# General properties of the electric Penrose process

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We consider the Penrose process with the charged particles in the Reissner-Nordström background. Let parent particle 0 decay to particles 1 and 2. With the assumption that all three particles move in the equatorial plane, the exact formulas for characteristics of particles 1 and 2 in terms of those of particle 0 are derived. We concentrate on scenarios in which particle 1 and 2 are ejected along the trajectory of particle 0. It is shown that such scenarios correspond to the extrema of energies $E_1$ or $E_2$ of daughter particles with respect to the angular momentum $L_1$ or $L_2$. We derive bounds on the values of angular momenta $L_1$ and $L_2$. We give classification of these scenarios and discuss their properties including decay in the near-horizon region. We find that the maximum of efficiency is achieved on the horizon for some of these scenarios but not for all of them and with additional constraints on particle parameters. The results are reformulated in terms of velocities of daughter particles in the center of mass frame. The approach is applicable also to collisional Penrose process, in which a combination of particles 1 and 2 is considered as one effective particle. If the mass of particle 0 $m_0 \to \infty$, then the situation corresponds to the Bañados-Silk-West effect, the results agree with the ones known in literature before. In addition, we consider special cases when decay occurs in the turning point for one or all three particles. The formalism developed in this work has a model-independent character and applies not only to the Reissner-Nordström metric.

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## I. INTRODUCTION

The Penrose process (hereafter PP) means that a parent particle 0 decays to two daughter ones 1 and 2 in such a way that one of them (say, 1) has a negative energy whereas particle 2 returns to infinity with an energy bigger than an initial energy $E_0$ of particle 0. This process becomes possible if in a space-time there exists the region (called ergoregion or ergosphere) inside which $g_{\mu\nu}\xi^{\mu}\xi^{\nu} > 0$, where $g_{\mu\nu}$ are metric coefficients and $\xi$ is the Killing vector responsible for time translation [the signature of a metric is chosen to be $(-,+,+,+)$] [1,2]. Meanwhile, a counterpart of the original PP in the Reissner-Nordström (RN) background was found due to the properties of particle dynamics even in spite of the absence of such an ergoregion [3,4]. Instead, in this metric negative energies are possible in some region (called generalized ergoregion) whose border depends on the electric charge, angular momentum and mass of a particle.

More recently, we have seen a new wave of interest to the PP in new contexts. This includes the collisional version of it [5], confined one [6,7], the relation between the so-called Bañados-Silk-West (BSW) effect [8] and the PP, the PP for spinning particles [9], binaries [10,11], the PP in the background of the Vaidya space-time [12]. In case of getting formally unbounded energies, the PP is called super-Penrose (SPP) effect (see, e.g. [13] and references therein). The subtleties connected with the difference between power and efficiency of the PP was discussed in [14].

As a rule, the investigations of the BSW effect were based on the careful analysis of the vicinity of the horizon. If we characterize the proximity to the horizon by the value of the lapse function $N$, then this implies that there is a small parameter $N \ll 1$. Meanwhile, there is another approach that is based on exact formulas describing decay. For neutral particles this was realized in [15]. The meaningful difference between (i) collisions near a rotating black hole and (ii) a static charged one consists in that in case (i) there is an upper bound forbidding the SPP near black holes but there is no such a bound for (ii) [16,17].

The emphasis in [3,4] and the main part of subsequent works of this trend (see Ref. [18] for review) was made on the properties of negative energy states for a given particle in the presence of the electromagnetic field for a concrete metric [19]. Meanwhile, we are going to focus on another aspect connected with the relation between an initial state

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and its products after decay. We elucidate how characteristics of particle 0 and those of particles 1 and 2 are related with each other, which scenarios are possible in principle and what are their properties.

Although for a realistic process in astrophysics it is necessary to take into account both rotation and electromagnetic field that includes not only the electric but also the magnetic one [20–24] (see also a useful list of references on the subject in [25]), it looks reasonable at the first stage to consider a simplified problem and separate two different factors—rotation and electric charge. In the previous work [26] we gave full classification of possible scenarios for the particle decay and PP with neutral particles in the rotating background, provided all processes occur within the equatorial plane. In the present work, we consider a similar problem for electrically charged particles in the static black hole metric with the charge, i.e. the Reissner-Nodström (RN) one. It is worth noting that a similar problem was considered recently in [27] but for weakly charged black holes only and one scenario whereas in our approach we take into account this charge exactly and give classification of all possible scenarios. What is more important, our results are qualitatively different (see discussion below).

Although the main area of application is the RN metric, practically all formulas do not use its particular form. This means that our results apply to the wide class of spherically symmetric static metrics described by Eq. (1) below. Although they do not apply to axially symmetric stationary ones directly, we hope that general idea is useful for them as well, even under the presence of the electromagnetic field. The reason is that our scheme is quite general. (i) We do not restrict ourselves by a particular scenario in which decay occurs in a turning point of radial motion but build classification scheme of all possible scenarios, (ii) we find explicitly the expression for a velocity of fragments after decay, (iii) we trace the relation between the electromagnetic version [28] of the Wald approach [29] and particle dynamics of decay, thus relating the velocities and masses, (iv) we find explicitly the angular momenta of daughter particles. We will see that the roles of rotation and electric charge are very different and some scenarios of the PP forbidden inside the ergosphere of the rotating metric [26] are allowed in the generalized ergosphere of the static charged metric. Thus the present work and the previous one [26] constitute the mutually complementary pair that we hope to combine in a unified picture later.

A separate question is the dependence of the efficiency of the process on the position of the decay point. There is a popular approach in which this point is chosen as close to the horizon as possible, with point of decay coinciding with a turning point of particle 0 (see, e.g., Sec. 5.1 of [20]). Meanwhile, we show that this is not always so and depends strongly on a type of scenario the list of which we discuss carefully case by case. Bearing in mind future application to astrophysical process, this seems to be important enough since it allows to describe different cases without additional artificial assumptions about a particular type of decay.

The paper is organized as follows. In Sec. II we list equations of particle motion in the RN background. In Sec. III we give the exact formulas that related characteristics of particle 0 and particles 1, 2. In Sec. IV we suggest classification of scenarios when particle 0 is ingoing. In Sec. VI we make emphasis on a type of decay when particles 1 and 2 are ejected along the trajectory of particle 0. In Sec. VII we consider a special case when an escaping particle is massless. In Sec. VIII we consider some properties of scenario I including bounds on the energy and angular momentum of an escaping particle and decay in the near-horizon region. In Sec. IX we discuss the situation when the point of decay coincides with the turning point of radial motion for particle 0, particle 2 or with a common turning point of all three particles. In Sec. X we derive the conditions necessary for the PP process in the scenarios under discussion. In Sec. XI we reformulate our result kinematically, using the velocities of particles and their Lorentz gamma factors. In Sec. XII we consider the scenario in which particle 0 is outgoing. In Sec. XIII we consider the conditions under which the maximum of the efficiency is achieved on the horizon depending on scenario. In Sec. XIV we summarize the main features of the considered scenarios with short comments. In Sec. XV we compare our results with some other ones recently published and explain the origin of discrepancy. In Sec. XVI we display the relation between our approach and that used in literature for the description of the BSW effect before. In Sec. XVII we give the summary of main results. In the Appendix we derive some useful inequalities relevant for the PP near the horizon.

