Spinor-helicity representations of particles of any mass in dS_4 and AdS_4 spacetimes

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The spinor-helicity representations of massive and (partially) massless particles in four-dimensional (anti–) de Sitter (A)dS spacetime are studied within the framework of the dual pair correspondence. We show that the dual groups (also known as "little groups") of the anti–de Sitter and de Sitter groups are, respectively, $O(2N)$ and $O^*(2N)$. For $N = 1$, the generator of the dual algebra $\mathfrak{so}(2) \cong \mathfrak{so}^*(2) \cong \mathfrak{u}(1)$ corresponds to the helicity operator, and the spinor-helicity representation describes massless particles in (A)dS_4. For $N = 2$, the dual algebra is composed of two ideals, $\mathfrak{s}$ and $\mathfrak{m}_\Lambda$. The former ideal $\mathfrak{s} \cong \mathfrak{so}(3)$ fixes the spin of the particle, while the mass is determined by the latter ideal $\mathfrak{m}_\Lambda$, which is isomorphic to $\mathfrak{so}(2,1)$, $\mathfrak{iso}(2)$, or $\mathfrak{so}(3)$ depending on the cosmological constant being positive, zero, or negative. In the case of a positive cosmological constant, namely dS_4, the spinor-helicity representation contains all massive particles corresponding to the principal series representations and the partially massless particles corresponding to the discrete series representations leaving out only the light massive particles corresponding to the complementary series representations. The zero and negative cosmological constant cases, which had been addressed in earlier references, are also discussed briefly. Finally, we consider the multilinear form of helicity spinors invariant under (A)dS group, which can serve as the (A)dS counterpart of the scattering amplitude, and discuss technical differences and difficulties of the (A)dS cases compared to the flat spacetime case.

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I. INTRODUCTION

The massless spinor-helicity (SH) representation[1] in flat spacetime ($Mink_4$) has proven very effective in expressing and determining scattering amplitudes (see e.g., [3–6] for reviews) and their massive counterpart is also prevalent in recent time (see [7,8] and [9,10], and more). Moreover, several attempts to generalize it to (anti–)de Sitter spaces [(A)dS_4] were undertaken in the literature (see e.g., [11–13] for dS_4 and [14–18] for AdS_4). In the series of references [15–17], the $Mink_4$ SH representation is deformed to (A)dS_4 ones with a term in translation generators proportional to the cosmological constant. Despite this deformation, the main salient structure of the scattering amplitude remains the same, while only the momentum conservation delta function is modified to a Λ-dependent function.

In this paper, we generalize the (A)dS_4 SH representation used in [12,15–17] to include massive and partially massless cases and carefully analyze their irreducible representation (irrep) content (see also [11] for the use of $SU(2)$ spinor in computing cosmological correlators in dS_4). Our analysis is systematic, using the reductive dual pair correspondence [19,20] (see [21–23] for physics-oriented reviews, and [24–27] for mathematics-oriented ones), the adequate mathematical framework responsible for most of the technical successes, yet always behind the curtain in the physicists' treatments of the subject.

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[1]Usually the term spinor-helicity refers to the simplifying parametrization of scattering amplitudes, as functions of kinematic variables and polarization tensors, in terms of complex spinors. This "parametrization" in terms of spinor variables can be understood as a specific representation of the Poincaré algebra (see, e.g., [1,2]), and hence we refer to this representation as the spinor-helicity representation, where each generator of the Poincaré algebra are given as differential operators of spinor variables.

We show that the dual groups of the AdS_4 and dS_4 groups are respectively $O(2N)$ and $O^*(2N)$.[2] For the $N = 1$ case, the generator of the dual algebra $\mathfrak{so}(2) \cong \mathfrak{so}^*(2) \cong \mathfrak{u}(1)$ corresponds to the standard helicity operator of the SH formalism, and the SH representation describes massless fields[3] in (A)dS_4. For the $N = 2$ case, the dual algebra is composed of two ideals, $\mathfrak{s}$ and $\mathfrak{m}_\Lambda$. The former ideal $\mathfrak{s} \cong \mathfrak{so}(3)$ fixes the spin of the (A)dS field, while the mass of the field is determined by the latter ideal $\mathfrak{m}_\Lambda$, which is isomorphic to $\mathfrak{so}(1,2)$, $\mathfrak{iso}(2)$ or $\mathfrak{so}(3)$ depending on the cosmological constant being positive, zero, or negative. In the case of positive cosmological constant, namely dS_4, the SH representation contains all massive fields corresponding to the principal series representations of $\mathfrak{so}(1,4)$ and the partially massless fields corresponding to the discrete series representations of $\mathfrak{so}(1,4)$. The only irreps left out are the *light* massive fields corresponding to the complementary series representations of $\mathfrak{so}(1,4)$. We also comment on the $Mink_4$ and the AdS_4 case, analyzed in earlier literature. The $Mink_4$ case was analyzed in detail in the earlier work [8] of the two of the authors. See also more widely known later work [9]. The AdS_4 case was analyzed in [23] in terms of creation/annihilation operators. We also briefly comment on the dual pairs responsible for the SH representations of (A)dS particles in other dimensions.

Remark that the dual group is also known as "little group". This terminology is misleading because the dual group differs from the little group of the induced representation *à la* Wigner: the actual little group is a subgroup of Lorentz, while the dual group commutes with the Lorentz. See the Appendix of [8] for the explicit comparison between the little group and dual group in the case of Poincaré algebra.

[2]The Lie group $O^*(2N)$ is a real form of the complex Lie group $O(2N,\mathbb{C})$ that can be defined in several different ways (see, e.g., [28]). It is a subgroup of $SU(N,N)$, containing elements $g \in SU(N,N)$ satisfying $g^T \eta g = \eta$ for

$$\eta = \begin{pmatrix} 0 & \mathbb{I}_N \\ \mathbb{I}_N & 0 \end{pmatrix},$$

where $\mathbb{I}_N$ is the $N \times N$ identity matrix. For $N > 1$, $SO^*(2N)$ is noncompact, and for $N > 2$ it is simple. The following isomorphisms hold for the corresponding Lie algebras for low values of N:

$$\mathfrak{so}^*(2) \cong \mathfrak{so}(2), \qquad \mathfrak{so}^*(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(1,2),$$
$$\mathfrak{so}^*(6) \cong \mathfrak{su}(1,3), \qquad \mathfrak{so}^*(8) \cong \mathfrak{so}(6,2).$$

[3]The SH representations describe single-particle states, but we will use the term "field" and "particle" interchangeably, as the most relevant context is scattering amplitudes in quantum field theory. Let us also note that the twisted-adjoint representation actively used by Vasiliev is of spinor-helicity type (see, e.g., [29]).

Finally, we consider the multilinear form of helicity spinors invariant under (A)dS_4 group, which can be used for the (A)dS counterpart of the scattering amplitude. Despite the similarity with the $Mink_4$ case, we find a few technical differences and difficulties in the (A)dS_4 cases. We discuss these points and propose potential resolutions.

II. SPINOR-HELICITY REPRESENTATIONS OF (A)DS FIELDS

The SH representation of massive $Mink_4$ fields [8,9] and that of massless (A)dS_4 fields [11,15–17] admit a common and simple generalization,

$$P_{a\dot{b}} = \lambda^I{}_a \tilde{\lambda}_{I\dot{b}} + \Lambda \frac{\partial}{\partial \lambda^{Ia}} \frac{\partial}{\partial \tilde{\lambda}_I{}^{\dot{b}}}, \tag{2.1}$$

$$L_{ab} = 2i\lambda^I{}_{(a} \frac{\partial}{\partial \lambda^{Ib)}}, \qquad \tilde{L}_{\dot{a}\dot{b}} = 2i\tilde{\lambda}_{I(\dot{a}} \frac{\partial}{\partial \tilde{\lambda}_I{}^{\dot{b})}}, \tag{2.2}$$

where $I = 1,\ldots,N$, and the $N = 1$ case corresponds to the massless case and the $\Lambda = 0$ limit corresponds to $Mink_4$ case. Here, $\tilde{\lambda}_{I\dot{a}}$ is the complex-conjugate of $\lambda^I{}_a$ for real "momenta". Round brackets indicate symmetrization with weight one. Both of the indices a, b and $\dot{a}$, $\dot{b}$ are raised and lowered by the two-dimensional Levi-Civita tensor.[4] We shall denote this (A)dS_4 isometry algebra as $\mathfrak{sym}_\Lambda$. It is straightforward to check that the commutators of the above operators satisfy the Lie brackets of the (A)dS_4 algebra with cosmological constant Λ: the generators L_{ab} and $\tilde{L}_{\dot{a}\dot{b}} = L_{ab}{}^\dagger$ form standard Lorentz subalgebra $\mathfrak{so}(1,3) \cong \mathfrak{sl}(2,\mathbb{C})$ with $[L_{ab}, \tilde{L}_{\dot{c}\dot{d}}] = 0$ and

$$[L_{ab}, L_{cd}] = -i(\epsilon_{ac}L_{bd} + \epsilon_{bc}L_{ad} + \epsilon_{ad}L_{bc} + \epsilon_{bd}L_{ac}). \tag{2.6}$$

[4]We follow notations and conventions of [8] with

$$(\sigma^\mu)_{a\dot{b}} = (1,\vec{\sigma})_{a\dot{b}}, \qquad (\bar{\sigma}^\mu)^{\dot{a}b} = \epsilon^{\dot{a}\dot{d}}\epsilon^{bc}(\sigma^\mu)_{c\dot{d}} = (1,-\vec{\sigma})^{\dot{a}b}, \tag{2.3}$$

where σ^i, $i = 1, 2, 3$, are the usual Pauli matrices, which verify $(\sigma^\mu)_{a\dot{a}}(\sigma_\mu)_{b\dot{b}} = -2\epsilon_{ab}\epsilon_{\dot{a}\dot{b}}$, and

$$(\sigma^{\mu\nu})_a{}^b = \frac{1}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)_a{}^b, \qquad (\bar{\sigma}^{\mu\nu})^{\dot{a}}{}_{\dot{b}} = -\frac{1}{4}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu)^{\dot{a}}{}_{\dot{b}}. \tag{2.4}$$

Indices are raised and lowered via

$$\psi_a = \epsilon_{ab}\psi^b, \qquad \psi^a = \epsilon^{ab}\psi_b, \qquad \epsilon^{ac}\epsilon_{cb} = \delta^a_b, \tag{2.5}$$

and similarly for dotted indices.

