Nonlinear rigid-body quantization of Skyrmions

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We consider rigid-body quantization of the Skyrmion in the most general four-derivative generalization of the Skyrme model with a potential giving pions a mass, as well as in a class of higher-order Skyrme models. We quantize the spin and isospin zero modes following the results of Pottinger and Rathske. Although one could hope that a one-parameter family of theories could provide a smaller spin contribution to the energy at some point in theory space—which would be welcome for Bogomol'nyi-Prasad-Sommerfield-type models, we find that the standard Skyrme model limit, with two time derivatives, gives rise to the smallest spin contribution to the energy. We speculate whether this tuning of the spin energy could be useful in the larger picture of quantizing vibrational and light massive modes of the Skyrmions. Finally, we establish a topological energy bound for the Pottinger-Rathske model with potential terms as well as new bounds for higher-order Skyrme models, with and without a potential.

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I. INTRODUCTION

The Skyrme model [1,2] is a low-energy effective field theory description of QCD in a pure pion theory, where baryons are solitons known as Skyrmions. Although the model was proposed already in the sixties by Skyrme, it first received serious attention after Witten showed that the Skyrmion is the nucleon of QCD in the large-N_c limit in the seminal papers [3,4]. Although the Skyrme model provides a qualitative description of the nucleon to about the 30% level of accuracy compared with experiments [5], a major obstacle in using the theory for nuclei is that it gets the binding energies wrong by roughly an order of magnitude, already at the classical level. The community has worked on solving this problem at the classical level, essentially by finding so-called Bogomol'nyi-Prasad-Sommerfield (BPS) limits of the theory, for which there exist solutions,[1] the idea being that once one have found the BPS limit, a small perturbation could create the tiny binding energies of about 1% of the nucleon mass per baryon. To list a few of the attempts to find a BPS-type model for Skyrmions, the Sutcliffe model [6,7] is a five-dimensional Yang-Mills theory that realizes a flat-space holographic description of Skyrmions as the holonomy of the gauge fields [8]; the BPS-Skyrme model is a radical change of the Skyrme model by eliminating the model and replacing it by the topological charge current squared as well as a suitable potential [9,10]; finally, the weakly-bound Skyrme model is based on an energy bound using the Hölder inequality for which the Skyrme term and a potential to the fourth power saturates the energy bound [11,12]. The Sutcliffe model requires an infinite number of vector bosons added to the Skyrme model in order to reach the BPS limit, i.e., the limit in which the classical energy is directly proportional to the baryon number, hence yielding vanishing classical binding energy. The advantage of the model, similarly to holographic constructions [13] and the hidden local symmetry approach [14], is that all the couplings to the infinite tower of vector bosons are determined. The BPS-Skyrme model has (analytic) solutions for any baryon number that saturates the Bogomol'nyi bound, but near-BPS solutions turn out to be a numerically challenging problem [12,15–17]. The weakly-coupled Skyrme model only saturates the Bogomol'nyi bound for a single baryon, hence all nuclei already have a nonvanishing albeit small classical binding energy. It turns out that all the solutions take the shape of lattices of point-particlelike Skyrmions [12]. For completeness, we can mention that a dielectric formulation of the Skyrme model [18] also provides small binding energies [18,19], but also in this case the solutions tend to be point-particlelike constellations like in the weakly-bound Skyrme model [19].

Now as put forward in the recent paper [20], although the BPS race that has taken place for over 10 years in the community has given rise to interesting ideas and some

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[1] There exists a Bogomol'nyi bound for the standard Skyrme model, but there are no solutions saturating the bound, which means all solutions have positive binding energy.

analytic solutions, the vanishing classical binding energy does not solve the problem of the binding energy of nuclei, simply due to the spin contribution. In order to illustrate this, let us consider the standard Skyrme model with the rational map approximation [21,22]:

$$E = \int d^3x[\lambda(a_1 + a_2 B)\mathcal{E}_2 + \lambda^{-1}(b_1 + b_2 B)B\mathcal{E}_4], \quad (1)$$

with $a_{1,2}$, $b_{1,2}$ being positive coefficients, B the baryon number and λ the length scale. Using the rational map approximation is a good approximation to Skyrmions with baryon numbers $B = 1, 2, \ldots, 7$ in the massive Skyrme model, i.e. once the pion mass term is turned on and overestimates the energies only by a few percent [23–25].[2] Finding the Derrick stability, we obtain

$$\lambda = \sqrt{\frac{e_4(b_1 + b_2 B)B}{e_2(a_1 + a_2 B)}}, \quad (2)$$

which for large B goes like $\lambda \propto \sqrt{B}$. The size of the Skyrmion hence grows like $\sqrt{B}$. Since the mass of the Skyrmion grows at least as fast as B, the moment of inertia scales like B^2 or higher. The spin contribution to the energy found in the seminal paper by Adkins-Nappi-Witten (ANW) thus goes like

$$E_{\rm spin} = \frac{J^2}{2\Lambda} \propto \frac{J^2}{B^2}. \quad (3)$$

Hence, even for nuclei whose ground state has a spin, the spin contribution is suppressed by roughly B^2 and quickly becomes negligible (for $B = 7$, the suppression is by a factor of $1/49$). Even more troublesome are the nuclei that are bosonic with spin 0 and isospin 0 in the ground state, as their contribution is just zero. These nuclei exist for B up to 40 for stable nuclei and 48 for "long-lived" nuclei. For nuclei with $B \gtrsim 40$, the Coulomb repulsion begins to be important, so the isospin quantum number is generically nonvanishing in the ground state of such nuclei.

Now let us contemplate a BPS model, which by definition has vanishing classical binding energy for the Skyrmions. Using only rigid-body quantization, we can thus compute the binding energy of e.g. ^{4}He as

$$\Delta = 4(M_1 + E_{\rm spin}) - (4M_1 + 0) = \frac{3}{2\Lambda}. \quad (4)$$

Computing this number within the standard Skyrme model as a rough estimate, one obtains of the order of $4 \times 4.6\%$ of the nucleon mass, which is about 175 MeV. The physical binding energy of ^{4}He is about 28.3 MeV. For nuclei with nonvanishing spin and/or isospin in the ground state, the problem is, of course, slightly less severe.

We can thus see that the BPS models cannot solve the binding energy problem of the Skyrme model in the scheme of rigid-body quantization, as also explained in Ref. [20]. In the latter reference, it was proposed that since the number of zero modes are fixed and independent of the baryon number,[3] the quantum contribution is underestimated for $B > 1$ nuclei and the resolution is to take more modes into account in the quantization procedure. We can now see that there are two ways to approach the problem: in the spirit of Ref. [20] one could take as many degrees of freedom as needed into account and quantize them to hopefully arrive at a cancellation of contributions that land just right on the nuclear physics scale of about 8 MeV per nucleon. Alternatively, one could believe in the semiclassical approximation of solitons being a good description of nature, with the classical contribution (mass) being the dominant one, and all quantum corrections being much smaller in magnitude, thus possibly avoiding too large cancellations (fine tuning) in the final result.

With the latter notion of naturalness in mind, which is also confirming the validity of using solitons in the first place, one may ask whether it is possible to lower the spin contribution to the nucleon. In this paper, we study this question in a generalization of the Skyrme model to the most general Lorentz-invariant Lagrangian written in terms of the chiral Lagrangian field $U \in SU(2)$ with up to four derivatives and up to second order in a polynomial potential. Unfortunately, our result is, as it turns out, the Skyrme model limit of the model at hand gives the smallest spin contribution to the energy for the nucleon. For the naturalness path forward, one would thus stick with the Skyrme term, whereas if one proceeded along the cancellation path to nuclear phenomenology, this model introduces an extra parameter that could be used to fine tune the binding energies.

The generalization of the Skyrme model to the most general fourth-order derivative term, has been studied previously in the literature in the context of the Skyrme model and the chiral Lagrangian. In the scheme of an EFT, there are only two different fourth-order derivative terms involving the pion matrix U [28–32].[4] Scrutinizing $\pi\pi$ scattering data in the D-wave using the chiral Lagrangian, Weinberg's renowned result [33] did not apply, and Gasser

[2] A multilayered rational map may be utilized for larger Skyrmions, where two or more different angular maps are utilized in radial layers of the soliton [26,27].

[3] To be more precise, the number of rotational and isorotation zero modes are 6 for $B > 1$, whereas spin and isospin are equal in magnitude for the $B = 1$ Skyrmion due to spherical symmetry.

[4] One might naively think that one could also write down the term $\mathrm{tr}(U^\dagger \Box U)^2$, but by using integration by parts and field redefinitions, it can be shown to be equivalent to the two fourth-order derivative terms studied in this paper as well as a combination of some higher-than-fourth-order terms.

and Leutwyler's result was that a nonvanishing coefficient of the squared kinetic term was favored [34,35]. With this result Donoghue, Golowich and Holstein used the Skyrme model with this new kinetic term squared to predict the proton mass from the $\pi\pi$ scattering data, obtaining 880 ± 300 MeV [36]. They calculated the contribution to the mass from the proton's spin by using ANW leading order formula [5], but with the full moment of inertia of the Skyrmion [36] (see also Ref. [37]). This was considered a good approximation, since the deviation from the Skyrme model limit (i.e., vanishing kinetic term squared) was experimentally quite small. The nucleon-nucleon potential, which turned out not to lead to an attractive force at medium distances in the standard Skyrme model, was computed for many generalizations in the search for this attractive property and also in the Skyrme model with the kinetic term squared [38,39]. They also did not find the exact contribution of the spin energy, but simply computed the moment of inertia and used it in the ANW formula.

The exact computation of the spin contribution to the energy of the Skyrmion was first done by Pottinger and Rathske [40], by solving the cubic equation relating the spin operator squared to the moments of inertia using Cardano's formula, which we will review. Although Donoghue, Golowich, and Holstein worked with the same model earlier, we will here denote the massless Skyrme model with the squared kinetic term as the Pottinger-Rathske (PR) model, since they treated the spin contribution to the energy exactly and not perturbatively, as other groups did.

The kinetic term squared, in contrast to the Skyrme term, contains four time derivatives, which in turn makes it lose SO(4) Euclidean symmetry in the Hamiltonian after performing the usual Legendre transformation from the Lorentz-invariant Lagrangian. It also gives the unpleasing side effect of opening up for runaway directions in the Hamiltonian energy. This is generically not unexpected for low-energy effective field theories at higher orders in the derivative expansion.

The kinetic term squared has also been considered in further literature, in the context of Skyrme-type models, see the review [41]. In particular, the nucleon-nucleon potential has been studied in an extension of the PR model with a sextic derivative term, being the baryon current squared [42]. Finite density computations of the energy in a hybrid model with both quarks and pions have been considered, using also the kinetic term squared [43]. Solitons with nonvanishing Hopf number were also studied in the Skyrme model with the squared kinetic term [44]. The Skyrmion and in particular the Skyrme model with the squared kinetic term can be related to the soliton in the Nambu-Jona-Lasinio model by a derivative expansion [45]. The Skyrme model, including the squared kinetic term, was generalized to include several sextic derivative terms and these terms were used to improve the fitted value of the pion decay constant [46]. The stability versus metastability aspects of the Skyrmion was ported from quantum chromodynamics (QCD) to electroweak Skyrmions in Ref. [47]. A nonsingular spacetime defect soliton has been studied on a nonsimply connected topology with nontrivial field solutions using the Skyrme model with the squared kinetic term [48]. Closed timelike curves were studied in the Skyrme model with the squared kinetic term—all coupled to Einstein gravity [49]. In all the papers of this paragraph, either the perturbative ANW formula for the spin contribution to the energy was used, or only the stability/metastability aspects of the Skyrmion were studied.

In this paper, we point out that the spin contribution to the energy changes with a positive definite correction upon including an arbitrary higher-order derivative term in the Lagrangian that contains 4 time derivatives, under these conditions: The higher-order derivative term is Lorentz invariant and its static energy is positive definite. This means that including 4 time derivatives in a Skyrme-type model, instead of 2 time derivatives, can only increase the spin contribution to the energy and hence exacerbate the binding energy problem. We illustrate this claim by considering generalizations of the Skyrme model with the kinetic term squared (the PR model) as well as with 8th, 10th, and 12th order derivative terms that contain four time derivatives (and no d'Alembertian).

The paper is organized as follows. In Sec. II we review the Pottinger-Rathske (PR) model and collective coordinate (or rigid-body) quantization therein. In Sec. II A we give our chosen calibration scheme and in Sec. II B we present the numerical results of the paper. In Sec. III we include a class of higher-order Skyrme models with four time derivatives and between 8 and 12 derivatives in total, whose quantization is identical to the model of Sec. II with modified moments of inertia. We conclude in Sec. IV with a discussion and outlook. We have delegated technical details of the models in the paper to appendices. In particular, the positivity of the static energy of the PR model is given in Appendix A 1 and of the higher-order models in Appendix A 2. A topological energy bound for the massless PR model is reviewed in Appendix B 1 and bounds are found for the case that includes nonderivative potentials. Finally, new topological energy bounds are found for the higher-order models in Appendix B 2.