We use system of units in which fundamental constants $G = c = 1$.

## II. CHARGED PARTICLES IN THE METRIC OF REISSNER-NORDSTRÖM BLACK HOLE

We consider the metric that has the form

$$ds^2 = -dt^2 f + \frac{dr^2}{f} + r^2 d\omega^2, \tag{1}$$

with $d\omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. We will mainly deal with the RN metric, then $f \equiv N^2 = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} = \frac{(r-r_+)(r-r_-)}{r^2}$. Here, $M$ is a black hole mass, $Q$ is its electric charge (for definiteness $Q > 0$), $r_+ = M + \sqrt{M^2 - Q^2}$ being the event horizon radius, $r_- = M - \sqrt{M^2 - Q^2}$ the inner horizon radius. Meanwhile, basic results are extendable to an arbitrary $f$ outside the event horizon.

Let us consider motion of a charged particle in this space-time without additional forces. Then, there are two integrals of motion. As the metric does not depend on time and angle $\phi$, the energy $E$ and angular momentum $L$ are conserved. Motion occurs within a plane which we choose to be $\theta = \frac{\pi}{2}$. Then, equations of motion read (dot denotes derivative with respect to the proper time $\tau$)

$$\dot{t} = \frac{X}{mf}, \tag{2}$$

$$X = E - q\varphi, \tag{3}$$

$$\dot{\phi} = \frac{L}{mr^2}, \tag{4}$$

$$m\dot{r} = \sigma P, \tag{5}$$

$$P = \sqrt{X^2 - N^2\tilde{m}^2}, \tag{6}$$

$$\tilde{m}^2 = m^2 + \frac{L^2}{r^2}. \tag{7}$$

Here, $q$ is a particle's charge, $\varphi = \frac{Q}{r}$ is the Coulomb potential of the RN metric, $\sigma = \pm 1$. The forward-in-time condition $\dot{t} > 0$ requires

$$X \geq 0, \tag{8}$$

outside the horizon $X > 0$.

From $P^2 \geq 0$ we have a more tight condition than (8):

$$X \geq \tilde{m}N. \tag{9}$$

## III. DECAY TO TWO PARTICLES: GENERAL SCENARIO

Let in some point $r = r_d$ particle 0 decay to particles 1 and 2. For simplicity, we assume that all three particles move within the same plane. In the point of decay the conservation laws give us

$$E_0 = E_1 + E_2, \tag{10}$$

$$L_0 = L_1 + L_2, \tag{11}$$

$$q_0 = q_1 + q_2. \tag{12}$$

It follows from (10)–(12) that

$$X_0 = X_1 + X_2. \tag{13}$$

The conservation of the radial component of momentum reads

$$\sigma_0 P_0 = \sigma_1 P_1 + \sigma_2 P_2. \tag{14}$$

Let particle 0 move with decreasing $r$, so $\sigma_0 = -1$. It is clear that combination $\sigma_1 = \sigma_2 = +1$ is impossible since this would contradict Eq. (14). For definiteness, we assume that particle 1 moves after decay with $\sigma_1 = -1$. Moreover, we imply that in the frame comoving with particle 0 (which is the center of mass frame of particles 1 and 2 in the point of decay), particle 1 moves in the inward direction as well. Meanwhile, $\sigma_2$ can have any sign. In the center of mass frame particle 2 moves in the outward direction with some velocity $v_2$. However, if we pass to the frame comoving with particle 0 that moves in the inward direction with the velocity $V_2$ in the static frame, both cases are possible: $\sigma_2 = -1$ or $\sigma_2 = +1$ depending on which value of the velocity is bigger (below we will discuss this issue in more detail).

Then we have

$$P_0 = P_1 - \sigma_2 P_2. \tag{15}$$

Solving equations of motion, we obtain

$$X_1 = \frac{X_0}{2\tilde{m}_0^2}\tilde{b}_1 - \delta\frac{P_0\sqrt{\tilde{d}}}{2\tilde{m}_0^2}, \tag{16}$$

$$X_2 = \frac{1}{2\tilde{m}_0^2}\left(X_0\tilde{b}_2 + P_0\delta\sqrt{\tilde{d}}\right), \tag{17}$$

$$P_1 = \left|\frac{P_0\tilde{b}_1 - \delta X_0\sqrt{\tilde{d}}}{2\tilde{m}_0^2}\right|, \tag{18}$$

$$P_2 = \left|\frac{P_0\tilde{b}_2 + \delta X_0\sqrt{\tilde{d}}}{2\tilde{m}_0^2}\right|. \tag{19}$$

Here, $\delta = \pm 1$,

$$\tilde{b}_{1,2} = \tilde{m}_0^2 + \tilde{m}_{1,2}^2 - \tilde{m}_{2,1}^2, \tag{20}$$

$$\tilde{d} = \tilde{b}_1^2 - 4\tilde{m}_0^2\tilde{m}_1^2 = \tilde{b}_2^2 - 4\tilde{m}_0^2\tilde{m}_2^2. \tag{21}$$

For what follows, it is also convenient to introduce also corresponding quantities without tilde:

$$b_{1,2} = m_0^2 + (m_{1,2}^2 - m_{2,1}^2), \tag{22}$$

$$d = b_1^2 - 4m_0^2m_1^2 = b_2^2 - 4m_0^2m_2^2. \tag{23}$$

Then, it can be obtained from (11) that

$$\tilde{b}_{1,2} = b_{1,2} + \frac{2L_0L_{1,2}}{g_\phi}, \tag{24}$$

$$\tilde{d} = d + 4b_1\frac{L_0L_1}{r^2} - \frac{4L_1^2}{r^2}m_0^2 - 4\frac{L_0^2}{r^2}m_1^2, \tag{25}$$

$$\tilde{d} = d + 4b_2\frac{L_0L_2}{r^2} - \frac{4L_2^2}{r^2}m_0^2 - 4\frac{L_0^2}{r^2}m_2^2. \tag{26}$$

Equations (16)–(19) are the counterpart of Eqs. (19), (20), (26), (27) in [15] obtained there for rotating metrics.

The essential difference consists in the expression for $X$. Now it is given by (3) and does not depend on $L$. Meanwhile, in [15] $X = E - \omega L$, the $\omega$ is the metric coefficient responsible for rotation. It is the fact that $L$ does not enter the expression for $X$ that allows some scenarios which were forbidden in the rotating case (see below).

