Field independent additive constant in Wilson actions

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We discuss the field independent additive constant in Wilson actions carefully within the exact renormalization group formalism. The additive constant does not affect the correlation functions of fields normalized by the partition function, and for that reason it is often ignored. But it is an essential part of the partition function, and in the limit where the UV cutoff goes to zero, the constant gives a renormalized vacuum energy density. We discuss some concrete examples: the massless and massive Gaussian theory for a single component scalar field and a Dirac fermion and the linear sigma model in the large N limit.

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I. INTRODUCTION

The Wilsonian renormalization group allows one to compute correlation functions by performing piecemeal functional integration: rather than integrating over all degrees of freedom at once, one lowers the momentum cutoff gradually. This procedure allows one to implement several approximation schemes, which can even be of nonperturbative nature. If one were able to make the calculation exactly, one would recover the correlation functions of the original microscopic model. This holds true also for the partition function.

A particularly powerful implementation of these ideas has been developed by Wilson [1], who put forward an exact differential equation for the action functional associated with the degrees of freedom that have not been integrated out yet. Since then, several exact renormalization group (ERG) equations have been derived and studied [2–6].

In this paper we focus on the Wilson action and study carefully how one can derive the partition function or equivalently the vacuum energy density from it. For the partition function to be preserved, the ERG must take the form of a continuity equation, i.e., the change of the exponentiated Wilson action under the change of the momentum cutoff Λ must be a total differential with respect to the fluctuating field:

$$-\Lambda\partial_\Lambda e^{S_\Lambda[\phi]} = \int d^D x \frac{\delta}{\delta\phi(x)}\Big[(\cdots)e^{S_\Lambda[\phi]}\Big]. \qquad (1)$$

Despite its being conceptually clear from the very beginning [1,2,7], in practice almost all of the works employing the Wilson action neglect the vacuum energy density and the relevant terms in the associated ERG equation.

In the present work several aspects of the vacuum energy density are studied and discussed in detail. In particular, following Wilson [1], we impose that an ERG equation have a diffusion-inspired form mentioned above. In so doing we shall introduce two cutoff functions. We highlight the role of different variables and their relation to the limit $\Lambda \to 0$ as well as their relation to other functionals, such as the associated effective average action. Furthermore, we relate equivalent Wilson actions, i.e., Wilson actions constructed via different cutoff functions whose partition functions and correlation functions are identical [8], by means of a functional integral formula.

The paper is organized as follows. In Sec. II we introduce our formalism for the case of a scalar field theory and discuss the introduction of different variables and the relation between equivalent Wilson actions. In Sec. III we generalize our discussion to include fermions. In Sec. IV we relate the Wilson action to the generating functional of connected correlation functions and to the effective average action. We show how the vacuum energy density appears in each of these functionals. In Secs. V and VI we consider two examples: the Gaussian theory for scalars and fermions and the large N limit of the O(N) linear sigma model. We summarize our

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findings in Sec. VII. Some technical discussions are relegated to Appendixes A and B.

Throughout the paper we work on D-dimensional Euclidean space, and employ the following shorthand notations:

$$\int_p = \int \frac{d^D p}{(2\pi)^D}, \qquad \delta(p) = (2\pi)^D \delta^{(D)}(p). \tag{2}$$

II. REAL SCALAR THEORY

We introduce an ultraviolet momentum cutoff Λ in terms of a cutoff function $R_\Lambda(p)$ that has the following properties:

(1) It is a decreasing but positive function of p^2.
(2) It is an increasing function of Λ.
(3) It can be written as

$$R_\Lambda(p) = \Lambda^2 R(p/\Lambda), \tag{3}$$

where

$$R(0) = 1, \tag{4a}$$

$$\lim_{x\to+\infty} R(x) = 0. \tag{4b}$$

$R_\Lambda(p)$ gives a degree of functional integration over the fluctuating fields of momentum p. The smaller it is, the amount of remaining integration is smaller. Since $R_\Lambda(p)$ is a decreasing function of p, the fluctuating fields of higher momenta are integrated more than those of lower momenta.

We start with a Wilson action $S_\Lambda[\phi]$ whose modified correlation functions [8]

$$\langle\langle \phi(p_1)\cdots\phi(p_n)\rangle\rangle \equiv \prod_1^n \frac{1}{K(p_i/\Lambda)} \left\langle \exp\left(-\frac{1}{2}\int_p \frac{k(p/\Lambda)}{p^2}\frac{\delta^2}{\delta\phi(p)\delta\phi(-p)}\right) \times \phi(p_1)\cdots\phi(p_n)\right\rangle_{S_\Lambda} \tag{5}$$

are independent of the cutoff Λ. The two cutoff functions $K(p/\Lambda)$ and $k(p/\Lambda)$ are related to $R_\Lambda(p)$ by

$$R_\Lambda(p) = \frac{p^2}{k(p/\Lambda)}\{K(p/\Lambda)\}^2. \tag{6}$$

For example, in [1] the following choice is made:

$$K(p/\Lambda) = \exp\left(-\frac{p^2}{\Lambda^2}\right), \qquad k(p/\Lambda) = \frac{p^2}{\Lambda^2},$$
$$R_\Lambda(p) = \Lambda^2 \exp\left(-2\frac{p^2}{\Lambda^2}\right). \tag{7}$$

The same $R_\Lambda(p/\Lambda)$ is also obtained from

$$K(p/\Lambda) = \frac{\Lambda^2 e^{-2\frac{p^2}{\Lambda^2}}}{p^2 + \Lambda^2 e^{-2\frac{p^2}{\Lambda^2}}},$$
$$k(p/\Lambda) = K(p/\Lambda)(1-K(p/\Lambda)) = p^2 \frac{\Lambda^2 e^{-2\frac{p^2}{\Lambda^2}}}{\left(p^2 + \Lambda^2 e^{-2\frac{p^2}{\Lambda^2}}\right)^2}. \tag{8}$$

Please note that the modified correlation functions (5) are not normalized by the partition function

$$Z_\Lambda \equiv \int [d\phi] \exp(S_\Lambda[\phi]). \tag{9}$$

Hence, for $n = 0$, the Λ independence of (5) amounts to the Λ independence of the partition function

$$\frac{d}{d\Lambda} Z_\Lambda = 0. \tag{10}$$

This implies that $\partial_\Lambda e^{S_\Lambda[\phi]}$ must be a total differential with respect to the field variables as we explained in Sec. I. Indeed we find the ERG differential equation as [8][1]

$$-\Lambda\partial_\Lambda e^{S_\Lambda[\phi]} = \int_p \frac{\delta}{\delta\phi(p)}\left[\left\{\Lambda \frac{\partial \ln K(p/\Lambda)}{\partial\Lambda}\phi(p) + \frac{1}{2}\Lambda\frac{\partial R_\Lambda(p)}{\partial\Lambda}\frac{K(p/\Lambda)^2}{R_\Lambda(p)^2}\frac{\delta}{\delta\phi(-p)}\right\} e^{S_\Lambda[\phi]}\right]. \tag{11}$$

