Dynamical branes on expanding orbifold and complex projective space

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We construct some new dynamical p-brane solutions to gravity theories on curved backgrounds. We discuss the relations between dynamical branes, a new time-dependent solution on complex projective space $\mathbb{C}\mathrm{P}^n$, and the static p-branes on the orbifold $\mathbb{C}^n/\mathbb{Z}_n$.

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I. INTRODUCTION

Dp-branes or more generally p-branes are $(p+1)$-dimensional nonperturbative solitonic objects appearing in string theory and supergravity [1–3]. Such brane configurations were applied in the contexts of brane world models and brane cosmology [4–20]. In particular, in applications to cosmology such as an inflationary universe, branes cannot be static anymore, and dynamics of branes is essential [5–7,9–13,15,16]. In black hole physics, dynamical branes are essential for black hole dynamics such as their collision [4,8,12,17,19]. The dynamical p-brane solutions in a higher-dimensional gravity theory were studied in Refs. [4–20] and have been widely discussed ever since. However, some aspects of the physical properties, such as having a quadratic order of time dependence and its dynamics in the context of string theory, have remained unclear.

Although a large number of static supergravity solutions have been investigated so far, there is still no systematic construction method for dynamic solutions. Hence we need as many examples as possible for dynamical solutions, which are especially important in applications to cosmology. We do not also have any dynamical orbifold solutions while there is a lot of work covering the static model on an orbifold due to not only theoretical interest but also applications to particle phenomenology or cosmology. Our aim in the present paper is to construct a cosmological model on an expanding orbifold.

There is also interesting recent work to find the dynamical p-brane solutions giving the dynamics of supersymmetry breaking [20] and the issues of spacetime singularity such as cosmic censorship conjecture [18]. Since some dynamical solutions preserve supersymmetry, we can find the relation deeply between the expansion of the Universe and breaking of supersymmetry [20]. The dynamical branes have been found by classical solutions of supergravities which are the low-energy effective theories of superstring theories or eleven-dimensional supergravity [6]. Since the dynamical p-branes are extensions to static p-branes in string theory that have been objects of intensive research, these objects have been treated as dynamical objects in general relativity as well as string theories. The dynamical p-brane solutions give interesting results and important descriptions of their dynamics in supergravities.

In this paper, we find new dynamical p-brane solutions, which are classified into the two classes. In the first, we promote p-brane solutions on the orbifolds $\mathbb{C}^n/\mathbb{Z}_n$ [21,22] to dynamical ones. In this case, the orbifolds expand in time. The second is a dynamical solution on the complex projective space $\mathbb{C}\mathrm{P}^n$.

This paper is organized as follows. Section II gives a brief introduction to dynamical p-branes in gravity theory. The ansatz of fields and the various field equations are then discussed. Section III is presented for constructing dynamical p-brane solutions on the orbifold. The setup is a simple extension of the static p-brane system. Section III is devoted to constructing new dynamical solutions carrying one antisymmetric tensor field charge. We discuss extremal (black) p-branes and dynamical solutions. When the space to which gauge potential does not extend is non-Ricci flat, the function in the metric is no longer linear in time like dynamical p-brane system but quadratic in it. We will show it in Sec. IV. Although solutions we find in Sec. IV do not describe a p-brane, it will allow us to obtain dynamical solutions in D-dimensional gravity theory. Finally, we conclude in Sec. V.

II. CHARGED EXTREMAL AND DYNAMICAL BLACK p-BRANES

We briefly summarize the results for $(p+2)$-form field strength in the D-dimensional theory. We consider a gauge field strength $F_{(p+2)}$ in the action [22]

$$S = \frac{1}{2\kappa^2}\int d^Dx\sqrt{-g}\left[R - \frac{1}{2(p+2)!}F^2_{(p+2)}\right], \quad (1)$$

where κ^2 is the D-dimensional gravitational constant and g and R denote the determinant, the Ricci scalar with respect to the D-dimensional metric g_{MN}, respectively. We assume that the expectation values of fermionic fields are vanishing in this paper. The D-dimensional action (1) describes the bosonic part of $D=10$ with trivial dilaton or $D=11$ supergravity.

