BPZ equations for higher degenerate fields and nonperturbative Dyson-Schwinger equations

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In the two-dimensional Liouville conformal field theory, correlation functions involving a degenerate field satisfy partial differential equations due to the decoupling of the null descendant field. On the other hand, the instanton partition function of a four-dimensional $\mathcal{N}=2$ supersymmetric theory in the Ω-background at a special point of the parameter space also satisfies a partial differential equation resulting from the constraints of the gauge field configurations. This partial differential equation can be proved using the nonperturbative Dyson-Schwinger equations. We show for the next-to-simplest case that the partial differential equations obtained from two different perspectives can be identified, thereby confirming an assertion of the Bogomol'nyi-Prasad-Sommerfield/conformal field theory correspondence.

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I. INTRODUCTION

Ever since the groundbreaking work of Seiberg and Witten on four-dimensional $\mathcal{N}=2$ supersymmetric gauge theories with gauge group SU(2) [1,2], there has been continuous progress in constructing and analyzing $\mathcal{N}=2$ supersymmetric theories. Although many impressive statements have already been made over the last few decades, we continue to discover new interesting results.

It has long been appreciated that when understanding a complicated system, it is helpful to explore its deformations and study the dependence on the deformation parameters. This general lesson has been convincingly demonstrated in [3]. The four-dimensional $\mathcal{N}=2$ supersymmetric gauge theories were studied in the Ω-background $\mathbb{R}^4_{\varepsilon 1,\varepsilon_2}$, with two deformation parameters ε_1 and ε_2. The deformed theory breaks Poincaré symmetry in a rotationally covariant way while still preserving a combination of the deformed supersymmetry. Applying the localization technique, the supersymmetric partition function $\mathcal{Z}$ and correlation functions of $\mathcal{N}=2$ chiral operators have been computed exactly for a large class of four-dimensional $\mathcal{N}=2$ supersymmetric gauge theories. At generic points in the parameter space, the low-energy effective prepotential $\mathcal{F}_{\text{eff}}$ can be extracted from $\mathcal{Z}$ by taking the flat-space limit [3,4],

$$\mathcal{F}_{\text{eff}} = -\lim_{\varepsilon_1,\varepsilon_2\to 0} \varepsilon_1\varepsilon_2 \log \mathcal{Z}. \tag{1.1}$$

Further investigation of the Ω-background with finite ε_1, ε_2 resulted in the proposal of a remarkable relation, the Bogomol'nyi-Prasad-Sommerfield/conformal field theory (BPS/CFT) correspondence, which identifies correlation functions of $\mathcal{N}=2$ chiral operators with quantities in two-dimensional conformal field theories or deformations thereof.

After establishing the BPS/CFT correspondence, we expect to gain insights into four-dimensional $\mathcal{N}=2$ supersymmetric gauge theories using the knowledge of two-dimensional theories, and vice versa. One particular implementation of the BPS/CFT correspondence is the Alday-Gaiotto-Tachikawa (AGT) correspondence, which was first conjectured as a relationship between a class of superconformal SU(2) quiver gauge theories and the Liouville conformal field theory [5], and was soon extended to a more general relationship between quiver gauge theories with other gauge groups and the Toda conformal field theory [6].

The four-dimensional $\mathcal{N}=2$ superconformal field theories considered in the AGT correspondence can be obtained by compactifying the six-dimensional $\mathcal{N}=(2,0)$ superconformal theory on a punctured Riemann surface $\mathcal{C}$ [7]. When the theory admits a weakly-coupled Lagrangian

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description, we can often compute its partition function in the Ω-background [3,8]. It was proposed that the instanton part of the partition function $\mathcal{Z}^{\text{instanton}}$ can be identified with a conformal block in the Liouville/Toda conformal field theory, and the partition function on a (squashed) sphere S_b^4 [9,10], which is given by the integral of the absolute value squared of the full partition function, can be identified with correlation functions in the Liouville/Toda conformal field theory on $\mathcal{C}$. Based on careful analysis of the structure of the instanton moduli space, some versions of the AGT correspondence have been proved [11–13].

In the study of the AGT correspondence, it is often assumed that the Coulomb branch parameters take generic values. However, it is also interesting to specialize these parameters and explore the consequence of the AGT correspondence in the $\mathcal{N}=2$ gauge theory context. In two-dimensional conformal field theory, we can make one of the fields in the correlation function degenerate. Belavin, Polyakov, and Zamolodchikov (BPZ) showed that such correlation functions satisfy partial differential equations as a result of the decoupling of the null descendant field [14,15]. The order of the differential equation is the level of the null field in the corresponding degenerate representation. In the case of Toda field theories, similar differential equations have been derived for certain four-point correlation functions in [16,17]. Correspondingly, the gauge field configurations in the four-dimensional $\mathcal{N}=2$ superconformal quiver gauge theories are constrained. We will show that the corresponding instanton partition functions also satisfy partial differential equations. In the context of the $\mathcal{N}=2$ gauge theory, the specialization of the Coulomb parameter initiate partial Higgsing of the theory, producing a half-BPS surface defect as a result [18,19]. The differential equation we obtained is the quantized chiral ring relation of the so-obtained 2D/4D coupled system [20]. This program was investigated carefully in the Nekrasov-Shatashvili limit [21] of the Ω-background, which corresponds to the classical limit $c \to \infty$ of two-dimensional conformal field theories [22–31]. However, previous methods become less powerful when we would like to go beyond such limits. Our results provide a new method to go beyond this limit.

In this paper, we shall follow the idea of [32] to provide a derivation of the differential equation using the nonperturbative Dyson-Schwinger (NPDS) equations, which result from the fact that the path integral of the instanton partition function is invariant with respect to the transformations changing topological sectors of the field space. We review the result of [32] and study the case of U(2) superconformal linear quiver gauge theories with the next-to-simplest constraint in this paper. Similar methods have also been applied to the study of Bethe/gauge correspondence [21,33,34] in [35].

The rest of the paper is organized as follows. In Sec. II, we recall some basic facts about two-dimensional Liouville field theory and review the derivation of BPZ equations on the degenerate correlation functions. In Sec. III, we review the relevant details of four-dimensional $\mathcal{N}=2$ quiver gauge theories in the Ω-background. We summarize the result of the partition function and review the NPDS equations. In Sec. IV, we study the superconformal gauge theory with gauge group $\mathrm{U}(N)$. We show that the instanton partition function at the simplest nontrivial degenerate point in the parameter space is a (generalized) hypergeometric function. After working out this simple warm-up example, we consider the U(2) superconformal linear quiver gauge theory in Sec. V. We review the second-order differential equation on the instanton partition function derived in [32] and derive the third-order differential equation for the next-to-simplest case. We also identify the differential equations derived from both sides using the AGT dictionary. Finally, we conclude in Sec. VI and discuss possible directions for future work. In Appendix A, we review some standard material on the (generalized) hypergeometric function. In Appendix B, we derive the partition function of the U(1) factor using the NPDS equations.

II. DEGENERATE CORRELATION FUNCTIONS IN THE LIOUVILLE FIELD THEORY

In this section, we recall some basic facts about two-dimensional Liouville field theory and present the derivation of the BPZ equations on the degenerate correlation functions.

A. Degenerate fields in the Liouville field theory

The two-dimensional Liouville conformal field theory is defined by the action,

$$S_{\text{Liouville}} = \int d^2\sigma\sqrt{g}\left(\frac{1}{4\pi}\partial_a\phi\partial^a\phi + \mu e^{2b\phi} + \frac{Q}{4\pi}R\phi\right), \tag{2.1}$$

where the background charge $Q = b + b^{-1}$, and R is the Ricci scalar of the Riemann surface. The symmetry algebra of the theory is two independent copies of the Virasoro algebra, with the central charge $c = 1 + 6Q^2$. In the following, we focus on the chiral part, which is spanned by generators L_n for $n \in \mathbb{Z}$ and the central charge c, satisfying

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}. \tag{2.2}$$

For the Virasoro algebra, a conformal primary field V_Δ with the conformal dimension Δ is defined to be

$$L_{n>0}V_\Delta = 0, \qquad L_0 V_\Delta = \Delta V_\Delta. \tag{2.3}$$

The descendant fields are obtained by taking the linear combinations of the basis vectors $L_{-\vec{n}}V_\Delta = L_{-n_1}L_{-n_2}\cdots L_{-n_l}V_\Delta$,

where $\vec{n} = \{1 \le n_1 \le n_2 \le \cdots \le n_l\}$. The conformal dimension of the basis vector $L_{-\vec{n}}V_\Delta$ is $\Delta + |\vec{n}|$, where the number $|\vec{n}| = \sum_{i=1}^{l} n_i$ is called the level of $L_{-\vec{n}}V_\alpha$.

A primary field V_Δ is called degenerate if it has a null descendant field $\tilde{V} = \sum_{\vec{n}} C_{\vec{n}} L_{-\vec{n}} V_\Delta \neq V_\Delta$, such that $L_n \tilde{V} = 0$ for $n > 0$. If the null field is at the level one, then

$$L_n(L_{-1}V_\Delta) = 0, \qquad n > 0. \tag{2.4}$$

This is automatically true for $n \ge 2$, and for $n = 1$ we have

$$0 = L_1 L_{-1} V_\Delta = 2L_0 V_\Delta = 2\Delta V_\Delta. \tag{2.5}$$

Thus the field $V_\Delta = 1$ with zero conformal dimension. If the level-two descendant field $\tilde{V} = (C_{1,1}L_{-1}^2 + C_2 L_{-2})V_\Delta$ is null, then $L_n\tilde{V} = 0$ for $n \ge 1$. The nontrivial constraints are

$$\begin{aligned} 0 &= L_1\tilde{V} = ((4\Delta+2)C_{1,1} + 3C_2)L_{-1}V_\Delta, \\ 0 &= L_2\tilde{V} = \left(6\Delta C_{1,1} + \left(4\Delta + \frac{c}{2}\right)C_2\right)V_\Delta. \end{aligned} \tag{2.6}$$

Therefore, we have

$$\begin{vmatrix} 4\Delta+2 & 3 \\ 6\Delta & 4\Delta + \frac{c}{2} \end{vmatrix} = 0, \tag{2.7}$$

which gives two solutions

$$\Delta_{(2,1)} = -\frac{1}{2} - \frac{3}{4}b^2, \quad \tilde{V}_{(2,1)} = \left(\frac{1}{b^2}L_{-1}^2 + L_{-2}\right)V_{\Delta_{(2,1)}}, \tag{2.8}$$

$$\Delta_{(1,2)} = -\frac{1}{2} - \frac{3}{4b^2}, \quad \tilde{V}_{(1,2)} = (b^2 L_{-1}^2 + L_{-2})V_{\Delta_{(1,2)}}. \tag{2.9}$$

If the level-three descendant field $\tilde{V} = (C_{1,1,1}L_{-1}^3 + C_{1,2}L_{-1}L_{-2} + C_3 L_{-3})V_\Delta$ is null, then $L_n\tilde{V} = 0$ for $n \ge 1$. Since $L_3 = [L_2, L_1]$, we only need

$$\begin{aligned} 0 &= L_1\tilde{V} = ((2\Delta+4)C_{1,2} + 4C_3)L_{-2}V_\Delta + ((6\Delta+6)C_{1,1,1} + 3C_{1,2})L_{-1}^2 V_\Delta, \\ 0 &= L_2\tilde{V} = \left((6+18\Delta)C_{1,1,1} + \left(4\Delta + 9 + \frac{c}{2}\right)C_{1,2} + 5C_3\right)L_{-1}V_\Delta. \end{aligned} \tag{2.10}$$

Therefore, we have

$$\begin{vmatrix} 0 & 2\Delta+4 & 4 \\ 6\Delta+6 & 3 & 0 \\ 6+18\Delta & 4\Delta+9+\frac{c}{2} & 5 \end{vmatrix} = 0, \tag{2.11}$$

which gives two solutions

$$\begin{aligned} &\Delta_{(3,1)} = -1 - 2b^2, \\ &\tilde{V}_{(3,1)} = \left(\frac{1}{4b^2}L_{-1}^3 + L_{-1}L_{-2} + \left(b^2 - \frac{1}{2}\right)L_{-3}\right)V_{\Delta_{(3,1)}}, \end{aligned} \tag{2.12}$$

$$\begin{aligned} &\Delta_{(1,3)} = -1 - \frac{2}{b^2}, \\ &\tilde{V}_{(1,3)} = \left(\frac{b^2}{4}L_{-1}^3 + L_{-1}L_{-2} + \left(\frac{1}{b^2} - \frac{1}{2}\right)L_{-3}\right)V_{\Delta_{(1,3)}}. \end{aligned} \tag{2.13}$$

Generally, the conformal dimension of a degenerate field can be read from the Kac determinant formula, and is given by

$$\Delta_{(m,n)} = \frac{Q^2 - (mb + nb^{-1})^2}{4}, \qquad m, n \in \mathbb{Z}^+, \tag{2.14}$$

with the null vector being at the level mn.