II. THE POTTINGER-RATHSKE SKYRME MODEL

We consider the chiral Lagrangian with the most general Lorentz-invariant Lagrangian, up to fourth order in derivatives [28–32], which was considered previously by Donoghue-Golowich-Holstein [36] and by Pottinger-Rathske (PR) [40]. In addition to the derivative terms, we include the standard pion mass term [50], as well as the loosely bound potential term [51,52]; hence the total Lagrangian is also the most general Lagrangian up to polynomials of order 2, in the chiral Lagrangian field, U [52]:

$$\mathcal{L}=\frac{F_\pi^2}{16}\mathrm{tr}(R_\mu R^\mu)+\frac{1}{32e^2}\mathrm{tr}([R_\mu,R_\nu][R^\mu,R^\nu])-\frac{\beta}{32}(\mathrm{tr}(R_\mu R^\mu))^2 -\frac{F_\pi^2 m_\pi^2}{8}\mathrm{tr}(\mathbf{1}_2-U)-\frac{F_\pi^2 M^2}{32}[\mathrm{tr}(\mathbf{1}_2-U)]^2, \tag{5}$$

where the right-invariant chiral current is

$$R_\mu = \partial_\mu U U^\dagger, \tag{6}$$

F_π is the pion decay constant, e is the Skyrme coupling constant, β is the dimensionless coupling of the other fourth-order derivative term which we shall dub the kinetic term squared, m_π is the pion mass, M is the mass parameter of the loosely bound potential term, the chiral Lagrangian or Skyrme field U is related to the pions via

$$U = \mathbf{1}_2\sigma + \mathrm{i}\boldsymbol{\pi}\cdot\boldsymbol{\tau}, \tag{7}$$

where $\boldsymbol{\tau}$ are the standard Pauli spin matrices, and finally we use the Minkowski metric with the mostly positive signature.

By the most general fourth-order derivative theory with only the field U, we mean that field redefinitions and integration-by-parts relations have been taken into account, leaving the two displayed terms as a representation of the two independent terms that exist, at this order in the derivative expansion [28–32].

The topological charge of the field U is known as the baryon number and can be computed as

$$B = -\frac{1}{24\pi^2}\int \mathrm{d}^3x\epsilon^{ijk}\mathrm{tr}(R_iR_jR_k), \tag{8}$$

which arises due to the finite-energy condition on U implying that $\lim_{|x|\to\infty} U = \mathbf{1}_2$ (or another constant, that however can be rotated into the unit matrix) which in turn effectively point-compactifies 3-space to S^3 and hence $\pi_3(\mathrm{SU}(2)_\mathrm{L}\times\mathrm{SU}(2)_\mathrm{R}/\mathrm{SU}(2)_\mathrm{diag}) = \mathbb{Z}\ni B$.

We first redefine new dimensionless coupling constants as

$$\frac{1}{e^2}=\alpha^2(2\eta-1), \qquad \beta=2\alpha^2(1-\eta), \qquad \eta\in[-1,1], \tag{9}$$

where η interpolates between the two fourth-order soliton-stabilizing terms and α is an overall positive coefficient. For the details of the positivity of the static energy in the PR model, see Appendix A 1. The Lagrangian now reads

$$\mathcal{L} = \frac{1}{2}\mathrm{tr}(R_\mu R^\mu)+\frac{2\eta-1}{16}\mathrm{tr}([R_\mu,R_\nu][R^\mu,R^\nu]) -\frac{1-\eta}{8}(\mathrm{tr}(R_\mu R^\mu))^2 - m_1^2\mathrm{tr}(\mathbf{1}_2-U) -\frac{m_2^2}{4}[\mathrm{tr}(\mathbf{1}_2-U)]^2, \tag{10}$$

where the energy and length units are rescaled as

$$\mu=\frac{F_\pi\alpha}{4}, \qquad \lambda=\frac{2\alpha}{F_\pi}, \tag{11}$$

and the dimensionless mass parameters are given by

$$m_1 \coloneqq \frac{2\alpha m_\pi}{F_\pi}, \qquad m_2 \coloneqq \frac{2\alpha M}{F_\pi}. \tag{12}$$

The real parameter η interpolates between three different models, see Table I.

The Lagrangian splits into potential and kinetic energy as

$$L = T^L - V, \tag{13}$$

$$V = \int \mathrm{d}^3x\left[-\frac{1}{2}\mathrm{tr}(R_i^2)-\frac{2\eta-1}{16}\mathrm{tr}([R_i,R_j]^2) +\frac{1-\eta}{8}(\mathrm{tr}(R_i^2))^2+m_1^2\mathrm{tr}(\mathbf{1}_2-U) +\frac{m_2^2}{4}[\mathrm{tr}(\mathbf{1}_2-U)]^2\right], \tag{14}$$

$$T^L = \int \mathrm{d}^3x\left[-\frac{1}{2}\mathrm{tr}(R_0^2)-\frac{2\eta-1}{8}\mathrm{tr}([R_0,R_i]^2) -\frac{1-\eta}{8}(\mathrm{tr}(R_0^2))^2+\frac{1-\eta}{4}\mathrm{tr}(R_0^2)\mathrm{tr}(R_i^2)\right]. \tag{15}$$

Using the hedgehog ansatz

$$U = \mathbf{1}_2\cos f(r) + \mathrm{i}\hat{\mathbf{x}}\cdot\boldsymbol{\tau}\sin f(r), \tag{16}$$

with $\hat{\mathbf{x}} = \mathbf{x}/r$ being a unit 3-vector in $\mathbb{R}^3$, $r = |\mathbf{x}|$ and the potential or static energy becomes

$$V = \int \mathrm{d}^3x\left[(f')^2+\frac{2\sin^2 f}{r^2}+\frac{1-\eta}{2}(f')^4+2\eta\frac{\sin^2(f)(f')^2}{r^2} +\frac{\sin^4 f}{r^4}+2m_1^2(1-\cos f)+m_2^2(1-\cos f)^2\right], \tag{17}$$

which can be seen to be positive (semi)definite, term by term, for $\eta\in[0,1]$. In order to see that this is still

TABLE I. A one-parameter family interpolating between 3 different models, with $\eta = 1$ being the Skyrme model limit.

	Skyrme term	Kinetic term squared
$\eta = 1$	1	0
$\eta = 1/2$	0	1/2
$\eta = -1$	−3	2

a positive static energy functional for $\eta \in [-1, 1]$, we rewrite it as

$$V = \int d^3x \left[(f')^2 + \frac{2\sin^2 f}{r^2} + \frac{1+\eta}{2} \left(\frac{2\sin^2(f)(f')^2}{r^2} + \frac{\sin^4 f}{r^4} \right) + \frac{1-\eta}{2} \left((f')^2 - \frac{\sin^2 f}{r^2} \right)^2 + 2m_1^2 (1 - \cos f) + m_2^2 (1 - \cos f)^2 \right], \tag{18}$$

which is indeed positive semidefinite for $\eta \in [-1, 1]$.

The static equation of motion for the profile function of the Skyrmion is found by varying the static potential energy (17) with respect to f, yielding:

$$f'' + \frac{2f'}{r} - \frac{\sin 2f}{r^2} + 3(1-\eta)(f')^2 f'' + 2(1-\eta)\frac{(f')^3}{r} + \frac{2\eta \sin^2(f) f''}{r^2} + \frac{\eta \sin(2f)(f')^2}{r^2} - \frac{\sin^2(f)\sin 2f}{r^4} - m_1^2 \sin f - m_2^2 (1 - \cos f)\sin f = 0, \tag{19}$$

which needs to be accompanied by the boundary conditions $f(0) = \pi$ and $f(\infty) = 0$ corresponding to a unit-Skyrmion ($B = 1$).

Introducing a classical rotation of the spherically symmetric hedgehog Skyrmion can be done in two ways, either by isospinning the soliton

$$U^{(A)}(\mathbf{x}, t) = A(t) U(\mathbf{x}) A(t)^\dagger, \tag{20}$$

which however is equivalent to spinning it via

$$U^{(A)}(\mathbf{x}, t) = U(\mathsf{R}(t)\mathbf{x}), \qquad \mathsf{R}_{ij} = \frac{1}{2}\mathrm{tr}(\tau^i A \tau^j A^\dagger), \tag{21}$$

where $U(\mathbf{x})$ is a static solution to the field equations. The (classical) kinetic part of the Lagrangian can now readily be written down

$$T^L = \int d^3x \left[-\frac{1}{2}\mathrm{tr}(T_i T_j) - \frac{2\eta - 1}{8}\mathrm{tr}\Big([T_i, R_k][T_j, R_k]\Big) + \frac{1-\eta}{4}\mathrm{tr}(T_i T_j)\mathrm{tr}(R_k^2) \right] a_i a_j + \int d^3x \left[-\frac{1-\eta}{8}\mathrm{tr}(T_i T_j)\mathrm{tr}(T_k T_l) \right] a_i a_j a_k a_l, \tag{22}$$

where $T_i = \frac{i}{2}[\tau^i, U]U^\dagger$ and the $\mathfrak{su}(2)$-valued angular momenta, $a_i \tau^i$, are defined as

$$a_i = -\mathrm{i}\,\mathrm{tr}(\tau^i A^\dagger \dot{A}). \tag{23}$$

In general, the kinetic energy is quite a complicated expression; however, for the spherically symmetric hedgehog ansatz (16), the integrals reduce to

$$T = J_i a_i - T^L = \frac{1}{2}\Lambda_1 a_i^2 - \frac{3}{4}\Lambda_2 a_i^2 a_j^2, \tag{24}$$

with the momentum conjugate to a_i:

$$J_i = \frac{\partial T^L}{\partial a_i} = \Lambda_1 a_i - \Lambda_2 (a_j)^2 a_i. \tag{25}$$

and

$$\Lambda_1 = \frac{16\pi}{3}\int dr\, r^2 \left[\sin^2 f + \eta \sin^2(f)(f')^2 + \frac{\sin^4 f}{r^2} \right], \tag{26}$$

$$\Lambda_2 = (1-\eta)\frac{64\pi}{15}\int dr\, r^2 \sin^4 f, \tag{27}$$

where we have used the following angular integrals

$$\int d\theta\, d\phi\, \sin\theta\, \hat{x}^i \hat{x}^j = \frac{4\pi}{3}\delta^{ij},$$
$$\int d\theta\, d\phi\, \sin\theta\, \hat{x}^i \hat{x}^j \hat{x}^k \hat{x}^l = \frac{4\pi}{15}(\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}).$$

Every term in Λ_1 is positive definite for $\eta \in [0, 1]$, but this is not the case for η in the full range of static positivity, namely $\eta \in [-1, 1]$. It is expected that Λ_1 takes the minimum value for $\eta = -1$ and the maximum value for $\eta = 1$ (ignoring the fact that the solution f depends on η), whereas the opposite is expected for Λ_2. There are two competing effects at play: $\eta \to 1$ (from below) increases the moment of inertia by increasing Λ_1, but it also decreases Λ_2, which reduces the moment of inertia by the quartic kinetic term, and vice versa for $\eta \to -1$.

The basic strategy for quantizing the isospin rotation, which for spherical symmetry is equal to a spatial rotation of the hedgehog Skyrmion, is to write the kinetic energy (24) in terms of $J_i J_i$, which is the square of the momentum conjugates. If this is possible, canonical quantization applies and we simply replace the operator $(2J)^2$ with $\frac{\partial^2}{\partial A_{ij}^2}$, with A_{ij} being the elements of the isospin rotation matrix A of Eq. (20); the latter is just the Laplacian operator on the 3-sphere [5], which has eigenvalues $\ell(\ell+2)$, with $\ell = 0, 1, 2, 3, \ldots$, $\ell = 2j$ and j is the isospin quantum number. As a result the lowest spin of a fermion $j = \frac{1}{2}$ corresponds to $\ell = 1$ and hence $J^2 = \frac{3}{4}$. Unfortunately, simply squaring Eq. (25) does not allow for the elimination of a_i, so we need to solve for a_i and insert the result into Eq. (24), which hopefully is a function of $J_i J_i$, i.e. the momentum conjugate squared. Although it is difficult to compute a_i in terms of J_i of Eq. (25), the squared quantities are easier to handle:

$$J^2 = \Lambda_1^2 a^2 + \Lambda_2^2 (a^2)^3 - 2\Lambda_1\Lambda_2 (a^2)^2. \tag{28}$$

We can now invert the equation by treating the equation as a classical cubic polynomial in a^2. Before solving the equation, it will prove useful to analyze the polynomial a bit before doing so. Let us write

$$y(x) = \Lambda_2^2 x^3 - 2\Lambda_1\Lambda_2 x^2 + \Lambda_1^2 x - J^2, \qquad x \coloneqq a^2. \tag{29}$$

We find the saddle points straightforwardly as $y'(x) = 0$:

$$x_\pm = \left(\frac{2}{3} \pm \frac{1}{3}\right)\frac{\Lambda_1}{\Lambda_2}, \tag{30}$$

which are both positive. The values of the polynomial function $y(x_\pm)$ at the saddle points determine the number and kind of roots the polynomial possesses. In particular, we get

$$y(x_-) = \frac{4\Lambda_1^3}{27\Lambda_2} - J^2, \qquad y(x_+) = -J^2. \tag{31}$$

The first (in the x-direction) saddle point value can change sign depending on the values of the integrals Λ_1 and Λ_2. Since Λ_1 is an increasing function with η and Λ_2 is a decreasing function with η, the ratio Λ_1^3/Λ_2 is an increasing function with η, that diverges at $\eta = 1$. This means that for large $\eta \sim 1$, $y(x_-)$ is positive and $y(x_+)$ is negative. In this case, there are 3 real roots of y, which is physically puzzling. It turns out that the 3 roots exists in most of the model's parameter space, but only 1 of the roots gives a positive spin contribution to the energy—this root connects to the Skyrme model limit and we will denote it the physical root [40].