The field equations are given by

$$R_{MN} = \frac{1}{2\cdot(p+2)!} \times [(p+2)F_{MA_1\cdots A_{p+1}}F_N{}^{A_1\cdots A_{p+1}} - g_{MN}F^2_{(p+2)}], \quad (2a)$$

$$d({}_*F_{(p+2)}) = 0. \quad (2b)$$

We review the properties of the p-brane to simplify the field equations. The p-brane has p spacelike directions which are longitudinal to the p-brane. It contains also $(D-p-1)$ spacelike directions that are characterized by transverse space to the p-brane.

The longitudinal spacetime to the p-brane thus gives the timelike direction. We will consider a single dynamical p-brane solution with a single charge. The dynamical p-brane does not have a translational invariant with respect to the longitudinal spacetime to the p-brane. Since they are localized at a point in the transverse space to the p-brane, there is also no translational invariance. We suppose spherical symmetry in the $(D-p-1)$-dimensional transverse space for the dynamical p-brane without any angular momentum.

We take a single p-brane ansatz for the D-dimensional metric

$$ds^2 = h^a(x,y)q_{\mu\nu}(x)dx^\mu dx^\nu + h^b(x,y)u_{ij}(y)dy^i dy^j, \quad (3)$$

where $q_{\mu\nu}(x)$ is a $(p+1)$-dimensional metric which depends only on the coordinates t, x^α with α being the spatial coordinates, and $u_{ij}(y)$ is the $(D-p-1)$-dimensional metric which depends only on the coordinates y^i. The coordinates of D-dimensional spacetime are divided by two sets, $x^M = (x^\mu, y^i)$, with $\mu = 0, \ldots, p$ and $i = 1, \ldots, D-p-1$. Here, the y^i's denote the coordinates of the transverse space. We divide again the coordinates x^μ into two parts, the time coordinate t and the spatial coordinates $x^\alpha (\alpha = 1, \ldots, p)$, where the x^α's span the directions longitudinal to the brane. We choose the timelike direction $x^0 = t$ and assume that the metric depends not only on t and y^i, but also on x^α. The metric form (3) is a straightforward generalization of the case of a static p-brane system [4–6].

The parameters a and b in the dynamical brane system are given by

$$a = -\frac{D-p-3}{D-2}, \qquad b = \frac{p+1}{D-2}, \quad (4)$$

while the gauge field strength $F_{(p+2)}$ is also assumed to be

$$F_{(p+2)} = d(h^{-1})\sqrt{-q} \wedge dt \wedge dx^1 \wedge \cdots \wedge dx^p, \quad (5)$$

where q denotes the determinant of the $(p+1)$-dimensional metric $q_{\mu\nu}$. Under our ansatz, the Einstein equations become

$$R_{\mu\nu}(\mathrm{X}) - h^{-1}D_\mu D_\nu h - \frac{a}{2}h^{-1}q_{\mu\nu}(\triangle_\mathrm{X} h + h^{-1}\triangle_\mathrm{Y} h) = 0, \quad (6a)$$

$$R_{ij}(\mathrm{Y}) - \frac{b}{2}u_{ij}(\triangle_\mathrm{X} h + h^{-1}\triangle_\mathrm{Y} h) = 0, \quad (6b)$$

$$\partial_\mu \partial_i h = 0, \quad (6c)$$

where D_μ denotes the covariant derivative with respect to the metric $q_{\mu\nu}$, $\triangle_\mathrm{X}$ and $\triangle_\mathrm{Y}$ are the Laplace operators on the $(p+1)$-dimensional world-volume spacetime X and $(D-p-1)$-dimensional space Y spaces, and $R_{\mu\nu}(\mathrm{X})$ and $R_{ij}(\mathrm{Y})$ are the Ricci tensors of the metrics $q_{\mu\nu}$ and u_{ij}, respectively. From Eq. (6c), the function $h(x,y)$ has to be in the form

$$h(x,y) = h_0(x) + h_1(y). \quad (7)$$

The other components of the Einstein equations (6a) and (6b) can be rewritten as

$$R_{\mu\nu}(\mathrm{X}) - h^{-1}D_\mu D_\nu h_0 - \frac{a}{2}h^{-1}q_{\mu\nu}(\triangle_\mathrm{X} h_0 + h^{-1}\triangle_\mathrm{Y} h_1) = 0, \quad (8a)$$

$$R_{ij}(\mathrm{Y}) - \frac{b}{2}u_{ij}(\triangle_\mathrm{X} h_0 + h^{-1}\triangle_\mathrm{Y} h_1) = 0. \quad (8b)$$

Next we consider the gauge field strength. From the assumption (5), we find that the Bianchi identity is automatically satisfied while the equation of motion for the gauge field (2b) becomes $\triangle_\mathrm{Y} h_1 = 0$, and $\partial_\mu \partial_i h = 0$.