B. BPZ equations

Now we are ready to derive the BPZ equations on the $(r+3)$-point correlation function of the conformal primary fields, with one of the primary fields being degenerate. In order to relate a correlation function involving Virasoro generators acting on a primary field with a correlation function of purely primary fields, we use the conformal Ward identities, which state that inserting the holomorphic energy-momentum tensor in a correlation function of primary fields yields,

$$\begin{aligned} &\left\langle T(z)\prod_{i=-1}^{r+1} V_{\Delta_i}(z_i)\right\rangle \\ &= \sum_{i=-1}^{r+1}\left(\frac{1}{z-z_i}\frac{\partial}{\partial z_i} + \frac{\Delta_i}{(z-z_i)^2}\right)\left\langle \prod_{i=-1}^{r+1} V_{\Delta_i}(z_i)\right\rangle. \end{aligned} \tag{2.15}$$

The simplest nontrivial example is the second order BPZ equation. We assume that $\Delta_0 = \Delta_{(2,1)}$. The decoupling of

the null descendant field (2.8) implies that the $(r+3)$-point correlation function satisfies,

$$\left[\frac{1}{b^2}\frac{\partial^2}{\partial z_0^2}+\sum_{i\neq 0}\left(\frac{1}{z_0-z_i}\frac{\partial}{\partial z_i}+\frac{\Delta_i}{(z_0-z_i)^2}\right)\right] \times\left\langle V_{\Delta_{(2,1)}}(z_0)\prod_{i\neq 0}V_{\Delta_i}(z_i)\right\rangle=0. \tag{2.16}$$

Similarly, the third-order BPZ equation with $\Delta_0=\Delta_{(3,1)}$ can be derived from the decoupling of the null vector (2.12),

$$\begin{aligned}0=&\left[\frac{1}{4b^2}\frac{\partial^3}{\partial z_0^3}+\frac{\partial}{\partial z_0}\sum_{i\neq 0}\left(\frac{1}{z_0-z_i}\frac{\partial}{\partial z_i}+\frac{\Delta_i}{(z_0-z_i)^2}\right)\right.\\&\left.-\left(b^2-\frac{1}{2}\right)\sum_{i\neq 0}\left(\frac{1}{(z_0-z_i)^2}\frac{\partial}{\partial z_i}+\frac{2\Delta_i}{(z_0-z_i)^3}\right)\right]\\&\times\left\langle V_{\Delta_{(3,1)}}(z_0)\prod_{i\neq 0}V_{\Delta_i}(z_i)\right\rangle.\end{aligned} \tag{2.17}$$

There are additional constraints on the correlation functions due to the global conformal symmetry. Using the holomorphy of the energy-momentum tensor at infinity, $T(z)=\mathcal{O}(z^{-4})$ as $z\to\infty$, we deduce the global conformal Ward identities,

$$\left[\sum_{i=-1}^{r+1}\frac{\partial}{\partial z_i}\right]\left\langle\prod_{i=-1}^{r+1}V_{\Delta_i}(z_i)\right\rangle=0, \tag{2.18}$$

$$\left[\sum_{i=-1}^{r+1}\left(z_i\frac{\partial}{\partial z_i}+\Delta_i\right)\right]\left\langle\prod_{i=-1}^{r+1}V_{\Delta_i}(z_i)\right\rangle=0, \tag{2.19}$$

$$\left[\sum_{i=-1}^{r+1}\left(z_i^2\frac{\partial}{\partial z_i}+2z_i\Delta_i\right)\right]\left\langle\prod_{i=-1}^{r+1}V_{\Delta_i}(z_i)\right\rangle=0. \tag{2.20}$$

For our purpose, it is convenient to get rid of all the ∂_{-1} and ∂_{r+1} terms using (2.18) and (2.20),

$$\begin{aligned}\frac{\partial}{\partial z_{-1}}\left\langle\prod_{i=-1}^{r+1}V_{\alpha_i}(z_i)\right\rangle&=\left[\sum_{i=0}^{r}\frac{z_i^2-z_{r+1}^2}{z_{r+1}^2-z_{-1}^2}\frac{\partial}{\partial z_i}+\sum_{i=-1}^{r+1}\frac{2z_i\Delta_i}{z_{r+1}^2-z_{-1}^2}\right]\left\langle\prod_{i=-1}^{r+1}V_{\Delta_i}(z_i)\right\rangle,\\\frac{\partial}{\partial z_{r+1}}\left\langle\prod_{i=-1}^{r+1}V_{\alpha_i}(z_i)\right\rangle&=\left[\sum_{i=0}^{r}\frac{z_i^2-z_{-1}^2}{z_{-1}^2-z_{r+1}^2}\frac{\partial}{\partial z_i}+\sum_{i=-1}^{r+1}\frac{2z_i\Delta_i}{z_{-1}^2-z_{r+1}^2}\right]\left\langle\prod_{i=-1}^{r+1}V_{\Delta_i}(z_i)\right\rangle.\end{aligned} \tag{2.21}$$

We then fix $z_{-1}=\infty$ and $z_{r+1}=0$, and the remaining global conformal Ward identity (2.19) gives

$$\left[\sum_{i=0}^{r}(\nabla_i+\Delta_i)-\Delta_{-1}+\Delta_{r+1}\right]\left\langle V_{\Delta_{-1}}(\infty)\prod_{i=0}^{r}V_{\Delta_i}(z_i)V_{\Delta_{r+1}}(0)\right\rangle=0. \tag{2.22}$$

Let us decouple a prefactor from the correlation function,

$$\left\langle V_{\Delta_{-1}}(\infty)V_{\Delta_{(m,n)}}(z_0)\prod_{i=1}^{r}V_{\Delta_i}(z_i)V_{\Delta_{r+1}}(0)\right\rangle=\left[\left(\prod_{i=0}^{r}z_i^{L_i}\right)\prod_{0\le i<j\le r}\left(1-\frac{z_j}{z_i}\right)^{T_{ij}}\right]\chi_{r+3}^{(m,n)}(\boldsymbol{z}), \tag{2.23}$$

where $\chi_{r+3}^{(m,n)}(\boldsymbol{z})$ only depends on the ratios of z_i, $i=0,\ldots,r$. The identity (2.22) is satisfied if

$$\sum_{i=0}^{r}(L_i+\Delta_i)-\Delta_{-1}+\Delta_{r+1}=0. \tag{2.24}$$

Using

$$\left[z_i\frac{\partial}{\partial z_i},\left(\prod_{i=0}^{r}z_i^{L_i}\right)\prod_{0\le i<j\le r}\left(1-\frac{z_j}{z_i}\right)^{T_{ij}}\right]=\left(L_i+\sum_{j=i+1}^{r}T_{ij}\frac{z_j}{z_i-z_j}+\sum_{j=0}^{i-1}T_{ji}\frac{z_i}{z_i-z_j}\right)\left(\prod_{i=0}^{r}z_i^{L_i}\right)\prod_{0\le i<j\le r}\left(1-\frac{z_j}{z_i}\right)^{T_{ij}}, \tag{2.25}$$

the second-order BPZ equation (2.16) can be express in terms of $\chi^{(m,n)}_{r+3}(z)$ as

$$0 = \left[\frac{1}{b^2}\left(\nabla_0 + L_0 + \sum_{j=1}^{r} T_{0j}\frac{z_j}{z_0 - z_j}\right)^2 - \left(1+\frac{1}{b^2}\right)\left(\nabla_0 + L_0 + \sum_{j=1}^{r} T_{0j}\frac{z_j}{z_0 - z_j}\right)\right.$$
$$+ \sum_{i=1}^{r}\frac{z_0}{z_0 - z_i}\left(\nabla_i + L_i + \sum_{j=i+1}^{r} T_{ij}\frac{z_j}{z_i - z_j} + \sum_{j=0}^{i-1} T_{ji}\frac{z_i}{z_i - z_j}\right)$$
$$\left. + \sum_{i=1}^{r}\frac{z_0^2\Delta_i}{(z_0 - z_i)^2} + \Delta_{r+1}\right]\chi^{(2,1)}_{r+3}(z), \tag{2.26}$$

and the third-order BPZ equation (2.17) becomes

$$0 = \left[\frac{1}{4b^2}\left(\nabla_0 + L_0 + \sum_{j=1}^{r} T_{0j}\frac{z_j}{z_0 - z_j}\right)^3 - \left(\frac{3}{4b^2}+1\right)\left(\nabla_0 + L_0 + \sum_{j=1}^{r} T_{0j}\frac{z_j}{z_0 - z_j}\right)^2\right.$$
$$+ \left(\frac{1}{b^2} + b^2 + \frac{3}{2} + \sum_{i=1}^{r}\frac{z_0^2\Delta_i}{(z_0 - z_i)^2} + \Delta_{r+1}\right)\left(\nabla_0 + L_0 + \sum_{j=1}^{r} T_{0j}\frac{z_j}{z_0 - z_j}\right)$$
$$+ \sum_{i=1}^{r}\frac{z_0}{z_0 - z_i}\left(\nabla_0 + L_0 + \sum_{j=1}^{r} T_{0j}\frac{z_j}{z_0 - z_j}\right)\left(\nabla_i + L_i + \sum_{j=i+1}^{r} T_{ij}\frac{z_j}{z_i - z_j} + \sum_{j=0}^{i-1} T_{ji}\frac{z_i}{z_i - z_j}\right)$$
$$- \left(b^2 + \frac{1}{2}\right)\sum_{i=1}^{r}\frac{z_0(2z_0 - z_i)}{(z_0 - z_i)^2}\left(\nabla_i + L_i + \sum_{j=i+1}^{r} T_{ij}\frac{z_j}{z_i - z_j} + \sum_{j=0}^{i-1} T_{ji}\frac{z_i}{z_i - z_j}\right)$$
$$\left. - (2b^2+1)\left(\sum_{i=1}^{r}\frac{z_0^3\Delta_i}{(z_0 - z_i)^3} + \Delta_{r+1}\right)\right]\chi^{(3,1)}_{r+3}(z), \tag{2.27}$$

where we denote

$$\nabla_i = z_i\frac{\partial}{\partial z_i}. \tag{2.28}$$

We should determine L_i and T_{ij} when we identify the BPZ equations with the differential equations derived in the corresponding gauge theories.

III. FOUR-DIMENSIONAL $\mathcal{N}=2$ QUIVER GAUGE THEORY IN THE Ω-BACKGROUND

In this section, we review some useful results of four-dimensional $\mathcal{N}=2$ quiver gauge theories in the Ω-background. A detailed discussion can be found in [8,29,36].

A. Partition function

Let us consider four-dimensional $\mathcal{N}=2$ superconformal linear quiver gauge theories with gauge group

$$G = \mathrm{U}(N_1)\times\mathrm{U}(N_2)\times\cdots\times\mathrm{U}(N_r), \tag{3.1}$$

where $N_1 = N_2 = \cdots = N_r = N$. The vector superfield splits into a collection of vector multiplets for each gauge factor $\mathrm{U}(N_i)$. The matter superfields consist of $r-1$ hypermultiplets transforming in the bifundamental representations $(\bar{\boldsymbol{N}}_i, \boldsymbol{N}_{i+1})$, N hypermultiplet transforming in the antifundamental representation of $\mathrm{U}(N_1)$, and N hypermultiplet transforming in the fundamental representation of $\mathrm{U}(N_r)$. Here we denote by $\boldsymbol{N}_i$ the representation of G in which the ith factor acts in the defining N-dimensional representation, while all other factors act trivially. It is convenient to extend the quiver by including two frozen nodes $\mathrm{U}(N_0)$ and $\mathrm{U}(N_{r+1})$ corresponding to the flavor symmetry $\mathrm{U}(N)\times\mathrm{U}(N)$. The Yang-Mills coupling constant g_i and the theta-angle ϑ_i for the gauge factor $\mathrm{U}(N_i)$ are combined into the complexified gauge couplings,

$$\tau_i = \frac{\vartheta_i}{2\pi} + \mathrm{i}\frac{4\pi}{g_i^2}, \quad i = 1,\ldots,r. \tag{3.2}$$

We introduce

$$q_i = \exp(2\pi\mathrm{i}\tau_i) = \frac{z_i}{z_{i-1}}, \quad i = 1,\ldots,r, \tag{3.3}$$

and $q_0 = q_{r+1} = 0$. Therefore, $z_{-1} = \infty$, $z_{r+1} = 0$, and $z_0, z_1, \ldots, z_r$ are defined up to an overall rescaling.