We would like to replace the field variables $\phi(p)$ by alternative field variables $\sigma(p)$ that simplify the Wilson action in the limit $\Lambda \to 0+$. We define

$$\sigma(p) \equiv \frac{\sqrt{R_\Lambda(p)}}{K(p/\Lambda)}\phi(p) = \sqrt{\frac{p^2}{k(p/\Lambda)}}\phi(p). \tag{12}$$

We assume that this is an analytic change of variables with respect to the momentum p. [That is the case with the two examples (7), (8).] The Jacobian of this change of variables is a field independent constant, and the Wilson action for the σ field is given by

$$e^{S'_\Lambda[\sigma]} = \frac{\int[d\phi]\exp\left(-\frac{1}{2}\int_p \frac{p^2}{k(p/\Lambda)}\phi(p)\phi(-p)\right)}{\int[d\sigma]\exp\left(-\frac{1}{2}\int_p \sigma(p)\sigma(-p)\right)} e^{S_\Lambda[\phi]}. \tag{13}$$

[1] We have added a constant to rewrite (26) of [8] as a total differential.

By definition, the partition function does not change:

$$\int [d\sigma] e^{S'_\Lambda[\sigma]} = \int [d\phi] e^{S_\Lambda[\phi]}. \tag{14}$$

In the following we adopt a particular convention of the Gaussian functional integral:

$$\int [d\sigma] \exp\left(-\frac{1}{2}\int_p \sigma(p)\sigma(-p)\right) = 1. \tag{15}$$

This implies

$$\int [d\sigma] \exp\left(-\frac{1}{2}\int_p A(p)\sigma(p)\sigma(-p)\right) = \exp\left(-\frac{1}{2}\delta(0)\int_p \ln A(p)\right), \tag{16}$$

where

$$\delta(0) = \int d^D x\, e^{ip\cdot 0} = VT \tag{17}$$

is the volume of spacetime.

In terms of the σ variables, the same correlation functions are given by

$$\langle\langle \phi(p_1)\cdots\phi(p_n)\rangle\rangle \equiv \prod_{i=1}^n \frac{1}{\sqrt{R_\Lambda(p_i)}} \times \left\langle \exp\left(-\frac{1}{2}\int_p \frac{\delta^2}{\delta\sigma(p)\delta\sigma(-p)}\right) \times \sigma(p_1)\cdots\sigma(p_n)\right\rangle_{S'_\Lambda}. \tag{18}$$

The ERG equation (11) that guarantees the Λ independence of the correlation functions is now given by

$$-\Lambda\frac{\partial}{\partial\Lambda} e^{S'_\Lambda[\sigma]} = \frac{1}{2}\int_p \Lambda \frac{\partial \ln R_\Lambda(p)}{\partial\Lambda}\frac{\delta}{\delta\sigma(p)} \times \left[\left(\sigma(p) + \frac{\delta}{\delta\sigma(-p)}\right) e^{S'_\Lambda[\sigma]}\right]. \tag{19}$$

The Wilson action $S'_\Lambda[\sigma]$ can be interpreted so that the field modes with $p < \Lambda$ are still to be integrated. As we decrease Λ toward 0, less and less number of degrees of freedom are to be integrated, and eventually we obtain

$$\lim_{\Lambda\to 0+} S'_\Lambda[\sigma] = -\varepsilon_{\rm vac}\delta(0) - \frac{1}{2}\int_p \sigma(p)\sigma(-p). \tag{20}$$

We will derive this limit in Appendix A. Let us note that this was already observed also by Wilson [1] in its zero dimensional version of the ERG before discussing the full fledged ERG in terms of "spin variables." The reader may be puzzled by such a simple form of $S'_\Lambda[\sigma]$ at $\Lambda = 0$ for any microscopic model; it seems that the physics is lost. The crucial observation here is that the choice of variables is very important. As we shall see in Sec. IV, $S'_\Lambda[\sigma]$ is related to the generating functional of connected correlation functions. Thus, the physics is not lost, but in order to obtain desired results one must adopt suitable variables before taking the limit $\Lambda \to 0$.

Using (15), we obtain the cutoff independent partition function as

$$\int [d\sigma] e^{S'_\Lambda[\sigma]} = \exp(-\varepsilon_{\rm vac}\delta(0)), \tag{21}$$

where $\varepsilon_{\rm vac}$ is the density of the vacuum energy. Taking $\delta(0)$ literally, it diverges, and the partition function is either infinite (if $\varepsilon_{\rm vac} < 0$) or zero (if $\varepsilon_{\rm vac} > 0$), but $\varepsilon_{\rm vac}$ is a finite physical quantity. We will drop the prime from the Wilson action $S'_\Lambda[\sigma]$ from now on.

Let us conclude this section by considering two Wilson actions associated with the same microscopic action and built via two different sets of cutoff functions, say K_1, R_1 and K_2, R_2. Augmenting the result from [8] by a constant multiple, one obtains

$$e^{S_2[\phi]} = \frac{1}{\int [d\phi''] \exp\left[-\frac{1}{2}\int_p \frac{1}{\frac{1}{R_2(p)} - \frac{1}{R_1(p)}} \frac{\phi''(p)}{K_2(p)}\frac{\phi''(-p)}{K_2(p)}\right]} \times \int [d\phi'] \exp\left[S_1[\phi'] - \frac{1}{2}\int_p \frac{1}{\frac{1}{R_2(p)} - \frac{1}{R_1(p)}} \times \left(\frac{\phi'(p)}{K_1(p)} - \frac{\phi(p)}{K_2(p)}\right)\left(\frac{\phi'(-p)}{K_1(p)} - \frac{\phi(-p)}{K_2(p)}\right)\right]. \tag{22}$$

This is consistent with

$$\int [d\phi] e^{S_1[\phi]} = \int [d\phi] e^{S_2[\phi]}. \tag{23}$$

III. DIRAC FERMION THEORY

It is straightforward to generalize what we have introduced for the scalar theory to the Dirac fermion theory. We introduce Dirac spinor fields $\sigma(p)$ and $\bar\sigma(-p)$ so that the modified correlation functions, defined by

$$\langle\langle \psi(p_1)\cdots\psi(p_n)\bar\psi(-q_1)\cdots\bar\psi(-q_n)\rangle\rangle \equiv \prod_{i=1}^n \frac{1}{\sqrt{R_\Lambda(p_i)R_\Lambda(q_i)}} \left\langle \sigma(p_1)\cdots\sigma(p_n) \times \exp\left(-\int_p \frac{\overleftarrow{\delta}}{\delta\sigma(p)}\frac{\overrightarrow{\delta}}{\delta\bar\sigma(-p)}\right)\bar\sigma(-q_1)\cdots\bar\sigma(-q_n)\right\rangle_{S_\Lambda}, \tag{24}$$

are independent of Λ. This implies the ERG equation

$$-\Lambda\frac{\partial}{\partial\Lambda}e^{S_\Lambda[\sigma,\bar\sigma]} = -\frac{1}{2}\int_p \Lambda\partial_\Lambda \log R_\Lambda(p) \tag{25}$$

$$\times \mathrm{Tr}\left[\frac{\vec\delta}{\delta\bar\sigma(-p)}\left\{e^{S_\Lambda}\left(\bar\sigma(-p)+\frac{\overleftarrow{\delta}}{\delta\sigma(p)}\right)\right\} + \left\{\left(\sigma(p)+\frac{\vec\delta}{\delta\bar\sigma(-p)}\right)e^{S_\Lambda}\right\}\frac{\overleftarrow{\delta}}{\delta\sigma(p)}\right], \tag{26}$$

where the right-hand side is again a total differential. As opposed to (3) for scalar fields, we use

$$R_\Lambda(p) = \Lambda R(p/\Lambda) \tag{27}$$

for the Dirac fermions.