The translation generators $P_{a\dot{b}}$ carry a vector representation of $\mathfrak{so}(1,3)$, that is a bifundamental representation of $\mathfrak{sl}(2,\mathbb{C})$,

$$[L_{ab}, P_{c\dot{d}}] = i(\epsilon_{ca}P_{b\dot{d}} + \epsilon_{cb}P_{a\dot{d}}),$$
$$[\tilde{L}_{\dot{a}\dot{b}}, P_{c\dot{d}}] = i(\epsilon_{\dot{d}\dot{a}}P_{c\dot{b}} + \epsilon_{\dot{d}\dot{b}}P_{c\dot{a}}). \tag{2.7}$$

With the cosmological constant Λ, the translation generators no longer commute but satisfy

$$[P_{a\dot{b}}, P_{c\dot{d}}] = i\Lambda(\epsilon_{ac}\tilde{L}_{\dot{b}\dot{d}} + \epsilon_{\dot{b}\dot{d}}L_{ac}). \tag{2.8}$$

Hence, we find $\mathfrak{sym}_\Lambda \simeq \mathfrak{so}(1,4)$ for $\Lambda > 0$ and $\mathfrak{sym}_\Lambda \simeq \mathfrak{so}(2,3)$ for $\Lambda < 0$.[5]

The (A)dS$_4$ algebra $\mathfrak{sym}_\Lambda$ is a subalgebra of $\mathfrak{sp}(8N,\mathbb{R})$ generated by all bilinears in λ^I_a, $\frac{\partial}{\partial\lambda^I_a}$ and their complex conjugates. The dual algebra, denoted by $\mathfrak{dual}^{(N)}_\Lambda$, is the stabilizer of $\mathfrak{sym}_\Lambda$ within $\mathfrak{sp}(8N,\mathbb{R})$, and is generated by

$$K^I{}_J = \lambda^I{}_a\frac{\partial}{\partial\lambda^J{}_a} - \tilde{\lambda}_{J\dot{a}}\frac{\partial}{\partial\tilde{\lambda}_{I\dot{a}}}, \tag{2.9a}$$

$$M^{IJ} = \lambda^I{}_a\lambda^{Ja} - \Lambda\frac{\partial}{\partial\tilde{\lambda}_I{}^{\dot{a}}}\frac{\partial}{\partial\tilde{\lambda}_{J\dot{a}}},$$
$$\tilde{M}_{IJ} = \tilde{\lambda}_{I\dot{a}}\tilde{\lambda}_J{}^{\dot{a}} - \Lambda\frac{\partial}{\partial\lambda^{Ia}}\frac{\partial}{\partial\lambda^J{}_a}. \tag{2.9b}$$

The SH representation of $\mathfrak{sym}_\Lambda$ is reducible and its decomposition into irreps can be carried out on the side of $\mathfrak{dual}^{(N)}_\Lambda$. In the following, we shall identify the dual algebra $\mathfrak{dual}^{(N)}_\Lambda$ and explain the intimate relation between $\mathfrak{sym}_\Lambda$ and $\mathfrak{dual}^{(N)}_\Lambda$, first through a preliminary analysis on the eigenvalues of Casimir operators, then using the more solid and powerful method of the dual pair correspondence.

III. PRELIMINARY ANALYSIS

In this section, we identify the dual algebra $\mathfrak{dual}^{(N)}_\Lambda$ for $N = 1, 2$, and establish its relation to $\mathfrak{sym}_\Lambda$ at the level of Casimir operators. By comparing the eigenvalues of the Casimir operators of $\mathfrak{sym}_\Lambda$ and $\mathfrak{dual}^{(N)}_\Lambda$, we provide a preliminary assessment of the correspondence between the irreps of $\mathfrak{sym}_\Lambda$ and $\mathfrak{dual}^{(N)}_\Lambda$.

A. $N = 1$

In the $N = 1$ case, considered in [15–17], the dual algebra $\mathfrak{dual}^{(1)}_\Lambda$ is simply isomorphic to $\mathfrak{u}(1)$ generated by

$$K = \lambda_a\frac{\partial}{\partial\lambda_a} - \tilde{\lambda}_{\dot{a}}\frac{\partial}{\partial\tilde{\lambda}_{\dot{a}}}, \tag{3.1}$$

which is nothing but the standard helicity operator. The $K = s$ state describes massless helicity s representations in Mink$_4$, AdS$_4$, and dS$_4$. This universal description is due to the conformal symmetry they enjoy; the SH representations of $\mathfrak{sym}_\Lambda$ can be lifted to a single irreducible representation, typically referred to as 'singleton', of the four-dimensional conformal group $\mathfrak{so}(2,4)$ [30–33] (see also [34–36] for the oscillator realization, where sometimes the representation is referred to as 'doubleton' for a historical reason). This special property of singleton can be easily understood in terms of the dual pair correspondence, as it was shown in [23]. We shall come back to this point in Sec. IV C.

B. $N = 2$

The $N = 2$ case will turn out to be sufficient to describe all massive spin representations in four dimensions. The generators $M = M^{12}$ and $\tilde{M} = \tilde{M}_{12}$ commute with the subalgebra $\mathfrak{so}(3) \simeq \mathfrak{su}(2) \subset \mathfrak{u}(2)$ generated by $\mathcal{K}^I{}_J = K^I{}_J - \frac{1}{2}\delta^I{}_J K^K{}_K$ while the $\mathfrak{u}(1)$ part $K = K^I{}_I$ satisfies,

$$[M,\tilde{M}] = -\Lambda K, \qquad [K,M] = 2M, \qquad [K,\tilde{M}] = -2\tilde{M}. \tag{3.2}$$

Taking into account that $M^\dagger = \tilde{M}$ and $K^\dagger = K$,[6] it is easy to show that the Hermitian generators $\frac{1}{2}K$, $\frac{1}{2}(M+\tilde{M})$ and $\frac{i}{2}(M-\tilde{M})$ form $\mathfrak{so}(2,1)$ for $\Lambda > 0$, $\mathfrak{so}(3)$ for $\Lambda < 0$ and $\mathfrak{iso}(2)$ for $\Lambda = 0$. The last case corresponds to the massive Mink$_4$ SH formulation [8,9]. To summarize, we find that for $N = 2$, the dual algebra is the direct sum,

$$\mathfrak{dual}^{(2)}_\Lambda \simeq \mathfrak{s} \oplus \mathfrak{m}_\Lambda, \tag{3.3}$$

where the two ideals $\mathfrak{s}$ and $\mathfrak{m}_\Lambda$ are

$$\mathfrak{s} = \mathfrak{so}(3), \qquad \mathfrak{m}_\Lambda = \begin{cases} \mathfrak{so}(2,1) & [\Lambda > 0] \\ \mathfrak{so}(3) & [\Lambda < 0] \\ \mathfrak{iso}(2) & [\Lambda = 0] \end{cases}. \tag{3.4}$$

Below, we will show that the common ideal $\mathfrak{s}$ for any Λ is responsible for the spin label of the $\mathfrak{sym}_\Lambda$ irreps, whereas the other subalgebra $\mathfrak{m}_\Lambda$ determines the mass. In order to see this identification, let us first exploit the relations between Casimir operators of $\mathfrak{sym}_\Lambda$ and $\mathfrak{dual}^{(2)}_\Lambda$.

Since $\mathfrak{sym}_\Lambda$ is a rank two Lie algebra $\mathfrak{so}(1,4)$ or $\mathfrak{so}(2,3)$ for $\Lambda \neq 0$, there are two independent Casimir

[5]Note here that Λ is related to the actual cosmological constant Λ_{cc} by $\Lambda_{cc} = 3\Lambda$.

[6]Note that the Hermitian conjugation $\dagger$ is defined with respect to the $L^2(\mathbb{C}^{2N})$ norm, and hence $\lambda^I{}_a{}^\dagger = (\lambda^I{}_a)^* = \tilde{\lambda}_{I\dot{a}}$ and $(\partial/\partial\lambda^I{}_a)^\dagger = -\partial/\partial\tilde{\lambda}_{I\dot{a}}$.

operators; the quadratic and quartic ones, whose expressions in vector notation read,

$$C_2(\mathfrak{sym}_\Lambda) = -\frac{1}{2} J^{A_1}{}_{A_2} J^{A_2}{}_{A_1}, \tag{3.5a}$$

$$C_4(\mathfrak{sym}_\Lambda) = \frac{1}{2} W_A W^A, \quad W_A = \frac{1}{2}\epsilon_{ABCDE} J^{BC} J^{DE}, \tag{3.5b}$$

where the capital indices take the values $A, B, \ldots = 0, 1, \ldots, 4$ and $J_{AB} = -J_{BA}$ are the generators of $\mathfrak{sym}_\Lambda$. Splitting J_{AB} into Lorentz and translation generators as $J_{4\mu} = P_\mu/\sqrt{|\Lambda|}$ and $J_{\mu\nu} = L_{\mu\nu}$, the two Casimirs are

$$C_2(\mathfrak{sym}_\Lambda) = -\frac{1}{2\Lambda} P^2 + \frac{1}{4}(L^2 + \tilde{L}^2), \tag{3.6a}$$

$$C_4(\mathfrak{sym}_\Lambda) = \frac{1}{4\Lambda} P^{a\dot{a}} P^{b\dot{b}} L_{ab} \tilde{L}_{\dot{a}\dot{b}} + \frac{1}{16\Lambda} P^2 (L^2 + \tilde{L}^2) - \frac{1}{4}(L^2 + \tilde{L}^2) - \frac{1}{64}(L^2 - \tilde{L}^2)^2. \tag{3.6b}$$

Here, $P^2 = P_{a\dot{b}} P^{a\dot{b}} = -2 P_\mu P^\mu$, $L^2 = L_{ab} L^{ab}$, and $L_{\mu\nu} L^{\mu\nu} = \frac{1}{2}(L^2 + \tilde{L}^2)$, where we use the mostly-plus signature for $\eta_{\mu\nu}$. Note that $\Lambda C_2(\mathfrak{sym}_\Lambda)$ and $\Lambda C_4(\mathfrak{sym}_\Lambda)$ reproduce the familiar quadratic Casimir and the Pauli-Lubański vector squared in the $\Lambda \to 0$ limit.