In order to reach the form for which Cardano's result applies, we shift the variable as $x = \xi + \frac{2}{3}\ell$, for which the polynomial reads

$$\frac{y}{\Lambda_2^2} = \xi^3 - \frac{1}{3}\ell^2\xi + \frac{2}{27}(\ell^3 - \varsigma^2), \tag{32}$$

where we have defined

$$\ell = \frac{\Lambda_1}{\Lambda_2}, \qquad \varsigma = \sqrt{\frac{27}{2}\frac{J}{\Lambda_2}}, \tag{33}$$

for which we have three roots of y (assuming that $2\ell^3 > \varsigma^2$) as

$$\xi_{0,\pm} = \frac{2\ell}{3}\cos\left(\frac{\theta - \pi \pm 2\pi}{3}\right), \qquad \theta = \arccos\left(1 - \frac{\varsigma^2}{\ell^3}\right), \tag{34}$$

where the index 0 means that $\pm 2\pi$ in the cosine is replaced by 0 and

$$x_{0,\pm} = \xi_{0,\pm} + \frac{2}{3}\ell, \tag{35}$$

with x_- being the smallest positive root.

Writing the kinetic quantum energy in terms of J^2, we get

$$T = \frac{1}{2}\Lambda_1 x_a - \frac{3}{4}\Lambda_2 x_a^2, \tag{36}$$

with x_a, $a = 0, \pm$ being one of the roots. Near the Skyrme model limit (η close to unity), it is safe to assume that $\varsigma^2 \ll \ell^3$, since Λ_1 is maximal and Λ_2 tends to zero. In this case, we can expand in $1/\ell$, obtaining

$$T(x_-) = \frac{J^2}{2\Lambda_1} + \frac{\Lambda_2 J^4}{4\Lambda_1^4} + \frac{\Lambda_2^2 J^6}{2\Lambda_1^7} + \frac{3\Lambda_2^3 J^8}{2\Lambda_1^{10}} + \mathcal{O}\left(\frac{J^{10}\Lambda_2^4}{\Lambda_1^{13}}\right), \tag{37}$$

$$T(x_{0,+}) = -\frac{\Lambda_1^2}{4\Lambda_2} \mp \sqrt{\frac{\Lambda_1}{\Lambda_2}}|J| - \frac{J^2}{4\Lambda_1} \pm \frac{\Lambda_2^{1/2}|J|^3}{8\Lambda_1^{5/2}} - \frac{\Lambda_2 J^4}{8\Lambda_1^4} + \mathcal{O}\left(\frac{|J|^5\Lambda_2^{3/2}}{\Lambda_1^{11/2}}\right), \tag{38}$$

yielding the order-Λ_2 correction to the usual Skyrme model's spin energy for the case of the smallest (physical) root, $T(x_-)$, plus higher-order corrections. Notice that $T(x_-)$ only increases once a nonvanishing Λ_2 is introduced. The only possibility of lowering the spin correction to the mass, would be if the change in η would increase Λ_1 much faster than it would increase Λ_2.

The term $-\Lambda_1^2/4\Lambda_2$ in $T(x_{0,+})$ signals that the two cases of x_0 and x_+ are unphysical; first, these roots do not connect to the Skyrme model limit, and second, they give a negative contribution to the energy (that diverges in the Skyrme model limit). This fact and also the N_c counting leads one to discard the two roots x_0 and x_+, see Ref. [40].

Returning to the physical root, x_-, it will prove convenient for numerical computations to rewrite the spin contribution to the energy as

$$T = \frac{\Lambda_1^2}{3\Lambda_2}\cos\frac{\theta}{3}\left(1 - \cos\frac{\theta}{3}\right), \tag{39}$$

with θ given by Eq. (34), which can also be written as [40]

$$\theta = \arccos\left(1 - \frac{\varsigma^2}{\ell^3}\right) = \arccos\left(1 - \frac{27\Lambda_2 J^2}{2\Lambda_1^3}\right). \tag{40}$$

The spin contribution (39) is positive definite and given in Ref. [40], but it is more difficult to see whether turning on

Λ_2 increases (or decreases) the spin contribution, which however can easily be seen from Eq. (37).

In order to restore the units, we have to substitute $\Lambda_1 \to \mu\lambda^2\Lambda_1$ and $\Lambda_2 \to \mu\lambda^4\Lambda_2$. Inserting these into the spin contribution (39), we obtain

$$T = \frac{\mu\Lambda_1^2}{3\Lambda_2}\cos\frac{\theta}{3}\left(1 - \cos\frac{\theta}{3}\right),$$
$$\theta = \arccos\left(1 - \frac{27\hbar^2\Lambda_2 J^2}{2\Lambda_1^3}\right), \tag{41}$$

where we have set $\hbar \coloneqq 1/(\mu\lambda) = 2\alpha^{-2}$ and we have used the energy and length units of Eq. (11). The spin contribution is thus directly proportional to the energy unit, as one would expect and we can also see that the angle θ depends on $\hbar$ or rather the model parameter α, which needs to be calibrated.

A. Calibration

In order to compare the quantum spin energy (41) to the classical mass of the Skyrmion μV, we must calibrate the model. The overall scale is simply μ [of Eq. (11)] for both energies, but their ratio depends also on α, as can be seen from Eq. (41) where $\hbar = 2\alpha^{-2}$. In order to fix α, we need to perform a calibration that entails fitting one energy quantity and one length scale, which we will do next.

We choose to fit the nucleon mass (M_N) to the classical Skyrmion mass V and the size of the Skyrmion R to the electric charge radius of the nucleon (R_N), so we obtain the following equations

$$\mu V = M_N, \qquad \lambda R = R_N, \tag{42}$$

where V is given by Eq. (14), the radius is computed as a weighted integral

$$R = \sqrt{\frac{-\frac{1}{24\pi^2}\int d^3x |x|^2 \epsilon^{ijk}\mathrm{tr}(R_i R_j R_k)}{B}}, \tag{43}$$

and the energy scale μ and the length scale λ are given in Eq. (11). We use the experimental values 939 MeV and 0.8783 fm for the nucleon mass and radius, respectively. Solving for the pion decay constant and α, we have

$$F_\pi = \frac{2R}{R_N}\alpha, \qquad \alpha = \sqrt{\frac{2M_N R_N}{VR}}. \tag{44}$$

We choose to fit to the classical Skyrmion mass, since including the spin correction is complicated by the fact that the nontrivial and nonlinear α dependence makes an analytic formula unavailable. If one were to choose to fit the total Skyrmion mass to the physical mass, only numerical methods would be able to do so. Since we know that there are further corrections to the Skyrmion energy, e.g., from vibrational modes [53], we simplify the problem and fit just to the classical Skyrmion mass here.

B. Numerical results

We solve the equation of motion (19) for the static Skyrmion, and use it to compute the two moments of inertia, $\Lambda_{1,2}$ of Eqs. (26) and (27). We use simple gradient flow as the numerical method. Then we use Eq. (44) to calibrate the model, which in turn determines the energy and length scales (11). With these at hand, and using the appropriate spin of the nucleon $j = 1/2$ yielding $J^2 = 3/4$, we can compute the spin correction to the energy using Eq. (41) and other observables of the model.

In Fig. 1, the moments of inertia $\Lambda_{1,2}$ are shown. $\Lambda_1/\Lambda_2 > 1$ holds for $\eta \in (-1, 1]$, which is necessary for the expansion in $1/\ell$ to be valid, but the ratio becomes ill defined in the limit $\eta \to -1$. In this limit, the stability of the Skyrmion also becomes questionable.

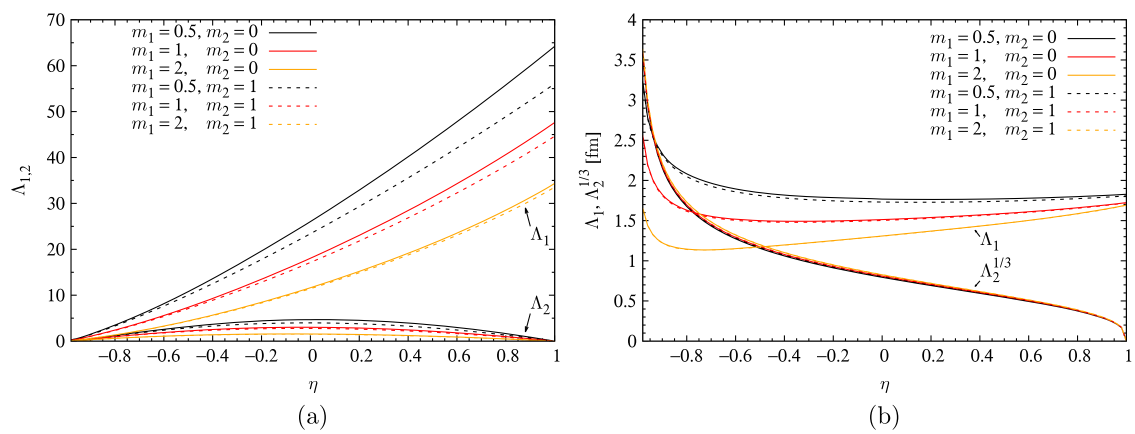

FIG. 1. (a) The moments of inertia Λ_1, Λ_2 as functions of η for a range of the pion mass parameter $m_1 = 0.5, 1, 2$ with and without the loosely bound potential turned on $m_2 = 0, 1$. (b) The moments of inertia after restoring physical units: $\mu\lambda^2\Lambda_1$ and $(\mu\lambda^4\Lambda_2)^{1/3}$.

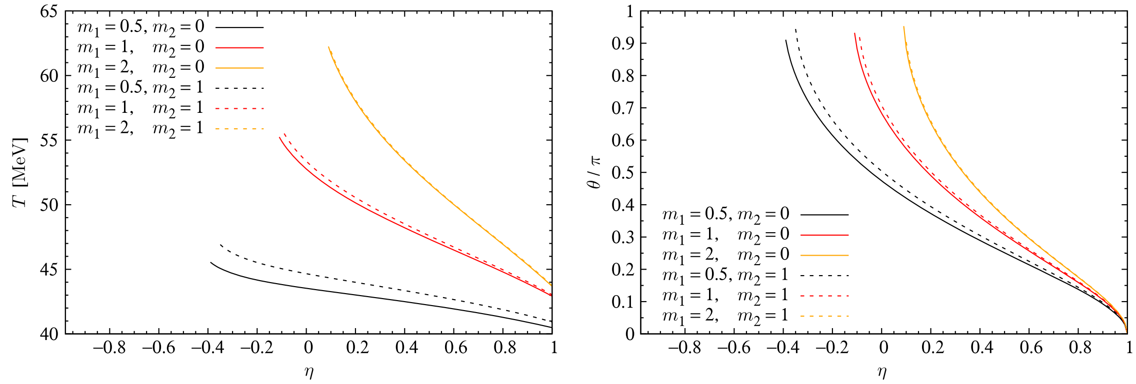

FIG. 2. (a) The physical spin correction to the energy T and (b) the angle θ of Eq. (34), both as functions of η for a range of the pion mass parameter $m_1 = 0.5, 1, 2$ with and without the loosely bound potential turned on $m_2 = 0, 1$.

In Fig. 2(a) we show the spin correction to the energy as a function of η for various pion mass and loosely bound potential parameters, whereas in Fig. 2(b) is shown the corresponding angles θ of Eq. (34). We notice that the angle θ tends to π before η reaches $\eta \to -1$ and hence the physical root becomes complex. There is still a real root of Eq. (32), but it is not connected to the physical root and it gives rise to a very large and negative spin contribution (it is one of the unphysical roots). This is also consistent with the results of Ref. [40].

The fit (44) employed here amounts to the classical mass of the Skyrmion always being at the experimental face value and so the spin correction should be as small as possible. Re-calibration could of course get the nucleon mass right, but as discussed in the introduction, the larger the spin energy is, the larger the binding energies are. Since they should physically be around 8 MeV per nucleon for larger nuclei, a spin correction to the energy much larger than that creates tension and warrants other (extended) quantization methods to provide physical spectra, see, e.g., Ref. [20]. We note that the dependence of the spin correction on the pion mass parameter m_1 is quite large, whereas the dependence on the loosely bound potential parameter m_2 is only mild.

Figure 3 shows the static classical mass and radius of the Skyrmion in dimensionless (model) units. We plot the figures only in the range $\eta \in [-0.98, 1]$, since we are unable to obtain trustable solutions in the limit $\eta \to -1$, where the Skyrmion size is also seen to shrink to zero [Fig. 3(b)], which is also anticipated in appendices A 1 and B 1. In the latter appendix, we also note that the topological energy bound goes to zero in the $\eta \to -1$ limit.

In physical units, the classical mass and radius are exactly equal to their experimental values, 939 MeV and 0.8783 fm, respectively, due to the calibration of the model (44).