The vacuum expectation values of the adjoint scalars in the vector multiplet for the gauge factor $\mathrm{U}(N_i)$ are $a_i = \mathrm{diag}(a_{i,1}, \ldots, a_{i,N})$. We also encode the masses of the (anti)fundamental hypermultiplets in $a_0 = \mathrm{diag}(a_{0,1}, \ldots, a_{0,N})$ and $a_{r+1} = \mathrm{diag}(a_{r+1,1}, \ldots, a_{r+1,N})$. We denote the collection of Coulomb parameters as

$$\boldsymbol{a} = \{a_{i,\alpha} | i = 0, \ldots, r+1, \alpha = 1, \ldots, N\}. \tag{3.4}$$

The partition function of the theory in the Ω-background $\mathbb{R}^4_{\varepsilon_1,\varepsilon_2}$ is given by a product of the classical, the one-loop, and the instanton contributions,

$$\mathcal{Z}(\boldsymbol{a};\boldsymbol{q};\varepsilon_1,\varepsilon_2) = \mathcal{Z}^{\mathrm{classical}}(\boldsymbol{a};\boldsymbol{q};\varepsilon_1,\varepsilon_2)\mathcal{Z}^{1-\mathrm{loop}}(\boldsymbol{a};\varepsilon_1,\varepsilon_2) \times \mathcal{Z}^{\mathrm{instanton}}(\boldsymbol{a};\boldsymbol{q};\varepsilon_1,\varepsilon_2), \tag{3.5}$$

where we denote the collection of Coulomb parameters as $\boldsymbol{a} = \{a_{i,\alpha} | i = 0, \ldots, r+1, \alpha = 1, \ldots, N\}$, and the collection of coupling constants as $\boldsymbol{q} = \{q_i | i = 1, \ldots, r\}$. The classical part of the partition function is simply

$$\mathcal{Z}^{\mathrm{classical}}(\boldsymbol{a};\boldsymbol{q};\varepsilon_1,\varepsilon_2) = \prod_{i=1}^{r}\prod_{\alpha=1}^{N} q_i^{-\frac{1}{2\varepsilon_1\varepsilon_2}a_{i,\alpha}^2}, \tag{3.6}$$

and the one-loop contribution to the partition function is

$$\mathcal{Z}^{1-\mathrm{loop}}(\boldsymbol{a};\varepsilon_1,\varepsilon_2) = \frac{\prod_{i=0}^{r}\prod_{\alpha,\beta}\Gamma_2(a_{i,\alpha} - a_{i+1,\beta} + \varepsilon | \varepsilon_1,\varepsilon_2)}{\prod_{i=1}^{r}\prod_{\alpha<\beta}\Gamma_2(a_{i,\alpha} - a_{i,\beta} + \varepsilon_1 | \varepsilon_1,\varepsilon_2)\Gamma_2(a_{i,\alpha} - a_{i,\beta} + \varepsilon_2 | \varepsilon_1,\varepsilon_2)}, \tag{3.7}$$

where $\varepsilon = \varepsilon_1 + \varepsilon_2$, and the Barnes double Gamma function $\Gamma_2(x|\varepsilon_1,\varepsilon_2)$ is defined by

$$\Gamma_2(x|\varepsilon_1,\varepsilon_2) = \exp\left\{\frac{d}{ds}\bigg|_{s=0}\left[\frac{1}{\Gamma(s)}\int_0^\infty \frac{dt}{t} t^s \frac{e^{-tx}}{(1-e^{-\varepsilon_1 t})(1-e^{-\varepsilon_2 t})}\right]\right\}. \tag{3.8}$$

The instanton part of the partition function is an equivariant integration on the instanton moduli space with respect to the maximal torus of the gauge group and the $SO(4)$ rotation. Applying the equivariant localization theorem, it is given by the fixed point formula as

$$\mathcal{Z}^{\mathrm{instanton}}(\boldsymbol{a};\boldsymbol{q};\varepsilon_1,\varepsilon_2) = \sum_{\boldsymbol{Y}} \mathcal{Q}_{\boldsymbol{Y}} \mathcal{Z}^{\mathrm{instanton}}(\boldsymbol{a};\boldsymbol{Y};\varepsilon_1,\varepsilon_2). \tag{3.9}$$

The sum is over all fixed points of instanton configurations, which are labeled by the collections of Young diagrams,

$$\boldsymbol{Y} = \{Y^{(i,\alpha)} | 0 \le i \le r+1, \alpha = 1, \ldots, N\}, \tag{3.10}$$

where $Y^{(0,\alpha)} = Y^{(r+1,\alpha)} = \emptyset$. Each Young diagram Y is a finite collection of boxes $\square = (u,v) \in Y$ arranged in left-justified rows, with the row lengths in nonincreasing order. The total number of boxes in the Young diagram Y is denoted by $|Y|$, and the number of boxes in each row gives a partition of $|Y|$,

$$Y = (Y_1 \ge Y_2 \ge \cdots \ge Y_{\ell(Y)} > Y_{\ell(Y)+1} = \cdots = 0). \tag{3.11}$$

We define the arm-length and leg-length as

$$A^Y_\square = Y_u - v, \qquad L^Y_\square = Y^T_v - u. \tag{3.12}$$

We also define the content $c_\square$ of a box $\square = (u,v) \in Y$ to be

$$c_\square = \varepsilon_1(u-1) + \varepsilon_2(v-1). \tag{3.13}$$

The weight $\mathcal{Q}_{\boldsymbol{Y}}$ is given by

$$\mathcal{Q}_{\boldsymbol{Y}} = \prod_{i=1}^{r} q_i^{k_i} = \prod_{i=0}^{r} z_i^{k_i - k_{i+1}}, \tag{3.14}$$

where k_i is the instanton charge associated with the gauge factor $\mathrm{U}(N_i)$, and

$$k_i = \sum_{\alpha=1}^{N} |Y^{(i,\alpha)}|. \tag{3.15}$$

The measure $\mathcal{Z}^{\mathrm{instanton}}(\boldsymbol{a};\boldsymbol{Y};\varepsilon_1,\varepsilon_2)$ is the product of factors corresponding to the field content of the theory,

$$\mathcal{Z}^{\mathrm{instanton}}(\boldsymbol{a};\boldsymbol{Y};\varepsilon_1,\varepsilon_2) = \left[\prod_{i=1}^{r} \mathcal{Z}^{\mathrm{instanton}}_{\mathrm{vector}}(\vec{a}_i, \vec{Y}^{(i)};\varepsilon_1,\varepsilon_2)\right] \times \left[\prod_{i=0}^{r} \mathcal{Z}^{\mathrm{instanton}}_{\mathrm{bifund}}(\vec{a}_i, \vec{Y}^{(i)}, \vec{a}_{i+1}, \vec{Y}^{(i+1)};\varepsilon_1,\varepsilon_2)\right], \tag{3.16}$$

where we denote $\vec{a}_i = \{a_{i,\alpha} | \alpha = 1, \ldots, N\}$ and $\vec{Y}^{(i)} = \{Y^{(i,\alpha)} | \alpha = 1, \ldots, N\}$. The contribution from a bifundamental hypermultiplet is

$$\mathcal{Z}^{\text{instanton}}_{\text{bifund}}(\vec{a},\vec{Y},\vec{b},\vec{W};\varepsilon_1,\varepsilon_2)=\prod_{\alpha,\beta=1}^{N}\prod_{\square\in Y^{(\alpha)}}[a_\alpha-b_\beta-\varepsilon_1 L_\square^{W^{(\beta)}}+\varepsilon_2(A_\square^{Y^{(\alpha)}}+1)]\times\prod_{\square\in W^{(\beta)}}[a_\alpha-b_\beta+\varepsilon_1(L_\square^{Y^{(\alpha)}}+1)-\varepsilon_2 A_\square^{W^{(\beta)}}]. \tag{3.17}$$

If we set one collection of Young diagrams to be empty, we obtain the contributions from an (anti)fundamental hypermultiplet,

$$\mathcal{Z}^{\text{instanton}}_{\text{bifund}}(\vec{a}_0,\emptyset,\vec{a}_1,\vec{Y}^{(1)};\varepsilon_1,\varepsilon_2)=\prod_{\alpha,\beta=1}^{N}\prod_{\square\in Y^{(1,\beta)}}(a_{0,\alpha}-a_{1,\beta}-c_\square), \tag{3.18}$$

$$\mathcal{Z}^{\text{instanton}}_{\text{bifund}}(\vec{a}_r,\vec{Y}^{(r)},\vec{a}_{r+1},\emptyset;\varepsilon_1,\varepsilon_2)=\prod_{\alpha,\beta=1}^{N}\prod_{\square\in Y^{(r,\alpha)}}(a_{r,\alpha}-a_{r+1,\beta}+c_\square+\varepsilon), \tag{3.19}$$

where $\varepsilon=\varepsilon_1+\varepsilon_2$. The contribution of the vector multiplet can be written in terms of that of the bifundamental hypermultiplet as

$$\mathcal{Z}^{\text{instanton}}_{\text{vector}}(\vec{a},\vec{Y};\varepsilon_1,\varepsilon_2)=\mathcal{Z}^{\text{instanton}}_{\text{bifund}}(\vec{a},\vec{Y},\vec{a},\vec{Y};\varepsilon_1,\varepsilon_2)^{-1}. \tag{3.20}$$

B. $\mathscr{Y}$-observables, qq-characters and NPDS equations

With the fixed point formula of instanton partition function (3.9), we can define the expectation value of certain BPS observables $\mathcal{O}$ in the Ω-background as

$$\langle\mathcal{O}\rangle=\mathcal{Z}^{\text{instanton}}(\boldsymbol{a};\boldsymbol{Y};\varepsilon_1,\varepsilon_2)^{-1}\times\sum_{\boldsymbol{Y}}\mathcal{Q}_{\boldsymbol{Y}}\mathcal{Z}^{\text{instanton}}(\boldsymbol{a};\boldsymbol{Y};\varepsilon_1,\varepsilon_2)\mathcal{O}[\boldsymbol{Y}], \tag{3.21}$$

where $\mathcal{O}[\boldsymbol{Y}]$ is the value of $\mathcal{O}$ evaluated at the instanton configuration labeled by $\boldsymbol{Y}$.

The standard local observables in the $\mathcal{N}=2$ theory on $\mathbb{R}^4$ are gauge-invariant polynomials of the scalar components of the vector multiplet, $\mathrm{Tr}\Phi_i^n(\boldsymbol{x})$. However, Poincaré symmetry is broken in the Ω-background, and the operators $\mathrm{Tr}\Phi_i^n(\boldsymbol{x})$ are invariant under the deformed supersymmetry of the Ω-background only at $\boldsymbol{x}=0$, the fixed point of the rotation. The $\mathscr{Y}$-observable is constructed as the generating function of such operators,

$$\mathscr{Y}_i(x)=x^N\exp\left(-\sum_{n=1}^{\infty}\frac{1}{nx^n}\mathrm{Tr}\Phi_i^n(0)\right),\ i=1,\ldots,r. \tag{3.22}$$

Classically, $\mathscr{Y}_i(x)$ is equal to the characteristic polynomial of the scalar component in the vector multiplet of the ith factor of the gauge group

$$\mathscr{Y}_i(x)^{\text{classical}}=\prod_{\alpha=1}^{N}(x-a_{i,\alpha})=\det(x-\Phi_i(0)). \tag{3.23}$$

Quantum mechanically, the $\mathscr{Y}$-observables receive corrections due to the mixing between the adjoint scalar and gluinos. The value of $\mathscr{Y}_i(x)$ evaluated at the instanton configuration labeled by $\boldsymbol{Y}$ is given by

$$\mathscr{Y}_i(x)[\boldsymbol{Y}]=\prod_{\alpha=1}^{N}\left[(x-a_{i,\alpha})\prod_{\square\in Y^{(i,\alpha)}}\frac{(x-\widehat{c_\square}-\varepsilon_1)(x-\widehat{c_\square}-\varepsilon_2)}{(x-\widehat{c_\square})(x-\widehat{c_\square}-\varepsilon)}\right], \tag{3.24}$$

where $\widehat{c_\square}=a_{i,\alpha}+c_\square$ is the shifted content of the box $\square\in Y^{(i,\alpha)}$. For $i=0$ and $i=r+1$, we define

$$\mathscr{Y}_0(x)=\prod_{\alpha=1}^{N}(x-a_{0,\alpha}),\qquad \mathscr{Y}_{r+1}(x)=\prod_{\alpha=1}^{N}(x-a_{r+1,\alpha}). \tag{3.25}$$

From the $\mathscr{Y}$-observables, we can build an important class of gauge-invariant composite operators, the so-called qq-characters. Let us denote

$$\Xi_i(x)=\frac{\mathscr{Y}_{i+1}(x+\varepsilon)}{\mathscr{Y}_i(x)}. \tag{3.26}$$

The ℓth fundamental qq-characters $\mathscr{X}_\ell(x)$ in linear quiver gauge theories can be written as

$$\mathscr{X}_\ell(x) = \frac{\mathscr{Y}_0(x+\varepsilon(1-\ell))}{z_0 z_1 \cdots z_{\ell-1}} \times \sum_{\substack{I\subset[0,r]\\|I|=\ell}} \prod_{i\in I} [z_i \Xi_i(x+\varepsilon(h_I(i)+1-\ell))], \quad \ell = 0,1,\ldots,r+1, \tag{3.27}$$

where $[0,r]=\{0,1,2,\ldots,r\}$, and $h_I(i)$ is the number of elements in I which is less than i. As demonstrated in [36], although the qq-characters $\mathscr{X}_\ell(x)$ has singularities in finite x, its expectation value $\langle \mathscr{X}_\ell(x)\rangle$ is a polynomial in x of degree N,

$$\langle \mathscr{X}_\ell(x)\rangle = T_\ell(x). \tag{3.28}$$

These equations are called NPDS equations, and contain nontrivial information of the instanton partition function of the theory. In particular, the x^{-n} coefficient $\mathscr{X}_\ell^{(-n)}$ of the large-x expansion of $\mathscr{X}_\ell(x)$ has zero expectation value when n is a positive integer,

$$\langle \mathscr{X}_\ell^{(-n)}\rangle = 0, \qquad n \geq 1. \tag{3.29}$$

C. Dictionary of AGT correspondence

It is useful to summarize the dictionary of AGT correspondence in order to make the paper self-contained. The main statement of the AGT correspondence is an identification between the $(r+3)$-point correlation function in the Liouville field theory with the partition function of superconformal quiver gauge theory with gauge group $\mathrm{SU}(2)^r$.