### A. Unconditional bounds on mass

From the condition $\tilde{d} \geq 0$ one can deduce that

$$\tilde{m}_0 \geq \tilde{m}_1 + \tilde{m}_2. \tag{27}$$

The same condition requires that, according to (25), (26)

$$L_{1,2}^{(-)} \leq L_{1,2} \leq L_{1,2}^{(+)}, \tag{28}$$

where

$$L_{1,2}^{(\pm)} = \frac{b_{1,2} L_0}{2m_0^2} \pm \frac{\sqrt{d} r_d}{2m_0^2} \tilde{m}_0. \tag{29}$$

If one takes into account the definition (7) and the conservation law (11), then one can obtain by straightforward algebraic manipulations that

$$m_0 \geq m_1 + m_2 \tag{30}$$

as well.

One can also use the conservation law for four-momenta in the covariant form

$$p_0^\mu = p_1^\mu + p_2^\mu. \tag{31}$$

Then, taking the square and taking into account that for future-directed four-vectors $p_{1\mu} p_2^\mu < 0$, it is easy to obtain

$$m_0^2 \geq m_1^2 + m_2^2. \tag{32}$$

This is just Eq. (4.11) in [20] and Eq. (3.11) in [21], written in our notations. Meanwhile, Eq. (30) is more tight than (32). It can be derived if one uses the property $p_{1\mu} p_2^\mu < -m_1 m_2$, which is more tight than $p_{1\mu} p_2^\mu < 0$ used in [20,21].

## IV. CLASSIFICATION OF SCENARIOS WITH RESPECT TO PARAMETERS IN GENERAL

Each scenario of decay can be characterized by the set $(\sigma_2, h_2, h_1, \delta)$, where

$$h_{1,2} = \mathrm{sign}(\tilde{b}_{1,2} N - 2\tilde{m}_{1,2} X_0). \tag{33}$$

The quantities $h_1$ and $h_2$ arise in a natural way, when we sort out combinations with different signs inside the absolute values in (18) and (19) and determine which term dominates. As a result, one obtains the following types of scenarios:

(I) $(\sigma_2 = +1,\ h_2 = 0 \text{ or } h = +1,\ \delta = -1)$

$$P_1 - P_2 = P_0, \tag{34}$$

$$P_2 = \frac{(X_0 \sqrt{\tilde{d}} - P_0 \tilde{b}_2)}{2\tilde{m}_0^2}, \tag{35}$$

$$P_1 = \frac{P_0 \tilde{b}_1 + X_0 \sqrt{\tilde{d}}}{2\tilde{m}_0^2}, \tag{36}$$

$$X_1 = \frac{X_0}{2\tilde{m}_0^2} \tilde{b}_1 + \frac{P_0 \sqrt{\tilde{d}}}{2\tilde{m}_0^2}, \tag{37}$$

$$X_2 = \frac{1}{2\tilde{m}_0^2} (X_0 \tilde{b}_2 - P_0 \sqrt{\tilde{d}}). \tag{38}$$

(II) $(\sigma_2 = -1,\ h_2 = -1,\ \delta = -1)$

$$P_1 + P_2 = P_0, \tag{39}$$

$$P_2 = \frac{\left(P_0 \tilde{b}_2 - X_0 \sqrt{\tilde{d}}\right)}{2\tilde{m}_0^2}, \tag{40}$$

$$P_1 = \frac{P_0 \tilde{b}_1 + X_0 \sqrt{\tilde{d}}}{2\tilde{m}_0^2}, \tag{41}$$

$$X_1 = \frac{X_0}{2\tilde{m}_0^2} \tilde{b}_1 + \frac{P_0 \sqrt{\tilde{d}}}{2\tilde{m}_0^2}, \tag{42}$$

$$X_2 = \frac{1}{2\tilde{m}_0^2} \left(X_0 \tilde{b}_2 - P_0 \sqrt{\tilde{d}}\right). \tag{43}$$

(III) $(\sigma_2 = -1,\ h_1 = 0 \text{ or } h = -1,\ \delta = +1)$

$$P_2 = \frac{X_0 \sqrt{\tilde{d}} + P_0 \tilde{b}_2}{2\tilde{m}_0^2}, \tag{44}$$

$$P_1 = \frac{P_0 \tilde{b}_1 - X_0 \sqrt{\tilde{d}}}{2\tilde{m}_0^2}, \tag{45}$$

$$X_1 = \frac{X_0}{2\tilde{m}_0^2} \tilde{b}_1 - \frac{P_0 \sqrt{\tilde{d}}}{2\tilde{m}_0^2}, \tag{46}$$

$$X_2 = \frac{1}{2\tilde{m}_0^2} (X_0 \tilde{b}_2 + P_0 \sqrt{\tilde{d}}). \tag{47}$$

Scenarios II and III can be obtained from each other by interchange of particles 1 and 2. In this sense, they are equivalent, so for definiteness we will consider scenario II.

## V. TYPES OF SCENARIOS AND THEIR MEANING

Formulas of the previous section describe general relations between a parent particle and daughter ones. Meanwhile, for physical purposes, we are interested in more concrete scenarios. The most simple and popular one consists in such a decay of particle 0 to 1 and 2 that all three particles are located in the corresponding turning points. Meanwhile, in any real process, one cannot guarantee the validity of this condition and, therefore, one is led to considering more expanded set of possible scenarios. In the first place, this concerns the situation when all three particles are moving and the products of decay are ejected along the trajectory. This is of special interest since this leads to the most effective (for a given state if particle 0) kind of process. (This will be proven in the next section.) In turn, here there are different cases depending on how a daughter particle move after decay, whether it moves towards a black hole or towards infinity. There are also intermediate cases when some of particles (but not all of them) are in their turning points.

The key ingredient of the Penrose process is the existence of the ergoregion (in the standard PP) or generalized ergoregion (in the electric version of PP), where particle energy can be, in principle, negative. In this respect it was noticed earlier [26] that for a rotating black hole some kinds of decay, which seem to be compatible with the existence of negative states for an individual particle, were forbidden in the ergoregion. It is of interest to elucidate, whether or not this happens for charged particles in the generalized ergoregion of a static charged black hole.

More precisely, we consider the following scenarios.

I: particle 2 moves in the outward direction, particle 1 moves in the inward one. Further, particle 1 can fall in a black hole while particle 2 can escape.

II: particles 1 and 2 move in the same direction as particle 0, i.e. inward direction.

In scenarios I and II it is implied that before decay particle 0 moved in the inward direction, so it follows from the momentum conservation that both particles 1 and 2 cannot move after decay in the outward direction.

