K}}{\tilde{\epsilon}^{5/2}} e^{-\frac{\pi k^2}{\sqrt{\tilde{\epsilon}}}} \left[1 - \frac{5}{2}\tilde{\epsilon} - 14.2215042621\tilde{\epsilon}^2 - 118.958452390\tilde{\epsilon}^3 \right.$$
$$\left. - 1823.20094973\tilde{\epsilon}^4 - 36515.2419905\tilde{\epsilon}^5 - 932270.777827\tilde{\epsilon}^6 - \ldots \right]. \tag{63}$$

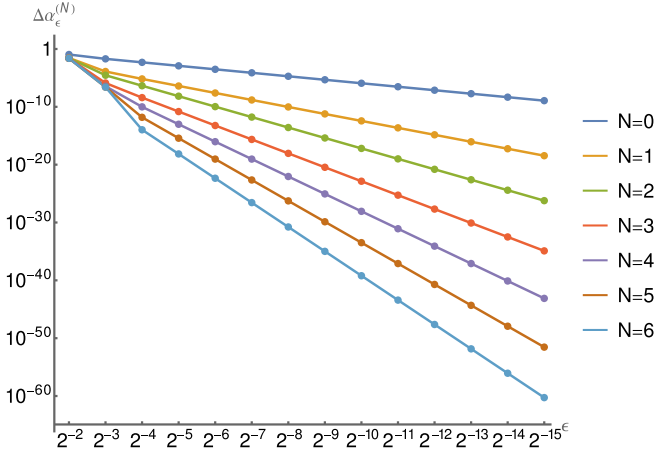FIG. 6.   Relative difference of the Nth order ϵ expansion result $\alpha_{\mathrm{m}}^{(N)}$ for the tail amplitude and a more precise high order WKB result, $\alpha_{\mathrm{WKB}}^{(N_\epsilon)}$.

In Eq. (63) k^2 in the exponential has not been expanded in $\tilde\epsilon$ to avoid the appearance of half integer powers of $\tilde\epsilon$, since

$$\mathrm{e}^{-\frac{\pi k^2}{\sqrt{\tilde\epsilon}}} = \mathrm{e}^{-\frac{\pi}{\sqrt{\tilde\epsilon}}}\left[1 - \pi\tilde\epsilon^{1/2} + \frac{\pi^2}{2}\tilde\epsilon - \frac{\pi}{6}(\pi^2 - 6)\tilde\epsilon^{3/2} + \dots\right]. \quad (64)$$

At this point we should like to point out, that the leading order term for $\tilde\epsilon \to 0$ of the asymmetry, is in complete agreement with a previous result of Byatt-Smith in Ref. [44]. In Ref. [44] it has been used to provide yet another proof of the nonexistence of a globally regular decaying solution on $\mathbb{R}$.

The perturbative results in Eqs. (62) and (63) are given in terms of asymptotic series; e.g., for $\tilde\epsilon = 2^{-4}$ the optimal truncation corresponds to keeping the first four terms in (63), up to order $\tilde\epsilon^3$. This way one obtains $y_{,3}(0) = 0.243033$, which is correct to four significant digits, the result up to seven significant digits being $Y_3 = 0.243445$. For $\tilde\epsilon \leq 2^{-8}$ the seven terms in (63) give at least six significant digits, hence we obtain the values of Y_3 listed in the first 16 lines of Table 1. of Ref. [26]. For $\tilde\epsilon \leq 2^{-20}$ the leading order approximation is already accurate to six decimal digits. The method presented in Sec. IV can be used for the calculation of the third derivative for essentially any $\tilde\epsilon$ up to hundreds of digits of precision. However, for $\tilde\epsilon \leq 2^{-6}$ it is much easier to use the simple expression in Eq. (63) to obtain a suitably precise value.