Having the classical mass and radius of the Skyrmion in hand, we thus readily calibrate the model according to Eq. (44), yielding the pion decay constant in Fig. 4(a), the four-derivative term coupling constant α in Fig. 4(b) or equivalently the energy scale μ in Fig. 5(a) and the length scale λ in Fig. 5(b). For completeness, we plot the pion

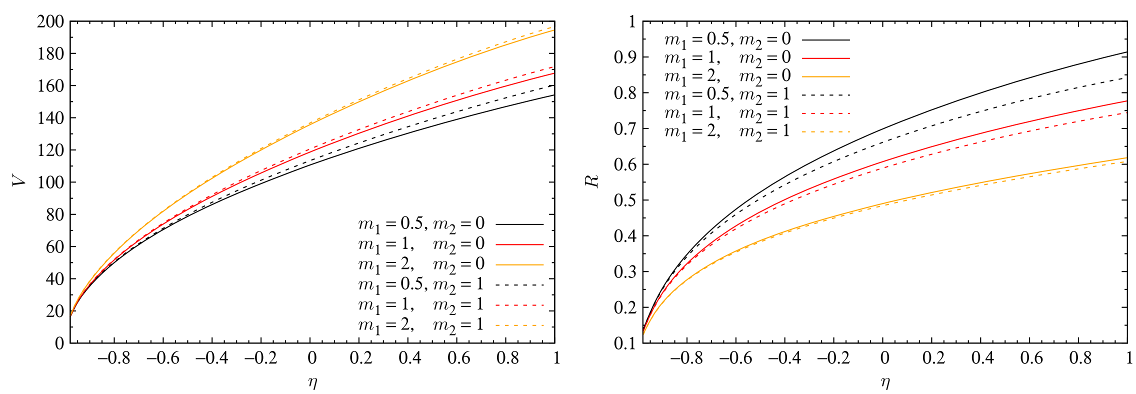

FIG. 3. (a) The static energy of the Skyrmion in Skyrme units and (b) the (charge) radius of the Skyrmion, both as functions of η for a range of the pion mass parameter $m_1 = 0.5, 1, 2$ with and without the loosely bound potential turned on $m_2 = 0, 1$.

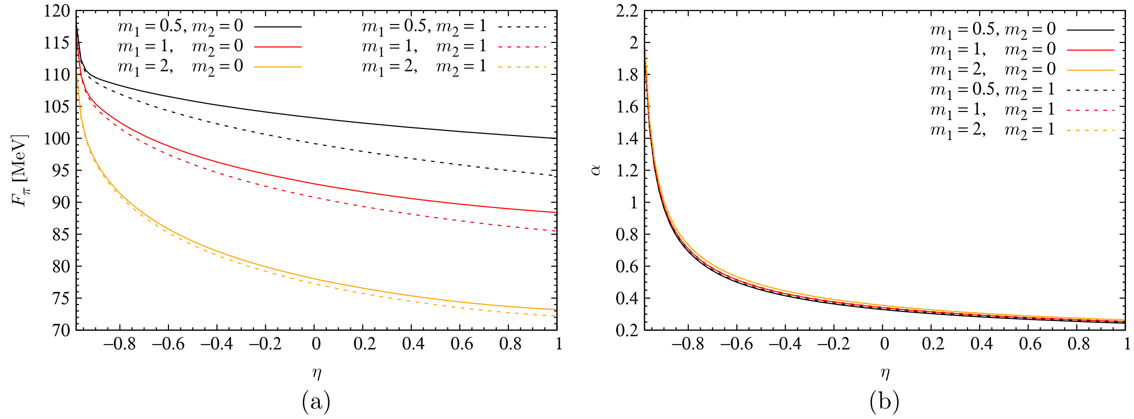

FIG. 4. (a) The pion decay constant and (b) the four-derivative coupling constant α, both as functions of η for a range of the pion mass parameter $m_1 = 0.5, 1, 2$ with and without the loosely bound potential turned on $m_2 = 0, 1$. The physical value of the pion decay constant in the normalization used in this paper is 186 MeV.

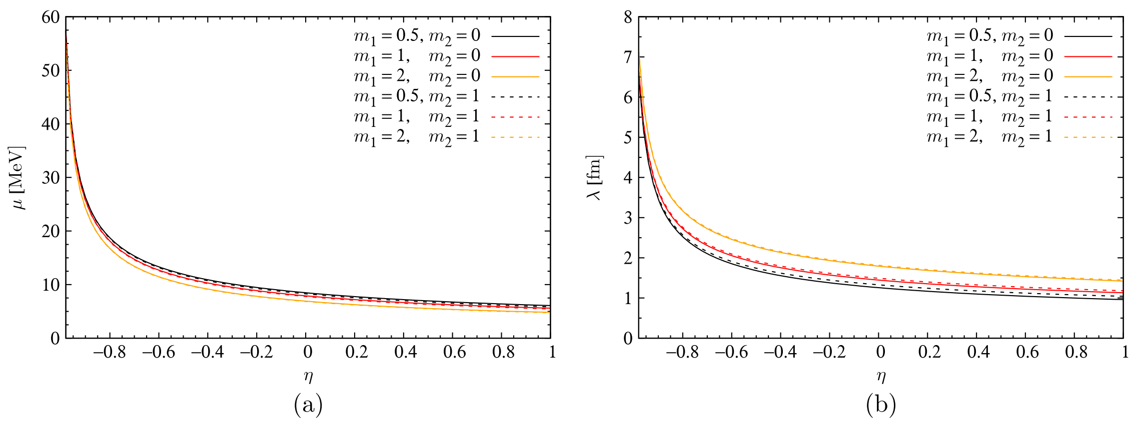

FIG. 5. The calibrated (a) energy scale and (b) length scale of the model, both as functions of η for a range of the pion mass parameter $m_1 = 0.5, 1, 2$ with and without the loosely bound potential turned on $m_2 = 0, 1$.

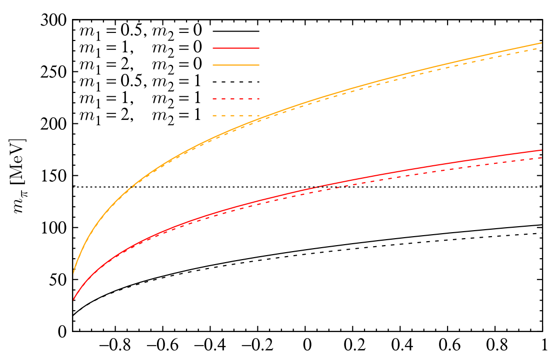

FIG. 6. The pion mass as a function of η for a range of the pion mass parameter $m_1 = 0.5, 1, 2$ with and without the loosely bound potential turned on $m_2 = 0, 1$. The physical value of the pion mass is around 139 MeV (ignoring isospin breaking effects) and is shown with a dotted horizontal black line.

mass in Fig. 6, from which we can see that the pion mass parameter should be taken somewhere between 0.75 and 1, depending on the values of η in its physical regime (see Fig. 2) and m_2, in order to reproduce the experimental value of roughly 139 MeV.

III. HIGHER-ORDER SKYRME MODELS

We now consider the cases of the higher-order models introduced in Ref. [54], i.e., higher-order derivative theories with four time derivatives and with eight to twelve derivatives in total. The Lagrangian reads

$$\mathcal{L} = \frac{F_\pi^2}{16}\mathrm{tr}(R_\mu R^\mu) + \frac{1}{32e^2}\mathrm{tr}([R_\mu, R_\nu][R^\mu, R^\nu]) + \mathcal{L}' - \frac{F_\pi^2 m_\pi^2}{8}\mathrm{tr}(\mathbf{1}_2 - U) - \frac{F_\pi^2 M^2}{32}[\mathrm{tr}(\mathbf{1}_2 - U)]^2, \qquad (45)$$

with the new higher-derivative term $\mathcal{L}'$ being one of the four possibilities:

$$\mathcal{L}_{8a} = -\frac{\beta_8}{1024F_\pi^4}(\mathrm{tr}([R_\mu, R_\nu][R^\mu, R^\nu]))^2, \quad (46)$$

$$\mathcal{L}_{8b} = -\frac{\beta_8}{768F_\pi^4}\mathrm{tr}(R_\sigma R^\sigma)\mathrm{tr}([R_\mu, R^\nu][R_\nu, R^\rho][R_\rho, R^\mu]), \quad (47)$$

$$\mathcal{L}_{10} = -\frac{\beta_{10}}{3072F_\pi^6}\mathrm{tr}([R_\sigma, R_\delta][R^\sigma, R^\delta]) \times \mathrm{tr}([R_\mu, R^\nu][R_\nu, R^\rho][R_\rho, R^\mu]), \quad (48)$$

$$\mathcal{L}_{12} = -\frac{\beta_{12}}{9216F_\pi^8}(\mathrm{tr}([R_\mu, R^\nu][R_\nu, R^\rho][R_\rho, R^\mu]))^2. \quad (49)$$

We first define the relations to the coupling η:

$$\frac{1}{e^2} = \alpha^2\eta, \qquad \beta_n = (\sqrt{2}\alpha)^{n-2}(1-\eta), \quad (50)$$

with $\eta \in [0, 1]$. For the analysis of the positivity of the static energy leading to the viable range of the parameter η, see Appendix A 2. Rescaling the Lagrangians to the energy and length units (11) yields the dimensionless Lagrangian

$$\mathcal{L} = \frac{1}{2}\mathrm{tr}(R_\mu R^\mu) + \frac{\eta}{16}\mathrm{tr}([R_\mu, R_\nu][R^\mu, R^\nu]) + \mathcal{L}' - m_1^2\mathrm{tr}(\mathbf{1}_2 - U) - \frac{m_2^2}{4}[\mathrm{tr}(\mathbf{1}_2 - U)]^2, \quad (51)$$

with the dimensionless higher-order term $\mathcal{L}'$ being one of the following four terms:

$$\mathcal{L}_{8a} = -\frac{1-\eta}{1024}(\mathrm{tr}([R_\mu, R_\nu][R^\mu, R^\nu]))^2, \quad (52)$$

$$\mathcal{L}_{8b} = -\frac{1-\eta}{768}\mathrm{tr}(R_\sigma R^\sigma)\mathrm{tr}([R_\mu, R^\nu][R_\nu, R^\rho][R_\rho, R^\mu]), \quad (53)$$

$$\mathcal{L}_{10} = -\frac{1-\eta}{6144}\mathrm{tr}([R_\sigma, R_\delta][R^\sigma, R^\delta])\mathrm{tr}([R_\mu, R^\nu][R_\nu, R^\rho][R_\rho, R^\mu]), \quad (54)$$

$$\mathcal{L}_{12} = -\frac{1-\eta}{36864}(\mathrm{tr}([R_\mu, R^\nu][R_\nu, R^\rho][R_\rho, R^\mu]))^2. \quad (55)$$

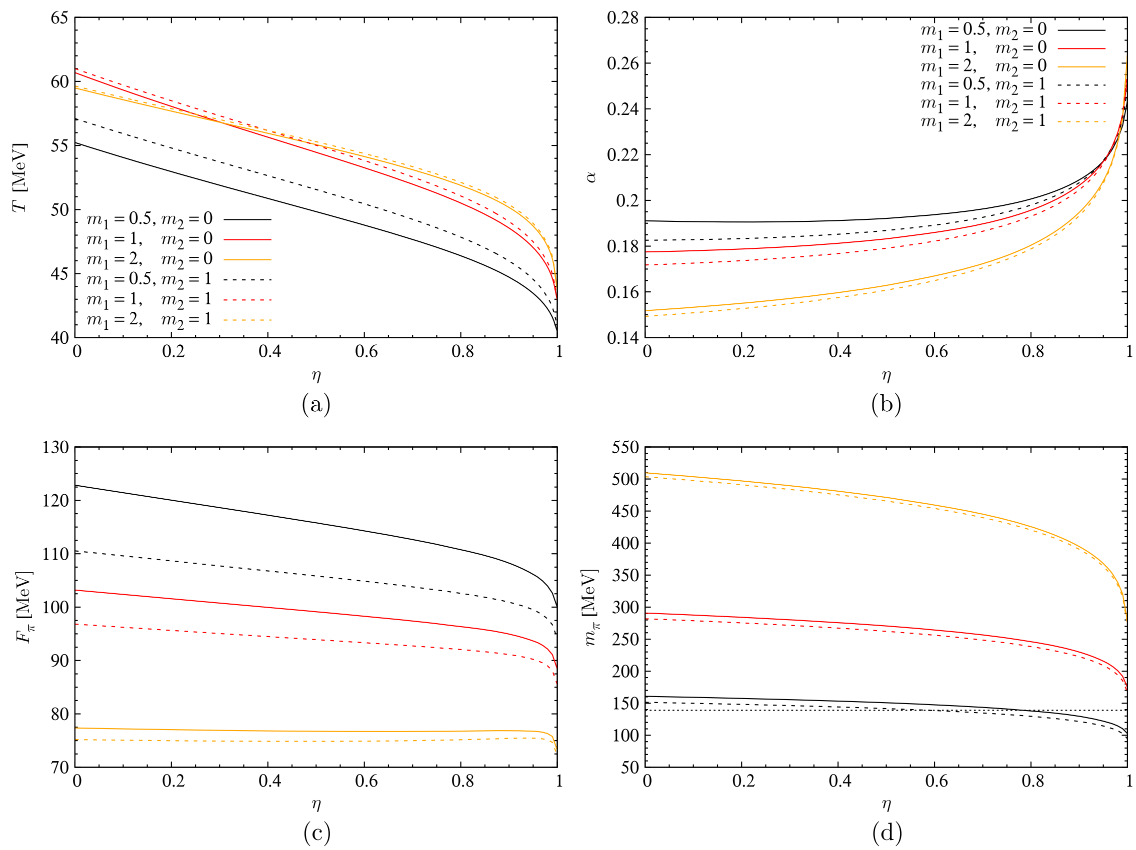

FIG. 7. The (a) spin correction to the energy, (b) the dimensional coupling constant of the Skyrme and higher-order terms, (c) the pion decay constant and (d) the pion mass, all as functions of $\eta \in [0, 1]$ which interpolates between the Skyrme model ($\eta = 1$) and the higher-order model ($\eta = 0$) $\mathcal{L}_{8a}$.