If $F_{(p+2)} \neq 0$, the function h_1 is nontrivial. The Einstein equations thus reduce to

$$R_{\mu\nu}(\mathrm{X}) = 0, \tag{9a}$$

$$R_{ij}(\mathrm{Y}) = \frac{1}{2} b(p+1)\lambda\, u_{ij}, \tag{9b}$$

$$D_\mu D_\nu h_0 = \lambda\, q_{\mu\nu}, \tag{9c}$$

$$\triangle_{\mathrm{Y}} h_1 = 0, \tag{9d}$$

where λ is a constant. We see that the space Y is not Ricci flat, but the Einstein space such as $\mathbb{C}\mathrm{P}^n$ if $\lambda \neq 0$, and the function h can be more nontrivial.

Let us consider the case

$$q_{\mu\nu} = \eta_{\mu\nu}, \tag{10}$$

where X is $(p+1)$-dimensional Minkowski spacetime. If $D_\mu h_0 \neq 0$ and $(D_\mu h_0)(D^\mu h_0) \neq 0$, the solution for h_0 is given by

$$h_0(x) = \frac{\lambda}{2} x_\mu x^\mu + \bar{a}_\mu x^\mu + \bar{a}. \tag{11}$$

Here we have introduced constants $\bar{a}_\mu$ and $\bar{a}$ satisfying the condition $\bar{a}_\mu \bar{a}^\mu \neq 0$. However, if $D_\mu h_0 \neq 0$ and $(D_\mu h_0)(D^\mu h_0) = 0$, there exists a solution only when $\lambda = 0$.

Before concluding this section, we should comment on he ansatz for fields (9). The simplification to the field equations (2) strongly depends on choosing parameters for the metric. With this choice, the metric can be written by the function $h(x, y)$ multiplying a flat metric for the $(p+1)$-dimensional longitudinal spacetime. Note that the function $h(x, y)$ depends on $(D-p-1)$-dimensional transverse space to the p-brane as well as the $(p+1)$-dimensional longitudinal coordinates. Hence, the p-brane is fully characterized by time. Moreover, in the context of cosmology, the dynamical p-brane is most of the time related to the fact that the solutions describe an expansion of the Universe.

III. DYNAMICAL p-BRANE ON ORBIFOLD

We now construct the solution of the dynamical p-brane on orbifold explicitly. This case is interesting because field equations are analytically solved (9).

The Einstein equations (9b) can be solved when we start with a $\mathbb{C}\mathrm{P}^{(D-p-3)/2}$ metric in $(D-p-1)$ dimensions, namely, a $\mathbb{Z}_{(D-p-1)/2}$ orbifold of $\mathbb{C}^{(D-p-1)/2}$ [23–27]:

$$u_{ij}(y) dy^i dy^j = dr^2 + r^2 \left[\left\{ d\rho + \sin^2\xi_{(D-p-3)/2} \left(d\psi_{(D-p-3)/2} + \frac{1}{(D-p-3)} \omega_{(D-p-5)/2} \right) \right\}^2 + ds^2_{\mathbb{C}\mathrm{P}^{(D-p-3)/2}} \right], \tag{12}$$

where we have used $\omega_0 = 0$, r is a radial coordinate, ρ is a coordinate of S^1, $\xi_{(D-p-3)/2}$ and $\psi_{(D-p-3)/2}$ are coordinates of the $\mathbb{C}\mathrm{P}^{(D-p-3)/2}$ space with the ranges $0 \leq \xi_{(D-p-3)/2} \leq \pi/2$, $0 \leq \psi_{(D-p-3)/2} \leq 2\pi$, and $\omega_{(D-p-3)/2}$ and $ds^2_{\mathbb{C}\mathrm{P}^{(D-p-3)/2}}$ state a one-form and a metric on the $\mathbb{C}\mathrm{P}^{(D-p-3)/2}$ space, recursively defined as [28–30]