On the other hand, the dual algebra is composed of two rank-one ideals, so we have one Casimir operator for each,

$$C_2(\mathfrak{s}) = \frac{1}{2} \mathcal{K}^I_J \mathcal{K}^J_I, \tag{3.7}$$

$$C_2(\mathfrak{m}_\Lambda) = -\frac{1}{2\Lambda}\{M, \tilde{M}\} + \frac{1}{4} K^2. \tag{3.8}$$

The SH representation of $\mathfrak{sym}_\Lambda$ (2.1) and (2.2) and that of $\mathfrak{dual}^{(2)}_\Lambda$ (2.9a) and (2.9b) relate these Casimir operators as

$$C_2(\mathfrak{sym}_\Lambda) = C_2(\mathfrak{m}_\Lambda) + C_2(\mathfrak{s}) - 2, \tag{3.9a}$$

$$C_4(\mathfrak{sym}_\Lambda) = -C_2(\mathfrak{m}_\Lambda) C_2(\mathfrak{s}). \tag{3.9b}$$

From the above relations, we can read off the Casimir eigenvalues of the unitary irreps of $\mathfrak{sym}_\Lambda$ by fixing an irrep of $\mathfrak{dual}^{(2)}_\Lambda \simeq \mathfrak{s} \oplus \mathfrak{m}_\Lambda$. For the ideal $\mathfrak{s} \simeq \mathfrak{so}(3)$, the $(2s+1)$-dimensional irreps with

$$C_2(\mathfrak{s}) = s(s+1), \tag{3.10}$$

account for all unitary irreps. About the ideal $\mathfrak{m}_\Lambda$, the quadratic Casimir operator can be parametrized as

$$C_2(\mathfrak{m}_\Lambda) = \mu(\mu+1), \tag{3.11}$$

which is invariant under

$$\mu \to -1 - \mu, \tag{3.12}$$

and we have the following options:

(1) For $\Lambda > 0$, apart from the trivial irrep with $\mu(\mu+1) = 0$, we have three series of unitary irreps for $\mathfrak{so}(2,1)$:
  (a) The principal series irreps $\mathcal{C}^\pm_\mu$ with complex μ satisfying

$$\mu(\mu+1) < -\frac{1}{4}, \tag{3.13}$$

  which is spanned by eigenstates of K with even/odd integer eigenvalues, related to the label $+/-$ respectively. We can parametrize irreps in this series via $\mu = -\frac{1}{2} + i\rho$ with $\rho \in \mathbb{R}$. In this case, the map (3.12), $\rho \to -\rho$, is an isomorphism, and hence we may restrict to the case $\rho > 0$.
  (b) The complementary series irrep $\mathcal{C}_\mu$ with $-1 < \mu < 0$ satisfying

$$-\frac{1}{4} \leq \mu(\mu+1) < 0, \tag{3.14}$$

  spanned by all even K-eigenstates. The map (3.12) is again an isomorphism.
  (c) The positive/negative discrete series irrep $\mathcal{D}^\pm_{2\mu+2}$ with

$$\mu = -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, \tag{3.15}$$

  spanned by the K-eigenstates with eigenvalues $\pm 2(\mu+1), \pm 2(\mu+2)$, etc. These are lowest/highest weight irreps.

(2) For $\Lambda \to 0$, the "bosonic/fermionic" irrep of $\mathfrak{iso}(2)$ with $|\mu| \to \infty$ while keeping finite

$$m = \sqrt{-\Lambda \mu^2}, \tag{3.16}$$

which is spanned by K-eigenstates with even/odd eigenvalues. These irreps can be thought of as the counterpart of the massive scalar and spinor representations of the Poincaré group (depending on the parity of the K-eigenstates).

The trivial representation, with $m = 0$, and which can be thought of as the counterpart of the zero-momentum irrep of the Poincaré group.

(3) For $\Lambda < 0$, the $(2\mu+1)$-dimensional irrep of $\mathfrak{so}(3)$ with

$$\mu = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, \tag{3.17}$$

with a basis composed of K-eigenstates with eigenvalues $-2\mu, -2\mu+2, \ldots, +2\mu$.

These irreps of $\mathfrak{dual}_\Lambda^{(2)}$ are in one-to-one correspondence with the irreps of $\mathfrak{sym}_\Lambda$ with

$$C_2(\mathfrak{sym}_\Lambda) = \mu(\mu+1) + s(s+1) - 2, \tag{3.18a}$$

$$C_4(\mathfrak{sym}_\Lambda) = -\mu(\mu+1)s(s+1), \tag{3.18b}$$

and we can compare these values with those of known irreps of $\mathfrak{sym}_\Lambda$.

1. $\mathbf{Mink_4}$

To begin with, let us consider the Poincaré case with $\Lambda = 0$ which has been treated in [7,9]. The quadratic Casimir,

$$\lim_{\Lambda\to 0} \Lambda C_2(\mathfrak{m}_\Lambda) = -M\tilde{M}, \tag{3.19}$$

of the dual algebra $\mathfrak{m}_0$ determines the mass,

$$M\tilde{M} = -P_\mu P^\mu = m^2, \tag{3.20}$$

while the 'spin s' representation of the dual algebra $\mathfrak{s}$ corresponds to the spin, thus defining a Poincaré representation of mass m and spin s. In fact, in all cases of $\mathfrak{sym}_\Lambda$, the irrep label s of the dual algebra $\mathfrak{s}$ simply corresponds to the spin of the four-dimensional field.

2. $\mathbf{dS_4}$

The unitary irreps of dS_4 Lie algebra, namely $\mathfrak{so}(1,4)$, were first classified in [37] where the eigenvalues of the Casimir operators are also given; see Appendix A for a summary, and [38] for the physical interpretations of these irreps. More recent treatments of dS representations can be found e.g., in [39–44].

Comparing the result (3.18b) with the Casimir eigenvalues identified in [37], we find that the irrep label μ of the dual algebra $\mathfrak{m}_\Lambda$ parametrizes the mass squared as[7]

$$m^2 = \Lambda[-\mu(\mu+1) + s(s-1)]. \tag{3.21}$$

Depending on the spin s, different ranges of mass are allowed for the unitarity of the $\mathfrak{sym}_\Lambda$ irreps:

(1) For the scalar case with $s = 0$, the allowed μ are
   (a) The complex values of μ with (3.13) corresponding to the principal series representations of $\mathfrak{so}(1,4)$, with the isomorphism (3.12).
   (b) The real values of $-2 < \mu < 1$ with

$$-\frac{1}{4} \leq \mu(\mu+1) < 2, \tag{3.22}$$

   corresponding to the complementary series representations of $\mathfrak{so}(1,4)$, with the isomorphism (3.12). The $\mu = 0$ case (or equivalently, the $\mu = -1$ case) corresponds to the conformally coupled scalar.
   (c) The positive integer values of μ corresponding to the discrete series representations of $\mathfrak{so}(1,4)$. The $\mu = 1$ case corresponds to the minimally coupled massless scalar, whereas $\mu = 2, 3, \ldots$ correspond to tachyonic scalars.

The unitarity of these $\mathfrak{sym}_\Lambda$ irreps includes not only all the $\mathfrak{m}_\Lambda$ unitary regions (3.13), (3.14) and the integer part of (3.15), but also the complementary series region $0 < \mu(\mu+1) < 2$ not allowed for the unitarity of $\mathfrak{m}_\Lambda$.

(2) For integral spins $s = 1, 2, \ldots$, the allowed μ are
   (a) The complex values with (3.13) corresponding to the principal series representations of $\mathfrak{so}(1,4)$, with the isomorphism (3.12).
   (b) The real values of $-1 < \mu < 0$ with (3.14) corresponding to the complementary series representations of $\mathfrak{so}(1,4)$, with the isomorphism (3.12).
   (c) The integer values $\mu = 0, 1, \ldots, s-1$. These integer values correspond to the partially massless fields of depth $s - \mu$, where the depth 1 corresponds to the massless field.

The unitarity of these $\mathfrak{sym}_\Lambda$ irreps allows the $\mathfrak{m}_\Lambda$ unitary regions (3.13) and (3.14), but restricts (3.15); integers greater than $s-1$ are excluded together with the half-integer values.

(3) For half-integral spins $s = \frac{1}{2}, \frac{3}{2}, \ldots$, the allowed μ are
   (a) The complex values of μ with (3.13) corresponding to the principal series representation of $\mathfrak{so}(1,4)$, with the isomorphism (3.12).
   (b) The half-integer values $\mu = -\frac{1}{2}, \frac{1}{2}, \ldots, s-1$ corresponding to the discrete series representations of $\mathfrak{so}(1,4)$. The positive half-integer values correspond to the partially massless fields of depth $s - \mu$.[8] Note that $\mu = -\frac{1}{2}$ corresponds to the endpoint of the continuous

[7]Here, we define the mass m^2 of a field φ of spin s in (A)dS_{d+1} via the wave equation,

$$\left(\nabla^2 + \frac{2\Lambda_{cc}}{d(d-1)}[(s-2)(s+d-2) - s] - m^2\right)\varphi = 0.$$

Parametrizing the eigenvalue of the quadratic Casimir operator of the irrep associated with φ as

$$C_2 = \Delta(\Delta - d) + s(s+d-2),$$

we can write the mass squared as

$$m^2 = \frac{2\Lambda_{cc}}{d(d-1)}(\Delta + s - 2)(s + d - 2 - \Delta),$$

which reproduces the formula (3.21) upon using $\mu = \Delta - 2$ (or $\mu = -\Delta + 1$) for $d = 3$. (See for instance [45] for an extended discussion of the dS_4 case, and [46,47] including also the AdS_4 case.)

[8]The partially massless fermion irreps are unitary only in dS_4 [48].

spectrum of massive fields, which we may refer to as *the lightest massive fermions*. For $s = \frac{1}{2}$, it simply corresponds to the massless spinor.

The unitarity of these $\mathfrak{sym}_\Lambda$ irreps includes the $\mathfrak{m}_\Lambda$ principal series (3.13) but entirely excludes the complementary series (3.14), and restrict the discrete series (3.15); any half-integers greater than $s-1$ are excluded together with the integer values.