Splitting the Lagrangians up into potential and kinetic terms, we get

$$L = T^L - V, \tag{56}$$

$$V = \int \mathrm{d}^3x \left[-\frac{1}{2}\mathrm{tr}(R_i^2) - \frac{\eta}{16}\mathrm{tr}([R_i, R_j]^2) + m_1^2 \mathrm{tr}(\mathbf{1}_2 - U) + \frac{m_2^2}{4}[\mathrm{tr}(\mathbf{1}_2 - U)]^2 \right] + V', \tag{57}$$

$$T^L = \int \mathrm{d}^3x \left[-\frac{1}{2}\mathrm{tr}(T_i T_j) - \frac{\eta}{8}\mathrm{tr}([T_i, R_k][T_j, R_k]) \right] a_i a_j + T^{L\prime}, \tag{58}$$

with potential terms

$$V_{8a} = \frac{1-\eta}{1024}\int \mathrm{d}^3x (\mathrm{tr}([R_i, R_j]^2))^2, \tag{59}$$

$$V_{8b} = \frac{1-\eta}{768}\int \mathrm{d}^3x\, \mathrm{tr}(R_l^2)\mathrm{tr}([R_i, R_j][R_j, R_k][R_k, R_i]), \tag{60}$$

$$V_{10} = \frac{1-\eta}{6144}\int \mathrm{d}^3x\, \mathrm{tr}([R_k, R_l]^2)\mathrm{tr}([R_i, R_j][R_j, R_k][R_k, R_i]), \tag{61}$$

$$V_{12} = \frac{1-\eta}{36864}\int \mathrm{d}^3x (\mathrm{tr}([R_i, R_j][R_j, R_k][R_k, R_i]))^2, \tag{62}$$

and kinetic terms

$$T_{8a}^L = \frac{1-\eta}{256}\int \mathrm{d}^3x\, \mathrm{tr}([T_i, R_k][T_j, R_k])\mathrm{tr}([R_l, R_m]^2) a_i a_j - \frac{1-\eta}{256}\int \mathrm{d}^3x\, \mathrm{tr}([T_i, R_m][T_j, R_m])\mathrm{tr}([T_k, R_n][T_l, R_n]) \times a_i a_j a_k a_l, \tag{63}$$

$$T_{8b}^L = \frac{1-\eta}{768}\int \mathrm{d}^3x [\mathrm{tr}(T_i T_j)\mathrm{tr}([R_k, R_l][R_l, R_m][R_m, R_k]) + 3\mathrm{tr}(R_m^2)\mathrm{tr}([T_i, R_k][R_k, R_l][R_l, T_j])] a_i a_j - \frac{1-\eta}{256}\int \mathrm{d}^3x\, \mathrm{tr}(T_i T_j)\mathrm{tr}([T_k, R_m][R_m, R_n][R_n, T_l]) \times a_i a_j a_k a_l, \tag{64}$$

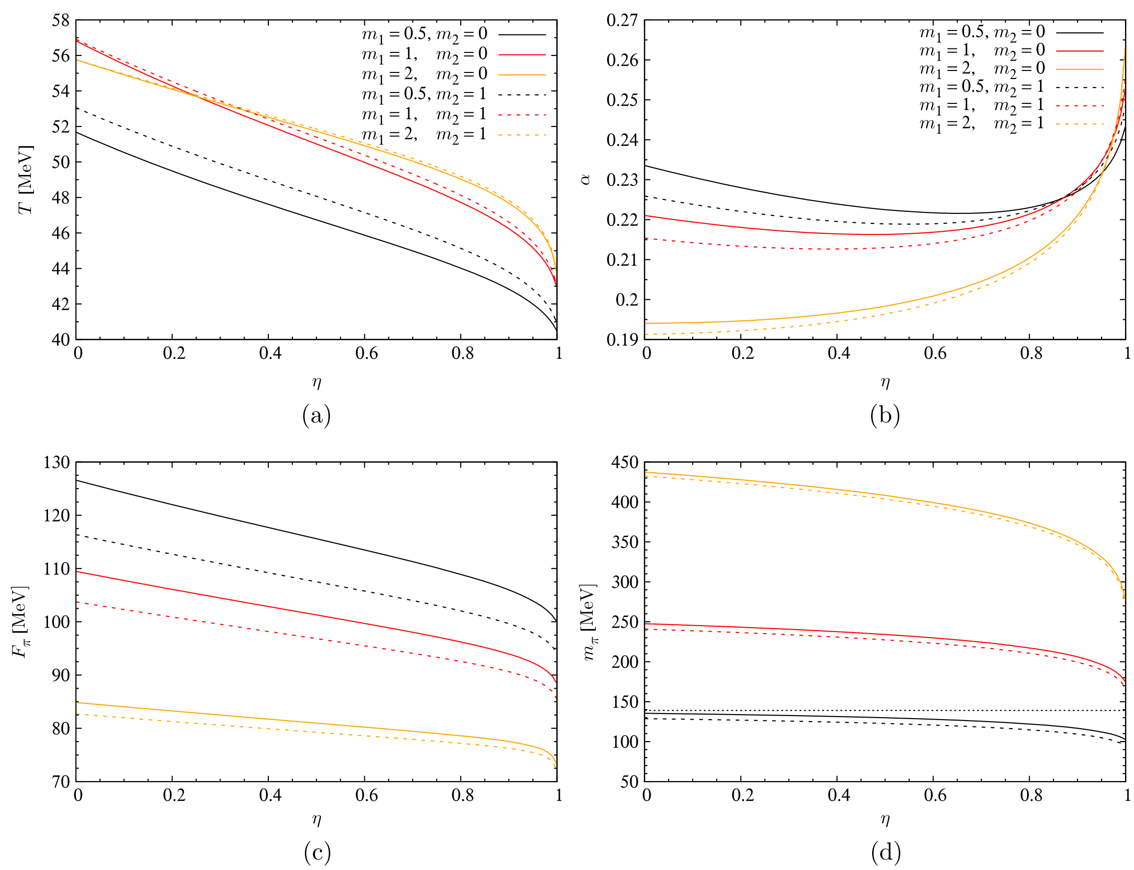

FIG. 8. The (a) spin correction to the energy, (b) the dimensional coupling constant of the Skyrme and higher-order terms, (c) the pion decay constant and (d) the pion mass, all as functions of $\eta \in [0, 1]$ which interpolates between the Skyrme model ($\eta = 1$) and the higher-order model ($\eta = 0$) $\mathcal{L}_{8b}$.

$$T_{10}^L = \frac{1-\eta}{6144}\int \mathrm{d}^3x[2\mathrm{tr}([T_i,R_k][T_j,R_k]) \times \mathrm{tr}([R_l,R_m][R_m,R_n][R_n,R_l]) + 3\mathrm{tr}([R_m,R_n]^2)\mathrm{tr}([T_i,R_k][R_k,R_l][R_l,T_j])]a_ia_j - \frac{1-\eta}{1024}\int \mathrm{d}^3x\,\mathrm{tr}([T_i,R_o][T_j,R_o]) \times \mathrm{tr}([T_k,R_m][R_m,R_n][R_n,T_l])a_ia_ja_ka_l, \quad (65)$$

$$T_{12}^L = \frac{1-\eta}{6144}\int \mathrm{d}^3x\,\mathrm{tr}([T_i,R_k][R_k,R_l][R_l,T_j]) \times \mathrm{tr}([R_m,R_n][R_o,R_p][R_p,R_m])a_ia_j - \frac{1-\eta}{4096}\int \mathrm{d}^3x\,\mathrm{tr}([T_i,R_m][R_m,R_n][R_n,T_j]) \times \mathrm{tr}([T_k,R_o][R_o,R_p][R_p,T_l])a_ia_ja_ka_l, \quad (66)$$

and $T_i = \frac{\mathrm{i}}{2}[\tau^i,U]U^\dagger$ as always. Inserting the hedgehog ansatz (16), we obtain the potential term

$$V = \int \mathrm{d}^3x\left[(f')^2 + \frac{2\sin^2 f}{r^2} + \eta\frac{\sin^2 f}{r^4}(\sin^2 f + 2r^2(f')^2) + 2m_1^2(1-\cos f) + m_2^2(1-\cos f)^2\right] + V' \quad (67)$$

with

$$V_{8a} = \frac{1-\eta}{4}\int \mathrm{d}^3x\frac{\sin^4 f}{r^8}(\sin^2 f + 2r^2(f')^2)^2, \quad (68)$$

$$V_{8b} = \frac{1-\eta}{4}\int \mathrm{d}^3x\frac{\sin^4(f)(f')^2}{r^6}(2\sin^2 f + r^2(f')^2), \quad (69)$$

$$V_{10} = \frac{1-\eta}{4}\int \mathrm{d}^3x\frac{\sin^6(f)(f')^2}{r^8}(\sin^2 f + 2r^2(f')^2), \quad (70)$$

$$V_{12} = \frac{1-\eta}{4}\int \mathrm{d}^3x\frac{\sin^8(f)(f')^4}{r^8}, \quad (71)$$

whereas for the kinetic term we get

$$T^L = \frac{1}{2}\Lambda_1 a_i^2 - \frac{1}{4}\Lambda_2' a_i^2 a_j^2, \quad (72)$$

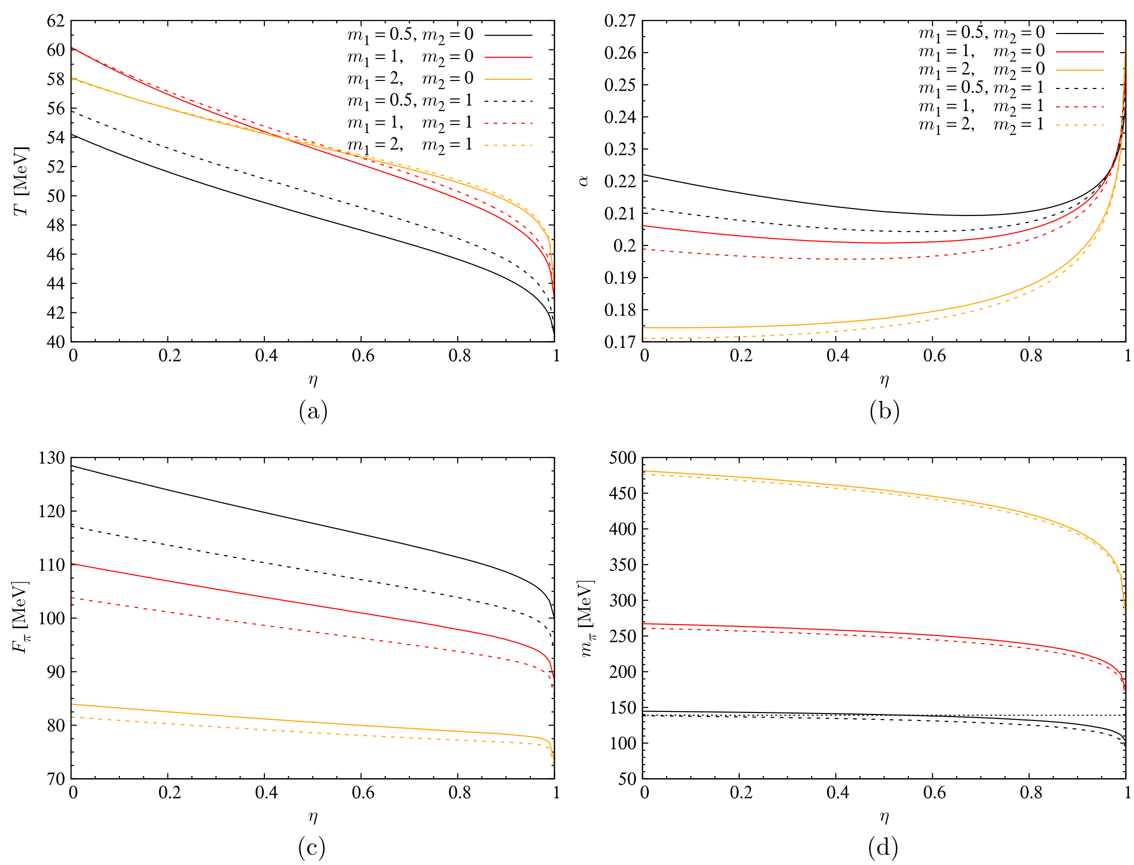

FIG. 9. The (a) spin correction to the energy, (b) the dimensional coupling constant of the Skyrme and higher-order terms, (c) the pion decay constant and (d) the pion mass, all as functions of $\eta \in [0,1]$ which interpolates between the Skyrme model ($\eta = 1$) and the higher-order model ($\eta = 0$) $\mathcal{L}_{10}$.

with

$$\Lambda_1 = \frac{16\pi}{3}\int \mathrm{d}r\, r^2\left[\sin^2 f + \eta\sin^2 f\left(\frac{\sin^2 f}{r^2} + (f')^2\right)\right] + \Lambda_1', \tag{73}$$

and

$$\Lambda_1^{(8a)} = \frac{8\pi(1-\eta)}{3}\int \mathrm{d}r\frac{\sin^4 f}{r^4}(\sin^2 f + r^2(f')^2) \times (\sin^2 f + 2r^2(f')^2), \tag{74}$$

$$\Lambda_1^{(8b)} = \frac{4\pi(1-\eta)}{3}\int \mathrm{d}r\frac{\sin^4(f)(f')^2}{r^2}(3\sin^2 f + r^2(f')^2), \tag{75}$$

$$\Lambda_1^{(10)} = \frac{4\pi(1-\eta)}{3}\int \mathrm{d}r\frac{\sin^6(f)(f')^2}{r^4}(2\sin^2 f + 3r^2(f')^2), \tag{76}$$

$$\Lambda_1^{(10)} = \frac{8\pi(1-\eta)}{3}\int \mathrm{d}r\frac{\sin^8(f)(f')^4}{r^4}, \tag{77}$$

as well as

$$\Lambda_2^{(8a)} = \frac{32\pi(1-\eta)}{15}\int \mathrm{d}r\frac{\sin^4 f}{r^2}(\sin^2 f + r^2(f')^2)^2, \tag{78}$$

$$\Lambda_2^{(8b)} = \frac{32\pi(1-\eta)}{15}\int \mathrm{d}r\sin^6(f)(f')^2, \tag{79}$$

$$\Lambda_2^{(10)} = \frac{32\pi(1-\eta)}{15}\int \mathrm{d}r\frac{\sin^6(f)(f')^2}{r^2}(\sin^2 f + r^2(f')^2), \tag{80}$$

$$\Lambda_2^{(12)} = \frac{32\pi(1-\eta)}{15}\int \mathrm{d}r\frac{\sin^8(f)(f')^4}{r^2}. \tag{81}$$

The quantum kinetic energy is then given by Eq. (24) and using the $\Lambda_{1,2}$ of the higher-order model of this section, we can readily use the result for the spin correction to the energy (41). This result follows through because both the PR model and the higher-order models have exactly 4 time derivatives. We will again calibrate the model in the same way using Eq. (44), but with the energy given by the potential (57) and the size computed from the solutions to the higher-order model, see Sec. II A.