$$\begin{aligned} ds^2_{\mathbb{C}\mathrm{P}^{(D-p-3)/2}} = (D-p-1) \Bigg[& d\xi^2_{(D-p-3)/2} + \sin^2\xi_{(D-p-3)/2} \cos^2\xi_{(D-p-3)/2} \left\{ d\psi_{(D-p-3)/2} + \frac{1}{(D-p-3)} \omega_{(D-p-5)/2} \right\}^2 \\ & + \frac{1}{(D-p-3)} \sin^2\xi_{(D-p-3)/2} ds^2_{\mathbb{C}\mathrm{P}^{(D-p-5)/2}} \Bigg], \end{aligned} \tag{13}$$

and

$$\omega_{(D-p-5)/2} = (D-p-3) \sin^2\xi_{(D-p-5)/2} \left[d\psi_{(D-p-5)/2} + \frac{1}{(D-p-5)} \omega_{(D-p-7)/2} \right], \tag{14a}$$

$$ds^2_{\mathbb{C}\mathrm{P}^1} = 4(d\xi_1^2 + \sin^2\xi_1 \cos^2\xi_1 d\psi_1^2), \tag{14b}$$

$$\omega_1 = 4 \sin^2\xi_1 d\psi_1. \tag{14c}$$

Here (r, ρ) describes a complex line, and ρ together with $\mathbb{C}\mathrm{P}^{(D-p-3)/2}$ denote a orbifold action of the $(D-p-2)$-dimensional sphere $S^{D-p-2}/\mathbb{Z}_{(D-p-1)/2}$, which is actually an event horizon [21,22].

One remark is in order before we continue. In Eq. (12), we assume $R(\mathrm{Y}) = 0$, which is constructed from the metric $u_{ij}(y)$. Such an assumption would give $\lambda = 0$ in the Einstein equations (9b).

We impose on the condition $h_1 = h_1(r)$ in the field equations. If we introduce the dependence of radial coordinate r for the function h_1, we have reduced the problem to the equation (9d)

$$\triangle_{\mathrm{Y}} h_1 = \frac{1}{r^{D-p-2}}\frac{d}{dr}\left(r^{D-p-2}\frac{d}{dr}h_1\right) = 0. \tag{15}$$

This is solved to give for $D-p-3 \neq 0$

$$h_1(r) = b_1 + \frac{b_2}{r^{D-p-3}}. \tag{16}$$

Here b_1 and b_2 are constant parameters. The gauge field strength is asymptotically vanishing according to the limit $r \to \infty$ in the function $h_1(r)$. We have assumed $D-p-3 > 0$ in Eqs. (15) and (16) giving zeroth of the gauge field strength asymptotically and a Kasner spacetime at infinity. We will discuss them more detail later. One can show that the solution (16), when $D-p-3 \neq 0$ is replaced for $D-p-3 = 0$, is a direct consequence of Eq. (15). The solution is then shown to be

$$h_1(r) = b_3 + b_4 \ln r, \tag{17}$$

where the function h_1 diverges both at $r \to \infty$ and $r \to 0$. Since there is no regular spacetime region near the p-brane, such solutions are not physically relevant. We will here consider the case $D-p-3 > 0$ in the following. We have constructed dynamical solutions depending on parameters, $\bar{a}_\mu$, $\hat{a}$, and b_2. The solutions are characterized by the function which is harmonic in $(D-p-3)$-dimensional space:

$$h(x,r) = \bar{a}_\mu x^\mu + \hat{a} + \frac{b_2}{r^{D-p-3}}, \tag{18}$$

where $\hat{a}$ is defined by $\hat{a} = \tilde{a} + b_1$.

The surfaces of constant t are spacelike everywhere. The geometry resembles the infinite throat familiar from the asymptotically flat extremal Reissner-Nordström solution near $r = 0$. This can be expressed by the spatial metric in spherical coordinates centered at $r = 0$. Near the origin of these coordinates, this metric becomes

$$ds^2 \approx \left(\frac{b_2}{r^{D-p-3}}\right)^a \eta_{\mu\nu}dx^\mu dx^\nu + \left(\frac{b_2}{r^{D-p-3}}\right)^b r^2\left(\frac{dr^2}{r^2} + d\Omega^2\right), \tag{19}$$

which is the metric for a warped cylinder of infinite spatial extent having cross sectional area. Here the metric $d\Omega^2$ takes the form of

$$d\Omega^2 = \left[d\rho + \sin^2\xi_{(D-p-3)/2}\left(d\psi_{(D-p-3)/2} + \frac{1}{(D-p-3)}\omega_{(D-p-5)/2}\right)\right]^2 + ds^2_{\mathbb{C}\mathrm{P}^{(D-p-3)/2}}. \tag{20}$$