C. AdS_4

In the AdS_4 case with $\Lambda < 0$, the irrep label μ of the dual algebra $\mathfrak{m}_\Lambda$ parametrizes the mass squared again as (3.21). The allowed μ for the unitarity of the lowest-energy irreps of $\mathfrak{sym}_\Lambda$ are $\mu = s-1, s, s+1, \ldots$ for spin $s = 0, \frac{1}{2} 1, \ldots$. The $\mu = s-1$ case corresponds to the massless spin s field, and higher μ cases correspond to massive fields. The reason that we have a discrete mass spectrum is due to the fact that μ is an eigenvalue of the generator of the compact subgroup $SO(2)$ associated with rotations in the plane of temporal directions, and hence is quantized. These representations can be interpreted as the irreps of 3d conformal group; $\Delta = \mu + 2$ and s correspond to the conformal weight and spin of the conformal primaries, respectively. In the scalar case, the $\mu = -1$ and $\mu = 0$ cases mapped by (3.12) are distinct irreps and correspond to different modes of the conformal scalar in AdS_4. Note that, moving to a covering group of $SO(1,4)$, the point $\mu = -\frac{3}{2}$ can be included for $s = 0$, and it corresponds to the conformal scalar in 3d.

The unitarity of the lowest energy irreps of $\mathfrak{sym}_\Lambda \cong \mathfrak{so}(2,3)$ excludes the lower μ values with $\mu < s-1$ from (3.17), corresponding to partially massless fields, together with all integer/half-integer values of μ for half-integral/integral spin.

Let us note that there are a few other types of $\mathfrak{sym}_\Lambda$ irreps with unbounded energy. These irreps would cover different ranges of $C_2(\mathfrak{sym}_\Lambda)$ and $C_4(\mathfrak{sym}_\Lambda)$.

IV. DUAL PAIR CORRESPONDENCE

In the previous section, we have identified the correspondences between the irreps of $\mathfrak{sym}_\Lambda$ and those of $\mathfrak{dual}^{(2)}_\Lambda$ through the Casimir eigenvalues. We have observed that the region of μ allowed by the $\mathfrak{sym}_\Lambda$ unitarity does not match the region allowed by the $\mathfrak{m}_\Lambda$ unitarity. This mismatch does not lead to a contradiction, because the SH representations cover only a part of unitary irreps of $\mathfrak{sym}_\Lambda \oplus \mathfrak{dual}^{(2)}_\Lambda$. In other words, the SH Fock space contains only a part of unitary irreps of $\mathfrak{sym}_\Lambda \oplus \mathfrak{dual}^{(2)}_\Lambda$. In order to identify the actual content of the unitary irreps that the Fock space contains, we need a more rigorous analysis using the dual pair correspondence.

For general N, the dual algebras (2.9) are $\mathfrak{dual}^{(N)}_{\Lambda>0} \simeq \mathfrak{so}^*(2N)$ and $\mathfrak{dual}^{(N)}_{\Lambda<0} \simeq \mathfrak{so}(2N)$, respectively. The interplay between the isometry and the dual algebras can be understood within the general framework of the dual pair correspondence, also known as Howe duality which amounts to the following; when a $Sp(2\mathcal{N},\mathbb{R})$ group contains a pair of reductive subgroups $(G,\tilde{G})$ which are mutual stabilizers,[9] there exists a one-to-one correspondence between the irreps of G and $\tilde{G}$ appearing in the decomposition of the oscillator (or metaplectic) representation of $Sp(2\mathcal{N},\mathbb{R})$ (see e.g., [23] for more details). In our context, the oscillator representation is simply the representation realized by the helicity spinors, or simply SH representation. Hence, the (A)dS_4 groups $\mathfrak{Sym}_{\Lambda>0} = Sp(1,1)$ and $\mathfrak{Sym}_{\Lambda<0} = Sp(4,\mathbb{R})$ and their respective dual groups $\mathfrak{Dual}^{(N)}_{\Lambda>0} = O^*(2N)$ and $\mathfrak{Dual}^{(N)}_{\Lambda<0} = O(2N)$ realized by helicity spinors as (2.8) and (2.9) form reductive dual pairs in $Sp(8N,\mathbb{R})$, the group generated by all quadratic operators in helicity spinors and their derivatives. Note that $Sp(1,1)$ and $Sp(4,\mathbb{R})$ are isomorphic to the double covers of $SO^\uparrow(1,4)$ and $SO^\uparrow(2,3)$, respectively. In fact, the flat space case with $\Lambda = 0$ can be viewed as the Inönü-Wigner contraction of the reductive dual pair $(Sp(1,1), O^*(2N))$ or $(Sp(4,\mathbb{R}), O(2N))$.

Let us remark once again that the dual group ought not to be confused with the standard little group of the induced representation *à la* Wigner; the former commutes with the isometry whereas the latter is a part of the isometry by definition. In the $\Lambda = 0$ case, the $SU(2)$ subgroup of the dual group and the little group are explicitly shown to be distinguished (see the Appendix of [8]) as they represent respectively left and right actions on $SU(2)$ which parametrizes a momentum eigenstate.

The dual pair correspondence assures that the irreps of the (A)dS_4 group, that is $Sp(1,1)$ and $Sp(4,\mathbb{R})$, realized by helicity spinors are in one-to-one correspondence with the irreps of the dual group $O^*(2N)$ or $O(2N)$. In other words, by singling out an irrep of the dual group, the reducible SH representation of the (A)dS_4 group (2.8) is restricted to an irrep. Then, the remaining task is to establish the dictionary between such irreps of the (A)dS_4 group and its dual group $O^*(2N)$ [or $O(2N)$]. For that, we once again focus on the cases of $N = 1$ and 2.

A. dS_4

Let us consider first the case with $\Lambda > 0$. We aim to obtain a dictionary between the irreps of $Sp(1,1)$ and $O^*(2N)$ appearing in the decomposition of the SH representation.

For $N = 1$, the dual pair correspondence between $Sp(1,1)$ and $O^*(2)$ has been explicitly established in [23]. Here, we just quote the result. Since $O^*(2)$ is isomorphic to $U(1)$, it has only one-dimensional irreps, each labeled by an integer. This integer corresponds to

[9] In other words, $\tilde{G}$ is the maximal subgroup of $Sp(2\mathcal{N},\mathbb{R})$ commuting with G, and vice versa.

twice the helicity of a $Sp(1,1)$ massless representation. The analysis is based on the decomposition of the $Sp(1,1)$ irrep into its maximal subgroup $Sp(1)\times Sp(1)$, and the SH representation restricted by the $O^*(2)$ irrep condition is shown to have a structure of the massless spin s irrep of $Sp(1,1)$ demonstrated e.g., in [37].

For the $N=2$ case, we need to begin with identifying irreps of the dual group $O^*(4)$. Thanks to the isomorphism $O^*(4)=[SU(2)\times SL(2,\mathbb{R})]/\mathbb{Z}_2$ [here, $SU(2)$ and $SL(2,\mathbb{R})$ are simply the Lie groups associated with $\mathfrak{s}=\mathfrak{so}(3)$ and $\mathfrak{m}_{\Lambda>0}=\mathfrak{so}(2,1)$], we know everything about the unitary irreps of $O^*(4)$: the irreps of $SU(2)$ are all given by $(2s+1)$-dimensional representation, which will be denoted by $[2s]$ henceforth, while $SL(2,\mathbb{R})$ has three classes of unitary irreps, namely $\mathcal{C}^{\pm}_{\mu=-\frac{1}{2}+i\rho}$ (3.13), $\mathcal{C}_\mu$ (3.14) and $\mathcal{D}^{\pm}_{2\mu+2}$ (3.15). We will denote these $O^*(4)$ irreps as $\tilde{\pi}_{s,\mu}$.

In the previous section, we have seen that not all $O^*(4)$ irreps correspond to irreps of $Sp(1,1)$ based on the match of Casimir operators. We shall see below how they are restricted. For that, we first consider the dual pair $(Sp(1),O^*(4))\subset Sp(8,\mathbb{R})$, whose representations are explicitly identified in Sec 5.4 in Ref. [23]; Since $Sp(1)\cong SU(2)$ the $Sp(1)$ irreps are again given by $[m]$ with non-negative integer m, and they correspond to the $O^*(4)$ irreps $[m]\otimes\mathcal{D}^{\pm}_{m+2}$. Note that only discrete series representations appear in the $SL(2,\mathbb{R})$ side, with the highest/lowest weight $m+2$ tied with the dimension $m+1$ of the $SU(2)$ irrep [which is a consequence of the fact that the Howe dual is a compact group, namely $Sp(1)$]. Whether the irrep $\mathcal{D}^{\pm}_{m+2}$ is a highest/lowest weight one is conventional at this stage, and only one sign is chosen depending on the convention of $SL(2,\mathbb{R})$.

Now we move on to the dS_4 group $Sp(1,1)$ and consider its maximal compact subgroup, which is $Sp(1)\times Sp(1)$. This subgroup forms its own dual pair in the same SH space [that is, in $Sp(16,\mathbb{R})$] with $O^*(4)\times O^*(4)$. The latter contains the original dual group $O^*(4)$ as the diagonal subgroup. The situation is conveniently depicted by the "seesaw" diagram,

$$\begin{array}{ccc} Sp(1,1) & & O^*(4)\times O^*(4) \\ \cup & \swarrow\!\!\!\!\nearrow\!\!\!\!\nwarrow\!\!\!\!\searrow & \cup \\ Sp(1)\times Sp(1) & & O^*(4) \end{array}, \tag{4.1}$$

where the arrows indicate the respective dual pairs. Any irrep of $Sp(1,1)$, say π_σ with some label σ, can be decomposed into irreps of $Sp(1)\times Sp(1)$ as

$$\pi_\sigma=\bigoplus_{m,n}N^{m,n}_\sigma[m]\otimes[n], \tag{4.2}$$

where $N^{m,n}_\sigma$ are the multiplicities of $[m]\otimes[n]$, and each of $[m]\otimes[n]$ correspond to the $O^*(4)\times O^*(4)$ irrep,

$$([m]\otimes\mathcal{D}^-_{m+2})\otimes([n]\otimes\mathcal{D}^+_{n+2}). \tag{4.3}$$

Here, we used the correspondence between the irreps of $Sp(1)$ and $O^*(4)$ that we introduced earlier. Note that the first $SL(2,\mathbb{R})$ irrep is a lowest-weight irrep, while the second is a heighest-weight irrep. This is because the $Sp(1)\times Sp(1)$ is embedded in the opposite signature parts of $Sp(1,1)$. The irrep (4.3) of $O^*(4)\times O^*(4)$ can be decomposed as well into the diagonal subgroup $O^*(4)$:

$$([m]\otimes\mathcal{D}^-_{m+2})\otimes([n]\otimes\mathcal{D}^+_{n+2})=\bigoplus_{s,\mu}\tilde{N}^{s,\mu}_{m,n}\tilde{\pi}_{s,\mu}, \tag{4.4}$$

where $\tilde{N}^{s,\mu}_{m,n}$ are the multiplicities of the $O^*(4)$ irrep $\tilde{\pi}_{s,\mu}$ that we have introduced before. The crucial point assured by the seesaw duality (see [24–26] and also Sec. 2.3 of Ref. [23]) is the equality between two multiplicities; for any $[m]\otimes[n]$,

$$N^{m,n}_{\sigma(s,\mu)}=\tilde{N}^{s,\mu}_{m,n}. \tag{4.5}$$

Here, $\sigma(s,\mu)$ is the label of the $Sp(1,1)$ irrep dual to the $O^*(4)$ irrep $\tilde{\pi}_{s,\mu}$.