In this case we find the asymptotic behavior

$$\lim_{\Lambda\to 0+} S_\Lambda[\sigma,\bar\sigma] = -\varepsilon_{\mathrm{vac}}\delta(0) - \int_p \bar\sigma(-p)\sigma(p). \tag{28}$$

Using the convention

$$\int[d\sigma d\bar\sigma]\exp\left[-\int_p \bar\sigma(-p)\sigma(p)\right] = 1, \tag{29}$$

we obtain

$$\int[d\sigma d\bar\sigma]e^{S_\Lambda[\sigma,\bar\sigma]} = \exp\left(-\varepsilon_{\mathrm{vac}}\delta(0)\right). \tag{30}$$

IV. EFFECTIVE POTENTIAL

So far we have discussed how to obtain the vacuum energy density as the limit of a Wilson action as $\Lambda\to 0+$. In literature it is more standard to calculate it as the minimum of the effective potential. Let us briefly explain the equivalence of the two approaches.

We define

$$W_\Lambda[J] \equiv S_\Lambda[\sigma] + \frac{1}{2}\int_p \sigma(p)\sigma(-p), \tag{31}$$

where

$$J(p) \equiv \sqrt{R_\Lambda(p)}\sigma(p). \tag{32}$$

The ERG equation for $W_\Lambda[J]$ is obtained from that for $S_\Lambda[\sigma]$ as

$$-\Lambda\frac{\partial}{\partial\Lambda}e^{W_\Lambda[J]} = \frac{1}{2}\int_p \Lambda\frac{\partial R_\Lambda(p)}{\partial\Lambda}\frac{\delta^2}{\delta J(p)\delta J(-p)}e^{W_\Lambda[J]}. \tag{33}$$

As is explained in Appendix A, $W_\Lambda[J]$ becomes the generating functional of the connected correlation functions in the limit $\Lambda\to 0+$. Expanding $W_\Lambda[J]$ in powers of J, we obtain

$$W_\Lambda[J] = c_\Lambda\delta(0) + \frac{1}{2}\int_p J(p)C_{2\Lambda}(p)J(-p) + \cdots, \tag{34}$$

where the constant part is the same as that in the expansion of $S_\Lambda[\sigma]$. The ERG equation (33) gives

$$-\Lambda\frac{\partial c_\Lambda}{\partial\Lambda} = \frac{1}{2}\int_p \Lambda\frac{\partial R_\Lambda(p)}{\partial\Lambda}C_{2\Lambda}(p). \tag{35}$$

If we know $C_{2\Lambda}(p)$, we can solve this to determine c_Λ. That is what we will do in Sec. VI.

What is usually studied in the context of the exact renormalization group is the one-particle-irreducible (1PI) Wilson action Γ_Λ, which is defined as the Legendre transform of $W_\Lambda[J]$:

$$\Gamma_\Lambda[\Phi] - \frac{1}{2}\int_p R_\Lambda(p)\Phi(p)\Phi(-p) \equiv W_\Lambda[J] - \int_p J(-p)\Phi(p), \tag{36}$$

where

$$\Phi(p) \equiv \frac{\delta W_\Lambda[J]}{\delta J(-p)}. \tag{37}$$

In the limit $\Lambda\to 0+$, we obtain the effective action

$$\lim_{\Lambda\to 0+}\Gamma_\Lambda[\Phi] = \Gamma_{\mathrm{eff}}[\Phi]. \tag{38}$$

For constant fields $J(p) = j\delta(p)$ and $\Phi(p) = \varphi\delta(p)$, we obtain

$$W_\Lambda[J] = w_\Lambda(j)\delta(0), \qquad \Gamma_\Lambda[\Phi] = G_\Lambda(\varphi)\delta(0). \tag{39}$$

Equation (36) reduces to the Legendre transformation

$$G_\Lambda(\varphi) - \frac{1}{2}\Lambda^2\varphi^2 = w_\Lambda(j) - j\varphi, \tag{40}$$

where

$$\varphi = w'_\Lambda(j). \tag{41}$$

The inverse Legendre transformation gives

$$j = -G'_\Lambda(\varphi). \tag{42}$$

We assume that $j = 0$ corresponds to the minimum, not the maximum, of $-G_\Lambda(\varphi)$. The effective potential is the limit

$$V_{\mathrm{eff}}(\varphi) = -\lim_{\Lambda\to 0+} G_\Lambda(\varphi). \tag{43}$$

Let v_Λ be the value of φ at the minimum of $-G_\Lambda(\varphi)$, satisfying

$$G'_\Lambda(v_\Lambda) = 0. \tag{44}$$

We then obtain

$$c_\Lambda = w_\Lambda(0) = G_\Lambda(v_\Lambda) - \frac{1}{2}\Lambda^2 v_\Lambda^2. \tag{45}$$

Hence, we obtain

$$\varepsilon_{\mathrm{vac}} = -\lim_{\Lambda\to 0+} c_\Lambda = -\lim_{\Lambda\to 0+} G_\Lambda(v_\Lambda) = V_{\mathrm{eff}}(v), \tag{46}$$

where v is at the minimum of the effective potential satisfying

$$V'_{\mathrm{eff}}(v) = 0. \tag{47}$$

V. THE GAUSSIAN THEORY

In this section we compute $\varepsilon_{\mathrm{vac}}$ for the Gaussian theory, both bosonic and fermionic.