Let us decompose the U(2) gauge group into the U(1) part and the SU(2) part,

$$\overline{a}_i = \frac{1}{2}\sum_{\alpha=1}^{2} a_{i,\alpha}, \qquad a'_{i,\alpha} = a_{i,\alpha} - \bar{a}_i. \tag{3.30}$$

From the point of view of an SU(2) linear quiver gauge theory, the masses of the anti-fundamental, fundamental and bifundamental hypermultiplets are given by

$$\bar{\mu}_\alpha = a_{0,\alpha} - \bar{a}_1, \quad \mu_\alpha = a_{r+1,\alpha} - \bar{a}_r, \quad \mu_{i,i+1} = \bar{a}_{i+1} - \bar{a}_i, \quad i = 1,\ldots,r-1. \tag{3.31}$$

If we identify the Liouville parameter $\boldsymbol{b}$ with the Ω-deformation parameters ε_1, ε_2 as

$$b^2 = \frac{\varepsilon_1}{\varepsilon_2}, \tag{3.32}$$

and relate the conformal dimensions Δ_i with the Coulomb parameters $\boldsymbol{a}$ in the following way:

$$\Delta_{-1} = \frac{\varepsilon^2 - (a_{0,1}-a_{0,2})^2}{4\varepsilon_1\varepsilon_2}, \quad \Delta_{r+1} = \frac{\varepsilon^2 - (a_{r+1,1}-a_{r+1,2})^2}{4\varepsilon_1\varepsilon_2}, \quad \Delta_i = \frac{(\bar{a}_{i+1}-\bar{a}_i)(\bar{a}_i - \bar{a}_{i+1}+\varepsilon)}{4\varepsilon_1\varepsilon_2}, \quad i = 0,\ldots,r, \tag{3.33}$$

then we have

$$\left\langle V_{\Delta_{-1}}(\infty)\prod_{i=0}^{r} V_{\Delta_i}(z_i) V_{\Delta_{r+1}}(0)\right\rangle = f(\Delta_{-1},\ldots,\Delta_{r+1})\left|\left(z_0^{\Delta_{-1}-\Delta_0-\frac{\varepsilon^2}{4\varepsilon_1\varepsilon_2}}\right)\left(\prod_{i=1}^{r-1} z_i^{-\Delta_i}\right)\left(z_r^{\frac{\varepsilon^2}{4\varepsilon_1\varepsilon_2}-\Delta_r-\Delta_{r+1}}\right)\right|^2 \times \int\prod_{i=1}^{r}[da'_i]\left|\frac{\mathcal{Z}(\boldsymbol{a};\boldsymbol{z};\varepsilon_1,\varepsilon_2)}{\mathcal{Z}^{\mathrm{U}(1)}(\boldsymbol{a};\boldsymbol{z};\varepsilon_1,\varepsilon_2)}\right|^2, \tag{3.34}$$

where the prefactor $f(\Delta_{-1},\ldots,\Delta_{r+1})$ is independent of $\boldsymbol{z}$, and $\mathcal{Z}^{\mathrm{U}(1)}(\boldsymbol{a};\boldsymbol{z};\varepsilon_1,\varepsilon_2)$ is the U(1) part of the partition function.

D. Degenerate partition function

Up to this point we assumed that the Coulomb moduli $\boldsymbol{a}$ are generic. Then the instanton partition function (3.9) contains an infinite sum over collections of Young diagrams $\boldsymbol{Y}$. However, we can tune some of the parameters to special values so as to force some of $Y^{(i,\alpha)}$ to have a constrained shape. For example, we can adjust

$$a_{0,\alpha} = \begin{cases} a_{1,1} + (m-1)\varepsilon_1 + (n-1)\varepsilon_2, & \alpha = 1, \\ a_{1,\alpha}, & \alpha \neq 1, \end{cases} \tag{3.35}$$

where $m,n\in\mathbb{Z}^+$. Since the measure of the instanton partition function contains a factor,

$$\prod_{\square=(u,v)\in Y^{(1,\alpha)}} (a_{0,\alpha} - a_{1,\alpha} - \varepsilon_1(u-1) - \varepsilon_2(v-1)), \tag{3.36}$$

the contribution to the instanton partition function vanishes unless the Young diagrams $Y^{(1,\alpha)} = \emptyset$ for $\alpha \neq 1$, and

$\square = (m,n) \notin Y^{(1,1)}$. Hence, the number of Young diagrams we need to sum over reduces drastically. In particular, when $m > 1$ and $n = 1$, the Young diagram $Y^{(1,1)}$ can have at most $m-1$ rows. According to the AGT dictionary, (3.35) corresponds to a degenerate field with the conformal dimension $\Delta_{(m,n)}$.

Physically, the tuning of the Coulomb parameters (3.35) initiates a partial Higgsing of the four-dimensional $\mathcal{N}=2$ gauge theory, after which only a part of the gauge symmetry is restored on the two-dimensional surfaces $\mathbb{C}_{\varepsilon_1}$ and $\mathbb{C}_{\varepsilon_2}$. Therefore, the partition function of the $\mathcal{N}=2$ gauge theory subject to the constraints yields the correlation function of the surface defects defined by coupling the two-dimensional degrees of freedom on these surfaces to the remaining four-dimensional gauge field. See Refs. [19,20] for more detail on the Higgsing construction of the surface defects in the $\mathcal{N}=2$ gauge theory.

IV. SUPERCONFORMAL THEORY WITH GAUGE GROUP U(N)

In this section, we take a simple example to illustrate the basic idea of deriving the differential equation on the instanton partition function at a special point in the parameter space. We consider the U(N) gauge theory with N fundamental hypermultiplets and N antifundamental hypermultiplets for general $N \geq 2$. At the degenerate point of parameter space,

$$a_{0,\alpha} = \begin{cases} a_{1,1} + \varepsilon_1, & \alpha = 1, \\ a_{1,\alpha}, & \alpha \neq 1, \end{cases} \tag{4.1}$$

the instanton partition function is only summed over the Young diagram $Y^{(1,1)}$ which has only one row,

$$Y^{(1)} = \left(\boxed{(1,1)}\boxed{(1,2)} \quad \cdots \quad \boxed{(1,k_1)}, \emptyset, \cdots, \emptyset \right). \tag{4.2}$$

Therefore, we can label the Young diagram $Y^{(1,1)}$ by the instanton charge k_1.

In this case, we face no obstruction in proving directly that the instanton partition function is a (generalized) hypergeometric function from the instanton partition function. The instanton partition function is

$$\mathcal{Z}^{\text{instanton}} = \sum_{k_1=0}^{\infty} \frac{q_1^{k_1}}{k_1!} \frac{\prod_{\alpha=1}^{N} \left(\frac{a_{0,1}-a_{2,\alpha}+\varepsilon_2}{\varepsilon_2}\right)^{\overline{k_1}}}{\prod_{\alpha=2}^{N} \left(\frac{a_{0,1}-a_{0,\alpha}+\varepsilon_2}{\varepsilon_2}\right)^{\overline{k_1}}} \tag{4.3}$$

$$= {}_N F_{N-1}\left(\left(\frac{a_{0,1}-a_{2,\alpha}+\varepsilon_2}{\varepsilon_2} \right)_{\alpha=1}^{N}; \left(\frac{a_{0,1}-a_{0,\alpha}+\varepsilon_2}{\varepsilon_2} \right)_{\alpha=2}^{N}; q_1 \right), \tag{4.4}$$

which is a (generalized) hypergeometric function, and satisfies the (generalized) hypergeometric differential equation (see the Appendix A for details),

$$0 = \left[q_1 \prod_{\alpha=1}^{N} \left(q_1 \frac{\partial}{\partial q_1} + \frac{a_{0,1}-a_{2,\alpha}+\varepsilon_2}{\varepsilon_2} \right) - q_1 \frac{\partial}{\partial q_1} \prod_{\alpha=2}^{N} \left(q_1 \frac{\partial}{\partial q_1} + \frac{a_{0,1}-a_{0,\alpha}+\varepsilon_2}{\varepsilon_2} - 1 \right) \right] \mathcal{Z}^{\text{instanton}}. \tag{4.5}$$

Now we would like to derive the above differential equation using the NPDS equations. There is only one fundamental qq-character in this theory,

$$\mathscr{X}_1(x) = \mathscr{Y}_1(x+\varepsilon) + q_1 \frac{\mathscr{Y}_0(x)\mathscr{Y}_2(x+\varepsilon)}{\mathscr{Y}_1(x)}. \tag{4.6}$$

At the degenerate point (4.1), the value $\mathscr{Y}_1(x)[\boldsymbol{Y}]$ simplifies to

$$\begin{aligned}\mathscr{Y}_1(x)[\boldsymbol{Y}] &= \left[\prod_{\alpha=1}^{N}(x-a_{1,\alpha})\right]\left[\prod_{v=1}^{k_1}\frac{(x-a_{1,1}-\varepsilon_2(v-1)-\varepsilon_1)(x-a_{1,1}-\varepsilon_2 v)}{(x-a_{1,1}-\varepsilon_2(v-1))(x-a_{1,1}-\varepsilon_2 v-\varepsilon_1)}\right]\\ &= \left[\prod_{\alpha=1}^{N}(x-a_{1,\alpha})\right]\left[\frac{(x-a_{1,1}-\varepsilon_1)(x-a_{1,1}-\varepsilon_2 k_1)}{(x-a_{1,1})(x-a_{1,1}-\varepsilon_2 k_1-\varepsilon_1)}\right]\\ &= \left[(x-a_{1,1}-\varepsilon_1)\prod_{\alpha=2}^{N}(x-a_{1,\alpha})\right]\left[\frac{x-a_{1,1}-\varepsilon_2 k_1}{x-a_{1,1}-\varepsilon_2 k_1-\varepsilon_1}\right]\\ &= \mathscr{Y}_0(x)\frac{x-a_{0,1}+\varepsilon_1-\varepsilon_2 k_1}{x-a_{0,1}-\varepsilon_2 k_1}.\end{aligned} \tag{4.7}$$

Accordingly, $\mathscr{X}_1(x)[\boldsymbol{Y}]$ becomes

$$\mathscr{X}_1(x)[\boldsymbol{Y}] = \mathscr{Y}_0(x+\varepsilon)\left(1+\frac{\varepsilon_1}{x+\varepsilon-a_{0,1}-\varepsilon_2 k_1}\right)+q_1\mathscr{Y}_2(x+\varepsilon)\left(1-\frac{\varepsilon_1}{x-a_{0,1}+\varepsilon_1-\varepsilon_2 k_1}\right). \tag{4.8}$$

The x^{-1} coefficient of the large-x expansion of $\mathscr{X}_1(x)[\boldsymbol{Y}]$ is given by

$$\mathscr{X}_1^{(-1)}[\boldsymbol{Y}] = \varepsilon_1\mathscr{Y}_0(a_{0,1}+\varepsilon_2 k_1)-q_1\varepsilon_1\mathscr{Y}_2(a_{0,1}+\varepsilon_2 k_1+\varepsilon_2). \tag{4.9}$$

Using the relation

$$\begin{aligned}\langle k_1^p\rangle &= \mathcal{Z}^{\text{instanton}}(\boldsymbol{a};\vec{Y};\varepsilon_1,\varepsilon_2)^{-1}\sum_{\vec{Y}=\{Y^{(\alpha)}\}}q_1^{k_1}\mathcal{Z}^{\text{instanton}}(\boldsymbol{a};\vec{Y};\varepsilon_1,\varepsilon_2)k_1^p\\ &= \mathcal{Z}^{\text{instanton}}(\boldsymbol{a};\vec{Y};\varepsilon_1,\varepsilon_2)^{-1}\left(q_1\frac{\partial}{\partial q_1}\right)^p\mathcal{Z}^{\text{instanton}}(\boldsymbol{a};q_1;\varepsilon_1,\varepsilon_2),\end{aligned} \tag{4.10}$$

the equation $\langle\mathscr{X}_1^{(-1)}\rangle = 0$ becomes

$$0 = \left[\prod_{\alpha=1}^{N}\left(a_{0,1}+\varepsilon_2 q_1\frac{\partial}{\partial q_1}-a_{0,\alpha}\right)-q_1\prod_{\alpha=1}^{N}\left(a_{0,1}+\varepsilon_2 q_1\frac{\partial}{\partial q_1}+\varepsilon_2-a_{2,\alpha}\right)\right]\mathcal{Z}^{\text{instanton}}, \tag{4.11}$$

which coincides with the differential equation (4.5).