Now let us identify the multiplicities $\tilde{N}^{s,\mu}_{m,n}$. The decomposition (4.4) comes in two parts; the decomposition of the $SU(2)$ irreps,

$$[m]\otimes[n]=[|m-n|]\oplus[|m-n|+2]\oplus\cdots\oplus[m+n], \tag{4.6}$$

and the decomposition of the $SL(2,\mathbb{R})$ irreps [49] (see also [50]),

$$\mathcal{D}^-_{m+2}\otimes\mathcal{D}^+_{n+2}=\int_0^\infty \mathrm{d}\rho\,\mathcal{C}^{(-1)^{m+n}}_{-\frac{1}{2}+i\rho}\oplus\bigoplus_{0\le k<\frac{|m-n|}{2}}\mathcal{D}^{\mathrm{sgn}(m-n)}_{|m-n|-2k}. \tag{4.7}$$

We see that the multiplicities are either 1 or 0. Hence, for a fixed $\tilde{\pi}_{s,\mu}$ the above decomposition simply restricts the possible $[m]\otimes[n]$ which appear in the decomposition (4.2) of $\pi_{\sigma(s,\mu)}$. Moreover, we find that certain $\tilde{\pi}_{s,\mu}$'s do not admit any $[m]\otimes[n]$ implying that such irreps cannot correspond to any (even trivial) $Sp(1,1)$ irrep. In other words, they are simply not contained in the SH representation. Let us see the details now. By choosing the $SU(2)$ irrep as $[2s]$, m and n are restricted as

$$|m-n|\le 2s\le m+n,\qquad m+n-2s\in 2\mathbb{Z}. \tag{4.8}$$

For the $SL(2,\mathbb{R})$ irreps with label μ, we have three choices, the principal series $\mathcal{C}^{\pm}_{\mu=-\frac{1}{2}+i\rho}$, the complementary series $\mathcal{C}_\mu$ and the discrete series $\mathcal{D}^{\pm}_{2\mu+2}$. We notice already that the complementary series is not available since it does not appear in the content of the tensor product decomposition, that is, in the rhs of (4.7).

If we select a principal series representation $\mathcal{C}^{(-1)^{m+n}}_{-\frac{1}{2}+i\rho}$, we do not have further restrictions on possible values of m and n. Therefore, we find

$$\pi_{\sigma(s,-\frac{1}{2}+i\rho)} = \bigoplus_{\substack{|m-n|\le 2s\le m+n \\ m+n-2s\in 2\mathbb{Z}}} [m]\otimes[n]. \tag{4.9}$$

These correspond to the spin s principal series representations of $Sp(1,1)$, describing massive spin s fields.

If we select a discrete representation $\mathcal{D}^{\pm}_{2\mu+2}$, we find a further restriction on the space and obtain,

$$\pi^{\pm}_{\sigma(s,\mu)} = \bigoplus_{\substack{|m-n|\le 2s\le m+n \\ m+n-2s\in 2\mathbb{Z} \\ 2\mu+2\le |m-n| \\ \pm(m-n)>0}} [m]\otimes[n]. \tag{4.10}$$

The additional bound on m and n restricts also possible values of μ. For integer s, we find $\mu=0,1,\ldots,s-1$, and for half-integer s, we find $\mu=-\frac{1}{2},\frac{1}{2},\ldots,s-1$. These irreps correspond to the spin s discrete series representation of $Sp(1,1)$ describing partially massless spin s fields and the lightest massive fermions. One can also see that they always come with two chiralities or helicities $\pm$.

To summarize, we find that the SH representations contain exactly all the unitary representations of $Sp(1,1)$ except for the complementary series ones; the $Sp(1,1)$ (not $SL(2,\mathbb{R})$) complementary series representations correspond to the interval $-\frac{1}{2}\le\mu<1$ for $s=0$ and $-\frac{1}{2}\le\mu<0$ for $s=1,2,\ldots$, respectively, while fermions do not appear in the complementary series. Interestingly, the SH representation with the dual pair $(Sp(1,1),O^*(4))$ contains also the massless spin s fields which can be realized by the $(Sp(1,1),O^*(2))$ dual pair. The conformal scalar with $\mu=0$ (equivalently $\mu=-1$) is in the field content of Vasiliev's higher spin gravity, together with all integer spin massless fields. This conformal scalar in dS_4 can be realized only by the latter dual pair. For a more formal treatment of the $(Sp(1,1),O^*(4))$ dual pair, one may consult with [51,52].

B. AdS_4

The $\Lambda<0$ case is more straightforward, and it was recently discussed in [23]. We use the seesaw diagram,

$$\begin{array}{ccc} Sp(4,\mathbb{R}) & & U(2N) \\ \cup & \times & \cup \\ U(2) & & O(2N) \end{array}, \tag{4.11}$$

relating the reductive dual pairs $(Sp(4,\mathbb{R}),O(2N))$ and $(U(2),U(2N))$ in $Sp(8N,\mathbb{R})$.

For $N=1$, the irreps of $O(2)$ are $[2s]_{O(2)}$ with $2s\in\mathbb{N}$ and $[1,1]_{O(2)}$. The one-dimensional irreps $[0]_{O(2)}$ and $[1,1]_{O(2)}$ corresponds to the scalar irreps of $Sp(4,\mathbb{R})$, whereas $[2s]_{O(2)}$ correspond to the massless spin s irreps of $Sp(4,\mathbb{R})$. The latter irreps are two-dimensional, composed of the helicity $\pm s$ irreps, which are related by the $\mathbb{Z}_2$ part of $O(2)\cong\mathbb{Z}_2\ltimes SO(2)$, so they assemble into a single irrep for $O(2)$.

For $N=2$, the dual representation of $[\mu+s,\mu-s]_{O(4)}=[s]_{O(3)}\otimes[\mu]_{O(3)}$ is the discrete series representation $\mathcal{D}_{Sp(4,\mathbb{R})}(\mu+2,s)$ with the lowest energy $\mu+2=s+2,s+3,\ldots$. Note that in this case the SH representation contains all the massive fields while excludes the massless fields, which can be realized by the $(Sp(4,\mathbb{R}),O(2))$ dual pair.

Above, we had mentioned that $Sp(4,\mathbb{R})$ contains many representations other than the discrete series ones. These irreps would correspond to rather exotic fields such as tachyon, continuous spin [53–55] and even the ones living in bitemporal counterpart of AdS_4 (see [56,57] for related discussions). These irreps might be also realized using proper SH representations, namely dual pairs with different dual groups $O(1,1)$, $O(2,1)$, $O(3,1)$, and $O(2,2)$. In the simplest $O(1)\cong\mathbb{Z}_2$ case, the dual pair describes the conformal scalar and spinor fields in $3d$, corresponding to the even and odd representation of $\mathbb{Z}_2$ respectively.[10] Let us remark also that this different signature variety is not available for dS_4 with $Sp(1,1)$ since $O^*(2N)$ does not allow any signature variations and $2N$ must be even.

C. Conformal group

As we had commented above, the four-dimensional conformal group $\mathfrak{so}(2,4)$ [30–33] has a special representation called 'singleton' which reduces to the massless irreps of $(A)dS_4$ with multiplicity one.[11] This can be easily seen from the dual pair correspondence, see Sec. 8.2 of Ref. [23]. First, within the SH representation, the conformal symmetry $SU(2,2)$ that the massless fields enjoy is enhanced to $U(2,2)$ with the dual group $U(1)$. The dS_4 group reduction can be understood from the dual pairs,

$$\begin{array}{ccc} U(2,2) & & O^*(2) \\ \cup & \times & \cup \\ Sp(1,1) & & U(1) \end{array}, \tag{4.12}$$

[10] Note that the dual pair correspondence works at the level of group representations, and hence is sensitive to finite subgroups as well. This aspect cannot be captured when presenting the generators of Lie algebras. See [23] for more details.

[11] In fact, the scalar irrep of $\mathfrak{so}(2,4)$ reduces into two irreps of $\mathfrak{so}(2,3)$ which can be interpreted as the two possible boundary conditions of the AdS_4 scalar field.

where the reduction of $O^*(2) \cong U(1)$ to $U(1)$ is trivial, thereby explaining the singleton property of the massless $Sp(1,1)$ irrep. Similarly, the AdS_4 group reduction follows the dual pairs,

$$\begin{array}{ccc} U(2,2) & & O(2) \\ \cup & \times & \cup \\ Sp(4,\mathbb{R}) & & U(1) \end{array}, \tag{4.13}$$

where again $O(2) \cong U(1) \rtimes \mathbb{Z}_2$ reduces to $U(1)$ trivially except for the scalar case, and hence the same mechanism works for the massless $Sp(4,\mathbb{R})$ irreps.