B ("bounce"). Particle 0 bounces back from the potential barrier, moves in the outward direction and decays afterwards. Preliminarily, this scenario gives us the most efficient possibility when a daughter particle moves in the same direction as particle 0. Formally, this is similar to scenario II but in scenario II both particles move towards a black hole whereas in scenario B they start to move to infinity.

For completeness, we take into account also the existence of a turning point.

TP2: point of decay of particle 0 coincides with the turning point of decay of particle 2.

TP0: point of decay of particle 0 coincides with its turning point but not with the turning points of particles 1 and 2.

TP3: decay occurs in the common point of decay for all three particles.

Below, we consider all these scenarios case by case. Bearing in mind potential applications to processes near a black hole, of primary interest is the near-horizon behavior of particles in these processes. The results are collected in Sec. XIV below where we also indicate near-horizon properties. A special questin about the role of a horizon in achieving the maximum of efficiency is considered in Sec. XIII.

## VI. EJECTION ALONG TRAJECTORY: MAXIMIZATION/MINIMIZATION OF ENERGY AND CLASSIFICATION OF SCENARIOS

Now, we consider a special type of decay in which products of decay are ejected along the trajectory of the original particle.

In doing so,

$$\frac{P_1}{P_0} = \frac{L_1}{L_0}, \tag{48}$$

whence

$$\left(P_0\tilde{b}_1 - \delta X_0\sqrt{\tilde{d}}\right)L_0 - 2\tilde{m}_0^2 L_1 P_0 = 0. \tag{49}$$

For definiteness, we assumed that signs of angular momenta of particles 0 and 1 coincide.

The interest to such a type of trajectory is motivated by the fact that it is the most efficient process (with other parameters of scenario fixed). The particle that moves in the same direction as particle 0 acquires the maximum possible energy, the other one that moves in the opposite direction has the minimum one. To show this, one can consider, say, energy $E_1$ as a function of $L_1$:

$$E_1 = \frac{X_0}{2\tilde{m}_0^2}\tilde{b}_1 - \delta\frac{P_0\sqrt{\tilde{d}}}{2\tilde{m}_0^2} + q_1\varphi. \tag{50}$$

In the case of extremum, we have

$$\frac{\partial E_1}{\partial L_1} = 0. \tag{51}$$

After taking the square and performing a number of algebraic manipulations, we obtain just expressions for $X_1$ and $X_2$ that can be obtained from (48). We list them below explicitly.

There are two different possible scenarios here. We consider them separately.

### A. Scenario I, $\sigma_2 = +1$, $\delta = -1$, $h_2 = +1$

Remarkably, it turns out that after somewhat long algebraic manipulation, one can get rid off tilted quantities and express the results in a rather simple form in terms of quantities without tilde:

$$X_2 = X_0 \frac{b_2}{2m_0^2} - \frac{\sqrt{d}\sqrt{X_0^2 - m_0^2 N^2}}{2m_0^2}, \tag{52}$$

$$X_1 = X_0 \frac{b_1}{2m_0^2} + \frac{\sqrt{d}\sqrt{X_0^2 - m_0^2 N^2}}{2m_0^2}, \tag{53}$$

$$P_2 = \frac{P_0}{2m_0^2\sqrt{X_0^2 - m_0^2 N^2}}\left(X_0\sqrt{d} - b_2\sqrt{X_0^2 - m_0^2 N^2}\right), \tag{54}$$

$$P_1 = \frac{P_0}{2m_0^2\sqrt{X_0^2 - m_0^2 N^2}}\left(b_1\sqrt{X_0^2 - m_0^2 N^2} + X_0\sqrt{d}\right). \tag{55}$$

In general in Eqs. (16)–(19) one of angular momenta (say, $L_1$) was a free parameter. However, now this is not so since there is an additional constraint (48) that leads to

$$L_2 = \frac{L_0}{2m_0^2}\left(b_2 - \frac{\sqrt{d}}{\sqrt{X_0^2 - N^2 m_0^2}} X_0\right), \tag{56}$$

$$L_1 = \frac{L_0}{2m_0^2}\left(b_1 + \frac{\sqrt{d}X_0}{\sqrt{X_0^2 - m_0^2 N^2}}\right). \tag{57}$$

One can check that Eqs. (6) and (48) do hold for each particle 1 and 2.

The requirement $d \geq 0$ entails the inequality

$$m_0 \geq m_1 + m_2, \tag{58}$$

similar to (27) and coinciding with (30).

It follows from $P_2 \geq 0$ that

$$b_2\sqrt{X_0^2 - m_0^2 N^2} \leq X_0\sqrt{d}, \tag{59}$$

whence

$$X_0 \leq \frac{N b_2}{2m_2}. \tag{60}$$

As the expression inside the square root should be nonnegative, $X_0 \geq m_0 N$. It holds automatically, if (9) does so. Combining the above inequalities and (9), we have

$$\tilde{m}_0 N \leq X_0 \leq \frac{N b_2}{2m_2}. \tag{61}$$

For $L_0 = 0$ (so $\tilde{m}_0 = m_0$) this is consistent with (60) automatically since

$$b_2 \geq 2m_0 m_2 \tag{62}$$

due to (58).

### B. Scenario II

$\sigma_2 = -1$, $\delta = -1$

$$P_1 + P_2 = P_0, \tag{63}$$

$$L_1 = \frac{L_0}{2m_0^2}\left(b_1 + \frac{\sqrt{d}X_0}{\sqrt{X_0^2 - m_0^2 N^2}}\right), \tag{64}$$

$$L_2 = \frac{L_0}{2m_0^2}\left(b_2 - \frac{\sqrt{d}X_0}{\sqrt{X_0^2 - m_0^2 N^2}}\right), \tag{65}$$

$$X_1 = \frac{X_0 b_1}{2m_0^2} + \frac{\sqrt{d}}{2m_0^2}\sqrt{X_0^2 - m_0^2 N^2}, \tag{66}$$

$$X_2 = \frac{b_2 X_0}{2m_0^2} - \frac{\sqrt{d}}{2m_0^2}\sqrt{X_0^2 - m_0^2 N^2}, \tag{67}$$

$$P_1 = \frac{P_0}{2m_0^2\sqrt{X_0^2 - N^2 m_0^2}}\left(b_1\sqrt{X_0^2 - m_0^2 N^2} + X_0\sqrt{d}\right), \tag{68}$$

$$P_2 = \frac{P_0}{2m_0^2\sqrt{X_0^2 - N^2 m_0^2}}\left(b_2\sqrt{X_0^2 - m_0^2 N^2} - X_0\sqrt{d}\right). \tag{69}$$

It follows from $P_2 \geq 0$ that

$$X_0 \geq \frac{b_2 N}{2m_2} \tag{70}$$

and

$$N m_2 \leq X_2 \leq \frac{b_2}{2m_0^2} X_0, \tag{71}$$

where we took into account (70) and (22), (23).