V. SUPERCONFORMAL LINEAR QUIVER GAUGE THEORIES

In this section, we would like to derive the differential equation on the instanton partition function of the superconformal linear quiver gauge theory using the NPDS equations.

A. Large-x expansion of fundamental $\mathscr{Y}$-observables

The first step is to compute the large-x expansion of the $\mathscr{Y}$-observables,

$$\mathscr{Y}_i(x)[\boldsymbol{Y}] = x^N\exp\left[\sum_{\alpha=1}^{N}\log\left(1-\frac{a_{i,\alpha}}{x}\right)+\sum_{\alpha=1}^{N}\sum_{\square\in Y^{(i,\alpha)}}\frac{(1-\frac{\widehat{c_\square}+\varepsilon_1}{x})(1-\frac{\widehat{c_\square}+\varepsilon_2}{x})}{(1-\frac{\widehat{c_\square}}{x})(1-\frac{\widehat{c_\square}+\varepsilon}{x})}\right] = x^N\exp\left(-\sum_{n=1}^{\infty}\frac{C_{i,n}[\boldsymbol{Y}]}{nx^n}\right), \tag{5.1}$$

where

$$C_{i,n}[\boldsymbol{Y}] = \mathrm{Tr}\Phi_i^n(0)[\boldsymbol{Y}] = \sum_{\alpha=1}^{N}\left\{a_{i,\alpha}^n+\sum_{\square\in Y^{(i,\alpha)}}[(\widehat{c_\square}+\varepsilon_1)^n+(\widehat{c_\square}+\varepsilon_2)^n-\widehat{c_\square}^n-(\widehat{c_\square}+\varepsilon)^n]\right\}. \tag{5.2}$$

In particular, we have

$$C_{i,1}[\boldsymbol{Y}] = \sum_{\alpha=1}^{N} a_{i,\alpha},$$
$$C_{i,2}[\boldsymbol{Y}] = \left(\sum_{\alpha=1}^{N} a_{i,\alpha}^2\right) - 2\varepsilon_1\varepsilon_2 k_i, \tag{5.3}$$

We also have the similar expression for $\mathscr{Y}_i(x+\varepsilon)[\boldsymbol{Y}]$,

$$\mathscr{Y}_i(x+\varepsilon)[\boldsymbol{Y}] = \prod_{\alpha=1}^{N}(x-(a_{i,\alpha}-\varepsilon)) \times \prod_{\square\in Y^{(i,\alpha)}} \frac{(x-\widehat{c_\square}+\varepsilon_1)(x-\widehat{c_\square}+\varepsilon_2)}{(x-\widehat{c_\square})(x-\widehat{c_\square}+\varepsilon)} = x^N \exp\left(-\sum_{n=1}^{\infty}\frac{C'_{i,n}[\boldsymbol{Y}]}{nx^n}\right), \tag{5.4}$$

where

$$C'_{i,n}[\boldsymbol{Y}] = \sum_{\alpha=1}^{N}\left\{(a_{i,\alpha}-\varepsilon)^n + \sum_{\square\in Y^{(i,\alpha)}}[(\widehat{c_\square}-\varepsilon_1)^n + (\widehat{c_\square}-\varepsilon_2)^n - \widehat{c_\square}^n - (\widehat{c_\square}-\varepsilon)^n]\right\}. \tag{5.5}$$

Therefore, we obtain the large-x expansion of $\Xi_i(x)[\boldsymbol{Y}]$,

$$\Xi_i(x)[\boldsymbol{Y}] = \frac{\mathscr{Y}_{i+1}(x+\varepsilon)[\boldsymbol{Y}]}{\mathscr{Y}_i(x)[\boldsymbol{Y}]} = 1 + \sum_{n=1}^{\infty}\frac{\zeta_{i,n}[\boldsymbol{Y}]}{x^n}. \tag{5.6}$$

The first two terms of $\zeta_{i,n}$ are given explicitly as

$$\zeta_{i,1} = \mathcal{A}_i^{(1)},$$
$$\zeta_{i,2} = \mathcal{A}_i^{(2)} - \varepsilon_1\varepsilon_2(k_i - k_{i+1}), \tag{5.7}$$

where

$$\mathcal{A}_i^{(1)} = \sum_{\alpha=1}^{N}(a_{i,\alpha} - a_{i+1,\alpha} + \varepsilon),$$
$$\mathcal{A}_i^{(2)} = \frac{1}{2}\sum_{\alpha=1}^{N}[a_{i,\alpha}^2 - (a_{i+1,\alpha}-\varepsilon)^2] + \frac{1}{2}(\mathcal{A}_i^{(1)})^2. \tag{5.8}$$

B. Generating function of the fundamental qq-characters

After expanding the $\mathscr{Y}$-observables, we would like to calculate the large-x expansion of the qq-characters. In order to deal with all of the fundamental qq-characters at the same time we introduce the generating function,

$$\mathscr{G}_r(x;t) = \mathscr{Y}_0(x)^{-1}\Delta_r^{-1}\sum_{\ell=0}^{r+1} z_0 z_1\cdots z_{\ell-1}t^\ell \mathscr{X}_\ell(x-\varepsilon(1-\ell)) = \Delta_r^{-1}\sum_{I\subset[0,r]}\left[\left(\prod_{i\in I} tz_i\right)\prod_{i\in I}\Xi_i(x+\varepsilon h_I(i))\right], \tag{5.9}$$

where

$$\Delta_r = \sum_{I\subset[0,r]}\left(\prod_{i\in I} tz_i\right) = \prod_{i=0}^{r}(1+tz_i). \tag{5.10}$$

In the following, we would like to sum over $I\subset[0,r]$ to obtain the large-x expansion of $\mathscr{G}_r(x;t)$,

$$\mathscr{G}_r(x;t) = \sum_{n=0}^{\infty}\frac{\mathscr{G}_r^{(-n)}(t)}{x^n}. \tag{5.11}$$

Let us define

$$u_i = \frac{tz_i}{1+tz_i}. \tag{5.12}$$

When $r=0$, $\mathscr{G}_0(t)$ is given by a sum over $I=\emptyset$ and $I=\{0\}$,

$$\mathscr{G}_0(x;t) = \frac{1}{1+tz_0} + \frac{tz_0}{1+tz_0}\Xi_0(x) = 1 + \sum_{n=1}^{\infty}\frac{u_0\zeta_{0,n}}{x^n}. \tag{5.13}$$

Hence,

$$\mathscr{G}_0^{(0)}(t) = 1, \qquad \mathscr{G}_0^{(-n)}(t) = u_0\zeta_{0,n}, \qquad n\in\mathbb{Z}^+. \tag{5.14}$$

For general $r\geq 1$, we can compute the value of the generating function (5.9) using the recurrence relation between $\mathscr{G}_r(x;t)$ and $\mathscr{G}_{r-1}(x;t)$. We divide the sum over $I\subset[0,r]$ into two classes: $r\notin I$ and $r\in I$,

$$\begin{aligned}\mathscr{G}_r(x;t) &= \frac{1}{1+tz_r}\Delta_{r-1}^{-1}\sum_{I'\subset[0,r-1]}\left[\left(\prod_{i\in I'}tz_i\right)\prod_{i\in I'}\Xi_i(x+\varepsilon h_{I'}(i))\right]\\ &\quad+\frac{tz_r}{1+tz_r}\Delta_{r-1}^{-1}\sum_{I'\subset[0,r-1]}\left[\left(\prod_{i\in I'}tz_i\right)\Xi_r(x+\varepsilon|I'|)\prod_{i\in I'}\Xi_i(x+\varepsilon h_{I'}(i))\right]\\ &= \mathscr{G}_{r-1}(x;t)+u_r\Delta_{r-1}^{-1}\left(\Xi_r\left(x+\varepsilon t\frac{\partial}{\partial t}\right)-1\right)(\Delta_{r-1}\mathscr{G}_{r-1}(x;t))\\ &= \mathscr{G}_{r-1}(x;t)+u_r\Delta_{r-1}^{-1}\sum_{n=1}^{\infty}\frac{\zeta_{r,n}}{(x+\varepsilon t\frac{\partial}{\partial t})^n}(\Delta_{r-1}\mathscr{G}_{r-1}(x;t))\\ &= \mathscr{G}_0(x;t)+\sum_{j=1}^{r}u_j\Delta_{j-1}^{-1}\sum_{n=1}^{\infty}\frac{\zeta_{j,n}}{(x+\varepsilon t\frac{\partial}{\partial t})^n}(\Delta_{j-1}\mathscr{G}_{j-1}(x;t)).\end{aligned}\tag{5.15}$$

Hence, we obtain the recursive relations,

$$\mathscr{G}_r^{(0)}(t)=\mathscr{G}_0^{(0)}(t)=1,\tag{5.16}$$

$$\mathscr{G}_r^{(-1)}(t)=\sum_{j=0}^{r}u_j\zeta_{j,1},\tag{5.17}$$

$$\mathscr{G}_r^{(-2)}(t)=\sum_{j=0}^{r}u_j\zeta_{j,2}+\sum_{j=1}^{r}u_j\zeta_{j,1}\mathscr{G}_{j-1}^{(-1)}(t)-\varepsilon\sum_{j=1}^{r}u_j\zeta_{j,1}\Delta_{j-1}^{-1}t\frac{\partial}{\partial t}\Delta_{j-1},\tag{5.18}$$

$$\begin{aligned}\mathscr{G}_r^{(-3)}(t) &= \sum_{j=0}^{r}u_j\zeta_{j,3}+\sum_{j=1}^{r}u_j[\zeta_{j,1}\mathscr{G}_{j-1}^{(-2)}(t)+\zeta_{j,2}\mathscr{G}_{j-1}^{(-1)}(t)]-\varepsilon\sum_{j=1}^{r}u_j\Delta_{j-1}^{-1}\left[\zeta_{j,1}t\frac{\partial}{\partial t}(\Delta_{j-1}\mathscr{G}_{j-1}^{(-1)}(t))+2\zeta_{j,2}t\frac{\partial}{\partial t}\Delta_{j-1}\right]\\ &\quad+\varepsilon^2\sum_{j=1}^{r}u_j\zeta_{j,1}\Delta_{j-1}^{-1}\left(t\frac{\partial}{\partial t}\right)^2\Delta_{j-1},\end{aligned}\tag{5.19}$$

$$\begin{aligned}\mathscr{G}_r^{(-4)}(t) &= \sum_{j=0}^{r}u_j\zeta_{j,4}+\sum_{j=1}^{r}u_j[\zeta_{j,1}\mathscr{G}_{j-1}^{(-3)}(t)+\zeta_{j,2}\mathscr{G}_{j-1}^{(-2)}(t)+\zeta_{j,3}\mathscr{G}_{j-1}^{(-1)}(t)]\\ &\quad-\varepsilon\sum_{j=1}^{r}u_j\Delta_{j-1}^{-1}\left[\zeta_{j,1}t\frac{\partial}{\partial t}(\Delta_{j-1}\mathscr{G}_{j-1}^{(-2)}(t))+2\zeta_{j,2}t\frac{\partial}{\partial t}(\Delta_{j-1}\mathscr{G}_{j-1}^{(-1)}(t))+3\zeta_{j,3}t\frac{\partial}{\partial t}\Delta_{j-1}\right]\\ &\quad+\varepsilon^2\sum_{j=1}^{r}u_j\Delta_{j-1}^{-1}\left[\zeta_{j,1}\left(t\frac{\partial}{\partial t}\right)^2(\Delta_{j-1}\mathscr{G}_{j-1}^{(-1)}(t))+3\zeta_{j,2}\left(t\frac{\partial}{\partial t}\right)^2\Delta_{j-1}\right]-\varepsilon^3\sum_{j=1}^{r}u_j\zeta_{j,1}\Delta_{j-1}^{-1}\left(t\frac{\partial}{\partial t}\right)^3\Delta_{j-1}.\end{aligned}\tag{5.20}$$