If we set $D-p-3 = -1$ and $\bar{a}_\alpha = 0 (\alpha = 1, 2, \ldots, p)$, then we have $h(t,r) = \bar{a}_0 t + \hat{a} + b_2 r$. Hence, any points on the branes are regular and time dependent. When we take the limit of $h(t,r) \to 0$ (or finite) as $r \to \infty$ for $D-p-3 > 1$ (or r is finite for $D-p-3 = 1$), the D-dimensional spacetime metric becomes

$$ds^2 = (\bar{a}_0 t)^a \eta_{\mu\nu}dx^\mu dx^\nu + (\bar{a}_0 t)^b (dr^2 + r^2 d\Omega^2), \tag{21}$$

where we set $\hat{a} = 0$ without loss of generality. One can note that the spacetime turns out to be time dependent. To see its dynamical behavior, we introduce a new time coordinate

$$\tau = \tau_0 (\bar{a}_0 t)^{\frac{a}{2}+1}, \tag{22}$$

where $\tau = \frac{2}{\bar{a}_0(a+2)}$. The asymptotic dynamical solution is rewritten as

$$ds^2 = -d\tau^2 + \left(\frac{\tau}{\tau_0}\right)^{a(\frac{a}{2}+1)^{-1}} \sum_\alpha (dx^\alpha)^2 + \left(\frac{\tau}{\tau_0}\right)^{b(\frac{a}{2}+1)^{-1}} u_{ij}dy^i dy^j. \tag{23}$$

Hence, we find a Kasner-like expansion:

$$\frac{a}{2}\left(\frac{a}{2}+1\right)^{-1} p + \frac{b}{2}\left(\frac{a}{2}+1\right)^{-1}(D-p-1) = 1, \tag{24a}$$

$$\frac{a^2}{4}\left(\frac{a}{2}+1\right)^{-2} p + \frac{b^2}{4}\left(\frac{a}{2}+1\right)^{-2}(D-p-1) = 1. \tag{24b}$$

Equation (24a) is always satisfied for any dynamical p-brane configuration while Eq. (24b) is true only for M theory or the D3-brane system because there is no or trivial dilaton in the background [6].

The dynamics of brane is also correct when we fix the position in the transverse space to the p-brane, even if the metric is locally inhomogeneous in the bulk space.

The curvature of Eqs. (3) and (18) can be singular at zeros of the metric function h. This can be seen from the square of the $(p+2)$-form field strength,

$$F^2_{(p+2)} \equiv F_{A_1\cdots A_{p+2}} F^{A_1\cdots A_{p+2}} = -h^\gamma (\partial_r h)^2, \tag{25}$$

where $\gamma = -4 + \frac{(D-p-4)(p+1)}{D-2} < 0$. If the function $h = 0$ and $\partial_r h$ does not vanish like h^2 or faster, then $F^2_{(p+2)}$ diverges and the curvature is singular.

According to these elements, we can find a behavior of how the background geometry develops in time. Here we assume $\bar{a}_0 \neq 0$ in the D-dimensional spacetime. One can note that the D-dimensional metric (3) exists for $h > 0$. Otherwise, the solution has curvature singularities at $h = 0$ while the metric gives complex geometry for $h < 0$. We expect a type of singularity may appear at $h(t, r) = 0$. Since $h(t, r)$ is a linear function of t for $\bar{a}_0 \neq 0$, it vanishes once for any position r at $t = -(\hat{a} + b_2 r^{-D+p+3})/\bar{a}_0$. Since the function h is positive everywhere for $\bar{a}_0 t + \hat{a} > 0$, the spatial surfaces are not singular. They are asymptotically time dependent spacetime and have the cylindrical form of an infinite throat near $r = 0$. The spatial metric is not singular and the cylindrical form everywhere. When $\bar{a}_0 t + \hat{a}$ is slightly increased, a singularity appears near $r = \infty$. As $\bar{a}_0 t + \hat{a}$ increases further, the singularity cuts off more and more of the cylinder.

IV. DYNAMICAL SOLUTION ON $\mathbb{C}\mathrm{P}^n$ SPACE

In this section, we present the dynamical solution on the $\mathbb{C}\mathrm{P}^n$ space which happens when the Einstein equations become Eq. (9). As seen from the Einstein equations, the internal space Y is not necessarily Ricci flat, and the function h_0 is no longer linear in the coordinates x^μ but quadratic in them.