It is worth noting that in scenario I $L_2$ has the sign opposite to that of $L_0$, in scenario II they coincide.

If $Q \to 0$, then we arrive at the Schwarzschild metric. Then, our formulas agree with [30]. In this case, there is no the PP and the energy of an escaping particle $E_2 < E_0$ but $E_2 \neq 0$.

## VII. MASSLESS CASE

A special case arises when particle 2 is massless since condition (70) is no longer valid if $m_2 = 0$. In turn, this means that scenario II fails. This can be explained as follows. In general, scenario II is realized when particle 2 is ejected in the outward direction in the frame comoving with particle 0 (that is also the center of mass frame for particles 1 and 2). However, when we pass to the stationary frame, it is drifted in the inward direction due to motion of particle 0 (see Sec. XI below for discussion of local Lorentz transformations). However, if particle 2 is massless, it moves with a speed of light and inward motion of particle 0 cannot overcome outward motion of particle 2. As a result, particle 2 moves outwardly that corresponds to scenario I, not II.

Therefore, we must use Eqs. (52) and (54), in which we put $m_2 = 0$ that gives us $b_2 = d = m_0^2 - m_1^2$. Thus we have

$$X_2 = \frac{b_2}{2m_0^2}\left(X_0 - \sqrt{X_0^2 - m_0^2 N^2}\right), \tag{72}$$

$$P_2 = \frac{P_0 b_2}{2m_0^2\sqrt{X_0^2 - m_0^2 N^2}}\left(X_0 - \sqrt{X_0^2 - N^2 m_0^2}\right). \tag{73}$$

In the horizon limit $N \to 0$,

$$X_2 \approx \frac{b_2 N^2}{4X_0}, \qquad E \approx \frac{q_2 Q}{r_+} + \frac{b_2 N^2}{4X_0}, \tag{74}$$

$$X_1 \approx X_0 - \frac{b_2 N^2}{4X_0}, \qquad E_1 \approx E_0 - \frac{q_2 Q}{r_+} - \frac{b_2 N^2}{4X_0}. \tag{75}$$

From another hand, if we put $m_1 = 0$, this does not give rise to any difficulties since particle 1 moves inwardly in both frames, so scenario II is valid as well.

## VIII. PROPERTIES OF SCENARIO I

We are interested mainly in scenario I since it is this scenario in which particle 2 can escape to infinity. First of all, we would like to stress the difference between scenario I and its counterpart for processes with neutral particles in the background of rotating black holes. In the latter case, decay of type I is impossible in the ergoregion at all [26]. Meanwhile, now it is allowed inside a generalized ergoregion. Below, we discuss some details concerning just scenario I.

### A. Upper bound on the angular momentum

For scenario I, there are restrictions (61). Both inequalities here are consistent with each other, provided

$$\tilde{m}_0 \le \frac{b_2}{2m_2}. \tag{76}$$

Taking the square, we obtain the restriction on a value of $L_0$:

$$L_0^2 \le L_m^2 = \frac{d r_d^2}{4m_2^2}. \tag{77}$$

Let $d \to 0$. This entails

$$m_0 = m_1 + m_2. \tag{78}$$

Then, scenario I is possible if $L_0 \to 0$ only. According to (56) and (57) this entails that $L_{1,2} \to 0$ as well, so all three particles move radially. This scenario is realized in the confined Penrose process [6,7]. There is no similar restriction on $L_0$ in scenario II.

### B. Upper bound on the energy of escaping particle

It follows from (60) that scenario I can be realized for $(X_0)_d \le \frac{N_d b_2}{2m_2}$ only. Then, from (52) and (22), (23) we have

$$N m_2 \le X_2 \le \frac{b_2^2 N}{4m_0^2 m_2}. \tag{79}$$

Thus for the escaping particle we have the upper bounds

$$q_2 \frac{Q}{r_d} + N m_2 \le E_2 \le q_2 \frac{Q}{r_d} + \frac{b_2^2 N}{4m_0^2 m_2}. \tag{80}$$

### C. Near-horizon limit

Assuming that $m_2 \neq 0$, let us consider

$$(X_0)_d = a\frac{N_d b_2}{2m_2}, \tag{81}$$

where according to (61) the coefficient $a$ (pure number) obeys inequalities

$$a_m \le a \le 1, \qquad a_m = 2\frac{m_2 \tilde{m}_0}{b_2} \tag{82}$$

in combinations with inequalities (76), (77).

If $N_d \to 0$ while keeping $a$ constant, then we obtain an approximate equality:

$$E_2 \approx q_2 \frac{Q}{r_+} + c m_2 N_d, \tag{83}$$

where

$$c = \frac{b_2^2 a - \sqrt{d}\sqrt{a^2 b_2^2 - 4m_0^2 m_2^2}}{4m_2^2 m_0^2}. \tag{84}$$

One can check that $\frac{\partial c}{\partial a} \le 0$. Therefore,

$$1 \le c \le c(a_m) = c_m, \tag{85}$$

where

$$c_m = \frac{\tilde{m}_0(r_+) b_2}{2m_2 m_0^2} - \frac{|L_0|\sqrt{d}}{2m_2 m_0^2 r_+}. \tag{86}$$

In particular, if $L_0 = 0$, then

$$c_m = \frac{b_2}{2m_2 m_0} \ge 1, \tag{87}$$

which agrees with (80). If $L_0 = L_m$, then we obtain from (77)

$$c_m = 1. \tag{88}$$

If $a = 1$, then it follows that $c = 1$ and we return to (80) with equality instead of inequality, so $E_2 \approx q_2 \frac{Q}{r_+} + m_2 N_d$.

Thus the correction to the main term in $X_2$ has the order $N$. We arrive at an important conclusion. If decay occurs near the horizon, the particle that moves to infinity should be near critical with $X_0 = O(N_d)$. Usual particles cannot escape from the horizon at all.

In the massless case $m_2 = 0$ the correction has the order $N^2$ according to (74).

For particle 1 we have in the near-horizon limit

$$E_1 \approx q_1 \frac{Q}{r_+} + N_d\left(\frac{ab_2}{2m_2} - c\right). \tag{89}$$

## IX. SPECIAL TYPE OF SCENARIO: DECAY IN TURNING POINT

There are special cases to be considered separately. They arise when decay occurs in the turning point. (Hereafter, by turning point we imply for brevity a turning point for radial motion, a particle can have in general an angular moment and nonzero angular velocity.) It follows from the conservation law (14) that the point of decay cannot coincide with the turning point for two particles precisely. Either it is the turning point for (i) only one particle or (ii) for all three at once. It follows from (14) or (55), (68) that in scenario I, case (i) is possible if there exists a turning point for particle 2, not for particle 1. Now, we will discuss (i) and (ii) case by case.