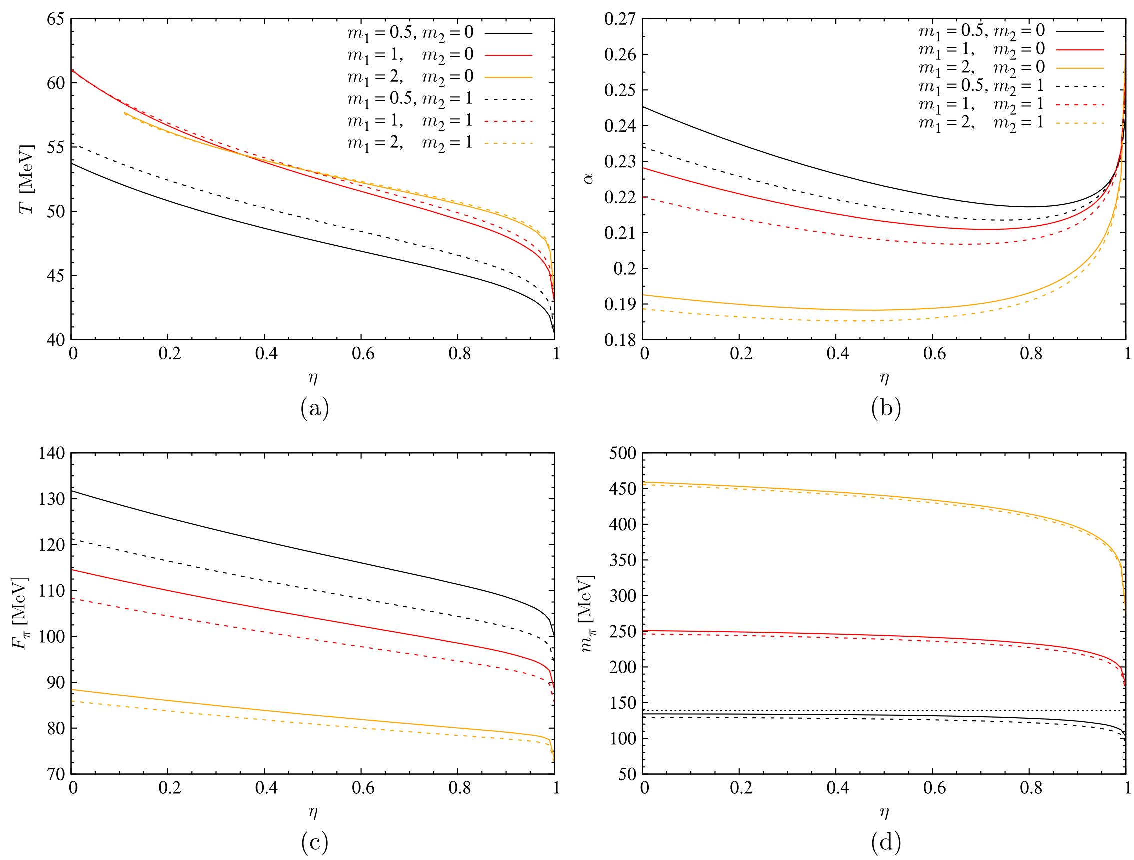

FIG. 10. The (a) spin correction to the energy, (b) the dimensional coupling constant of the Skyrme and higher-order terms, (c) the pion decay constant and (d) the pion mass, all as functions of $\eta \in [0, 1]$ which interpolates between the Skyrme model ($\eta = 1$) and the higher-order model ($\eta = 0$) $\mathcal{L}_{12}$.

We are now ready to present the numerical results for the higher-order Skyrme models (52)–(55), which are shown in Figs. 7–10. In the figures, the spin corrections to the energy is shown in panels (a), the dimensionless coupling constant α in panels (b), the pion decay constant in panels (c) and finally, the pion mass in panels (d). It is seen also for all the higher-order models, that the spin correction to the energy is smallest in the Skyrme model limit, i.e., for $\eta = 1$. We also notice that all the models have viable (calculable) spin contributions to the energy in the entire parameter space, i.e. $\eta \in [0, 1]$, except in the case of the 12th-order model (55), where the spin contribution to the energy disappears for $\eta \lesssim 0.12$ for $m_1 = 2$ (very large pion mass), since the root in Eq. (32) turns complex and hence unphysical. This should not be too worrisome, since m_1 so large in this model gives a physical pion mass above 450 MeV.

IV. DISCUSSION AND OUTLOOK

In this paper, we have studied the Skyrme model with the addition of the other possible fourth-order derivative term—the kinetic term squared, which however gives rise to more than two time derivatives in the model; this model has been considered previously in the literature (e.g., [36–39]) and studied in detail by Pottinger and Rathske [40], who performed exact collective coordinate quantization using Cardano's formula.

We parametrized the model with a parameter η that interpolates between the Skyrme model ($\eta = 1$), the pure kinetic term squared model ($\eta = 1/2$) and a new model with a negative Skyrme term that completely cancels off the kinetic term squared part ($\eta = 0$) and all the way down to a limit where there is no Skyrme term left ($\eta = -1$), but only a term that vanishes for spherically symmetric solitons. Our result is that the spin contribution only increases once a four-time derivative term is taken into account, independently of how many derivatives the term has in total. This statement holds under the condition of Lorentz-invariant terms that have a positive definite static energy. We show this is true by expanding the spin contribution to the energy in a power series (37), where every term is positive. We further illustrate that this holds for any kind of theory, by computing the spin contribution to the energy for 4 different higher-order models.

We further establish topological energy bounds for the models under consideration in this paper. In particular, we extend the topological energy bound for the PR model to include two (nonderivative) potentials, being the pion mass term and the pion mass term squared or loosely bound potential and we compute new bounds for the higher-order models. These results are given in Appendix B.

Unfortunately, it turns out that the ambition of being able to reduce the spin contribution to the energy in the class of generalized Skyrme models with four time derivatives, is not possible. Hence, the Skyrme model in the class of theories studied in this paper, is the model with the smallest spin contribution to the energy and hence the model giving rise to the smallest lower bound on the binding energy.

The result can have two opposite implications for future work on achieving physical binding energies. In one direction, one could go the BPS way and try to reduce the classical binding energies as well as the spin correction to the energy as much as possible. This is in the spirit of the assumption that the main contribution to the nucleon energy is the classical Skyrmion energy, and quantum corrections are small. If one chooses to go in another direction of acknowledging that the classical binding energies should not be small, but only the total binding energies must sum up to values that are at the percent level of the mass scale in question, then one could look at this extra degree of freedom of increasing the spin energy, if needed, as a tuning parameter. This latter approach to quantization of Skyrmions is discussed in Ref. [20] for the standard Skyrme model.

An issue with the current model, which we have not solved in this work, is to perform the quantization for Skyrmions that do not enjoy spherical symmetry. This becomes complicated because the moment of inertia tensor will no longer be proportional to the unit matrix; this implies that the cubic equation that we solve becomes a cubic matrix equation, that presumably is harder to solve. One may consider, as a first step, to solve the problem with axial symmetry, for which two eigenvalues of the tensors are equal. Nevertheless, if the complete model of quantization of Skyrmions requires the smallest possible spin contribution, then the Skyrme model is the best option, to the fourth order in the derivative expansion. We will leave for future work, a possible investigation of the spin contribution for nonspherically symmetric inertia tensors in this model or other models with more than two time derivatives. Although more complicated, we believe that the smallest root(s) of the cubic matrix equation can be found with numerical methods, if it is not possible to write down analytic expressions.

Another comment in store about theories with four or more time derivatives, is the problem of the Ostrogradsky instability [55]. In the formulation of Woodard [56], the dynamics of the Hamiltonian is generally described by two or more conjugate momenta, but only if the Lagrangian is nondegenerate in the double time derivative of some field. Luckily, although we have four time derivatives, they each act on their own field, making the theory highly nonlinear but not inducing the Ostrogradsky instability. One may wonder why there is no term like $\mathrm{tr}(U^\dagger \Box U)^2$ in the most general Lagrangian of pions, but as shown in the literature such term can be eliminated by a field redefinition and it will be described at the four-derivative level just by the two terms included in the Lagrangian (5) plus higher-order terms in the derivative expansion [28–32]. The Ostrogradsky instability, which exists for nondegenerate double time derivatives, is due to the fact that the corresponding Hamiltonian will depend only linearly on one of the two conjugate momenta—this makes it possible to drive the theory into larger and larger energies with either sign. The theory thus has no lower or upper bound on the energy.[5] In the case of the Ostrogradsky instability, arguments have been made that it is not an issue for EFTs, since the energy needed to excite a mode that possesses a runaway behavior is larger than the (cutoff) scale of the EFT and hence anyway beyond the validity of the EFT [57]. It has also been argued that in a certain class of asymptotically free theories, the effective mass of the unstable modes becomes infinitely heavy in the UV limit [58]. At some higher order in the derivative expansion, it will no longer be possible to eliminate all d'Alembertian operators from the EFT Lagrangian even using field redefinitions and integration-by-parts relations; at such order the Ostrogradsky instability is inevitable, although it is possible that it will not cause problems for the EFT observables at sufficiently small energies.

Finally, one may consider the possibility of going to a higher order in the derivative expansion. In the literature, the sextic term has been studied extensively [9,10,59–62], which however, like the Skyrme term, contains only 2 time derivatives. A natural generalization of the PR model would be to consider models with 6 or more time derivatives, which however would give rise to a higher-order polynomial equation than the cubic equation (32). In particular, in the case of theories with 6 time derivatives, the corresponding order of the polynomial equation is of 5th order. For a theory with $2n$ time derivatives, the corresponding polynomial equation for the squared spin operator would then be of order $2n-1$. In such cases, it is probably necessary to turn to numerical methods for finding the (smallest/physical) roots. We leave such problems for future work.

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[5]This runaway of the energy is different from the dynamical instability of the PR model. That is, in the PR model the instability comes from a negative contribution to the energy from the four time derivatives, whereas the Ostrogradsky instability has a linear dependence on one of the conjugate momenta that can classically cause a runaway at any values of the kinetic energy.

APPENDIX A: POSITIVITY OF STATIC ENERGY

1. The Pottinger-Rathske model

In order to prove the entire suitable range of the couplings while retaining a positive definite static energy, we rewrite the derivative part of the static part of the Lagrangian (14) in terms of the 4-vector field $\mathbf{n}=(n^0,n^1,n^2,n^3)$:

$$U=n^0\mathbf{1}_2+\mathrm{i}\tau^a n^a,\quad a=1,2,3,\tag{A1}$$

yielding

$$\begin{aligned}\mathcal{E}&=(\partial_i\mathbf{n}\cdot\partial_i\mathbf{n})+\frac{2\eta-1}{2}(\partial_i\mathbf{n}\cdot\partial_i\mathbf{n})^2-\frac{2\eta-1}{2}(\partial_i\mathbf{n}\cdot\partial_j\mathbf{n})^2\\&\quad+\frac{1-\eta}{2}(\partial_i\mathbf{n}\cdot\partial_i\mathbf{n})^2.\end{aligned}\tag{A2}$$

Using the eigenvalues, $\{\lambda_i\}$, of the strain tensor [63]

$$\begin{aligned}D_{ij}&=-\frac{1}{2}\mathrm{tr}[R_iR_j]=(\partial_i\mathbf{n}\cdot\partial_j\mathbf{n})\\&=\left[V\begin{pmatrix}\lambda_1^2&&\\&\lambda_2^2&\\&&\lambda_3^2\end{pmatrix}V^{\mathrm{T}}\right]_{ij},\qquad V\in\mathrm{SO}(3),\end{aligned}\tag{A3}$$

we obtain

$$\begin{aligned}\mathcal{E}&=(\lambda_1^2+\lambda_2^2+\lambda_3^2)+\eta(\lambda_1^2\lambda_2^2+\lambda_1^2\lambda_3^2+\lambda_2^2\lambda_3^2)\\&\quad+\frac{1-\eta}{2}(\lambda_1^4+\lambda_2^4+\lambda_3^4)\\&=(\lambda_1^2+\lambda_2^2+\lambda_3^2)\\&\quad+\frac{1-\eta}{4}[(\lambda_1^2-\lambda_2^2)^2+(\lambda_1^2-\lambda_3^2)^2+(\lambda_2^2-\lambda_3^2)^2]\\&\quad+\frac{1+\eta}{2}(\lambda_1^2\lambda_2^2+\lambda_1^2\lambda_3^2+\lambda_2^2\lambda_3^2).\end{aligned}\tag{A4}$$

This (derivative part) of the static energy density is thus positive definite for $\eta\in[-1,1]$. We expect, however, that the spherically symmetric Skyrmion, which has $\lambda_1=\lambda_2=\lambda_3$, to be unstable at the point $\eta=-1$.