We further introduce the notation,

$$U_r[s_1,s_2,\ldots,s_\ell]\equiv\sum_{0\le i_1<\cdots<i_\ell\le r}\prod_{n=1}^{\ell}(u_{i_n}\zeta_{i_n,s_n}),\tag{5.21}$$

where $[s_1,\ldots,s_\ell]$ is a sequence of non-negative integers, and we adopt the convention that $\zeta_{i,0}=1$. We have the following useful relations from the definition,

$$t\frac{\partial}{\partial t}\Delta_r=\Delta_r U_r[0],\tag{5.22}$$

$$t\frac{\partial}{\partial t}(\Delta_r U_r[s_1,\ldots,s_\ell])=\Delta_r(\ell U_r[s_1,\ldots,s_\ell]+U_r^{\oplus}[s_1,\ldots,s_\ell]),\tag{5.23}$$

$$\sum_{j=1}^{r}u_j\zeta_{j,m}U_{j-1}[s_1,\ldots,s_\ell]=U_r[s_1,s_2,\ldots,s_\ell,m],\tag{5.24}$$

where

$$U_r^{\oplus}[s_1,\ldots,s_\ell] \equiv U_r[0,s_1,\ldots,s_\ell] + U_r[s_1,0,\ldots,s_\ell] + \cdots + U_r[s_1,\ldots,s_\ell,0]. \tag{5.25}$$

We also have

$$\Delta_r^{-1} t\frac{\partial}{\partial t}\Delta_r = U_r[0], \tag{5.26}$$

$$\Delta_r^{-1}\left(t\frac{\partial}{\partial t}\right)^2\Delta_r = U_r[0] + 2U_r[0,0], \tag{5.27}$$

$$\Delta_r^{-1}\left(t\frac{\partial}{\partial t}\right)^3\Delta_r = U_r[0] + 6U_r[0,0] + 6U_r[0,0,0], \tag{5.28}$$

After solving the recurrence relations, the first few terms of $\mathscr{G}_r^{(-n)}(t)$ can be written as

$$\mathscr{G}_r^{(0)}(t) = 1, \tag{5.29}$$

$$\mathscr{G}_r^{(-1)}(t) = U_r[1], \tag{5.30}$$

$$\mathscr{G}_r^{(-2)}(t) = U_r[2] + U_r[1,1] - \varepsilon U_r[0,1], \tag{5.31}$$

$$\begin{aligned}\mathscr{G}_r^{(-3)}(t) = {} & U_r[3] + U_r[2,1] + U_r[1,2] - \varepsilon(U_r[1,1] + 2U_r[0,2]) + \varepsilon^2 U_r[0,1] \\ & + U_r[1,1,1] - \varepsilon(2U_r[0,1,1] + U_r[1,0,1]) + 2\varepsilon^2 U_r[0,0,1],\end{aligned} \tag{5.32}$$

$$\begin{aligned}\mathscr{G}_r^{(-4)}(t) = {} & U_r[4] + U_r[1,3] + U_r[2,2] + U_r[3,1] \\ & - \varepsilon(U_r[2,1] + 2U_r[1,2] + 3U_r[0,3]) + \varepsilon^2(U_r[1,1] + U_r[0,2]) - \varepsilon^3 U_r[0,1] \\ & + U_r[2,1,1] + U_r[1,2,1] + U_r[1,1,2] \\ & - \varepsilon(3U_r[1,1,1] + 3U_r[0,2,1] + 3U_r[0,1,2] + 2U_r[1,0,2] + U_r[2,0,1]) \\ & + \varepsilon^2(6U_r[0,1,1] + 3U_r[1,0,1] + 6U_r[0,0,2]) - 6\varepsilon^3 U_r[0,0,1] \\ & + U_r[1,1,1,1] - \varepsilon(3U_r[0,1,1,1] + 2U_r[1,0,1,1] + U_r[1,1,0,1]) \\ & + \varepsilon^2(6U_r[0,0,1,1] + 3U_r[0,1,0,1] + 2U_r[1,0,0,1]) - 6\varepsilon^3 U_r[0,0,0,1].\end{aligned} \tag{5.33}$$

In this paper, we are interested in the special case $N = 2$, with $\mathscr{Y}_0(x) = x^2 - (a_{0,1} + a_{0,2})x + a_{0,1}a_{0,2}$. We can deduce from the NPDS equations (3.28) that $\langle \mathscr{Y}_0(x)\mathscr{G}_r(x;t)\rangle$ is a polynomial in x for arbitrary t. In particular, we have

$$\begin{aligned}0 &= \langle \mathscr{G}_r^{(-3)}(t)\rangle - (a_{0,1}+a_{0,2})\langle \mathscr{G}_r^{(-2)}(t)\rangle + a_{0,1}a_{0,2}\langle \mathscr{G}_r^{(-1)}(t)\rangle \\ &= \langle U_r[3]\rangle - (a_{0,1}+a_{0,2})\langle U_r[2]\rangle + a_{0,1}a_{0,2}\langle U_r[1]\rangle \\ &\quad + \langle U_r[2,1]\rangle + \langle U_r[1,2]\rangle - 2\varepsilon\langle U_r[0,2]\rangle - (a_{0,1}+a_{0,2}+\varepsilon)\langle U_r[1,1]\rangle + \varepsilon(a_{0,1}+a_{0,2}+\varepsilon)\langle U_r[0,1]\rangle \\ &\quad + \langle U_r[1,1,1]\rangle - \varepsilon\langle U_r[1,0,1]\rangle - 2\varepsilon\langle U_r[0,1,1]\rangle + 2\varepsilon^2\langle U_r[0,0,1]\rangle,\end{aligned} \tag{5.34}$$

and

$$\begin{aligned}0 &= \langle \mathscr{G}_r^{(-4)}(t)\rangle - (a_{0,1}+a_{0,2})\langle \mathscr{G}_r^{(-3)}(t)\rangle + a_{0,1}a_{0,2}\langle \mathscr{G}_r^{(-2)}(t)\rangle \\ &= \langle U_r[4]\rangle - (a_{0,1}+a_{0,2})\langle U_r[3]\rangle + a_{0,1}a_{0,2}\langle U_r[2]\rangle + \langle U_r[1,3]\rangle + \langle U_r[3,1]\rangle - 3\varepsilon\langle U_r[0,3]\rangle + \langle U_r[2,2]\rangle \\ &\quad - (a_{0,1}+a_{0,2}+\varepsilon)\langle U_r[2,1]\rangle - (a_{0,1}+a_{0,2}+2\varepsilon)\langle U_r[1,2]\rangle + \varepsilon(\varepsilon+2a_{0,1}+2a_{0,2})\langle U_r[0,2]\rangle \\ &\quad + (a_{0,1}+\varepsilon)(a_{0,2}+\varepsilon)\langle U_r[1,1]\rangle - \varepsilon(a_{0,1}+\varepsilon)(a_{0,2}+\varepsilon)\langle U_r[0,1]\rangle + \langle U_r[2,1,1]\rangle + \langle U_r[1,2,1]\rangle + \langle U_r[1,1,2]\rangle \\ &\quad - \varepsilon\langle U_r[2,0,1]\rangle - 2\varepsilon\langle U_r[1,0,2]\rangle - 3\varepsilon\langle U_r[0,2,1]\rangle - 3\varepsilon\langle U_r[0,1,2]\rangle + 6\varepsilon^2\langle U_r[0,0,2]\rangle \\ &\quad - (a_{0,1}+a_{0,2}+3\varepsilon)\langle U_r[1,1,1]\rangle + \varepsilon(a_{0,1}+a_{0,2}+3\varepsilon)\langle U_r[1,0,1]\rangle \\ &\quad + 2\varepsilon(a_{0,1}+a_{0,2}+3\varepsilon)\langle U_r[0,1,1]\rangle - 2\varepsilon^2(a_{0,1}+a_{0,2}+3\varepsilon)\langle U_r[0,0,1]\rangle \\ &\quad + \langle U_r[1,1,1,1]\rangle - \varepsilon(3\langle U_r[0,1,1,1]\rangle + 2\langle U_r[1,0,1,1]\rangle + \langle U_r[1,1,0,1]\rangle) \\ &\quad + \varepsilon^2(6\langle U_r[0,0,1,1]\rangle + 3\langle U_r[0,1,0,1]\rangle + 2\langle U_r[1,0,0,1]\rangle) - 6\varepsilon^3\langle U_r[0,0,0,1]\rangle.\end{aligned} \tag{5.35}$$

By taking the residue of (5.34) at $t=-z_i^{-1}$, we have

$$\begin{aligned}0=&\langle\zeta_{i,3}\rangle+\left[-a_{0,1}-a_{0,2}+\sum_{j=0}^{i-1}\frac{z_j}{z_j-z_i}(\mathcal{A}_j^{(1)}-2\varepsilon)+\sum_{j=i+1}^{r}\frac{z_j}{z_j-z_i}\mathcal{A}_j^{(1)}\right]\langle\zeta_{i,2}\rangle+a_{0,1}a_{0,2}\mathcal{A}_i^{(1)}\\&+\sum_{j=0}^{i-1}\frac{z_j}{z_j-z_i}[\mathcal{A}_i^{(1)}\langle\zeta_{j,2}\rangle-(a_{0,1}+a_{0,2}+\varepsilon)(\mathcal{A}_j^{(1)}-\varepsilon)\mathcal{A}_i^{(1)}]\\&+\sum_{j=i+1}^{r}\frac{z_j}{z_j-z_i}[(\mathcal{A}_i^{(1)}-2\varepsilon)\langle\zeta_{j,2}\rangle-(a_{0,1}+a_{0,2}+\varepsilon)(\mathcal{A}_i^{(1)}-\varepsilon)\mathcal{A}_j^{(1)}]\\&+\sum_{0\le i_1<i_2<i}\frac{z_{i_1}z_{i_2}}{(z_{i_1}-z_i)(z_{i_2}-z_i)}(\mathcal{A}_{i_1}^{(1)}-2\varepsilon)(\mathcal{A}_{i_2}^{(1)}-\varepsilon)\mathcal{A}_i^{(1)}\\&+\sum_{i_1=0}^{i-1}\sum_{i_2=i+1}^{r}\frac{z_{i_1}z_{i_2}}{(z_{i_1}-z_i)(z_{i_2}-z_i)}(\mathcal{A}_{i_1}^{(1)}-2\varepsilon)(\mathcal{A}_i^{(1)}-\varepsilon)\mathcal{A}_{i_2}^{(1)}\\&+\sum_{i<i_1<i_2\le r}\frac{z_{i_1}z_{i_2}}{(z_{i_1}-z_i)(z_{i_2}-z_i)}(\mathcal{A}_i^{(1)}-2\varepsilon)(\mathcal{A}_{i_1}^{(1)}-\varepsilon)\mathcal{A}_{i_2}^{(1)}.\end{aligned}\tag{5.36}$$

In particular, when $j=0$, we have

$$\begin{aligned}0=&\langle\zeta_{0,3}\rangle-\left(a_{0,1}+a_{0,2}+\sum_{i=1}^{r}\frac{z_i}{z_0-z_i}\mathcal{A}_i^{(1)}\right)\langle\zeta_{0,2}\rangle+a_{0,1}a_{0,2}\mathcal{A}_0^{(1)}\\&-\sum_{i=1}^{r}\frac{z_i}{z_0-z_i}[(\mathcal{A}_0^{(1)}-2\varepsilon)\langle\zeta_{i,2}\rangle-(a_{0,1}+a_{0,2}+\varepsilon)(\mathcal{A}_0^{(1)}-\varepsilon)\mathcal{A}_i^{(1)}]\\&+\sum_{1\le i_1<i_2\le r}\frac{z_{i_1}z_{i_2}}{(z_0-z_{i_1})(z_0-z_{i_2})}(\mathcal{A}_0^{(1)}-2\varepsilon)(\mathcal{A}_{i_1}^{(1)}-\varepsilon)\mathcal{A}_{i_2}^{(1)}.\end{aligned}\tag{5.37}$$

We also need the equation obtained by taking residue of (5.35) at $t=-z_0^{-1}$,

$$\begin{aligned}0=&\langle\zeta_{0,4}\rangle-\left(a_{0,1}+a_{0,2}+\sum_{i=1}^{r}\frac{z_i}{z_0-z_i}\mathcal{A}_i^{(1)}\right)\langle\zeta_{0,3}\rangle+a_{0,1}a_{0,2}\langle\zeta_{0,2}\rangle\\&-\sum_{i=1}^{r}\frac{z_i}{z_0-z_i}[(\mathcal{A}_0^{(1)}-3\varepsilon)\langle\zeta_{i,3}\rangle+\langle\zeta_{0,2}\zeta_{i,2}\rangle-(a_{0,1}+a_{0,2}+\varepsilon)\mathcal{A}_i^{(1)}\langle\zeta_{0,2}\rangle\\&+(\varepsilon(\varepsilon+2a_{0,1}+2a_{0,2})-(a_{0,1}+a_{0,2}+2\varepsilon)\mathcal{A}_0^{(1)})\langle\zeta_{i,2}\rangle+(a_{0,1}+\varepsilon)(a_{0,2}+\varepsilon)(\mathcal{A}_0^{(1)}-\varepsilon)\mathcal{A}_i^{(1)}]\\&+\sum_{1\le i_1<i_2\le r}\frac{z_{i_1}z_{i_2}}{(z_0-z_{i_1})(z_0-z_{i_2})}[(\mathcal{A}_{i_1}^{(1)}-\varepsilon)\mathcal{A}_{i_2}^{(1)}\langle\zeta_{0,2}\rangle+(\mathcal{A}_0^{(1)}-3\varepsilon)\mathcal{A}_{i_2}^{(1)}\langle\zeta_{i_1,2}\rangle\\&+(\mathcal{A}_0^{(1)}-3\varepsilon)(\mathcal{A}_{i_1}^{(1)}-2\varepsilon)\langle\zeta_{i_2,2}\rangle-(a_{0,1}+a_{0,2}+3\varepsilon)(\mathcal{A}_0^{(1)}-2\varepsilon)(\mathcal{A}_{i_1}^{(1)}-\varepsilon)\mathcal{A}_{i_2}^{(1)}]\\&-\sum_{1\le i_1<i_2<i_3\le r}\frac{z_{i_1}z_{i_2}z_{i_3}}{(z_0-z_{i_1})(z_0-z_{i_2})(z_0-z_{i_3})}(\mathcal{A}_0^{(1)}-3\varepsilon)(\mathcal{A}_{i_1}^{(1)}-2\varepsilon)(\mathcal{A}_{i_2}^{(1)}-\varepsilon)\mathcal{A}_{i_3}^{(1)}.\end{aligned}\tag{5.38}$$

C. Derivation of the differential equations

Now we are ready to derive the differential equations satisfied by the instanton partition function using the NPDS equations. The key point is that $\zeta_{0,n}$ take special values at a degenerate point in the parameter space.