### A. Case (i), turning point for particle 2 (TP2)

Let particle 2 arise in its own turning point, so $P_2 = 0$ but $P_0 \neq 0$. We call it scenario TP2. It follows from (16)–(19) that

$$X_2 = \frac{2\tilde{m}_2^2 X_0}{\tilde{b}_2}, \tag{90}$$

$$X_1 = \frac{X_0}{\tilde{b}_2}(\tilde{m}_0^2 - \tilde{m}_1^2 - \tilde{m}_2^2), \tag{91}$$

$$X_0 = \frac{\tilde{b}_2 N}{2\tilde{m}_2}. \tag{92}$$

For ejection along the trajectory, the case under discussion is realized in scenario II and we obtain from (54) and (69) with $P_2 = 0$, $P_0 \neq 0$ that

$$X_0 = \frac{b_2 N}{2m_2}. \tag{93}$$

$$X_2 = m_2 N = \frac{2m_2^2}{b_2} X_0. \tag{94}$$

Then, (13) gives us

$$X_1 = \frac{X_0}{b_2}(m_0^2 - m_1^2 - m_2^2). \tag{95}$$

From (56) and (65) we have

$$L_2 = 0, \quad L_1 = L_0. \tag{96}$$

One can check using (93) and (94) that we can also rewrite $X_2$ in the form

$$X_2 = \frac{b_2}{2m_0^2} X_0 - \frac{\sqrt{d}}{2m_0^2}\sqrt{X_0^2 - m_0^2 N^2}. \tag{97}$$

### B. Case (i), turning point for particle 0 only (TP0)

In general scenario we take the limit $P_0 \to 0$. This means that decay occurs in the turning point of particle 0. For shortness, we call it TP0. Particles 1 and 2 may have nonzero $P_1$ and $P_2$, so the point $r_d$ is not the turning point for them. It is clear from the conservation law (14) that scenario TP0 can be realized in scenario I but not in II. We obtain from (16)–(19) that

$$P_1 = P_2 = \frac{N\sqrt{\tilde{d}}}{2\tilde{m}_0}, \tag{98}$$

$$X_1 = \frac{N}{2\tilde{m}_0}\tilde{b}_1, \tag{99}$$

$$X_2 = \frac{N}{2\tilde{m}_0}\tilde{b}_2. \tag{100}$$

Three quantities $E_0$, $L_0$, $N$ are related by one equation

$$X_0 = \tilde{m}_0 N. \tag{101}$$

It follows from (101) immediately that

$$L_0^2 = r_d^2 \frac{X_0^2 - m_0^2 N^2}{N^2}. \tag{102}$$

In general, $L_1$ (or $L_2 = L_0 - L_1$) is a free parameter. It is only restricted by the condition (28).

Let particles 1 and 2 be ejected in point $r_d$ along the trajectory of particle 0. Now, it follows from (48) that either $L_0 = 0$ or $P_1 = 0 = P_2$. By assumption, the second situation is now impossible and will be considered in the next subsection. Now, we put $L_0 = 0$, whence $\tilde{m}_0 = m_0$, so

$$X_0 = m_0 N. \tag{103}$$

Then, it follows from (52)–(57) that

$$X_1 = X_0 \frac{b_1}{2m_0^2} = \frac{b_1}{2m_0} N, \tag{104}$$

$$X_2 = X_0 \frac{b_2}{2m_0^2} = \frac{b_2}{2m_0} N. \tag{105}$$

### C. Case (ii), turning point for all three particles (TP3)

Let us call this scenario TP3. It follows from (6) that

$$X_0 = \tilde{m}_0 N, \tag{106}$$

$$X_1 = \tilde{m}_1 N, \tag{107}$$

$$X_2 = \tilde{m}_2 N. \tag{108}$$

With (13) taken into account, this leads to

$$\tilde{m}_0 = \tilde{m}_1 + \tilde{m}_2. \tag{109}$$

Then, after some algebraic manipulations, we can again obtain the same expressions for $X_1$ and $X_2$ as in scenarios I and II, but with one important difference. In scenario I, the expression for $X_2$ and, correspondingly, energy $E_2$ for particle 2 that is enable to escape to infinity, contained sign "minus" before the square root. This was due to the necessity to have non-negative factor in (54) inside parentheses. Meanwhile, now this is irrelevant since $P_2 = 0$ due to the factor $P_0 = 0$. As a result, escaping particle 2 can have not only sign "minus" but also sign "plus," so we are free to take

$$E_2 = q_2\varphi + \frac{b_2}{2m_0^2} X_0 + \frac{\sqrt{d}}{2m_0^2}\sqrt{X_0^2 - m_0^2 N^2}. \tag{110}$$

If so, for particle 1 we have

$$E_1 = q_1\varphi + \frac{b_1}{2m_0^2} X_0 - \frac{\sqrt{d}}{2m_0^2}\sqrt{X_0^2 - m_0^2 N^2}. \tag{111}$$

Equivalently,

$$E_2 = q_2\varphi + \frac{b_2}{2m_0^2}\tilde{m}_0 N + \frac{\sqrt{d}|L_0|}{2m_0^2 r_d} N, \tag{112}$$

$$E_1 = q_1\varphi + \frac{b_1}{2m_0^2}\tilde{m}_0 N - \frac{\sqrt{d}|L_0|}{2m_0^2 r_d} N. \tag{113}$$

From (109) we have, taking the square and solving the quadratic equation that

$$L_2 = \frac{L_0 b_2 + \mathrm{sign}L_0 \sqrt{d}\tilde{m}_0 r_d}{2m_0^2} = L_2^{(+)}, \tag{114}$$

$$L_1 = \frac{L_0 b_1 - \mathrm{sign}L_0 \sqrt{d}\tilde{m}_0 r_d}{2m_0^2} = L_1^{(-)}, \tag{115}$$

where $L^{(+)}$, $L^{(-)}$ are defined in (29). Distinction between particles 1 and 2 is almost conditional now but with the reservation that the sign in the second term in (112) or in (113) correlates with that in (114), (115).

In a sense, when from scenarios I and II we pass to scenario TP3, there is an exchange of branches "plus" and "minus" between both particles in the point where $P_0 = 0$ for nonradial motion. The aforementioned difference disappears if $L_0 = 0$. Then, it follows from (106) that $X_0 = m_0 N$. If also $d = 0$, then we return to the process described in [6,7]. However, for nonzero $L_0$ the case under discussion is more general.

When both $P_0 \to 0$ and $\tilde{d} \to 0$, all scenarios discussed in this section agree with each other.