A. Scalar theory

We assume a quadratic form

$$S_\Lambda[\sigma] = c_\Lambda \delta(0) - \frac{1}{2}\int_p \sigma(p) C_\Lambda(p) \sigma(-p). \tag{48}$$

Substituting this into (19), we obtain

$$-\Lambda\frac{\partial}{\partial\Lambda} c_\Lambda = \frac{1}{2}\int_p \Lambda\frac{\partial \ln R_\Lambda(p)}{\partial\Lambda}(1 - C_\Lambda(p)), \tag{49a}$$

$$\Lambda\frac{\partial}{\partial\Lambda} C_\Lambda(p) = -\Lambda\frac{\partial \ln R_\Lambda(p)}{\partial\Lambda}(1 - C_\Lambda(p))C_\Lambda(p). \tag{49b}$$

The second equation can be solved as

$$C_\Lambda(p) = \frac{F(p)}{F(p) + R_\Lambda(p)}, \tag{50}$$

where $F(p)$ is independent of Λ. This corresponds to the correlation function

$$\langle\langle \phi(p)\phi(q) \rangle\rangle = \delta(p+q)\frac{1}{F(p)} \times e^{-\varepsilon_{\mathrm{vac}}\delta(0)}. \tag{51}$$

The Gaussian theory is given by

$$F(p) = p^2 + m^2, \tag{52}$$

where m^2 is a constant squared mass.

Equation (49a) now gives

$$-\Lambda\frac{\partial}{\partial\Lambda} c_\Lambda = \frac{1}{2}\int_p \Lambda\frac{\partial R_\Lambda(p)}{\partial\Lambda}\frac{1}{p^2 + m^2 + R_\Lambda(p)}. \tag{53}$$

We will solve this first for $2 < D < 4$, and then for $D = 4$.

For the scalar theory, $R_\Lambda(p) = \Lambda^2 R(p/\Lambda)$, and

$$\Lambda\frac{\partial R_\Lambda(p)}{\partial\Lambda} = (2 - p\cdot\partial_p) R_\Lambda(p). \tag{54}$$

For $2 < D < 4$, we can rewrite (53) as

$$\begin{aligned} -\Lambda\frac{\partial c_\Lambda}{\partial\Lambda} &= \Lambda^D \int_p \left\{\left(1 - \frac{1}{2}p\cdot\partial_p\right)R(p)\right\}\frac{1}{p^2 + \frac{m^2}{\Lambda^2} + R(p)} \\ &= \Lambda^D \int_p \left\{\left(1 - \frac{1}{2}p\cdot\partial_p\right)R(p)\right\}\left[\frac{1}{p^2 + R(p)} - \frac{m^2}{\Lambda^2}\frac{1}{(p^2 + R(p))^2}\right] \\ &\quad + \frac{1}{2}\int_p \Lambda\frac{\partial R_\Lambda(p)}{\partial\Lambda}\left[\frac{1}{p^2 + m^2 + R_\Lambda(p)} - \frac{1}{p^2 + R_\Lambda(p)} + \frac{m^2}{(p^2 + R_\Lambda(p))^2}\right]. \end{aligned} \tag{55}$$

Integrating this, we obtain

$$\begin{aligned} c_\Lambda = c &- \frac{1}{D}\Lambda^D \int_p \left\{\left(1 - \frac{1}{2}p\cdot\partial_p\right)R(p)\right\}\frac{1}{p^2 + R(p)} + \frac{1}{D-2}m^2\Lambda^{D-2}\int_p \left\{\left(1 - \frac{1}{2}p\cdot\partial_p\right)R(p)\right\}\frac{1}{(p^2 + R(p))^2} \\ &- \frac{1}{2}\int_p \left[\ln\frac{p^2 + m^2 + R_\Lambda(p)}{p^2 + R_\Lambda(p)} - \frac{m^2}{p^2 + R_\Lambda(p)}\right], \end{aligned} \tag{56}$$

where c is a constant of integration with mass dimension D, and the last integral is UV finite. Hence, we obtain

$$\varepsilon_{\rm vac} = -\lim_{\Lambda\to 0+} c_\Lambda = -c + \frac{1}{2}\int_p \left[\ln\frac{p^2+m^2}{p^2} - \frac{m^2}{p^2}\right], \tag{57}$$

where the integral is UV finite, and we obtain

$$\frac{1}{2}\int_p \left[\ln\frac{p^2+m^2}{p^2} - \frac{m^2}{p^2}\right] = \frac{m^2}{D}\int_p \left[\frac{1}{p^2+m^2} - \frac{1}{p^2}\right] = -\frac{1}{2}\frac{1}{(4\pi)^{\frac{D}{2}}}\Gamma\left(-\frac{D}{2}\right)(m^2)^{\frac{D}{2}}, \tag{58}$$

which is negative for $2 < D < 4$. $\varepsilon_{\rm vac}$ is not analytic with respect to m^2. This is equal to the zero-point energy calculated with dimensional regularization:

$$\frac{1}{2}\int \frac{d^{D-1}p}{(2\pi)^{D-1}}\sqrt{\vec{p}^2+m^2} = -\frac{1}{2}\frac{1}{(4\pi)^{\frac{D}{2}}}\Gamma\left(-\frac{D}{2}\right)(m^2)^{\frac{D}{2}}. \tag{59}$$

For $D = 4$, we can rewrite (53) as

$$-\Lambda\frac{\partial c_\Lambda}{\partial\Lambda} = \int_p \left\{\left(1-\frac{1}{2}p\cdot\partial_p\right)R(p)\right\}\left[\Lambda^4\frac{1}{p^2+R(p)} - m^2\Lambda^2\frac{1}{(p^2+R(p))^2} + m^4\frac{1}{(p^2+R(p))^3}\right] + \frac{1}{2}\int_p \Lambda\frac{\partial R_\Lambda(p)}{\partial\Lambda}\left[\frac{1}{p^2+m^2+R_\Lambda(p)} - \frac{1}{p^2+R_\Lambda(p)} + \frac{m^2}{(p^2+R_\Lambda(p))^2} - \frac{m^4}{(p^2+R_\Lambda(p))^3}\right], \tag{60}$$

where the integral multiplied by m^4 is obtained as

$$\int_p \left\{\left(1-\frac{1}{2}p\cdot\partial_p\right)R(p)\right\}\frac{1}{(p^2+R(p))^3} = \frac{1}{4}\int_p (p\cdot\partial_p + 4)\frac{1}{(p^2+R(p))^2} = \frac{1}{4}\frac{2}{(4\pi)^2}\int_0^\infty dp^2\frac{d}{dp^2}\frac{p^4}{(p^2+R(p))^2} = \frac{1}{2(4\pi)^2}. \tag{61}$$

Integrating (60) over Λ, we obtain

$$c_\Lambda = c - \frac{\Lambda^4}{4}\int_p \left\{\left(1-\frac{1}{2}p\cdot\partial_p\right)R(p)\right\}\frac{1}{p^2+R(p)} + \frac{m^2\Lambda^2}{2}\int_p \left\{\left(1-\frac{1}{2}p\cdot\partial_p\right)R(p)\right\}\frac{1}{(p^2+R(p))^2} - \frac{1}{2(4\pi)^2}m^4\ln\frac{\Lambda}{\mu} + \Lambda^4 F\left(\frac{m^2}{\Lambda^2}\right), \tag{62}$$

where c is a constant of integration, μ is an arbitrary mass parameter, and

$$\Lambda^4 F\left(\frac{m^2}{\Lambda^2}\right) \equiv \frac{1}{2}\int_\Lambda^\infty d\Lambda'\frac{\partial}{\partial\Lambda'}\int_p\left[\ln\frac{p^2+R_{\Lambda'}(p)+m^2}{p^2+R_{\Lambda'}(p)} - \frac{m^2}{p^2+R_{\Lambda'}(p)} + \frac{1}{2}\frac{m^4}{(p^2+R_{\Lambda'}(p))^2}\right] = -\frac{1}{2}\int_p\left[\ln\frac{p^2+R_\Lambda(p)+m^2}{p^2+R_\Lambda(p)} - \frac{m^2}{p^2+R_\Lambda(p)} + \frac{1}{2}\frac{m^4}{(p^2+R_\Lambda(p))^2}\right]. \tag{63}$$