D. Other dimensions

The SH formalism for massless fields in Mink_4 can be extended to $3d$ [58], $5d$ [59], $6d$ [60] and 10d [61] (see also [62]). These do not cover all massless representations, but only the "singleton" ones. In the case of three and six dimensions, such SH representations can be uplifted to the irreps of conformal groups $\widetilde{SO}^{\uparrow}(2,3) \cong Sp(4,\mathbb{R})$ and $\widetilde{SO}^{\uparrow}(2,6) \cong O^*(8)$. Together with the four-dimensional one $\widetilde{SO}^{\uparrow}(2,4) \cong SU(2,2)$, the conformal groups can be regarded as symplectic groups $\mathsf{Sp}(4,\mathbb{F})$[12] over $\mathbb{F} = \mathbb{R}, \mathbb{C}$, and $\mathbb{H}$,

$$\mathsf{Sp}(4,\mathbb{R}) = Sp(4,\mathbb{R}), \qquad \mathsf{Sp}(4,\mathbb{C}) \cong U(2,2), \qquad \mathsf{Sp}(4,\mathbb{H}) \cong O^*(8). \tag{4.14}$$

These groups naturally include as subgroups the three-, four-, and six-dimensional Lorentz groups isomorphic to $SL(2,\mathbb{R})$, $SL(2,\mathbb{C})$, and $SL(2,\mathbb{H})$, respectively.

For the SH representations of (A)dS fields, the (A)dS groups in the spinor representation need to contain the Lorentz group in the spinor representation. In four dimensions, this was possible thanks to the embedding of the Lorentz group $Sp(2,\mathbb{C})$ into $Sp(4,\mathbb{R})$ as well as $Sp(1,1)$. We can summarize the situation in the following diagram where the middle column corresponds to the Lorentz group and its dual, while the left and right columns correspond to the AdS_4 and dS_4 groups and their duals, respectively,

$$\begin{array}{ccccc} Sp(4,\mathbb{R}) & \supset & Sp(2,\mathbb{C}) & \subset & Sp(1,1) \\ \updownarrow & & \updownarrow & & \updownarrow \\ O(2n) & \subset & O(2n,\mathbb{C}) & \supset & O*(2n) \end{array} \tag{4.15}$$

In three dimensions, we find an analogous structure that ensures the SH representations of $(\mathrm{A})\mathrm{dS}_3$ fields. The relevant diagram is the following:

$$\begin{array}{ccccc} Sp(2,\mathbb{R}) \times Sp(2,\mathbb{R}) & \supset & Sp(2,\mathbb{R}) & \subset & Sp(2,\mathbb{C}) \\ \updownarrow & & \updownarrow & & \updownarrow \\ O(n) \times O(n) & \subset & O(2n) & \supset & O(n,\mathbb{C}) \end{array} \tag{4.16}$$

In five dimensions, we find the following structure ($SU^*(4) \cong \widetilde{SO}^{\uparrow}(1,5)$ is the dS_5 group).

$$\begin{array}{ccccc} U(2,2) & \supset & Sp(1,1) & \subset & U^*(4) \\ \updownarrow & & \updownarrow & & \updownarrow \\ U(2n) & \subset & O^*(4n) & \supset & U^*(2n) \end{array} \tag{4.17}$$

Note that the flat limit of the above should agree with the $5d$ SH representations constructed in [59].

V. MULTILINEAR INVARIANTS

A. Generalities

The $(\mathrm{A})\mathrm{dS}_4$ SH representation can be utilized in physical observables like scattering amplitudes in flat space. Of course, n-particle scattering amplitudes in $(\mathrm{A})\mathrm{dS}_4$ would not make a literal sense, and one should regard them rather as boundary n-point correlation functions. See e.g., [64–68] for the recent application of SH formalism to conformal field theory correlators. At the technical level, they are nothing but the functions of n helicity spinors invariant under $\mathfrak{Sym}_\Lambda$, which is essentially the branching rule under the restriction $\mathfrak{Sym}_\Lambda^{\times n} \downarrow \mathfrak{Sym}_\Lambda$. This leads to the dual pair,

$$\begin{array}{ccc} \mathfrak{Sym}_\Lambda \times \cdots \times \mathfrak{Sym}_\Lambda & & \mathfrak{Dual}_\Lambda^{(N_1,\ldots,N_n)} \\ \cup & \times & \cup \\ \mathfrak{Sym}_\Lambda & & \mathfrak{Dual}_\Lambda^{(N_1)} \times \cdots \times \mathfrak{Dual}_\Lambda^{(N_n)} \end{array}, \tag{5.1}$$

[12] Here, the symplectic group $\mathsf{Sp}(4,\mathbb{F})$ is defined as the matrices $A \in GL(4,\mathbb{F})$ satisfying $A^\dagger \Omega_{(4)} A = \Omega_{(4)}$ where $\dagger$ is the conjugation with respect to $\mathbb{F}$ and $\Omega_{(4)}$ is the four-dimensional symplectic matrix [63]. This definition differs from the standard definition of symplectic groups.

where $\mathfrak{Dual}_{\Lambda}^{(N_1,\ldots,N_n)}$ is given by

$$\begin{aligned}\mathfrak{Dual}_{\Lambda>0}^{(N_1,\ldots,N_n)} &= O^*(2(N_1+\cdots+N_n)),\\ \mathfrak{Dual}_{\Lambda<0}^{(N_1,\ldots,N_n)} &= O(2(N_1+\cdots+N_p),\\ &\qquad 2(N_{p+1}+\cdots+N_n)),\end{aligned}\tag{5.2}$$

where p and $n-p$ are respectively the number of incoming and outgoing particles. Note that in dS_4 case, there is no distinction between incoming and outgoing particles as the energy of a particle is not positive definite.

In (5.1), we require that the down-right factor $\mathfrak{Dual}_{\Lambda}^{(N_1)}\times\cdots\times\mathfrak{Dual}_{\Lambda}^{(N_n)}$ carry an irrep correspondingly to the particle species entering the scattering, and the down-left $\mathfrak{Sym}_{\Lambda}$ carry the trivial representation, that is invariance under (A)dS_4 symmetry. The translation invariance condition is deformed by the derivative part in $P_{a\dot{b}}$ (2.8), and its solution becomes more involved, while the Lorentz invariance can be easily achieved, like in the flat space case, by assuming that the amplitude is a function of the contracted variables,

$$\langle iIjJ\rangle = \lambda^{iI}{}_{a}\lambda^{jJa}, \qquad [iIjJ] = \tilde{\lambda}_{iI\dot{a}}\tilde{\lambda}_{jJ}{}^{\dot{a}}. \tag{5.3}$$

Here, iI, jJ are collective indices in which $i,j=1,2,\ldots,n$ label the particles entering to the scattering, whereas $I=1,2,\ldots,N_i$ and $J=1,2,\ldots,N_j$ are the dual group indices for each particle. The $\mathfrak{Dual}_{\Lambda}^{(N)}$ irrep condition depends on N, and it is sufficient for us to consider $N=1$ and $N=2$. For $N=1$, it is the usual helicity condition. For $N=2$ with $\mathfrak{dual}_{\Lambda}^{(2)}=\mathfrak{s}\oplus\mathfrak{m}_{\Lambda}$, the irrep condition of $\mathfrak{s}$ can be imposed like in the flat space case as in [7,9], and we need to impose the irrep condition of $\mathfrak{m}_{\Lambda}$ which becomes involved due to the derivative parts of M and $\tilde{M}$ given in (2.9).

As a side remark, let us point out that the complex positive Grassmannian structure of scattering amplitudes of n massless fields [69–71] naturally appears within the framework of the dual pair correspondence, as explained in Sec. 7 of [23]. When the scattering particles are all massless, that is $N_1=\cdots=N_n=1$, the spacetime symmetry $\mathfrak{Sym}_{\Lambda}$ is enhanced to $U(2,2)$, while the dual group $\mathfrak{Dual}_{\Lambda}^{(1,\ldots,1)}$ becomes the indefinite unitary group $U(p,n-p)$ in the dual pairs (5.1). In this enhanced setting, we do not require the full invariance under $U(2,2)$ but only under the subgroup $\mathfrak{Sym}_{\Lambda}$, which contains the Lorentz subgroup $SL(2,\mathbb{C})$. Together with the diagonal subgroup $\mathbb{C}^{\times}$ generated by the total helicity and the dilation operator, the Lorentz $SL(2,\mathbb{C})$ can be uplifted to $GL(2,\mathbb{C})$, which has $GL(n,\mathbb{C})$ as its dual group. The situation can be again summarized by the following seesaw diagram.

$$\begin{array}{ccccc} U(2,2)\times\cdots\times U(2,2) & & & & GL(n,\mathbb{C})\\ \cup & \nwarrow & & \nearrow & \cup\\ U(2,2) & \leftarrow & & \rightarrow & U(p,n-p)\\ \cup & \swarrow & & \searrow & \cup\\ GL(2,\mathbb{C}) & & & & U(1)\times\cdots\times U(1) \end{array}\;. \tag{5.4}$$

The Lorentz invariance is equivalent to the condition that under restriction to $GL(2,\mathbb{C})$, the amplitudes carry a one-dimensional representation, wherein $SL(2,\mathbb{C})$ acts trivially, and $GL(1,\mathbb{C})\cong\mathbb{C}^{\times}$ acts diagonally. The corresponding $GL(n,\mathbb{C})$ representation is a degenerate principal series representation (see e.g., [72]), which is realized as the space of functions on the complex positive Grassmannian manifold $Gr_{2,n}(\mathbb{C})$.

Coming back to the picture (5.1), the only nontrivial part of the conditions are the translational invariance condition, and the irrep condition of $\mathfrak{m}_{\Lambda}$ for $N=2$. When $\Lambda=0$, both of these conditions are algebraic and could be solved by imposing the helicity spinors to be constrained on the shell of the momentum conservation and constant mass-squared. When $\Lambda\neq 0$, both of these conditions become differential equations.