## X. THRESHOLD FOR THE PENROSE PROCESS

Now, we are going to elucidate, when the Penrose process is possible. For rotating black holes, this requires the existence of the ergoregion where, by definition, $g_{00} > 0$ [1]. In the RN case, $g_{00}$ does not change the sign outside the horizon and there is no ergoregion in a usual sense but there exists its analog—so-called generalized ergosphere [3,4] where negative energy are allowed. It is sensitive to the properties of particles. When $E_1 < 0$, the PP becomes possible and particle 2 with an excess of energy goes to infinity. Meanwhile, in scenario II both particles fall in a black hole. Therefore, we consider scenario I. The condition $E_1 < 0$ leads to

$$X_0 \frac{b_1}{2m_0^2} + q_1\varphi + \frac{\sqrt{d}\sqrt{X_0^2 - m_0^2 N^2}}{2m_0^2} \le 0. \tag{116}$$

For $q_1 \ge 0$ this is impossible. Let $q_1 = -|q_1| < 0$. Then this condition takes the form

$$2|q_1| m_0^2 \varphi \ge X_0 b_1 + \sqrt{d}\sqrt{X_0^2 - m_0^2 N^2}. \tag{117}$$

Moreover, the SPP is now possible as well. Indeed, for a fixed $q_0$, the energy $E_2 = X_2 + q_2\varphi$ formally grows unbounded when $q_2 \to \infty$. The fact that the electromagnetic field can significantly enhance the efficiency of the Penrose process was pointed in [20,21]. Meanwhile, for sufficiently high $q_2$ this is quite generic feature that does not require the presence of a black hole. This can happen even in the flat space-time [31].

Let us consider an important particular case. If, in the framework of scenario I, decay occurs in the turning point of particle 2, we can substitute here (93) in the point of decay and obtain

$$E_2 = q_2\varphi + Nm_2, \tag{118}$$

$$E_1 = q_1\varphi + N\frac{(b_1b_2+d)}{4m_0^2m_2} = q_1\varphi + N\frac{m_0^2-m_1^2-m_2^2}{2m_2}. \tag{119}$$

It follows from (119) and (22), (23) that in this case (117) can be rewritten in the form

$$|q_1|\varphi > N\frac{m_0^2-m_1^2-m_2^2}{2m_2}. \tag{120}$$

If all $q_i \to 0$, then the results coincide with those obtained in the static limit of a rotating black hole—see Eq. (111) in [29,30], Eq. (3.30) in [26,32]. Then, the Penrose process is impossible as it should be. For $|q_1| \neq 0$, the PP is possible in the scenario under discussion, if the point of decay, according to (120), is located sufficiently close to the horizon.

In the particular case, when $d = 0$, we have $m_0 = m_1 + m_2$ [6] and the above condition turns into

$$|q_1|\varphi > N_d m_1, \tag{121}$$

typical of the PP for pure radial motion, if decay occurs in the turning point [6].

Equation (121) can be interpreted as the statement that the electrostatic energy of particle in an external field should be bigger than the red-shifted energy measured by a local observer. It is worth noting that in the case under discussion $r_d$ is the turning point for particle 2, but not for particles 0 and 1. In particular, Eq. (78) is not satisfied in general.

It is instructive to note, for comparison, that in the case of rotating metrics extraction of energy in scenario I inside the ergoregion cannot be realized at all (see Ref. [26], Sec. VI A).

## XI. VELOCITIES AND GAMMA FACTORS FOR SCENARIO I

The above results are given in terms of particles' masses. Meanwhile, the approach can be reformulated kinematically in terms of velocities. For definiteness, let us consider scenario I, so $\sigma_2 = +1$. Then, one can check using (130) and (52) the validity of the equation

$$V_2 = \frac{v_2 - V_0}{1 - v_2V_0} \tag{122}$$

that is nothing but the Lorentz law of adding velocities. Now, the gamma factor of relative motion for particles 0 and 2 $\gamma_{02} = -u_{0\mu}u_2^{\mu}$, whence

$$m_2m_0\gamma_{02} = \frac{X_0X_2 + P_0P_2}{N^2} - \frac{L_0L_2}{r^2}. \tag{123}$$

In (123) we used equations of motion (2)–(6) and took into account that both particles move in the opposite direction that gives us sign "plus" in the numerator of Eq. (123). Then, one obtains

$$\gamma_{02} = \frac{b_2}{2m_0m_2}, \qquad v_2 = \frac{\sqrt{d}}{b_2}. \tag{124}$$

Thus we can rewrite the formula for $X_2$ in the form

$$X_2 = \frac{m_2}{m_0}\gamma_{02}\left(X_0 - v_2\sqrt{X_0^2 - m_0^2N^2}\right). \tag{125}$$

In a similar manner, we find

$$m_2m_0\gamma_{01} = \frac{X_0X_1 - P_0P_1}{N^2} - \frac{L_0L_1}{r^2}, \tag{126}$$

$$\gamma_{01} = \frac{b_1}{2m_0m_1}, \qquad v_1 = \frac{\sqrt{d}}{b_1}, \tag{127}$$

$$V_1 = \frac{v_1 + V_0}{1 + V_1V_0}, \tag{128}$$

$$X_1 = \frac{m_1}{m_0}\gamma_{01}\left(X_0 + v_1\sqrt{X_0^2 - m_0^2N^2}\right). \tag{129}$$

Here, $\gamma_{0i}$ $(i = 1, 2)$ has the meaning of the standard Lorentz factor of relative motion. For an individual particle we have

$$X_i = m\gamma_iN, \qquad \gamma_i = \frac{1}{\sqrt{1 - V_i^2}}, \quad i = 0, 1, 2, \tag{130}$$

so

$$V_i = \sqrt{1 - \left(\frac{m_iN}{X_i}\right)^2}, \tag{131}$$

$V_i$ is the velocity is measured in the stationary frame.

The expressions for the energy can be rewritten as

$$E_2 = q_2\varphi + \gamma_{02}\frac{m_2}{m_0}\left(X_0 - v_2\sqrt{X_0^2 - m_0^2N^2}\right), \tag{132}$$

$$E_1 = q_1\varphi + \gamma_{01}\frac{m_1}{m_0}\left(X_0 + v_1\sqrt{X_0^2 - m_0^2N^2}\right). \tag{133}$$

For the existence of the PP, the key restriction on the electric charge (117) is required. It can be written in terms of velocities

$$\frac{m_1}{m_0}\left(X_0 + v_1\sqrt{X_0^2 - m_0^2 N^2}\right) < |q_1|\varphi\sqrt{1 - v_1^2}, \quad (134)$$

whence

$$1 + v_1 V_0 < \rho\sqrt{1 - v_1^2}, \quad \rho = \frac{m_0}{X_0 m_1}|q_1|\varphi. \quad (135)$$

Using (130), we can also write $\rho = \frac{|q_1|\varphi}{m_1 N \gamma_0}$. Taking the square of (135), we obtain

$$v_1^2 + \frac{2v_1 V_0}{V_0^2 + \rho^2} + \frac{1 - \rho^2}{V_0^2 + \rho^2 < 0} = (v_1 - v_+)(v_1 - v_-) < 0, \quad (136)$$

$$v_+ = -\frac{V_0}{V_0^2 + \rho^2} + \frac{\rho}{V_0^2 + \rho^2}\sqrt{\rho^2 + V_0^2 - 1}, \quad (137)$$

$$v_- = -\frac{V_0}{V_0^2 + \rho^2} - \frac{\rho}{V_0^2 + \rho^2}\sqrt{\rho^2 + V_0^2 - 1} < 0, \quad (138)$$

whence

$$v_1 < v_+. \quad (139)$$

One can check easily that $v_+ < 1$.