By definition we find

$$\Lambda^4 F\left(\frac{m^2}{\Lambda^2}\right) \overset{\Lambda\to\infty}{\longrightarrow} 0. \tag{64}$$

Since c_Λ remains finite as $\Lambda \to 0+$, we obtain

$$\Lambda^4 F\left(\frac{m^2}{\Lambda^2}\right) \overset{\Lambda\to 0+}{\longrightarrow} -\frac{1}{4(4\pi)^2}m^4\ln\frac{m^2}{\Lambda^2} + \text{const}\times m^4. \tag{65}$$

Hence, we obtain

$$\varepsilon_{\rm vac} = -\lim_{\Lambda\to 0+} c_\Lambda = -c + \frac{1}{4(4\pi)^2}m^4\ln\frac{m^2}{\mu^2}, \tag{66}$$

where we have redefined c by absorbing a constant multiple of m^4. The change of μ can be compensated by a change of c. This result is consistent with the zero-point energy obtained by dimensional regularization with a minimal subtraction ($D = 4 - \epsilon$):

$$\mu^\epsilon\frac{1}{2}\int\frac{d^{D-1}p}{(2\pi)^{D-1}}\sqrt{\vec{p}^2+m^2} + \frac{1}{\epsilon}\frac{m^4}{2(4\pi)^2} \overset{\epsilon\to 0}{\longrightarrow} \frac{m^4}{4(4\pi)^2}\ln\frac{m^2 e^{\gamma-\frac{3}{2}}}{4\pi\mu^2}. \tag{67}$$

Let us conclude this section by commenting on the massless case which would be the Gaussian fixed point in the dimensionless framework. From Eqs. (57) and (66), we expect that the vacuum energy density vanishes for the fixed point theory. We find it instructive to see how this happens in some details. In the dimensionless framework, where all the physical quantities are rendered dimensionless by using appropriate powers of the cutoff Λ, the Gaussian theory is given by

$$\bar{S}_t[\bar{\sigma}] = \bar{c}_t\delta(0) - \frac{1}{2}\int_p \bar{\sigma}(-p)\frac{p^2+\bar{m}_t^2}{p^2+\bar{m}_t^2+R(p)}\bar{\sigma}(p), \quad (68)$$

where the logarithmic scale t is introduced by $\Lambda = \mu e^{-t}$, and

$$\bar{c}_t \equiv \frac{c_\Lambda}{\Lambda^D}, \qquad \bar{m}_t^2 \equiv \frac{m^2}{\Lambda^2} = \frac{m^2}{\mu^2}e^{2t}. \quad (69)$$

Equation (53) gives

$$\left(\frac{d}{dt} - D\right)\bar{c}_t = \int_p \left\{\left(1-\frac{1}{2}p\cdot\partial_p\right)R(p)\right\}\frac{1}{p^2+\bar{m}_t^2+R(p)}. \quad (70)$$

At the fixed point $\bar{m}_t^2 = 0$, we obtain

$$\bar{S}^*[\bar{\sigma}] = \bar{c}^*\delta(0) - \frac{1}{2}\int_p \bar{\sigma}(-p)\frac{p^2}{p^2+R(p)}\bar{\sigma}(p), \quad (71)$$

where the fixed point value $\bar{c}^*$ is given by[2]

$$\bar{c}^* = -\frac{1}{D}\int_p \left\{\left(1-\frac{1}{2}p\cdot\partial_p\right)R(p)\right\}\frac{1}{p^2+R(p)}. \quad (72)$$

By integrating over the fluctuating field $\bar{\sigma}$ in (71), we obtain the vanishing vacuum energy density

$$\bar{\varepsilon}^* = -\bar{c}^* + \frac{1}{2}\int_p \log\frac{p^2}{p^2+R(p)} = 0. \quad (73)$$

Though this result is expected, it is still nice to get it by integrating out the fluctuating field.

B. Dirac theory

We assume a quadratic form

$$S_\Lambda[\sigma,\bar{\sigma}] = c_{F,\Lambda}\delta(0) - \int_p \bar{\sigma}(-p)C_{F,\Lambda}(p)\sigma(p). \quad (74)$$

We obtain

$$C_{F,\Lambda}(p) = \frac{i(\not{p}+im)}{R_\Lambda(p)+i(\not{p}+im)}, \quad (75)$$

where

$$R_\Lambda(p) = \Lambda R(p/\Lambda). \quad (76)$$

This corresponds to

$$\langle\langle\psi(p)\bar{\psi}(-q)\rangle\rangle = \delta(p-q)\frac{1}{i(\not{p}+im)}e^{-\varepsilon_{\mathrm{vac}}\delta(0)}. \quad (77)$$

The ERG equation for $c_{F,\Lambda}$ is given by

$$\Lambda\frac{\partial c_{F,\Lambda}}{\partial\Lambda} = \int_p \Lambda\frac{\partial R_\Lambda(p)}{\partial\Lambda}\mathrm{Tr}\frac{1}{R_\Lambda(p)+i(\not{p}+im)} \quad (78)$$

$$= \mathrm{Tr}1\int_p \Lambda\frac{\partial R_\Lambda(p)}{\partial\Lambda}\frac{R_\Lambda(p)-m}{(R_\Lambda(p)-m)^2+p^2}. \quad (79)$$