1. Second-order differential equation

In order to derive a second-order differential equation, we should tune the parameters in the following way:

$$a_{0,1} = a_{1,1} + \varepsilon_1, \qquad a_{0,2} = a_{1,2}. \tag{5.39}$$

The configuration of the gauge fields are constrained so that the Young diagram $Y^{(1,1)}$ has only one row and $Y^{(1,2)} = \emptyset$,

$$Y^{(1)} = \left(\boxed{(1,1)}\boxed{(1,2)} \quad \cdots \quad \boxed{(1,k_1)}, \emptyset \right). \tag{5.40}$$

Hence, the Young diagram $Y^{(1,1)}$ is completely determined by the instanton charge k_1, and

$$\begin{aligned}\Xi_0(x)[k_1] &= \frac{\mathscr{Y}_1(x+\varepsilon)[k_1]}{\mathscr{Y}_0(x)} \\ &= \frac{(x+\varepsilon-a_{0,1})(x+\varepsilon-a_{0,2})}{(x-a_{0,1})(x-a_{0,2})} \\ &\quad \times \frac{x-a_{0,1}+2\varepsilon_1-\varepsilon_2(k_1-1)}{x-a_{0,1}+\varepsilon_1-\varepsilon_2(k_1-1)} \\ &= 1+\sum_{n=1}^{\infty}\frac{\zeta_{0,n}[k_1]}{x^n},\end{aligned} \tag{5.41}$$

which gives

$$\begin{aligned}\zeta_{0,1}[k_1] &= 3\varepsilon_1+2\varepsilon_2, \\ \zeta_{0,2}[k_1] &= (2\varepsilon_1+\varepsilon_2)a_{0,1}+\varepsilon a_{0,2}+\varepsilon(2\varepsilon_1+\varepsilon_2)+\varepsilon_1\varepsilon_2 k_1, \\ \zeta_{0,3}[k_1] &= (2\varepsilon_1+\varepsilon_2)a_{0,1}^2+\varepsilon(2\varepsilon_1+\varepsilon_2)a_{0,1}+\varepsilon a_{0,2}^2 \\ &\quad +\varepsilon(2\varepsilon_1+\varepsilon_2)a_{0,2}+2\varepsilon_1\varepsilon_2 a_{0,1}k_1+\varepsilon_1\varepsilon_2^2 k_1^2.\end{aligned} \tag{5.42}$$

Hence, from (5.37), we have

$$\begin{aligned}0 &= \varepsilon_1\varepsilon_2(a_{0,1}-a_{0,2})\langle k_1\rangle+\varepsilon_1\varepsilon_2^2\langle k_1^2\rangle \\ &\quad -\sum_{i=1}^{r}\frac{z_i}{z_0-z_i}\varepsilon_1[-a_{0,2}\mathcal{A}_i^{(1)}+\varepsilon_2\mathcal{A}_i^{(1)}\langle k_1\rangle \\ &\quad +\mathcal{A}_i^{(2)}-\varepsilon_1\varepsilon_2\langle k_i-k_{i+1}\rangle] \\ &\quad +\sum_{1\le i_1<i_2\le r}\frac{z_{i_1}z_{i_2}}{(z_0-z_{i_1})(z_0-z_{i_2})}\varepsilon_1(\mathcal{A}_{i_1}^{(1)}-\varepsilon)\mathcal{A}_{i_2}^{(1)}.\end{aligned} \tag{5.43}$$

Using

$$\begin{aligned}\mathcal{Z}^{\text{instanton}}\langle k_1\rangle &= -\nabla_0\mathcal{Z}^{\text{instanton}}, \\ \mathcal{Z}^{\text{instanton}}\langle k_i-k_{i+1}\rangle &= \nabla_i\mathcal{Z}^{\text{instanton}},\end{aligned} \tag{5.44}$$

we obtain a differential equation on the instanton partition function,

$$\begin{aligned}0 &= \left\{\varepsilon_2^2\nabla_0^2-\varepsilon_2\left(a_{0,1}-a_{0,2}-\sum_{i=1}^{r}\frac{z_i}{z_0-z_i}\mathcal{A}_i^{(1)}\right)\nabla_0\right. \\ &\quad +\sum_{i=1}^{r}\frac{z_i}{z_0-z_i}[a_{0,2}\mathcal{A}_i^{(1)}-\mathcal{A}_i^{(2)}+\varepsilon_1\varepsilon_2\nabla_i] \\ &\quad \left.+\sum_{1\le i_1<i_2\le r}\frac{z_{i_1}z_{i_2}}{(z_0-z_{i_1})(z_0-z_{i_2})}(\mathcal{A}_{i_1}^{(1)}-\varepsilon)\mathcal{A}_{i_2}^{(1)}\right\}\mathcal{Z}^{\text{instanton}}.\end{aligned} \tag{5.45}$$

This is the equation that was derived in [32] to confirm the BPS/CFT correspondence for this particular case.

2. Third-order differential equation

The derivation can be extended to the next-to-simplest case, as we now explain. To obtain a third-order differential equation, we tune the parameters

$$a_{0,1} = a_{1,1} + 2\varepsilon_1, \qquad a_{0,2} = a_{1,2}. \tag{5.46}$$

In this case, the configurations of the gauge field are required to satisfy that the Young diagram $Y^{(1,1)}$ has at most two rows and $Y^{(1,2)} = \emptyset$,

$$Y^{(1)} = \left(\begin{array}{|c|c|c|c|c|c|c|} \hline (1,1) & (1,2) & \cdots & (1,y_2) & \cdots & (1,y_1) \\ \hline (2,1) & (2,2) & \cdots & (2,y_2) \\ \cline{1-4} \end{array}, \emptyset \right), \tag{5.47}$$

where we denote the number of boxes in the first and the second row of the Young diagram $Y^{(1,1)}$ as y_1 and y_2, respectively. The instanton charge $k_1 = y_1 + y_2$. Then, we have

$$\begin{aligned}\Xi_0(x)[\boldsymbol{Y}] &= \frac{\mathscr{Y}_1(x+\varepsilon)[\boldsymbol{Y}]}{\mathscr{Y}_0(x)} \\ &= \frac{(x+\varepsilon-a_{0,1})(x+\varepsilon-a_{0,2})}{(x-a_{0,1})(x-a_{0,2})} \\ &\quad \times \frac{x+\varepsilon-a_{0,1}+2\varepsilon_1-\varepsilon_2 y_1}{x+\varepsilon-a_{0,1}+\varepsilon_1-\varepsilon_2 y_1}\frac{x+\varepsilon-a_{0,1}+\varepsilon_1-\varepsilon_2 y_2}{x+\varepsilon-a_{0,1}-\varepsilon_2 y_2} \\ &= 1+\sum_{n=1}^{\infty}\frac{\zeta_{0,n}[\boldsymbol{Y}]}{x^n}.\end{aligned} \tag{5.48}$$

We have

$$\begin{aligned}\zeta_{0,1} &= 4\varepsilon_1+2\varepsilon_2, \\ \zeta_{0,2} &= (3\varepsilon_1+\varepsilon_2)a_{0,1}+\varepsilon a_{0,2}+\varepsilon(3\varepsilon_1+\varepsilon_2)+\varepsilon_1\varepsilon_2 k_1,\end{aligned} \tag{5.49}$$

while $\zeta_{0,4}$ are related to $\zeta_{0,3}$ as

$$\begin{aligned}\zeta_{0,4} = &\left(3a_{0,1} - 2\varepsilon_1 - \varepsilon_2 + \frac{3}{2}\varepsilon_2 k_1\right)\zeta_{0,3} - \frac{1}{2}\varepsilon_1\varepsilon_2^3 k_1^3 - \frac{1}{2}\varepsilon_1\varepsilon_2^2(6a_{0,1} - \varepsilon_1)k_1^2 \\ &- \varepsilon_2\left(\frac{3}{2}(5\varepsilon_1 + \varepsilon_2)a_{0,1}^2 + \frac{1}{2}(\varepsilon_1 + 3\varepsilon_2)(3\varepsilon_1 + \varepsilon_2)a_{0,1} + \frac{3}{2}\varepsilon a_{0,2}^2 + \frac{1}{2}\varepsilon(7\varepsilon_1 + 3\varepsilon_2)a_{0,2}\right)k_1 \\ &- 2(3\varepsilon_1 + \varepsilon_2)a_{0,1}^3 - \varepsilon_2(3\varepsilon_1 + \varepsilon_2)a_{0,1}^2 + \varepsilon(2\varepsilon_1 + \varepsilon_2)(3\varepsilon_1 + \varepsilon_2)a_{0,1} - 3\varepsilon a_{0,1}a_{0,2}^2 \\ &- 2\varepsilon(3\varepsilon_1 + \varepsilon_2)a_{0,1}a_{0,2} + \varepsilon a_{0,2}^3 + \varepsilon(5\varepsilon_1 + 2\varepsilon_2)a_{0,2} + \varepsilon(2\varepsilon_1 + \varepsilon_2)(3\varepsilon_1 + \varepsilon_2)a_{0,2}. \end{aligned} \tag{5.50}$$

Using (5.37), (5.50), and (5.36), we can get rid of all terms with $\langle\zeta_{0,3}\rangle$, $\langle\zeta_{0,4}\rangle$, and $\langle\zeta_{i,3}\rangle$ in (5.38), and we obtain a differential equation on the instanton partition function,