A. Dynamical solution on $\mathbb{C}\mathrm{P}^1$ space

First, we consider the case in which Y is a simple $\mathbb{C}\mathrm{P}^1$ space

$$ds^2_{\mathbb{C}\mathrm{P}^1} = (1 + \tilde{r}^2)^{-2}(d\tilde{r}^2 + \tilde{r}^2 d\tilde{\theta}^2). \tag{26}$$

Note that $\mathbb{C}\mathrm{P}^1$ space can be expressed by the Fubini-Study metric because of a diffeomorphism $\mathbb{C}\mathrm{P}^1 \cong S^2$. The metric of four-dimensional bulk transverse space is thus given by

$$u_{ij}(y)dy^i dy^j = dr^2 + r^2\left[\left\{d\rho + \frac{1}{2}\sin^2\left(\frac{1}{2}\arctan\tilde{r}\right)d\tilde{\theta}\right\}^2 + ds^2_{\mathbb{C}\mathrm{P}^1}\right], \tag{27}$$

where we have used $\omega_0 = 0$. Let $h_1(\tilde{r}, \tilde{\theta})$ be a function on Y of the form

$$h_1(\tilde{r}, \tilde{\theta}) = \tilde{H}(\tilde{r}) + \tilde{K}(\tilde{\theta}). \tag{28}$$

Then, the equation $\triangle_{\mathrm{Y}} h_1 = 0$ gives

$$\partial_{\tilde{r}}(\tilde{r}\partial_{\tilde{r}}\tilde{H}) + \frac{1}{\tilde{r}}\partial^2_{\tilde{\theta}}\tilde{K} = 0. \tag{29}$$

If we assume that functions $\tilde{H}(\tilde{r})$ and $\tilde{K}(\tilde{\theta})$ obey

$$\tilde{r}\partial_{\tilde{r}}(\tilde{r}\partial_{\tilde{r}}\tilde{H}) - \varepsilon = 0, \qquad \partial^2_{\tilde{\theta}}\tilde{K} + \varepsilon = 0, \tag{30}$$

we find

$$h_1(\tilde{r}, \tilde{\theta}) = \frac{1}{2}\varepsilon(\ln\tilde{r})^2 + \tilde{c}_1 \ln\tilde{r} - \frac{1}{2}\varepsilon\tilde{\theta}^2 + \tilde{c}_2\tilde{\theta} + \tilde{c}_3. \tag{31}$$

Here ε and $\tilde{c}_i (i = 1, \ldots, 3)$ are constants.

The metric we found as the solution (11) and (31) is not of the product type. The existence of a nontrivial gauge field strength forces the function $h(x, y)$ to be a linear combination of a function of x^μ and a function of y^i, which is not the conventional assumption. The function in Eq. (11) implies that we cannot drop the dependence on the world volume coordinate for a nonvanishing Ricci scalar $R(\mathrm{Y})$. This solution gives the inhomogeneous universe due to the function h_1 when we regard the bulk transverse space as four-dimensional space.

B. Dynamical solution on $\mathbb{C}\mathrm{P}^2$ space

Next, we discuss solution on $\mathbb{C}\mathrm{P}^2$ space, whose metric is given by [31]

$$ds^2_{\mathbb{C}\mathrm{P}^2} = (1 + \bar{\rho}^2)^{-2}d\bar{\rho}^2 + \frac{\bar{\rho}^2}{4}(1 + \bar{\rho}^2)^{-2}(d\psi + \cos\theta d\phi)^2 + \frac{\bar{\rho}^2}{4}(1 + \bar{\rho}^2)^{-1}(d\theta^2 + \sin^2\theta d\phi^2). \tag{32}$$

If we set

$$h_1(\bar{\rho}, \theta) = \bar{H}(\bar{\rho}) + \bar{K}(\theta), \tag{33}$$

the equation $\triangle_{\mathrm{Y}} h_1 = 0$ yields

$$\frac{(1 + \bar{\rho}^2)^3}{\bar{\rho}^3}\partial_{\bar{\rho}}\left(\frac{\bar{\rho}^3}{1 + \bar{\rho}^2}\partial_{\bar{\rho}}\bar{H}\right) + \frac{1}{\sin\theta}\partial_\theta\left[\frac{4(1 + \bar{\rho}^2)}{\bar{\rho}^2}\sin\theta\partial_\theta\bar{K}\right] = 0. \tag{34}$$