The solution to this equation is analogous to the scalar case. We give it only for $3 < D < 4$:

$$\begin{aligned} c_{F,\Lambda} = c_F + \mathrm{Tr}1\Bigg[&\frac{\Lambda^D}{D}\int_p\left\{(1-p\cdot\partial_p)R(p)\right\}\frac{R(p)}{p^2+R(p)^2} + \frac{m\Lambda^{D-1}}{D-1}\int_p\left\{(1-p\cdot\partial_p)R(p)\right\}\frac{R(p)^2-p^2}{(p^2+R(p)^2)^2} \\ &+ m^2\frac{\Lambda^{D-2}}{D-2}\int_p\left\{(1-p\cdot\partial_p)R(p)\right\}\frac{R(p)^3-3p^2R(p)}{(p^2+R(p)^2)^3} + m^3\frac{\Lambda^{D-3}}{D-3}\int_p\left\{(1-p\cdot\partial_p)R(p)\right\}\frac{p^4-6p^2R(p)^2+R(p)^4}{(p^2+R(p)^2)^4} \\ &+ \int_p\left\{\frac{1}{2}\ln\frac{(R_\Lambda(p)-m)^2+p^2}{p^2+R_\Lambda(p)^2} + m\frac{R_\Lambda(p)}{p^2+R_\Lambda(p)^2} + m^2\frac{-p^2+R_\Lambda(p)^2}{2(p^2+R_\Lambda(p)^2)^2} + m^3\frac{R_\Lambda(p)(R_\Lambda(p)^2-3p^2)}{3(p^2+R_\Lambda(p)^2)^3}\right\}\Bigg], \end{aligned} \quad (80)$$

where c_F is a constant of integration. Taking the limit $\Lambda \to 0+$, we obtain

$$\varepsilon_{\mathrm{vac}} = -c_F - \mathrm{Tr}1\frac{1}{2}\int_p\left(\ln\frac{p^2+m^2}{p^2} - \frac{m^2}{p^2}\right) = -c_F + \mathrm{Tr}1\frac{1}{2}\frac{1}{(4\pi)^{\frac{D}{2}}}\Gamma\left(-\frac{D}{2}\right)(m^2)^{\frac{D}{2}}. \quad (81)$$

[2] This result is reminiscent of the fixed point value for the cosmological constant. See, for example, [9–11].

This is consistent with the energy density of the Dirac vacuum:

$$-\frac{1}{2}\mathrm{Tr}1\int\frac{d^{D-1}p}{(2\pi)^{D-1}}\sqrt{\vec{p}^2+m^2}. \tag{82}$$

The factor $\frac{1}{2}$ is there because only the negative energy states contribute to the vacuum.

VI. THE LARGE N LIMIT

We consider the O(N) linear sigma model in D dimensions ($2 < D < 4$). Let $S_\Lambda[\sigma]$ be the Wilson action in terms of the spin variables $\sigma^I(p) (I = 1, \ldots, N)$. Expanding $S_\Lambda[\sigma]$ in powers of fields, we obtain

$$S_\Lambda[\sigma] = c_\Lambda\delta(0) + \frac{1}{2}\int_p \sigma^I(p)\sigma^I(-p)c_{2\Lambda}(p) + \cdots, \tag{83}$$

where the repeated indices are summed. The ERG equation

$$-\Lambda\frac{\partial}{\partial\Lambda}e^{S_\Lambda[\sigma]} = \frac{1}{2}\int_p \Lambda\frac{\partial\log R_\Lambda(p)}{\partial\Lambda}\frac{\delta}{\delta\sigma^I(p)} \times\left[\left(\sigma^I(p)+\frac{\delta}{\delta\sigma^I(-p)}\right)e^{S_\Lambda[\sigma]}\right] \tag{84}$$

gives

$$-\Lambda\frac{\partial}{\partial\Lambda}c_\Lambda = \frac{1}{2}\int_p \Lambda\frac{\partial\ln R_\Lambda(p)}{\partial\Lambda}N(1+c_{2\Lambda}(p)). \tag{85}$$

In the large N limit, we find

$$1 + c_{2\Lambda}(p) = \frac{R_\Lambda(p)}{p^2+R_\Lambda(p)+m_\Lambda^2}, \tag{86}$$

where the squared mass m_Λ^2 satisfies

$$-\Lambda\frac{\partial m_\Lambda^2}{\partial\Lambda} = \frac{\frac{1}{2}\int_p \Lambda\frac{\partial R_\Lambda(p)}{\partial\Lambda}\frac{1}{(p^2+R_\Lambda(p)+m_\Lambda^2)^2}}{\frac{1}{\lambda}+\frac{1}{2}\int_p\frac{1}{(p^2+R_\Lambda(p)+m_\Lambda^2)^2}}. \tag{87}$$

The positive constant λ is a ϕ^4 self-coupling. Equation (85), combined with (86), is the same as Eq. (53) for the Gaussian theory, except for the cutoff dependence of the squared mass. We refer the reader to Appendix B for a derivation of Eqs. (85)–(87).

The solution to (85) is given by

$$\frac{1}{N}c_\Lambda = c - \frac{1}{2\lambda}m_\Lambda^4 + m_\Lambda^2\frac{1}{2}\int_p\left(\frac{1}{p^2+R_\Lambda(p)+m_\Lambda^2}-\frac{1}{p^2}\right) - \frac{1}{2}\int_p\left(\ln\frac{p^2+R_\Lambda(p)+m_\Lambda^2}{p^2}-\frac{m_\Lambda^2}{p^2}\right), \tag{88}$$

where c is a constant of integration. Denoting the physical squared mass by

$$m_{\mathrm{ph}}^2 = \lim_{\Lambda\to0+}m_\Lambda^2, \tag{89}$$

we obtain the vacuum energy density as

$$\begin{aligned}\frac{1}{N}\varepsilon_{\mathrm{vac}} &= -\lim_{\Lambda\to0+}\frac{1}{N}c_\Lambda\\ &= -c + \frac{1}{2\lambda}m_{\mathrm{ph}}^4 - m_{\mathrm{ph}}^2\frac{1}{2}\int_p\left(\frac{1}{p^2+m_{\mathrm{ph}}^2}-\frac{1}{p^2}\right)\\ &\quad+\frac{1}{2}\int_p\left(\ln\frac{p^2+m_{\mathrm{ph}}^2}{p^2}-\frac{m_{\mathrm{ph}}^2}{p^2}\right)\\ &= -c + \frac{1}{2\lambda}m_{\mathrm{ph}}^4 + \frac{D-2}{4}\frac{\Gamma\left(-\frac{D}{2}\right)}{(4\pi)^{\frac{D}{2}}}(m_{\mathrm{ph}}^2)^{\frac{D}{2}},\end{aligned} \tag{90}$$

where $2 < D < 4$. With c and m_{ph}^2 fixed, this has a well-defined strong coupling limit

$$\frac{1}{N}\varepsilon_{\mathrm{vac}} \overset{\lambda\to\infty}{\longrightarrow} c + \frac{D-2}{4}\frac{\Gamma(-\frac{D}{2})}{(4\pi)^{\frac{D}{2}}}(m_{\mathrm{ph}}^2)^{\frac{D}{2}}. \tag{91}$$

For (90) to have a good weak coupling limit, we must introduce appropriate λ dependence to c. We note

$$k \equiv -\frac{m_\Lambda^2}{\lambda} + \frac{1}{2}\int_p\left(\frac{1}{p^2+R_\Lambda(p)+m_\Lambda^2}-\frac{1}{p^2}\right) \tag{92}$$

is independent of Λ. Writing

$$c = c' + \frac{\lambda}{2}k^2, \tag{93}$$

we obtain an alternative expression for c_Λ:

$$\frac{1}{N}c_\Lambda = c' + \frac{\lambda}{2}\left\{\frac{1}{2}\int_p\left(\frac{1}{p^2+R_\Lambda(p)+m_\Lambda^2}-\frac{1}{p^2}\right)\right\}^2 - \frac{1}{2}\int_p\left(\ln\frac{p^2+R_\Lambda(p)+m_\Lambda^2}{p^2}-\frac{m_\Lambda^2}{p^2}\right). \tag{94}$$