$$\begin{aligned}0 = &\left\{\frac{\varepsilon_2^3}{2}\nabla_0^3 + \frac{\varepsilon_2^2}{2}\left(-3a_{0,1} + 3a_{0,2} + \varepsilon_1 + 3\sum_{i=1}^{r}\frac{z_i}{z_0 - z_i}\mathcal{A}_i^{(1)}\right)\nabla_0^2\right. \\ &+ \varepsilon_2\left[(a_{0,1} - a_{0,2})(a_{0,1} - a_{0,2} - \varepsilon_1) - \sum_{i=1}^{r}\frac{z_i}{z_0 - z_i}\left(2\mathcal{A}_i^{(2)} - 2\varepsilon_1\varepsilon_2\nabla_i + \mathcal{A}_i^{(1)}\left(2a_{0,1} - 4a_{0,2} + \frac{\varepsilon_2}{2}\right)\right)\right. \\ &\left.+ \sum_{i=1}^{r}\frac{z_i^2}{(z_0 - z_i)^2}\mathcal{A}_i^{(1)}\left(\mathcal{A}_i^{(1)} - \varepsilon_1 - \frac{\varepsilon_2}{2}\right) + 2\sum_{1\le i_1<i_2\le r}\frac{z_{i_1}z_{i_2}}{(z_0 - z_{i_1})(z_0 - z_{i_2})}(2\mathcal{A}_{i_1}^{(1)} - \varepsilon)\mathcal{A}_{i_2}^{(1)}\right]\nabla_0 \\ &- \sum_{i=1}^{r}\frac{z_i}{z_0 - z_i}(2a_{0,1} - 2a_{0,2} + \varepsilon_2)(a_{0,2}\mathcal{A}_i^{(1)} - \mathcal{A}_i^{(2)} + \varepsilon_1\varepsilon_2\nabla_i) \\ &+ \sum_{i=1}^{r}\frac{z_i^2}{(z_0 - z_i)^2}(2\mathcal{A}_i^{(1)} - 2\varepsilon_1 - \varepsilon_2)(a_{0,2}\mathcal{A}_i^{(1)} - \mathcal{A}_i^{(2)} + \varepsilon_1\varepsilon_2\nabla_i) \\ &+ 2\sum_{1\le i_1<i_2\le r}\frac{z_{i_1}z_{i_2}}{(z_0 - z_{i_1})(z_0 - z_{i_2})}[(\mathcal{A}_{i_1}^{(1)} - \varepsilon)\mathcal{A}_{i_2}^{(1)}(-a_{0,1} + a_{0,2} - \varepsilon) \\ &+ \mathcal{A}_{i_1}^{(1)}(-\mathcal{A}_{i_2}^{(2)} + a_{0,2}\mathcal{A}_{i_2}^{(1)} + \varepsilon_1\varepsilon_2\nabla_i) + \mathcal{A}_{i_2}^{(1)}(-\mathcal{A}_{i_1}^{(2)} + a_{0,2}\mathcal{A}_{i_1}^{(1)} + \varepsilon_1\varepsilon_2\nabla_i)] \\ &+ 2\sum_{1\le i_1<i_2<i_3\le r}\frac{z_{i_1}z_{i_2}z_{i_3}}{(z_0 - z_{i_1})(z_0 - z_{i_2})(z_0 - z_{i_3})}(\mathcal{A}_{i_1}^{(1)}(3\mathcal{A}_{i_2}^{(1)} - \varepsilon)\mathcal{A}_{i_3}^{(1)} - 2\varepsilon\mathcal{A}_{i_2}^{(1)}\mathcal{A}_{i_3}^{(1)}) \\ &+ \sum_{1\le i_1<i_2\le r}\frac{z_{i_1}^2 z_{i_2}}{(z_0 - z_{i_1})^2(z_0 - z_{i_2})}(\mathcal{A}_{i_1}^{(1)} - \varepsilon)(2\mathcal{A}_{i_1}^{(1)} - 2\varepsilon_1 - \varepsilon_2)\mathcal{A}_{i_2}^{(1)} \\ &\left.+ \sum_{1\le i_1<i_2\le r}\frac{z_{i_1}z_{i_2}^2}{(z_0 - z_{i_1})(z_0 - z_{i_2})^2}(\mathcal{A}_{i_1}^{(1)} - \varepsilon)\mathcal{A}_{i_2}^{(1)}(2\mathcal{A}_{i_2}^{(1)} - 2\varepsilon_1 - \varepsilon_2)\right\}\mathcal{Z}^{\text{instanton}}. \end{aligned} \tag{5.51}$$

D. Identification with the BPZ equations

The final step is to identify the differential equations we derived in the gauge theory side with the BPZ equations by solving the undetermined parameters L_i and T_{ij}. We find that the following solution exists:

$$\begin{aligned}L_0 &= \Delta_{-1} - \Delta_0 - \frac{\varepsilon^2 - (a_{1,1} - a_{1,2})^2}{4\varepsilon_1\varepsilon_2}, \\ L_i &= -\Delta_i - \frac{(a_{i,1} - a_{i,2})^2}{4\varepsilon_1\varepsilon_2} + \frac{(a_{i+1,1} - a_{i+1,2})^2}{4\varepsilon_1\varepsilon_2}, \quad i = 1, \ldots, r-1, \\ L_r &= -\Delta_r - \Delta_{r+1} + \frac{\varepsilon^2 - (a_{r,1} - a_{r,2})^2}{4\varepsilon_1\varepsilon_2}, \\ T_{ij} &= \frac{(\mathcal{A}_i^{(1)} - 2\varepsilon)\mathcal{A}_j^{(1)}}{2\varepsilon_1\varepsilon_2}. \end{aligned} \tag{5.52}$$

With this identification of parameters, we observe the precise agreement between (2.26), (2.27), and (5.45), (5.51). It is easy to check that (2.24) is also satisfied. Notice that the prefactor can also be written as

$$\begin{aligned}&\left(\prod_{i=0}^{r} z_i^{L_i}\right)\prod_{0\le i<j\le r}\left(1-\frac{z_j}{z_i}\right)^{T_{ij}}\\&=\left(z_0^{\Delta_{-1}-\Delta_0-\frac{\varepsilon^2}{4\varepsilon_1\varepsilon_2}}\right)\left(\prod_{i=1}^{r-1} z_i^{-\Delta_i}\right)\left(z_r^{\frac{\varepsilon^2}{4\varepsilon_1\varepsilon_2}-\Delta_r-\Delta_{r+1}}\right)\\&\times\prod_{i=1}^{r} q_i^{-\frac{(a_{i,1}-a_{i,2})^2}{4\varepsilon_1\varepsilon_2}}\prod_{0\le i<j\le r}\left(1-\frac{z_j}{z_i}\right)^{\frac{2(\bar{a}_i-\bar{a}_{i+1})(\bar{a}_j-\bar{a}_{j+1}+\epsilon)}{\varepsilon_1\varepsilon_2}}.\end{aligned}\tag{5.53}$$

which give the expected tree-level partition function and the U(1) part of the partition function (see Appendix B for details). Therefore, we confirm the BPS/CFT correspondence.

VI. CONCLUSIONS AND FUTURE DIRECTIONS

In this paper, we perform the derivation of the differential equation on the instanton partition function at a special point in the parameter space using the method of NPDS equations and identify the differential equations with the BPZ equations in the Liouville field theory. Therefore, we confirm the main assertion of the BPS/CFT correspondence. It is interesting to notice that this application of NPDS equations is complementary to the study of chiral trace relations in [37], where parameters are taken to be generic.

There are several obvious generalizations of our paper. First of all, it is natural to consider the general degenerate fields with conformal dimension $\Delta_{(m,n)}$ and derive the differential equation of order mn on the instanton partition function. The computation will be unavoidably lengthy, but the basic idea is the same. To simplify the derivation, it is sometimes useful to consider the NPDS equations of both fundamental and nonfundamental qq-characters.

We can also generalize the discussion to the U(N) superconformal linear quiver gauge theories. However, from the knowledge of corresponding Toda field theory, we do not expect to obtain a differential equation on the instanton partition function. Instead, the equations derived from the NPDS equations will generally relate the instanton partition function with the expectation values of certain BPS observables. Only if we take the Nekrasov-Shatashvili limit can we get a differential equation on the instanton partition function. The nonconformal A_2-quiver SU(3) gauge theory and the degenerate irregular conformal block in the A_2 Toda field theory were studied in [35] along this direction. The detailed discussion on the general quiver will appear in a separate paper.

In spite of the successful application of the NPDS equations to derive the BPZ equations, there are still some open problems. From the point of view of conformal field theory, it is equally good to choose any one of the fields to be degenerate, and we have the BPZ equation for every choice. In the corresponding four-dimensional theory, we need to tune the parameters in the following way for arbitrary $i=0,\ldots,r$:

$$a_{i,\alpha}=\begin{cases}a_{i+1,1}+(m-1)\varepsilon_1+(n-1)\varepsilon_2, & \alpha=1,\\ a_{i+1,\alpha}, & \alpha\neq 1.\end{cases}\tag{6.1}$$

However, we do not get the expected constraints of the form (6.1). For example, the constraint is $Y^{(i+1,\alpha)}\subset Y^{(i,\alpha)}$ rather than $Y^{(i+1,\alpha)}=Y^{(i,\alpha)}$ for $\alpha\neq 1$. This problem is associated with the annoying U(1) factor in the AGT dictionary. We may have to figure out how to factor out the U(1) factor at the level of the measure $\mathcal{Z}^{\text{instanton}}(\boldsymbol{a};\boldsymbol{Y};\varepsilon_1,\varepsilon_2)$. Progress in this direction will also lead us immediately to a derivation of the BPZ equation for the conformal field theory on a torus.

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APPENDIX A: GENERALIZED HYPERGEOMETRIC FUNCTION

A generalized hypergeometric function ${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z)$ is defined as the power series,

$${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z)=\sum_{n=0}^{\infty}\frac{a_1^{\bar{n}}\cdots a_p^{\bar{n}}}{b_1^{\bar{n}}\cdots b_q^{\bar{n}}}\frac{z^n}{n!},\tag{A1}$$

where the parameters $a_1,\ldots,a_p,b_1,\ldots,b_q$ are complex numbers. The notation $x^{\bar{n}}$ is the rising factorial or Pochhammer function, which is defined for real values of n using the Gamma function provided that x and $x+n$ are real numbers that are not negative integers,

$$x^{\bar{n}}=\frac{\Gamma(x+n)}{\Gamma(x)}.\tag{A2}$$

The generalized hypergeometric function ${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z)$ is a solution to the generalized hypergeometric function,

$$\left[z\prod_{n=1}^{p}\left(z\frac{d}{dz}+a_n\right)-z\frac{d}{dz}\prod_{n=1}^{q}\left(z\frac{d}{dz}+b_n-1\right)\right] \times {}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z)=0. \tag{A3}$$

Clearly, the order of the parameters $\{a_1,\ldots,a_p\}$, or the order of the parameters $\{b_1,\ldots,b_q\}$ can be changed without changing the value of the function. The standard hypergeometric function ${}_2F_1(a,b;c;z)$ is simply a special case of the generalized hypergeometric function when $p=2$ and $q=1$.

If any a_j is a nonpositive integer, then the series (A1) only has a finite number of terms and becomes a polynomial of degree $-a_j$. If any b_k is a nonpositive integer (excepting the previous case with $b_k<a_j$), then the series (A1) is undefined. Excluding these special cases for which the numerator or the denominator of the coefficients can be 0, the radius of convergence can be determined using the ratio test. In this paper, we are interested in the case $p=q+1$. The ratio of coefficients tends to one, implying that the series (A1) converges for $|z|<1$ and diverges for $|z|>1$.

APPENDIX B: U(1) FACTOR

In this appendix, we derive the U(1) factor using the NPDS equations.

When $N=1$, we have $\mathscr{Y}_0(x)=x-\bar{a}_0$, and the NPDS equations lead to

$$\begin{aligned}0&=\langle\!\langle[\mathscr{Y}_0(x)\mathscr{G}_r(x;t)]^{(-1)}\rangle\!\rangle\\&=\langle\mathscr{G}_r^{(-2)}(t)\rangle-\bar{a}_0\langle\mathscr{G}_r^{(-1)}(t)\rangle\\&=\langle U_r[2]\rangle+\langle U_r[1,1]\rangle-\varepsilon\langle U_r[0,1]\rangle-\bar{a}_0\langle U_r[1]\rangle\\&=\sum_{i=0}^{r}u_i(\langle\zeta_{i,2}\rangle-\bar{a}_0\langle\zeta_{i,1}\rangle)\\&\quad+\sum_{0\le i_1<i_2\le r}u_{i_1}u_{i_2}(\langle\zeta_{i_1,1}\zeta_{i_2,1}\rangle-\varepsilon\langle\zeta_{i_2,1}\rangle),\end{aligned} \tag{B1}$$

where

$$\zeta_{j,1}=\bar{a}_j-\bar{a}_{j+1}+\varepsilon, \tag{B2}$$

$$\zeta_{j,2}=\bar{a}_j(\bar{a}_j-\bar{a}_{j+1}+\varepsilon)-\varepsilon_1\varepsilon_2(k_i-k_{i+1}). \tag{B3}$$

Picking up the residue at $t=-z_j^{-1}$ we obtain,

$$\begin{aligned}\varepsilon_1\varepsilon_2\langle k_j-k_{j+1}\rangle&=(\bar{a}_j-\bar{a}_0)(\bar{a}_j-\bar{a}_{j+1}+\varepsilon)+\sum_{i=j+1}^{r}\frac{z_i}{z_i-z_j}(\bar{a}_j-\bar{a}_{j+1})(\bar{a}_i-\bar{a}_{i+1}+\varepsilon)+\sum_{i=0}^{j-1}\frac{z_i}{z_i-z_j}(\bar{a}_i-\bar{a}_{i+1})(\bar{a}_j-\bar{a}_{j+1}+\varepsilon)\\&=\sum_{i=j+1}^{r}\frac{z_i}{z_i-z_j}(\bar{a}_j-\bar{a}_{j+1})(\bar{a}_i-\bar{a}_{i+1}+\varepsilon)+\sum_{i=0}^{j-1}\frac{z_j}{z_i-z_j}(\bar{a}_i-\bar{a}_{i+1})(\bar{a}_j-\bar{a}_{j+1}+\varepsilon).\end{aligned} \tag{B4}$$

From the structure of the instanton partition function, we see that

$$\langle k_j-k_{j+1}\rangle=z_j\frac{d}{dz_j}\log Z^{\text{instanton}}. \tag{B5}$$

Therefore, we have the U(1) part of the instanton partition function

$$Z^{\text{instanton}}=\prod_{0\le i<j\le r}\left(1-\frac{z_j}{z_i}\right)^{-\frac{(\bar{a}_i-\bar{a}_{i+1})(\bar{a}_j-\bar{a}_{j+1}+\varepsilon)}{\varepsilon_1\varepsilon_2}}. \tag{B6}$$

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