2. The higher-order models

In the higher-order models, the static energy is given in Eq. (57) with V' of Eq. (59) for the $8a$ term, Eq. (60) for the $8b$ term, Eq. (61) for the 10 term and Eq. (62) for the 12 term. Using the relations in Ref. [54], we can write the derivative part of the static energy density as

$$\begin{aligned}
\mathcal{E}^{248a} &= (\lambda_1^2+\lambda_2^2+\lambda_3^2)+\eta(\lambda_1^2\lambda_2^2+\lambda_1^2\lambda_3^2+\lambda_2^2\lambda_3^2)\\
&\quad+\frac{1-\eta}{4}(\lambda_1^2\lambda_2^2+\lambda_1^2\lambda_3^2+\lambda_2^2\lambda_3^2)^2,\\
\mathcal{E}^{248b} &= (\lambda_1^2+\lambda_2^2+\lambda_3^2)+\eta(\lambda_1^2\lambda_2^2+\lambda_1^2\lambda_3^2+\lambda_2^2\lambda_3^2)\\
&\quad+\frac{1-\eta}{4}(\lambda_1^2+\lambda_2^2+\lambda_3^2)(\lambda_1^2\lambda_2^2\lambda_3^2),\\
\mathcal{E}^{24(10)} &= (\lambda_1^2+\lambda_2^2+\lambda_3^2)+\eta(\lambda_1^2\lambda_2^2+\lambda_1^2\lambda_3^2+\lambda_2^2\lambda_3^2)\\
&\quad+\frac{1-\eta}{4}(\lambda_1^2\lambda_2^2+\lambda_1^2\lambda_3^2+\lambda_2^2\lambda_3^2)(\lambda_1^2\lambda_2^2\lambda_3^2),\\
\mathcal{E}^{24(12)} &= (\lambda_1^2+\lambda_2^2+\lambda_3^2)+\eta(\lambda_1^2\lambda_2^2+\lambda_1^2\lambda_3^2+\lambda_2^2\lambda_3^2)\\
&\quad+\frac{1-\eta}{4}(\lambda_1^4\lambda_2^4\lambda_3^4). \qquad \text{(A5)}
\end{aligned}$$

Positivity of all these static energy functionals can only be guaranteed if $\eta\in[0,1]$.

For the $8a$ case, we can see this by considering a very small Skyrme-energy contribution, for which the last term is negligible. This requires $\eta\geq 0$. For regions with a large Skyrme-energy contribution, the last term is dominant and we need $1-\eta\geq 0$.

For the $8b$, 10, and 12 cases, one can consider regions in space where the baryon density vanishes, but two of the strain tensor eigenvalues do not (i.e., $\lambda_1\neq 0$, $\lambda_2\neq 0$ and $\lambda_3=0$). In this case, the last term vanishes and $\eta\geq 0$ is a necessity, hence in general it is.

APPENDIX B: TOPOLOGICAL ENERGY BOUND

1. The Pottinger-Rathske model

Let us consider the topological energy bound for the derivative terms of the static energy (14). Using the second line of Eq. (A4), we can write the topological bound on the static energy [40]

$$\begin{aligned}
E &= \int \mathrm{d}^3x\left[(\lambda_1^2+\lambda_2^2+\lambda_3^2)+\frac{1-\eta}{4}\left[(\lambda_1^2-\lambda_2^2)^2+(\lambda_1^2-\lambda_3^2)^2+(\lambda_2^2-\lambda_3^2)^2\right]+\frac{1+\eta}{2}(\lambda_1^2\lambda_2^2+\lambda_1^2\lambda_3^2+\lambda_2^2\lambda_3^2)\right]\\
&= \int \mathrm{d}^3x\left[\left(\lambda_1\mp\sqrt{\frac{1+\eta}{2}}\lambda_2\lambda_3\right)^2+\left(\lambda_2\mp\sqrt{\frac{1+\eta}{2}}\lambda_3\lambda_1\right)^2+\left(\lambda_3\mp\sqrt{\frac{1+\eta}{2}}\lambda_1\lambda_2\right)^2\right.\\
&\quad\left.+\frac{1-\eta}{4}\left[(\lambda_1^2-\lambda_2^2)^2+(\lambda_1^2-\lambda_3^2)^2+(\lambda_2^2-\lambda_3^2)^2\right]\pm 6\sqrt{\frac{1+\eta}{2}}\lambda_1\lambda_2\lambda_3\right]\\
&\geq 6\sqrt{\frac{1+\eta}{2}}\int \mathrm{d}^3x|\lambda_1\lambda_2\lambda_3|\\
&\geq 12\pi^2\sqrt{\frac{1+\eta}{2}}|B|, \qquad \text{(B1)}
\end{aligned}$$

where B is the baryon number of Eq. (8) and we have used the fact that $\int \mathrm{d}^3x\lambda_1\lambda_2\lambda_3=2\pi^2 B$. We can see that the Skyrme bound on the energy ($12\pi^2|B|$) [2] is recovered for $\eta=1$ and that the bound goes to zero for $\eta\to -1$, signaling a possible instability for the spherically symmetric case.

Taking into account the nonderivative part of the potential energy can be done following Harland [11], see also Refs. [25,64]. We define the following functions

$$E_2=\int \mathrm{d}^3x(\lambda_1^2+\lambda_2^2+\lambda_3^2), \qquad \text{(B2)}$$

$$E_4=\int \mathrm{d}^3x(\lambda_1^2\lambda_2^2+\lambda_1^2\lambda_3^2+\lambda_2^2\lambda_3^2), \qquad \text{(B3)}$$

$$E_{01}=\frac{1}{2}\int \mathrm{d}^3x\,\mathrm{tr}(\mathbf{1}_2-U), \qquad \text{(B4)}$$

$$E_{02}=\frac{1}{4}\int \mathrm{d}^3x[\mathrm{tr}(\mathbf{1}_2-U)]^2. \qquad \text{(B5)}$$

We write the maximization problem as

$$\begin{aligned}
E &= \left(E_2+\alpha\frac{1+\eta}{2}E_4\right)+\left(2m_1^2E_{01}+\beta(1-\alpha)\frac{1+\eta}{2}E_4\right)\\
&\quad+\left(m_2^2E_{02}+(1-\beta)(1-\alpha)\frac{1+\eta}{2}E_4\right)\\
&\geq 12\pi^2\left[\sqrt{\frac{\alpha(1+\eta)}{2}}+\frac{128\sqrt{m_1}\beta^{\frac{3}{4}}(1-\alpha)^{\frac{3}{4}}(1+\eta)^{\frac{3}{4}}\Gamma^2\left(\frac{3}{4}\right)}{45\times 2^{\frac{3}{4}}\pi^{\frac{3}{2}}}\right.\\
&\quad\left.+\frac{64\sqrt{m_2}(1-\beta)^{\frac{3}{4}}(1-\alpha)^{\frac{3}{4}}(1+\eta)^{\frac{3}{4}}}{45\pi}\right]|B|, \qquad \text{(B6)}
\end{aligned}$$

with $\alpha\in[0,1]$ and $\beta\in[0,1]$, which is a maximization problem in two real variables on the unit interval, where we have used the bound [11]:

$$\begin{aligned}
&2m_1^2\int \mathrm{d}^3x\,v(\mathrm{tr}U)+\mu E_4\\
&\quad\geq 16\pi\times 2^{\frac{1}{4}}\sqrt{m_1}\mu^{\frac{3}{4}}|B|\int_0^\pi \sin^2(f)[v(2\cos f)]^{\frac{1}{4}}\mathrm{d}f. \qquad \text{(B7)}
\end{aligned}$$

The result of the maximization of Eq. (B6) does not only depend on the masses m_1 and m_2, but also on the chosen value of η. First we extremize with respect to β, obtaining

$$E \geq 12\pi^2\left[\sqrt{\frac{\alpha(1+\eta)}{2}} + \frac{64(1-\alpha)^{\frac{3}{4}}(1+\eta)^{\frac{3}{4}}}{45\pi}\left(2^{\frac{1}{4}}\sqrt{\frac{m_1}{\pi}}\Gamma^2\left(\frac{3}{4}\right)F(\zeta) + \sqrt{m_2}F(\zeta^{-1})\right)\right]|B|, \tag{B8}$$

with

$$F(\zeta) = \left(\frac{1}{1+\zeta}\right)^{\frac{3}{4}}, \qquad \zeta = \frac{\pi^2 m_2^2}{2m_1^2\Gamma^8\left(\frac{3}{4}\right)}. \tag{B9}$$

On the other hand, maximization with respect to α yields

$$\alpha = \frac{a^2}{2}\left(\sqrt{1+\frac{4}{a^2}} - 1\right), \qquad a = \frac{225\pi^2(1+\eta)}{2048(1+\eta)^{\frac{3}{2}}\left(2^{\frac{1}{4}}\sqrt{\frac{m_1}{\pi}}\Gamma^2\left(\frac{3}{4}\right)F(\zeta) + \sqrt{m_2}F(\zeta^{-1})\right)^2}, \tag{B10}$$

which is valid for $\eta \in (-1, 1]$.

2. The higher-order models

We start by considering the derivative part of the higher-order models, which have the terms (59)–(62). We start by determining new bounds on higher-order terms, that have not been derived earlier, to the best of our knowledge. Starting with the term (59), we write

$$\begin{aligned} E &= \int d^3x[\mu(\lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2)^2 + 2m_1^2 v(\mathrm{tr}\,U)] \\ &= \frac{3}{8}\left[\frac{8}{3}\mu\int d^3x(\lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2)^2\right] + \frac{5}{8}\left[\frac{8}{5}\int d^3x\, 2m_1^2 v(\mathrm{tr}\,U)\right] \\ &\geq \left[\frac{8}{3}\mu\int d^3x(\lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2)^2\right]^{\frac{3}{8}}\left[\frac{8}{5}\int d^3x\, 2m_1^2 v(\mathrm{tr}\,U)\right]^{\frac{5}{8}}, \end{aligned} \tag{B11}$$

where we have used the inequality of the arithmetic and geometric means

$$\sum_{a=1}^{n} w_a x_a \geq \prod_{a=1}^{n} x_a^{w_a}, \tag{B12}$$

which holds for non-negative x_a and $w_1 + w_2 + \cdots w_n = 1$, all positive as well. Using the same inequality again, we have [11]

$$\frac{1}{3}(\lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2) \geq |\lambda_1\lambda_2\lambda_2|^{\frac{4}{3}}, \tag{B13}$$

and consequently

$$\frac{1}{9}(\lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2)^2 \geq |\lambda_1\lambda_2\lambda_2|^{\frac{8}{3}}. \tag{B14}$$

We can now write

$$E \geq 8\left[3\mu\int d^3x|\lambda_1\lambda_2\lambda_3|^{\frac{8}{3}}\right]^{\frac{3}{8}}\left[\frac{1}{5}\int d^3x 2m_1^2 v(\mathrm{tr}\,U)\right]^{\frac{5}{8}}, \tag{B15}$$

where everything has been chosen carefully such that the power $\frac{3}{8}$ is the inverse of $\frac{8}{3}$.