This gives

$$\frac{1}{N}\varepsilon_{\mathrm{vac}} = -c' - \frac{\lambda}{2}\left(\frac{1}{(4\pi)^{D/2}}\frac{D}{4}\Gamma\left(-\frac{D}{2}\right)(m_{\mathrm{ph}}^2)^{\frac{D-2}{2}}\right)^2 - \frac{1}{2}\frac{\Gamma(-\frac{D}{2})}{(4\pi)^{\frac{D}{2}}}(m_{\mathrm{ph}}^2)^{\frac{D}{2}}. \tag{95}$$

In the weak coupling limit, where c' is fixed instead of c, we reproduce the Gaussian result (58):

$$\frac{1}{N}\varepsilon_{\mathrm{vac}} \xrightarrow{\lambda\to 0} -c' - \frac{1}{2}\frac{\Gamma\left(-\frac{D}{2}\right)}{(4\pi)^{\frac{D}{2}}}(m_{\mathrm{ph}}^2)^{\frac{D}{2}}. \tag{96}$$

VII. CONCLUSIONS

In this work we have studied the field independent part of the Wilson action that is fundamental in the calculation of the partition function. Let us summarize our results.

We have considered a general ERG equation for the Wilson action involving two cutoff functions. This equation has a diffusionlike form for both scalar fields and fermions. We have introduced the field variables σ that simplify the form of the Wilson action in the limit $\Lambda \to 0$, as given by (20) and (28). Moreover, in Eq. (22), we have formally related two equivalent Wilson actions, constructed by two different sets of cutoff functions, that give rise to the same correlation functions and partition function. We have also related our formalism to the generating functional for connected correlation functions and to the 1PI generating functional in Sec. IV. Finally, we have made our discussion concrete by considering two examples: the Gaussian models for scalar and fermionic fields (Sec. V) and the large N limit of the $\mathrm{O}(N)$ linear sigma model (Sec. VI). We hope this work lays a ground for further generalizations. For instance, it may be natural to go beyond the flat space and work on more general background spacetimes.

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APPENDIX A: THE LIMIT OF $S_\Lambda[\sigma]$ AS $\Lambda \to 0+$

In the limit $\Lambda \to 0+$ we expect $S_\Lambda[\sigma]$ to become "trivial" since below the UV cutoff $\Lambda = 0$ there is no mode left to be integrated. To make this statement precise let us compute

$$\lim_{\Lambda\to 0+} S_\Lambda[\sigma]$$

explicitly, where σ is a special choice of field variables defined by (12).

For this goal, we take a rather roundabout path. We first consider the generating functional $\mathcal{W}[\mathcal{J}]$ of the connected correlation functions defined by [12]

$$\begin{aligned} e^{\mathcal{W}[\mathcal{J}]} &\equiv \sum_0^\infty \frac{1}{n!}\int_{p_1,\ldots,p_n} \mathcal{J}(-p_1)\cdots\mathcal{J}(-p_n) \\ &\quad\times \langle\langle \phi(p_1)\cdots\phi(p_n)\rangle\rangle \qquad \text{(A1)} \\ &= \sum_0^\infty \frac{1}{n!}\int_{p_1,\ldots,p_n}\prod_1^n \frac{\mathcal{J}(-p_i)}{\sqrt{R_\Lambda(p_i)}}\left\langle \exp\left(-\frac{1}{2}\int_p \frac{\delta^2}{\delta\sigma(p)\delta\sigma(-p)}\right)\sigma(p_1)\cdots\sigma(p_n)\right\rangle_{S_\Lambda} \\ &= \int [d\sigma] e^{S_\Lambda[\sigma]}\exp\left(-\frac{1}{2}\int_p \frac{\delta^2}{\delta\sigma(p)\delta\sigma(-p)}\right)\exp\left(\int_p \frac{\mathcal{J}(-p)}{\sqrt{R_\Lambda(p)}}\sigma(p)\right) \\ &= \int [d\sigma]\exp\left[S_\Lambda[\sigma] - \frac{1}{2}\int_p \frac{\mathcal{J}(p)\mathcal{J}(-p)}{R_\Lambda(p)} + \int_p \frac{\mathcal{J}(-p)}{\sqrt{R_\Lambda(p)}}\sigma(p)\right]. \qquad \text{(A2)} \end{aligned}$$

We now introduce new field variables by

$$J(p) \equiv \sqrt{R_\Lambda(p)}\sigma(p). \tag{A3}$$

Taking the Jacobian into account, we obtain

$$\begin{aligned} e^{\mathcal{W}[\mathcal{J}]} &= \frac{1}{\int [dJ']\exp\left(-\frac{1}{2}\int_p \frac{J'(p)J'(-p)}{R_\Lambda(p)}\right)}\int [dJ]\exp\Bigg[W_\Lambda[J] \\ &\quad - \frac{1}{2}\int_p \frac{1}{R_\Lambda(p)}(\mathcal{J}(p)-J(p))(\mathcal{J}(-p)-J(-p))\Bigg], \end{aligned} \tag{A4}$$

where

$$\begin{aligned} W_\Lambda[J] &\equiv S_\Lambda[\sigma] + \frac{1}{2}\int_p \sigma(p)\sigma(-p) = S_\Lambda\left[\frac{J}{\sqrt{R_\Lambda}}\right] \\ &\quad + \frac{1}{2}\int_p \frac{J(p)J(-p)}{R_\Lambda(p)}, \end{aligned} \tag{A5}$$

and we have used the triviality of the Gaussian functional integral

$$\int [d\sigma]\exp\left(-\frac{1}{2}\int_p \sigma(p)\sigma(-p)\right) = 1. \tag{A6}$$

The right-hand side of (A4) is independent of Λ. In the limit $\Lambda \to 0+$ the functional integral becomes one over the delta functional

$$\lim_{\Lambda\to 0+}\frac{1}{\int[dJ']\exp\left(-\frac{1}{2}\int_p\frac{J'(p)J'(-p)}{R_\Lambda(p)}\right)}\times\exp\left(-\frac{1}{2}\int_p\frac{1}{R_\Lambda(p)}(\mathcal{J}(p)-J(p))(\mathcal{J}(-p)-J(-p))\right)=\prod_p\delta(\mathcal{J}(p)-J(p)), \tag{A7}$$

and we obtain

$$\lim_{\Lambda\to 0+}W_\Lambda[J]=\mathcal{W}[J]. \tag{A8}$$

Fixing $\sigma(p)$, we obtain

$$\lim_{\Lambda\to 0+}J(p)=0. \tag{A9}$$

Hence, we obtain

$$\lim_{\Lambda\to 0+}W_\Lambda[\sqrt{R_\Lambda}\sigma]=\mathcal{W}[0]=-\varepsilon_{\mathrm{vac}}\delta(0). \tag{A10}$$

Thus, Eq. (A5) gives the desired limit

$$\lim_{\Lambda\to 0+}S_\Lambda[\sigma]=-\varepsilon_{\mathrm{vac}}\delta(0)-\frac{1}{2}\int_p\sigma(p)\sigma(-p). \tag{A11}$$

APPENDIX B: BRIEF REVIEW OF THE LARGE N LIMIT

In this appendix we briefly review the large N approximation in the ERG by following the method adopted in [13], which is based on the method introduced in [14] for the effective potentials. The large N limit of the ERG has been discussed by several authors; we refer the interested reader to [14–17] and references therein for a complete list of references that certainly includes pioneering works such as [18,2].