VII. CONCLUSIONS

In the limit $\epsilon \to 0$, a connection has been established between the amplitude of the minimal wave tail of weakly delocalized solitons of the fKdV equation and the asymmetry of (smooth) solutions on $\mathbb{R}^+$ tending to zero for $x \to \infty$, constructed in Ref. [26]. The asymmetry of the decaying solutions of Ref. [26] is an exact, albeit somewhat implicit result. We have simplified a bit the numerical calculation of the asymmetry, and on a small laptop computer it is easily computed to many significant digits, as it can be verified from our Supplemental Material [39]. On the basis of our high order perturbative results for the tail amplitude, a small ϵ expansion of the asymmetry has been derived and verified numerically. Our results make the computation of the tail amplitudes possible for arbitrarily small values of ϵ to extremely high precision. Exploiting this correspondence in combination with high order perturbative calculations and precise numerical methods, we have been able to compute for such small values as $\epsilon \leq 2^{-15}$, in which case the amplitude of the minimal wave tail is $\sim 10^{-44698}$. Somewhat surprisingly our method works quite well up to as "large" values of ϵ as $2^{-3/2} \approx 0.353553$.

[1] D. J. Kaup and A. C. Newell, Solitons as particles and oscillators and in slowly varying media: A singular perturbation theory, Proc. R. Soc. A **361**, 413 (1978).

[2] V. I. Karpman and E. M. Maslov, Structure of tails produced under the action of perturbations on solitons, Zh. Eksp. Teor. Fiz. **75**, 504 (1978).

[3] C. J. Knickerbocker and A. C. Newell, Shelves and the Korteweg-de Vries equation, J. Fluid Mech. **98**, 803 (1980).

[4] I. L. Bogolyubskii and V. G. Makhan'kov, Dynamics of spherically symmetrical pulsons of large amplitude, JETP Lett. **25**, 107 (1977).

[5] M. Gleiser, Pseudostable bubbles, Phys. Rev. D **49**, 2978 (1994).

[6] E. Seidel and W.-M. Suen, Formation of solitonic stars through gravitational cooling, Phys. Rev. Lett. **72**, 2516 (1994).

[7] E. J. Copeland, M. Gleiser, and H.-R. Müller, Oscillons: Resonant configurations during bubble collapse, Phys. Rev. D **52**, 1920 (1995).

[8] E. P. Honda and M. W. Choptuik, Fine structure of oscillons in the spherically symmetric ϕ^4 Klein-Gordon model, Phys. Rev. D **65**, 084037 (2002).

[9] G. Fodor, P. Forgács, P. Grandclément, and I. Rácz, Oscillons and quasibreathers in the ϕ^4 Klein-Gordon model, Phys. Rev. D **74**, 124003 (2006).

[10] P. M. Saffin and A. Tranberg, Oscillons and quasi-breathers in $D + 1$ dimensions, J. High Energy Phys. 01 (2007) 030.

[11] D. Cyncynates and T. Giurgica-Tiron, Structure of the oscillon: The dynamics of attractive self-interaction, Phys. Rev. D **103**, 116011 (2021).

[12] G. Fodor, A review on radiation of oscillons and oscillatons, arXiv:1911.03340.

[13] L. Visinelli, Boson stars and oscillatons: A review, Int. J. Mod. Phys. D **30**, 2130006 (2021).

[14] M. D. Kruskal and H. Segur, Asymptotics beyond all orders in a model of crystal growth, Stud. Appl. Math. **85**, 129 (1991).

[15] H. Segur and M. D. Kruskal, Nonexistence of small-amplitude breather solutions in ϕ^4 theory, Phys. Rev. Lett. **58**, 747 (1987).

[16] Y. Pomeau, A. Ramani, and B. Grammaticos, Structural stability of the Korteweg-de Vries solitons under a singular perturbation, Physica (Amsterdam) **31D**, 127 (1988).

[17] T. Kakutani and H. Ono, Weak non-linear hydromagnetic waves in a cold collision-free plasma, J. Phys. Soc. Jpn. **26**, 1305 (1969).

[18] T. Kawahara, Oscillatory solitary waves in dispersive media, J. Phys. Soc. Jpn. **33**, 260 (1972).

[19] J. K. Hunter and J. Scheurle, Existence of perturbed solitary wave solutions to a model equation for water waves, Physica (Amsterdam) **32D**, 253 (1988).

[20] E. Benilov, R. Grimshaw, and E. Kuznetsova, The generation of radiating waves in a singularly-perturbed Korteweg-de Vries equation, Physica (Amsterdam) **69D**, 270 (1993).

[21] R. Grimshaw and N. Joshi, Weakly nonlocal solitary waves in a singularly perturbed Korteweg–De Vries equation, SIAM J. Appl. Math. **55**, 124 (1995).