For example, we require that the functions $\bar{H}(\bar{\rho})$ and $\bar{K}(\theta)$ satisfy

$$\frac{(1 + \bar{\rho}^2)^2}{\bar{\rho}}\partial_{\bar{\rho}}\left(\frac{\bar{\rho}^3}{1 + \bar{\rho}^2}\partial_{\bar{\rho}}\bar{H}\right) - \bar{\varepsilon} = 0,$$
$$\frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta\bar{K}) + \bar{\varepsilon} = 0. \tag{35}$$

Here, $\bar{\varepsilon}$ denotes constant. The solution to these equations is thus

$$\bar{H}(\bar{\rho}) = \frac{2\bar{\varepsilon} - \bar{c}_1}{2\bar{\rho}^2} + \bar{c}_1 \ln\bar{\rho} + \bar{c}_2, \tag{36a}$$

$$\bar{K}(\theta) = \frac{1}{2}(\bar{\varepsilon} + \bar{c}_3)\ln(1 - \cos\theta) + \frac{1}{2}(\bar{\varepsilon} - \bar{c}_3)\ln(1 + \cos\theta) + \bar{c}_4, \tag{36b}$$

where $\bar{c}_i (i = 1, \ldots, 4)$ are constants.

The scale factor of the Universe again includes the inhomogeneity due to functions h_0 and h_1. We live in the $(p+1)$-dimensional spacetime. In this case, since we fix our Universe at some position in the $\mathbb{C}\mathrm{P}^2$ space, the line element is given by

$$ds^2 = \left[\frac{\lambda}{2}(-t^2 + x^\alpha x_\alpha)\right]^a (-dt^2 + d\bar{r}^2 + \bar{r}^2 d\Omega^2_{(p-1)}) + \left[\frac{\lambda}{2}(-t^2 + x^\alpha x_\alpha)\right]^b ds^2(\mathrm{Y}), \tag{37}$$

where we set $\bar{a}_\mu = 0$, $\hat{a} = 0$, $d\Omega^2_{(p-1)}$ denotes the metric of $(p-1)$-dimensional sphere S^{p-1}, and

$$\eta_{\mu\nu} dx^\mu dx^\nu = -dt^2 + \delta_{\alpha\beta} dx^\alpha dx^\beta = -dt^2 + d\bar{r}^2 + \bar{r}^2 d\Omega^2_{(p-1)}, \tag{38a}$$

$$ds^2(\mathrm{Y}) = dr^2 + r^2\left[\left\{d\rho + 6\sin^2\xi_2\left(d\psi_2 + \frac{1}{4}\omega_1\right)\right\}^2\right]. \tag{38b}$$

The D-dimensional spacetime (37) is regular for $h_0 = \frac{\lambda}{2}(-t^2 + x^\alpha x_\alpha) > 0$. We find again curvature singularities if $h_0 = 0$. Since the function h_0 changes sign somewhere in the D-dimensional spacetime, the metric is restricted to the $h_0 > 0$ region bounded by curvature singularities.

Although it looks inhomogeneous at first glance, in terms of the coordinate transformation for $\lambda > 0$,

$$t = \sqrt{\frac{2}{\lambda}}T \sinh \bar{R}, \qquad \bar{r} = \sqrt{\frac{2}{\lambda}}T \cosh \bar{R}, \tag{39}$$

we can rewrite the metric (37) as

$$ds^2 = \frac{2}{\lambda}T^{2a}[-dT^2 + T^2\{d\bar{R}^2 + \bar{R}^2 d\Omega^2_{(p-1)} + d\bar{s}^2(\mathrm{Y})\}] = \frac{2}{\lambda}\left[-d\bar{T}^2 + \left(\frac{p+1}{D-2}\right)^2 \bar{T}^2\{d\bar{R}^2 + \bar{R}^2 d\Omega^2_{(p-1)} + d\bar{s}^2(\mathrm{Y})\}\right], \tag{40}$$

where we have defined

$$\bar{T} = \left(\frac{D-2}{p+1}\right)T^{(p+1)/(D-2)}, \qquad d\bar{s}^2(\mathrm{Y}) = \frac{\lambda}{2}ds^2(\mathrm{Y}). \tag{41}$$

One can note that the metric (40) represents an isotropic and homogeneous spacetime. The scale factor of the Universe is thus proportional to the function $\bar{T}$ of the cosmic time, which is known as the Milne universe.