The main aim of this appendix is to derive Eqs. (85)–(87) in the main text. We work in D dimensions with $2<D<4$. To obtain the large N limit it is convenient to adopt the 1PI formalism. Let us consider the ansatz

$$\Gamma_\Lambda=-\frac{1}{2}\int_p\phi^i(-p)p^2\phi^i(p)+N\Gamma_{I\Lambda}\left[\frac{\phi^i\phi^i}{2N}\right]. \tag{B1}$$

This is consistent at leading order in large N. We next introduce the variable φ defined by $\varphi\equiv\frac{\phi^i\phi^i}{2N}$. In the large N limit, the ERG equation for $\Gamma_{I\Lambda}$ reads

$$-\Lambda\partial_\Lambda\Gamma_{I\Lambda}=\frac{1}{2}\int_p\Lambda\frac{\partial R_\Lambda(p)}{\partial\Lambda}\mathcal{G}_{\Lambda;p,-p}[\varphi], \tag{B2}$$

where

$$\int_q\mathcal{G}_{\Lambda;p,-q}[\varphi]\left\{(q^2+R_\Lambda(q))\delta(q-r)-\frac{\delta\Gamma_{I\Lambda}[\varphi]}{\delta\varphi(q-r)}\right\}=\delta(p-r). \tag{B3}$$

It turns out simpler to work with the Legendre transform of $\Gamma_{I\Lambda}$. We introduce the functional

$$F_\Lambda[\sigma]=\Gamma_{I\Lambda}[\varphi]-\int_p\sigma(p)\varphi(-p), \tag{B4}$$

where $\sigma=\frac{\delta\Gamma_{I\Lambda}}{\delta\varphi}[\varphi]$. It follows that

$$\varphi(p)=-\frac{\delta F_\Lambda}{\delta\sigma(-p)}. \tag{B5}$$

The ERG equation for $F_\Lambda[\sigma]$ is then given by

$$-\Lambda\partial_\Lambda F_\Lambda[\sigma]=\frac{1}{2}\int_p\Lambda\frac{\partial R_\Lambda(p)}{\partial\Lambda}\mathcal{G}_{\Lambda;p,-p}[\sigma], \tag{B6}$$

where $\mathcal{G}_\Lambda$, regarded as a functional of σ, satisfies

$$\int_q\mathcal{G}_{\Lambda;p,-q}[\sigma]\left\{(q^2+R_\Lambda(q))\delta(q-r)-\sigma(r-q)\right\}=\delta(p-r), \tag{B7}$$

and can be expressed by a geometric series of σ.

A particular solution to (B6) is given by

$$I_\Lambda[\sigma]=c_\Lambda\delta(0)+c_{1\Lambda}\sigma(0)+\sum_{i=2}^{\infty}\frac{1}{2n}\int_{p_1\cdots p_n}\sigma(p_1)\cdots\sigma(p_n)\delta\left(\sum_{i=1}^n p_i\right)I_{n\Lambda}(p_1,\ldots,p_n), \tag{B8}$$

where

$$c_\Lambda=-\frac{1}{2}\int_q\log\left(\frac{q^2+R_\Lambda(q)}{q^2}\right), \tag{B9a}$$

$$c_{1\Lambda}=\frac{1}{2}\int_q\left(\frac{1}{q^2+R_\Lambda(q)}-\frac{1}{q^2}\right), \tag{B9b}$$

$$I_{n\Lambda}=\int_q h_\Lambda(q)h_\Lambda(q+p_1)\cdots h_\Lambda(q+p_1+\cdots+p_{n-1}), \tag{B9c}$$

with $h_\Lambda(q)\equiv 1/(q^2+R_\Lambda(q))$.

The general solution to (B6) is obtained as

$$F_\Lambda[\sigma] = \tilde{F}[\sigma] + I_\Lambda[\sigma], \tag{B10}$$

where $\tilde{F}[\sigma]$ is an arbitrary functional independent of Λ. In the present case we consider

$$\tilde{F}[\sigma] \equiv f_0\delta(0) + f_1\sigma(0) + \frac{1}{\lambda}\int_p \frac{1}{2}\sigma(p)\sigma(-p), \tag{B11}$$

where λ is a positive constant reminiscent of the ϕ^4-interaction coupling. Finally, let us also write down in a compact form the solution for the case of constant field σ. We find

$$F_\Lambda(\sigma) = \tilde{F}(\sigma) - \frac{1}{2}\int_q \left[\log\frac{q^2 - \sigma + R_\Lambda}{q^2} + \frac{\sigma}{q^2}\right], \tag{B12}$$

where

$$F_\Lambda[\sigma] = F_\Lambda(\sigma)\delta(0), \qquad \tilde{F}[\sigma] = \tilde{F}(\sigma)\delta(0). \tag{B13}$$

The main aim of this appendix has been to derive Eqs. (85)–(87) in the main text. Let us work in the symmetric phase, where $c_\Lambda = G_\Lambda(0)$ [see Eq. (45)]. According to the relations among the various functionals detailed in Sec. IV, we can write the right-hand side of (85) by employing the following equation:

$$1 + c_{2\Lambda}(p) = R_\Lambda(p)\left(p^2 - \frac{\delta\Gamma_{I\Lambda}}{\delta\varphi} + R_\Lambda(p)\right)^{-1}\Bigg|_{\varphi=0}, \tag{B14}$$

which implies Eq. (86) after we identify m_Λ^2 with

$$-\sigma_{\mathrm{os}} = -\frac{\delta\Gamma_{I\Lambda}}{\delta\varphi}[\varphi = 0] = m_\Lambda^2. \tag{B15}$$

The ERG equation associated with σ_{os} reads

$$-\Lambda\frac{d}{d\Lambda}\sigma_{\mathrm{os}} = \frac{1}{2}(-\partial_\sigma^2 F_\Lambda)^{-1}\int_q \frac{\Lambda\partial_\Lambda R_\Lambda}{(q^2 - \sigma_{\mathrm{os}} + R_\Lambda)^2}, \tag{B16}$$

which reproduces Eq. (87) once expressed in terms of m_Λ^2.

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