[22] S. M. Sun, On the oscillatory tails with arbitrary phase shift for solutions of the perturbed Korteweg–de Vries equation, SIAM J. Appl. Math. **58**, 1163 (1998).

[23] J. P. Boyd, A numerical calculation of a weakly non-local solitary wave: The ϕ^4 breather, Nonlinearity **3**, 177 (1990).

[24] J. P. Boyd, *Weakly Nonlocal Solitary Waves and Beyond-All-Orders Asymptotics: Generalized Solitons and Hyperasymptotic Perturbation Theory* (Springer US, Boston, MA, 1998), 10.1007/978-1-4615-5825-5.

[25] G. Fodor, P. Forgács, and M. Mushtaq, Higher order corrections to "beyond all order" effects in a fifth order Korteweg-de Vries equation, Phys. Rev. D **107**, 105002 (2023).

[26] J. M. Hammersley and G. Mazzarino, Computational aspects of some autonomous differential equations, Proc. R. Soc. A **424**, 19 (1989).

[27] C. J. Amick and J. B. McLeod, A singular perturbation problem in water-waves, Stab. Appl. Anal. Contin. Media **1**, 127 (1991).

[28] B. T. N. Gunney, Y. A. Li, and P. J. Olver, Solitary waves in the critical surface-tension model, J. Eng. Math. **36**, 99 (1999).

[29] G. Fodor, P. Forgács, Z. Horváth, and M. Mezei, Radiation of scalar oscillons in 2 and 3 dimensions, Phys. Lett. B **674**, 319 (2009).

[30] G. Fodor, P. Forgács, Z. Horváth, and M. Mezei, Oscillons in dilaton-scalar theories, J. High Energy Phys. 08 (2009) 106.

[31] G. Fodor, P. Forgács, and M. Mezei, Mass loss and longevity of gravitationally bound oscillating scalar lumps (oscillatons) in D dimensions, Phys. Rev. D **81**, 064029 (2010).

[32] E. Lombardi, Orbits homoclinic to exponentially small periodic orbits for a class of reversible systems. Application to water waves, Arch. Ration. Mech. Anal. **137**, 227 (1997).

[33] J. M. Hammersley and G. Mazzarino, A differential equation connected with the dendritic growth of crystals, IMA J. Appl. Math. **42**, 43 (1989).

[34] P. Flajolet and A. Odlyzko, Singularity analysis of generating functions, SIAM J. Discrete Math. **3**, 216 (1990).

[35] C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers I* (Springer, New York, NY, Boston, MA, 2010), 10.1007/978-1-4757-3069-2.

[36] ARB—a C library for arbitrary-precision ball arithmetic, https://arblib.org/.

[37] F. Johansson, ARB: Efficient arbitrary-precision midpoint-radius interval arithmetic, IEEE Trans. Comput. **66**, 1281 (2017).

[38] CLN—Class Library for Numbers, https://www.ginac.de/CLN/.

[39] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevD.109.125011. A *Mathematica* notebook, thdcalc.nb, and an equivalent C language file, thdcalc.c, which uses the ARB library, are provided as Supplemental Material to compute Y_3. In addition, a *Mathematica* notebook, wkb.nb, is provided as Supplemental Material to compute α_{WKB}.

[40] J. Boyd, *Chebyshev and Fourier Spectral Methods: Second Revised Edition*, Dover Books on Mathematics (Dover Publications, New York, 2013).

[41] C. J. Amick and J. F. Toland, Solitary waves with surface tension I: Trajectories homoclinic to periodic orbits in four dimensions, Arch. Ration. Mech. Anal. **118**, 37 (1992).

[42] J. P. Boyd, Weakly non-local solitons for capillary-gravity waves: Fifth-degree Korteweg-de Vries equation, Physica (Amsterdam) **48D**, 129 (1991).

[43] J. P. Boyd, Multiple precision pseudospectral computations of the radiation coefficient for weakly nonlocal solitary waves: Fifth-order Korteweg–DeVries equation, Comput. Phys. **9**, 324 (1995).

[44] J. G. Byatt-Smith, On the existence of homoclinic and heteroclinic orbits for differential equations with a small parameter, Eur. J. Appl. Math. **2**, 133 (1991).