If $\lambda < 0$, we should use the coordinate transformation:

$$t = \sqrt{-\frac{2}{\lambda}}\bar{R} \cosh T, \qquad \bar{r} = \sqrt{-\frac{2}{\lambda}}\bar{R} \sinh T. \tag{42}$$

Then we have

$$ds^2 = \frac{2}{|\lambda|}\bar{R}^{2a}[d\bar{R}^2 + \bar{R}^2\{-dT^2 + \bar{R}^2 d\Omega^2_{(p-1)} + d\hat{s}^2(\mathrm{Y})\}] = \frac{2}{|\lambda|}\left[d\hat{R}^2 + \left(\frac{p+1}{D-2}\right)^2 \hat{R}^2\{-dT^2 + \hat{R}^2 d\Omega^2_{(p-1)} + d\hat{s}^2(\mathrm{Y})\}\right], \tag{43}$$

where $\hat{R}$ and $d\hat{s}^2(\mathrm{Y})$ are given by

$$\hat{R} = \left(\frac{D-2}{p+1}\right)\bar{R}^{(p+1)/(D-2)}, \qquad d\hat{s}^2(\mathrm{Y}) = \frac{|\lambda|}{2}ds^2(\mathrm{Y}). \tag{44}$$

The metric (43) describes a conformally flat and inhomogeneous spacetime, but it is different from a Milne universe. The D-dimensional background is described by static spacetime when we fix a position in the $\mathbb{C}\mathrm{P}^2$ space. The solution gives a parameter λ coming from the curvature of Y space. Note that it makes a contribution similar to the cosmological constant in the Einstein equations. Since λ is not a D-dimensional cosmological constant, it does not actually give de Sitter (dS) or anti–de Sitter (AdS) space. In our setup, the solution for $\lambda > 0$ gives an expanding universe similar to dS while we find a conformal flat static solution like AdS spacetime in the case of $\lambda < 0$.

V. CONCLUSION AND REMARKS

We have found two new dynamical p-brane solutions. The first is p-brane solutions on the orbifolds $\mathbb{C}^n/\mathbb{Z}_n$ [21,22] expanding in time. The second is dynamical solutions on the complex projective space $\mathbb{C}\mathrm{P}^n$.

Our new solutions have been obtained by replacing a constant c in the function $h = c + h_1$ of a static solution with a quadratic function of the coordinates x^μ. We have obtained dynamical p-brane solutions on the orbifold whose spacetime metric depends on the coordinates of both the world volume and the space transverse to the p-brane. The field equations normally indicate that dynamical solutions can be found while two functions in the metric depends on both the time and overall transverse space coordinates. We have constructed a solution explicitly in the case of $\lambda \neq 0$ beyond the examples considered in the previous works. The ansatz for fields to solve the field equations have been chosen by the extension to the static solution or the supersymmetric static p-brane solution, which is the extremal case. We have proceeded the construction further with respect to the D-dimensional action (1) and considered a time-dependent gauge field strength in the background. Since the field equation with our ansatz of fields allows the time-dependent solution, the supergravity theories, for instance, realize the dynamical p-brane at the classical level. We could present dynamical solution explicitly in Eqs. (31) and (36). We note that the no-force condition for the dynamical p-brane on the orbifold is the same as dynamical branes which have been discussed in [16].

Constructing dynamical p-brane solutions on the orbifold are most interesting issues of the string cosmology because the evolution of universe is derived from brane configurations. We then find cosmological models from those solutions by smearing some dimensions. We have the cosmological solutions with a power-law expansion. However, the solutions of Einstein equations cannot give a realistic expansion law. Although our solution gives the dynamics of the various branes in D dimensions, we have to specify the compactification to construct the four-dimensional cosmology. The time-dependent solution we have obtained here would give a key to construct in more realistic cosmological models.

The metric in Eq. (12) in our paper denotes the orbifolds $\mathbb{C}^n/\Gamma$ with a discrete group Γ. When $\Gamma = \mathbb{Z}_n$ the orbifold metric is well known including the case that its orbifold singularity is resolved, in which case the space is a certain complex line bundle over $\mathbb{CP}^{n-1}$. The metric with an arbitrary Γ is in general difficult to solve or unknown. In this paper, we concentrated on the simplest case, but it remains a future problem to consider more general cases.

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