At this point, it will prove convenient to write the expression with a positive integer n as

$$E \geq n\left[\frac{\mu'}{3}\int d^3x|\lambda_1\lambda_2\lambda_3|^{\frac{n}{3}}\right]^{\frac{3}{n}}\left[\frac{1}{n-3}\int d^3x\, 2m_1^2 v(\mathrm{tr}\,U)\right]^{\frac{n-3}{n}}, \quad n > 3. \tag{B16}$$

Now we utilize the Hölder's inequality ($\frac{1}{p} + \frac{1}{q} = 1$, $p > 0$, $q > 0$):

$$\left(\int d^3x|f_1|^p\right)^{\frac{1}{p}}\left(\int d^3x|f_2|^q\right)^{\frac{1}{q}} \geq \int d^3x|f_1 f_2|, \tag{B17}$$

to obtain

$$E \geq n\left(\frac{\mu'}{3}\right)^{\frac{3}{n}}\left(\frac{2m_1^2}{n-3}\right)^{\frac{n-3}{n}}\int \mathrm{d}^3x|v(\mathrm{tr}U)|^{\frac{n-3}{n}}|\lambda_1\lambda_2\lambda_3|, \tag{B18}$$

which can be written in terms of the topological degree B as

$$E \geq 4n\pi\left(\frac{\mu'}{3}\right)^{\frac{3}{n}}\left(\frac{2m_1^2}{n-3}\right)^{\frac{n-3}{n}}|B|\int_0^\pi \sin^2(f)|v(2\cos f)|^{\frac{n-3}{n}}\mathrm{d}f. \tag{B19}$$

In particular, for the case of Eq. (B15) with $n=8$ and $\mu'=9\mu$, we have for the pion mass term

$$\int \mathrm{d}^3x[\mu(\lambda_1^2\lambda_2^2+\lambda_1^2\lambda_3^2+\lambda_2^2\lambda_3^2)^2+m_1^2\mathrm{tr}(\mathbf{1}_2-U)] \geq 12\pi^2\frac{256\sqrt{\pi}2^{\frac{1}{4}}(3\mu)^{\frac{3}{8}}\left(\frac{m_1^2}{5}\right)^{\frac{5}{8}}}{455\sin\left(\frac{\pi}{8}\right)\Gamma\left(\frac{5}{8}\right)\Gamma\left(\frac{7}{8}\right)}|B|. \tag{B20}$$

We will now turn to the term (60):

$$\begin{aligned}
E &= \int \mathrm{d}^3x[\mu(\lambda_1^2+\lambda_2^2+\lambda_3^2)(\lambda_1^2\lambda_2^2\lambda_3^2)+2m_1^2v(\mathrm{tr}\,U)]\\
&\geq 8\left[\frac{\mu}{3}\int \mathrm{d}^3x(\lambda_1^2+\lambda_2^2+\lambda_3^2)(\lambda_1^2\lambda_2^2\lambda_3^2)\right]^{\frac{3}{8}}\left[\frac{1}{5}\int \mathrm{d}^3x\,2m_1^2v(\mathrm{tr}\,U)\right]^{\frac{5}{8}}\\
&\geq 8\left[\mu\int \mathrm{d}^3x|\lambda_1\lambda_2\lambda_3|^{\frac{8}{3}}\right]^{\frac{3}{8}}\left[\frac{1}{5}\int \mathrm{d}^3x\,2m_1^2v(\mathrm{tr}\,U)\right]^{\frac{5}{8}}\\
&\geq 32\pi\mu^{\frac{3}{8}}\left(\frac{2m_1^2}{5}\right)^{\frac{5}{8}}|B|\int_0^\pi \sin^2(f)|v(2\cos f)|^{\frac{5}{8}}\mathrm{d}f,
\end{aligned} \tag{B21}$$

where we have used the step of Eq. (B11), the inequality

$$\frac{1}{3}(\lambda_1^2+\lambda_2^2+\lambda_3^2) \geq |\lambda_1\lambda_2\lambda_3|^{\frac{2}{3}}, \tag{B22}$$

as well as Eq. (B19). Using the pion mass term as the nonderivative potential, we get

$$\int \mathrm{d}^3x[\mu(\lambda_1^2+\lambda_2^2+\lambda_3^2)(\lambda_1^2\lambda_2^2\lambda_3^2)+m_1^2\,\mathrm{tr}(\mathbf{1}_2-U)] \geq 12\pi^2\frac{256\sqrt{\pi}2^{\frac{1}{4}}\mu^{\frac{3}{8}}\left(\frac{m_1^2}{5}\right)^{\frac{5}{8}}}{455\sin\left(\frac{\pi}{8}\right)\Gamma\left(\frac{5}{8}\right)\Gamma\left(\frac{7}{8}\right)}|B|. \tag{B23}$$

The next higher-order term is (61), which is a 10th order derivative term:

$$\begin{aligned}
E &= \int \mathrm{d}^3x[\mu(\lambda_1^2\lambda_2^2+\lambda_1^2\lambda_3^2+\lambda_2^2\lambda_3^2)(\lambda_1^2\lambda_2^2\lambda_3^2)+2m_1^2v(\mathrm{tr}\,U)]\\
&\geq 10\left[\frac{\mu}{3}\int \mathrm{d}^3x(\lambda_1^2\lambda_2^2+\lambda_1^2\lambda_3^2+\lambda_2^2\lambda_3^2)(\lambda_1^2\lambda_2^2\lambda_3^2)\right]^{\frac{3}{10}}\left[\frac{1}{7}\int \mathrm{d}^3x\,2m_1^2v(\mathrm{tr}\,U)\right]^{\frac{7}{10}}\\
&\geq 10\left[\mu\int \mathrm{d}^3x|\lambda_1\lambda_2\lambda_3|^{\frac{10}{3}}\right]^{\frac{3}{10}}\left[\frac{1}{7}\int \mathrm{d}^3x\,2m_1^2v(\mathrm{tr}\,U)\right]^{\frac{7}{10}}\\
&\geq 40\pi\mu^{\frac{3}{10}}\left(\frac{2m_1^2}{7}\right)^{\frac{7}{10}}|B|\int_0^\pi \sin^2(f)|v(2\cos f)|^{\frac{7}{10}}\mathrm{d}f,
\end{aligned} \tag{B24}$$

where we have used the step of Eq. (B11), the inequality (B13) as well as Eq. (B19). Using the pion mass term as the nonderivative potential, we get

$$\int \mathrm{d}^3x[\mu(\lambda_1^2\lambda_2^2+\lambda_1^2\lambda_3^2+\lambda_2^2\lambda_3^2)(\lambda_1^2\lambda_2^2\lambda_3^2)+m_1^2\mathrm{tr}(\mathbf{1}_2-U)] \geq 12\pi^2\frac{640\sqrt{5(5+\sqrt{5})\pi}(8\mu)^{\frac{3}{10}}\left(\frac{m_1^2}{7}\right)^{\frac{7}{10}}}{3213\Gamma\left(\frac{7}{10}\right)\Gamma\left(\frac{4}{5}\right)}|B|. \tag{B25}$$

The last higher-order term is (62), which is a 12th order derivative term:

$$\begin{aligned} E &= \int \mathrm{d}^3x[\mu(\lambda_1^4\lambda_2^4\lambda_3^4)+2m_1^2v(\mathrm{tr}\,U)] \\ &\geq 4\left[\mu\int \mathrm{d}^3x|\lambda_1\lambda_2\lambda_3|^4\right]^{\frac{1}{4}}\left[\frac{1}{3}\int \mathrm{d}^3x\,2m_1^2v(\mathrm{tr}\,U)\right]^{\frac{3}{4}} \\ &\geq 16\pi\mu^{\frac{1}{4}}\left(\frac{2m_1^2}{3}\right)^{\frac{3}{4}}|B|\int_0^\pi \sin^2(f)|v(2\cos f)|^{\frac{3}{4}}\mathrm{d}f, \end{aligned} \tag{B26}$$

where we have used the step of Eqs. (B11) and (B19). Using the pion mass term as the nonderivative potential, we get

$$\int \mathrm{d}^3x[\mu(\lambda_1^4\lambda_2^4\lambda_3^4)+m_1^2\mathrm{tr}(\mathbf{1}_2-U)] \geq 12\pi^2\frac{1280\mu^{\frac{1}{4}}\left(\frac{m_1^2}{3}\right)^{\frac{3}{4}}K\left(\frac{1}{2}\right)}{693\pi}|B|, \tag{B27}$$

where $K(m)$ is the complete elliptic integral of the first kind and $K(0.5)\approx 1.85407$.

We will now write the complete topological energy bounds for the higher-order models, using the new results for the subbounds that involve the higher-order derivative term and a potential term. For simplicity, we will turn on only the pion mass term here, and switch off the other nonderivative potential, i.e., setting $m_2 \coloneqq 0$. Writing the higher-order model (57) as

$$E_{\mathrm{HO}}^{(X)} = (E_2+\alpha\eta E_4)+(2\beta m_1^2E_{01}+(1-\alpha)\eta E_4)+\left(2(1-\beta)m_1^2E_{01}+\frac{1-\eta}{4}E_X\right), \tag{B28}$$

where E_{01}, E_2 and E_4 are given by Eqs. (B4), (B2) and (B3), respectively, and E_X is the static energy of a higher-order derivative term. We can now write the full topological energy bounds for all the higher-order models as

$$\begin{aligned} E_{\mathrm{HO}}^{(8a)} &\geq 12\pi^2\left[\sqrt{\alpha\eta}+\frac{128\beta^{\frac{1}{4}}\sqrt{m_1}((1-\alpha)\eta)^{\frac{3}{4}}\Gamma^2\left(\frac{3}{4}\right)}{45\pi^{\frac{3}{2}}}+\frac{128\sqrt{2\pi}(3(1-\eta))^{\frac{3}{8}}\left(\frac{(1-\beta)m_1^2}{5}\right)^{\frac{5}{8}}}{455\sin\left(\frac{\pi}{8}\right)\Gamma\left(\frac{5}{8}\right)\Gamma\left(\frac{7}{8}\right)}\right]|B|, \\ E_{\mathrm{HO}}^{(8b)} &\geq 12\pi^2\left[\sqrt{\alpha\eta}+\frac{128\beta^{\frac{1}{4}}\sqrt{m_1}((1-\alpha)\eta)^{\frac{3}{4}}\Gamma^2\left(\frac{3}{4}\right)}{45\pi^{\frac{3}{2}}}+\frac{128\sqrt{2\pi}((1-\eta))^{\frac{3}{8}}\left(\frac{(1-\beta)m_1^2}{5}\right)^{\frac{5}{8}}}{455\sin\left(\frac{\pi}{8}\right)\Gamma\left(\frac{5}{8}\right)\Gamma\left(\frac{7}{8}\right)}\right]|B|, \\ E_{\mathrm{HO}}^{(10)} &\geq 12\pi^2\left[\sqrt{\alpha\eta}+\frac{128\beta^{\frac{1}{4}}\sqrt{m_1}((1-\alpha)\eta)^{\frac{3}{4}}\Gamma^2\left(\frac{3}{4}\right)}{45\pi^{\frac{3}{2}}}+\frac{640\sqrt{5(5+\sqrt{5})\pi}(2(1-\eta))^{\frac{3}{10}}\left(\frac{(1-\beta)m_1^2}{7}\right)^{\frac{7}{10}}}{3213\Gamma\left(\frac{7}{10}\right)\Gamma\left(\frac{4}{5}\right)}\right]|B|, \\ E_{\mathrm{HO}}^{(12)} &\geq 12\pi^2\left[\sqrt{\alpha\eta}+\frac{128\beta^{\frac{1}{4}}\sqrt{m_1}((1-\alpha)\eta)^{\frac{3}{4}}\Gamma^2\left(\frac{3}{4}\right)}{45\pi^{\frac{3}{2}}}+\frac{640(1-\eta)^{\frac{1}{4}}\left(\frac{(1-\beta)m_1^2}{3}\right)^{\frac{3}{4}}}{693\pi}\right]|B|, \end{aligned} \tag{B29}$$

and we have defined

$$E_{8a} = \int d^3x(\lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2)^2,$$
$$E_{8b} = \int d^3x(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)(\lambda_1^2\lambda_2^2\lambda_3^2),$$
$$E_{10} = \int d^3x(\lambda_1^2\lambda_2^2 + \lambda_1^2\lambda_3^2 + \lambda_2^2\lambda_3^2)(\lambda_1^2\lambda_2^2\lambda_3^2),$$
$$E_{12} = \int d^3x(\lambda_1^4\lambda_2^4\lambda_3^4). \tag{B30}$$

The maximization of the bounds (B29) with respect to α can be carried out easily as in the Pottinger-Rathske model [see Appendix B 1] if it is done first, but since the $(1-\alpha)$-dependence is only included in the Skyrme term and not in the higher-order term, the extremization with respect to α is changed and possibly becomes more difficult after maximization has been done with respect to β.

Starting with the $8a$ and $8b$ cases, extremization with respect to β can be carried out as:

$$\beta = \frac{2}{25}\left(\frac{2}{5}\right)^{\frac{2}{3}}\xi^{\frac{8}{3}}\left[\sqrt{1+25\left(\frac{5}{2}\right)^{\frac{2}{3}}\xi^{-\frac{8}{3}}} - 1\right], \tag{B31}$$

$$\xi = \frac{91\sqrt{m_1}((1-\alpha)\eta)^{\frac{3}{4}}\Gamma\left(\frac{5}{8}\right)\Gamma^2\left(\frac{3}{4}\right)\Gamma\left(\frac{7}{8}\right)\sin\left(\frac{\pi}{8}\right)}{9\sqrt{2}\pi^2\left(\frac{(1-\beta)m_1^2}{5}\right)^{\frac{5}{8}}(3(1-\eta))^{\frac{3}{8}}}. \tag{B32}$$

We are, however, unable to perform the maximization with respect to α analytically at this point. For the 10 case, the extremization with respect to β requires the root to 5th order polynomial equation, that we do not know in closed form. For the 12 case, maximization with respect to β is given by a real root to a 3rd order polynomial equation, which can be found by Cardano's formula. However, we are also in this case unable to perform the analytic maximization with respect to α after having maximized with respect to β.

We are thus not able to find analytic expressions for both α and β in the cases of the higher-order models that maximize their respective bounds, but the maximization in the two variables α and β on the unit interval can for fixed values of m_1 and η easily be done numerically.

Taking into account both nonderivative potentials ($m_1 > 0$ and $m_2 > 0$) can straightforwardly be done as in the Pottinger-Rathske model by introducing a third parameter (see Appendix B 1), which we will leave as an exercise. Again only a numerical solution to the three parameters will be possible.

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