B. Translational invariance

Let us consider first the condition of translation invariance,

$$P_{a\dot{b}}\mathcal{A} = \left(\lambda^{\mathcal{I}}{}_{a}\tilde{\lambda}_{\mathcal{I}\dot{b}} + \Lambda\frac{\partial}{\partial\lambda^{\mathcal{I}a}}\frac{\partial}{\partial\tilde{\lambda}_{\mathcal{I}}{}^{\dot{b}}}\right)\mathcal{A} = 0, \tag{5.5}$$

where $\mathcal{I}=iI$, $\mathcal{J}=jJ$ denote the collective indices. In the Mink_4 case, the solution is nothing but the momentum conservation delta distribution $\delta^4(p)$ with

$$p_{a\dot{b}} = \lambda^{\mathcal{I}}{}_{a}\tilde{\lambda}_{\mathcal{I}\dot{b}}, \tag{5.6}$$

and hence we expect a similar kind of distributional property for the $\Lambda\neq 0$ solution. For the massless 3 pt case, this equation has been analyzed in detail in Appendix E in Ref. [16], where the authors made an ansatz as a function of $\langle 12\rangle[12]$, $\langle 23\rangle[23]$, and $\langle 31\rangle[31]$ and derived a system of four partial differential equations (PDEs). Instead of solving these equations directly, they checked that the amplitudes obtained from field theoretical approach (that is, spacetime integral of three AdS plane wave solutions) solve the equations. The solution is spanned by four independent distributions of $\langle 12\rangle[12]+\langle 23\rangle[23]+\langle 31\rangle[31]=\frac{1}{2}p_{a\dot{b}}p^{a\dot{b}}$.

Let us revisit the problem slightly differently for the general case (massive or massless n-pt). Since $\mathcal{A}$ should involve the momentum conservation delta function in the flat limit, we consider the ansatz $\mathcal{A}=\mathcal{A}(p_{a\dot{b}},\langle\mathcal{I}\mathcal{J}\rangle,[\mathcal{I}\mathcal{J}])$.[13] Then the condition (5.5) sets up the differential equation,

$$\left[p_{a\dot{b}}+\Lambda\left(p^{c\dot{d}}\frac{\partial}{\partial p^{a\dot{d}}}\frac{\partial}{\partial p^{c\dot{b}}}+H\frac{\partial}{\partial p^{a\dot{b}}}+\lambda^{\mathcal{J}}{}_a\tilde{\lambda}_{\mathcal{K}\dot{b}}\frac{\partial}{\partial\langle\mathcal{I}\mathcal{J}\rangle}\frac{\partial}{\partial[\mathcal{I}\mathcal{K}]}\right)\right]\mathcal{A}=0, \tag{5.7}$$

with the number operator H,

$$H=\mathcal{N}+\frac{1}{2}\langle\mathcal{I}\mathcal{J}\rangle\frac{\partial}{\partial\langle\mathcal{I}\mathcal{J}\rangle}+\frac{1}{2}[\mathcal{I}\mathcal{J}]\frac{\partial}{\partial[\mathcal{I}\mathcal{J}]}, \tag{5.8}$$

where $\mathcal{N}=\sum_{i=1}^{n}N_i$ is the sum of the ranks of the dual groups for all n particles, and the factor $1/2$ has been introduced to take the antisymmetry of $\mathcal{I}\mathcal{J}$ into account. The last term of the differential equation (5.7) is problematic since it is *not* expressed in terms of the variables $p_{a\dot{b}}$, $\langle\mathcal{I}\mathcal{J}\rangle$ and $[\mathcal{I}\mathcal{J}]$ only. We can bypass the problem by focusing on the "longitudinal part" of the equation; contracting (5.7) with $p^{a\dot{b}}$, we find

$$\left[p^{a\dot{b}}p_{a\dot{b}}+\Lambda\left(p^{a\dot{b}}p^{c\dot{d}}\frac{\partial}{\partial p^{a\dot{d}}}\frac{\partial}{\partial p^{c\dot{b}}}+Hp^{a\dot{b}}\frac{\partial}{\partial p^{a\dot{b}}}+2R\right)\right]\mathcal{A}=0, \tag{5.9}$$

where R is a differential operator acting on the Lorentz invariant variables as

$$R=\frac{1}{2}\langle\mathcal{I}\mathcal{J}\rangle\frac{\partial}{\partial\langle\mathcal{J}\mathcal{L}\rangle}[\mathcal{I}\mathcal{K}]\frac{\partial}{\partial[\mathcal{K}\mathcal{L}]}. \tag{5.10}$$

Viewing Λ as a deformation parameter, our aim is to find the deformation of the delta distribution solution of the $\Lambda=0$ case. We can better control the situation by going to the Fourier space $q^{a\dot{b}}$ where the constant solution corresponds to the correct delta distribution in the $\Lambda=0$ case. Since the constant solution is isotropic, we assume that $\tilde{\mathcal{A}}$ is a function of $t=\frac{1}{2}q^{a\dot{b}}q_{a\dot{b}}$, and this reduces the equation to the simple second-order differential equation in the t variable,

$$\left[\left(t\frac{\partial}{\partial t}+2\right)\left((1-\Lambda t)\frac{\partial}{\partial t}+\Lambda(H-4)\right)-\Lambda R\right]\tilde{\mathcal{A}}=0. \tag{5.11}$$

We can use the separation of variables,

$$R\tilde{\mathcal{A}}_r=r\tilde{\mathcal{A}}_r, \tag{5.12}$$

to decompose the PDE (5.11) into hypergeometric differential equations with two types of solutions, the first one being,

$$\tilde{\mathcal{A}}_r={}_2F_1(a_+,a_-,2;\Lambda t)f_r, \tag{5.13}$$

where $a_\pm$ are

$$a_\pm=\frac{1}{2}\left(6-H\pm\sqrt{(H-2)^2-r}\right), \tag{5.14}$$

and f_r is an arbitrary function of $\langle\mathcal{I}\mathcal{J}\rangle$ and $[\mathcal{I}\mathcal{J}]$. The second solution of the hypergeometric differential equation takes the form,

$$\frac{1}{t}+\Lambda\sum_{n=0}^{\infty}\frac{(a_+-1)_{n+1}(a_--1)_{n+1}}{(n+1)!n!}\times[\ln(\Lambda t)(\Lambda t)^n+c_n(\Lambda t)^{n+1}], \tag{5.15}$$

with

$$c_n=\sum_{m=0}^{n-1}\left(\frac{1}{a_++m}+\frac{1}{a_-+m}-\frac{1}{m+2}-\frac{1}{m+1}\right). \tag{5.16}$$

The hypergeometric function ${}_2F_1(a_+,a_-,2;\Lambda t)$ reduces to 1 for $\Lambda=r=0$, while the second solution (5.15) to $1/t$. Since the constant solution corresponds to the desired delta distribution, we retain only the hypergeometric function. Remark that for $r=0$, the hypergeometric function gets simplified to give

$$\tilde{\mathcal{A}}_0=(1-\Lambda t)^{H-4}f_0. \tag{5.17}$$

This is consistent with the expressions obtained in [15–17] for massless 3 pt. We remark also that the hypergeometric function (5.13) has a branch point at $\Lambda t=1$,[14] which might be interpreted as the cosmological horizon and related to the alpha vacua.[15] Eventually, the most general invariant will be linear combinations of $\mathcal{A}_r$ with different r values.

C. Mass condition

Let us move on to the irrep condition of $\mathfrak{m}_\Lambda$ for each of the n particles, fixing their masses. For the discrete series irreps of $\mathfrak{dual}^{(2)}_{\Lambda>0}$ in the dS_4 case and the finite-dimensional irreps of $\mathfrak{dual}^{(2)}_{\Lambda<0}$ in the AdS_4 case, we can impose the highest-weight condition $M_i\mathcal{A}=0$ or the lowest-weight condition $\tilde{M}_i\mathcal{A}=0$, on the K_i eigenstate with

[13]Note that the variables $p_{a\dot{b}}$ are not independent from $\langle\mathcal{I}\mathcal{J}\rangle$ and $[\mathcal{I}\mathcal{J}]$, as they are related by $p_{a\dot{b}}p^{a\dot{b}}=\langle\mathcal{I}\mathcal{J}\rangle[\mathcal{I}\mathcal{J}]$. Therefore, whenever the latter combination appears, we have to regard them as a function of $p_{a\dot{b}}$ to avoid the related ambiguities.

[14]In [15–17], the $q_{a\dot{b}}$ variables carry a spacetime coordinate interpretation, and the branch point corresponds to the boundary of the coordinate chart.

[15]In de Sitter space, there is a one-parameter family of dS invariant vacuum states [73]. This vacuum ambiguity would lead to an analogous ambiguity in n-point correlation functions.

$$K_i f = \mp 2(\mu_i + 1) f, \tag{5.18}$$

for dS_4 and

$$K_i f = \pm 2\mu_i f, \tag{5.19}$$

for AdS_4. Here, the K_i, upon acting on f, reduces to the differential operator,

$$K_i f = \left(\langle i I \mathcal{J} \rangle \frac{\partial}{\partial \langle i I \mathcal{J} \rangle} - [i I \mathcal{J}] \frac{\partial}{\partial [i I \mathcal{J}]} \right) f, \tag{5.20}$$

where the repeated indices $\mathcal{J}$ and I are summed over except for the particle label i. The highest-weight condition $M\mathcal{A} = 0$ can be translated as well into the differential equations,

$$M_i f = \left[\langle i1 i2 \rangle + \Lambda \left(2 \frac{\partial}{\partial [i1 i2]} + [\mathcal{J}\mathcal{K}] \frac{\partial}{\partial [i1 \mathcal{J}]} \frac{\partial}{\partial [i2 \mathcal{K}]} \right) \right] f = 0,$$

where the repeated indices $\mathcal{J}$, $\mathcal{K}$ include ith particle's values iI, and the lowest-weight condition is simply given by the complex conjugate of the above.

Note that the K eigenstate conditions (5.18) and (5.19) become singular in the flat limit where μ is sent to infinity while $\mu\sqrt{|\Lambda|}$ held fixed. Moreover, the principal series irreps of dS_4 have neither a highest nor a lowest weight state. Therefore, the above conditions are inapplicable in that case. We may consider to use the K eigenstate with eigenvalue 0 or ± 1 to avoid this problem, but in that case we cannot use any more the simple condition $M = 0$ (or $\tilde{M} = 0$). Instead we need to use the Casimir condition involving the anticommutator $\{M, \tilde{M}\}$ resulting in a fourth-order differential equation instead of (5.21).

In fact, for the principal series irreps, it is more natural to impose,

$$(M_i - \tilde{M}_i) f = 2\sqrt{\Lambda} \mu_i f, \tag{5.21a}$$

$$(M_i + \tilde{M}_i - \sqrt{\Lambda} K_i) f = 0, \tag{5.21b}$$

which has also a well-defined flat limit, and can be expressed as second-order differential equations in $\langle \mathcal{I}\mathcal{J} \rangle$ and $[\mathcal{I}\mathcal{J}]$. Solving these conditions is beyond the scope of the current work. Instead, let us make a few remarks on the change of basis where the $O^*(4)$ actions become more natural.

For the change of basis, we fix the convention as $a, b, \dot{a}, \dot{b} = +, -$ and $\epsilon_{-+} = \epsilon^{+-} = 1$ and perform Fourier transform with respect to λ^I_- and its complex conjugate as

$$\left(\frac{\lambda^I{}_-}{\sqrt{\Lambda}}, \sqrt{\Lambda} \frac{\partial}{\partial \lambda^I{}_-}, \frac{\tilde{\lambda}_{I-}}{\sqrt{\Lambda}}, \sqrt{\Lambda} \frac{\partial}{\partial \tilde{\lambda}_{I-}} \right) \longrightarrow i \left(\frac{\partial}{\partial \zeta_I}, \zeta_I, \frac{\partial}{\partial \tilde{\zeta}^I}, \tilde{\zeta}^I \right). \tag{5.22}$$

Then, the dual algebra generators read,

$$K^I{}_J = \lambda^I \frac{\partial}{\partial \lambda^J} - \zeta_J \frac{\partial}{\partial \zeta_I} - \tilde{\lambda}_J \frac{\partial}{\partial \tilde{\lambda}_I} + \tilde{\zeta}^I \frac{\partial}{\partial \tilde{\zeta}^J}, \tag{5.23a}$$

$$\frac{M^{IJ}}{\sqrt{\Lambda}} = i \left(\lambda^I \frac{\partial}{\partial \zeta_J} - \lambda^J \frac{\partial}{\partial \zeta_I} + \tilde{\zeta}^I \frac{\partial}{\partial \tilde{\lambda}_J} - \tilde{\zeta}^J \frac{\partial}{\partial \tilde{\lambda}_I} \right), \tag{5.23b}$$

where we used $\lambda^I = \lambda^I_+$ and $\tilde{\lambda}_I = \tilde{\lambda}_{I+}$. In this basis, the $\mathfrak{o}^*(2N)$ generators become first-order differential operators, and hence can be easily integrated to a Lie group. This basis admits in fact a natural realization in terms of quaternions; see Appendix B for the details. While the new basis (5.23a) and (5.23b) renders the dual algebra as simple first-order differential operators, the Lorentz algebra becomes second order instead. In other words, in the basis where the dual algebra is linearly realized, the dS_4 algebra is not. And vice versa; we can go to another basis where the dS_4 algebra is realized linearly, but then the dual algebra is not.

For $N = 2$, we can consider a different Fourier transformation,

$$\left(\frac{\lambda^2{}_a}{\sqrt{\Lambda}}, \sqrt{\Lambda} \frac{\partial}{\partial \lambda^2{}_a}, \frac{\tilde{\lambda}_{2\dot{a}}}{\sqrt{\Lambda}}, \sqrt{\Lambda} \frac{\partial}{\partial \tilde{\lambda}_{2\dot{a}}} \right) \longrightarrow i \left(\frac{\partial}{\partial \xi^a}, \xi^a, \frac{\partial}{\partial \tilde{\xi}^{\dot{a}}}, \tilde{\xi}^{\dot{a}} \right), \tag{5.24}$$

where only the $I = 2$ variables are transformed. Upon a further change of basis,

$$z^a = \frac{\xi^a - i\lambda^{1a}}{2}, \qquad w^a = \frac{\xi^a + i\lambda^{1a}}{2},$$
$$\tilde{z}^{\dot{a}} = \frac{\tilde{\xi}^{\dot{a}} + i\tilde{\lambda}_1{}^{\dot{a}}}{2}, \qquad \tilde{w}^{\dot{a}} = \frac{\tilde{\xi}^{\dot{a}} - i\tilde{\lambda}_1{}^{\dot{a}}}{2}, \tag{5.25}$$

the conditions (5.21a) and (5.21b) become simple,

$$\frac{M - \tilde{M}}{\sqrt{\Lambda}} = z^a \frac{\partial}{\partial z^a} + \tilde{z}^{\dot{a}} \frac{\partial}{\partial \tilde{z}^{\dot{a}}} - w^a \frac{\partial}{\partial w^a} - \tilde{w}^{\dot{a}} \frac{\partial}{\partial \tilde{w}^{\dot{a}}}, \tag{5.26a}$$

$$\frac{M + \tilde{M}}{\sqrt{\Lambda}} - K = 2 \left(z^a \frac{\partial}{\partial w^a} - \tilde{z}^{\dot{a}} \frac{\partial}{\partial \tilde{w}^{\dot{a}}} \right). \tag{5.26b}$$

The condition (5.21b) can be solved by an arbitrary function of z^a, $\tilde{z}^{\dot{a}}$, and $z^a \tilde{w}^{\dot{b}} + w^a \tilde{z}^{\dot{b}}$. Furthermore, the condition (5.21a), which fixes the principal series label, becomes a simple homogeneity condition with respect to the number operator (5.26a). The variables $z^a \tilde{w}^{\dot{b}} + w^a \tilde{z}^{\dot{b}}$ have weight zero and hence are not constrained, while the homogeneity of $|z| = \sqrt{z^a \tilde{z}^{\dot{a}}}$ is restricted to μ. The spin condition further restricts the variables $z^a\tilde{w}^{\dot{b}} + w^a\tilde{z}^{\dot{b}}$ and $z^a/|z|$. In the end, the remaining freedom corresponds to the massive irrep of $Sp(1,1)$. However, in this basis, the spin part $\mathfrak{s}$ of the dual algebra, that is generated by $K^I{}_J$, is realized by second-order differentials.

Therefore, the dS_4 invariance condition and the $O^*(4)$ irrep condition for each of the particles cannot be solved within a single basis, but by employing multiple bases that are related by Fourier transformations. These conditions may be solved for concrete examples of interest. We leave this to future investigations.

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APPENDIX A: UNITARY AND IRREDUCIBLE REPRESENTATIONS OF dS$_4$ GROUP AND CASIMIRS

Unitary and irreducible representations (UIRs) of the dS_4 isometry group were first classified by J. Dixmier in [37]. In this appendix, we recall this classification and the eigenvalues of the quadratic and quartic Casimir operators,

(i) $\pi^{\pm}_{p,q}$: $[p = \frac{1}{2}, 1, \frac{3}{2}, \ldots;\ q = p, p-1, \ldots, 1 \text{ or } \frac{1}{2}]$ with

$$\begin{aligned} C_2 &= -p(p+1) - (q-1)q + 2 \\ &= -p(p+1) - (q+1)(q-2), \\ C_4 &= -p(p+1)(q-1)q, \end{aligned} \tag{A1}$$

corresponding to the discrete series.

(ii) $\pi_{p,0}$: $[p = 1, 2, \ldots]$ with the quadratic and quartic Casimir operators taking the values

$$\begin{aligned} C_2 &= p(p+1) - 2, \\ C_4 &= 0. \end{aligned} \tag{A2}$$

These UIRs form the discrete series.

(iii) $\nu_{p,\sigma}$: $[p = 0;\ \sigma > -2]$ and $[p = 1, 2, \ldots;\ \sigma > 0]$ and $[p = \frac{1}{2}, \frac{3}{2}, \ldots;\ \sigma > \frac{1}{4}]$ with

$$\begin{aligned} C_2 &= p(p+1) - \sigma - 2, \\ C_4 &= -p(p+1)\sigma, \end{aligned} \tag{A3}$$

corresponding to the principal and complementary series.

APPENDIX B: QUATERNION REALIZATION OF dS$_4$ GROUP

The dual pair $(Sp(M,M), O^*(2N))$ can be naturally realized in terms of quaternions. The oscillator representation is the space of functions on $\mathbb{H}^{MN}$, where $O^*(2N)$ acts on a function Φ by right multiplication,

$$\langle \mathsf{Q}|U_{O^*(2N)}(\mathsf{A})\Phi\rangle = \langle \mathsf{QA}|\Phi\rangle, \tag{B1}$$

where Q is an $M \times N$ quaternionic matrix and A is an $N \times N$ quaternionic matrix satisfying,[16]

$$\mathsf{A}^\dagger \mathsf{j} \mathsf{A} = \mathsf{j}, \tag{B2}$$

thereby representing an arbitrary element of $O^*(2N)$. For $M = 1$, each of the quaternionic elements of $\mathsf{Q} = (\mathsf{q}_I)$, seen as a 2×2 complex matrix, can be parametrized by two complex numbers as

$$\mathsf{q}_I = \begin{pmatrix} \lambda^I + i\zeta_I & \zeta_I + i\lambda^I \\ -\tilde{\zeta}^I + i\tilde{\lambda}_I & \tilde{\lambda}_I - i\tilde{\zeta}^I \end{pmatrix}. \tag{B3}$$

Note that we recover the expressions (5.23a) and (5.23b) from the above parametrization of q_I.

For even $N = 2L$, the $Sp(M,M)$ action can also be represented by the left multiplication of a quaternionic matrix,

$$\left\langle \begin{pmatrix} \mathsf{Q}_1 \\ \mathsf{P}_2 \end{pmatrix} \middle| U_{Sp(M,M)}(\mathsf{B})\Phi \right\rangle = \left\langle \mathsf{B}^t \begin{pmatrix} \mathsf{Q}_1 \\ \mathsf{P}_2 \end{pmatrix} \middle| \Phi \right\rangle, \tag{B4}$$

where B is an element of $Sp(M,M)$, and hence a $2M \times 2M$ quaternionic matrix satisfying

$$\mathsf{B}^\dagger \begin{pmatrix} 0 & I_M \\ I_M & 0 \end{pmatrix} \mathsf{B} = \begin{pmatrix} 0 & I_M \\ I_M & 0 \end{pmatrix}. \tag{B5}$$

The submatrices Q_1 and P_2 are $M \times L$ quaternionic matrices, and P_2 is the Fourier conjugate of Q_2 where Q_1 and Q_2 form the $M \times 2L$ matrix $\mathsf{Q} = (\mathsf{Q}_1\mathsf{Q}_2)$.

[16] Here, j denotes the basis element of quarternions that can be represented by the Pauli matrix $i\sigma_2$.

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