The requirement $v_+ \geq 0$ gives us

$$\rho^2 \geq 1. \quad (140)$$

In particular, for the Schwarzschild metric $\rho = 0$, $v_+ < 0$ and the PP is impossible as it should be.

Equation (140) can be rewritten in the form

$$|q_1|\varphi \geq \frac{X_0}{m_0} m_1. \quad (141)$$

Thus in the present section we suggested description of decay and PP in kinematic language by analogy with the rotating case [29]. However, there is qualitative difference. For the existence of the PP, in the rotating case the relative velocity between a new fragment and particle 0 should be quite high, the particle being ultrarelativistic [29]. In our case, there is no restriction on velocity that can be arbitrarily low. Moreover, instead of the lower bound typical of the rotating metric and process with neutral particles, now there exists the upper bound (139).

## XII. SCENARIO B AND THE MAXIMUM OF EFFICIENCY

Both scenarios I and II imply that the parent particle 0 moves from infinity towards a black hole. Meanwhile, of interest is also another situation when particle 0 bounces back from its turning point and only afterwards decays to particles 1 and 2. Particle 1 flies towards a back hole, particle 2 moves in the outward direction. We call this scenario B (the first letter of the word "bounce"). Then, the formulas for the energies read

$$E_2 = q_2\varphi + \gamma_{02}\frac{m_2}{m_0}\left(X_0 + v_2\sqrt{X_0^2 - m_0^2 N^2}\right), \quad (142)$$

$$E_1 = q_1\varphi + \gamma_{01}\frac{m_1}{m_0}\left(X_0 - v_1\sqrt{X_0^2 - m_0^2 N^2}\right). \quad (143)$$

If we assume that in the frame comoving with particle 0, then it is particle 2 that moves outwardly but is drifted in the inward direction due to motion of particle 0, then instead of (122) we have

$$V_2 = \frac{V_0 + v_2}{1 + v_2 V_0}. \quad (144)$$

One can check using (142) and (130) that (144) is indeed satisfied.

For particle 1 now

$$V_1 = \frac{v_1 - V_0}{1 - V_1 V_0}, \quad (145)$$

Here, as before, particle 2 moves to infinity (now in the same direction as particle 0) and particle 1 moves in the inward direction (opposite to particle 0). However, now the signs before the radicals are opposite to those in (132) and (133). The condition for the PP gives us now

$$\gamma_{01}\frac{m_1}{m_0}(X_0 - v_1\sqrt{X_0^2 - m_0^2 N^2}) < |q_1|\varphi \quad (146)$$

instead of (134). This entails

$$v_1^2 - \frac{2v_1 V_0}{V_0^2 + \rho^2} + \frac{1 - \rho^2}{V_0^2 + \rho^2 < 0} = (v_1 - v_-)(v_1 - v_+) < 0, \quad (147)$$

$$v_\pm = \frac{V_0}{V_0^2 + \rho^2} \pm \frac{\rho}{V_0^2 + \rho^2}\sqrt{\rho^2 + V_0^2 - 1}. \quad (148)$$

It is easy to check that $v_+ < 1$. Here, there are two different cases.

If

$$1 - V_0^2 \leq \rho^2 \leq 1, \quad (149)$$

then $v_- \geq 0$ and

$$v_- < v_1 < v_+. \quad (150)$$

If

$$\rho^2 > 1, \quad (151)$$

then $v_- < 0$ and

$$v_1 < v_+, \tag{152}$$

so we have an upper bound on $v_1$.

If $\rho \to \infty$, then $v_+ \to 1$, so actually bound (152) is satisfied automatically.

One can also choose TP3 as a scenario intermediate between I and B. Then, again signs can be arranged according to (112) and (113) with the same conclusions about restrictions on the velocity of particle 1 that are necessary for the PP to exist.

The significance of scenario B consists in that it enables us to attain the maximum efficiency $\eta = \frac{E_2}{E_0}$ of the process. Indeed, in this case the energy of particle 2 is given by Eq. (142) with sign "plus" before the radical.

It is seen from (110) or (142) that $E_2$ is monotonically decreasing function of $r$ since $\varphi = \frac{Qq_2}{r}$ with $q_2 > 0$ and $\frac{dN}{dr} > 0$. Therefore, the most possible maximum of $E_2$ is attained when decay happens near the horizon. One should bear in mind that in this case $X_0$ should have the order $N_d$, so this is possible for near-critical particle 0 only.

However, some important reservations about near-horizon decay are in order. For a nonextremal black hole there is a potential barrier of a finite height that prevents a near-critical particle with $X = O(N_d)$ in the near-horizon region from approaching the horizon. Moreover, if a particle is exactly critical near the turning point $r_t$ from which a particle bounces back, $X^2 = O(r_t - r_+)^2$ near the horizon. But for nonextremal black holes $N^2 \sim (r - r_+)$ near the horizon, so $X^2 \sim N^4$. As a result, it is seen from (6) that the condition $P^2 \geq 0$ cannot be satisfied and a particle cannot penetrate into the near-horizon region at all. If it is not exactly critical but near critical, a particle is able to move in the near-horizon region but such a particle can exist only between the horizon and the turning point, so it cannot arrive from infinity anyway. The situation is completely similar to that for rotating black holes [33–36]. However, if a black hole is extremal (or at least near extremal) this becomes possible—see Sec. III of Ref. [16] for more details about motion near turning points in the case of the extremal RN black hole.

### A. Near-horizon limit for scenario B

It is worth noting what happens in the near-horizon limit $N \to 0$ within the scenario under discussion. On the first glance, it follows from (142) that $X_2 \equiv E_2 - q_2\varphi$ in this limit can be arbitrary nonzero. However, this is not the case. The point is the correlation between initial conditions and the type of a horizon. Let a particle move away from the nonextremal horizon with finite nonzero $X_2$ (so-called usual particle). Then, if we continue its trajectory in the past, it turns out that it appeared there some small proper time $\tau$ ago from the region behind the horizon. But this would be a white, not a black hole and is beyond the scope of our work. See also on details [13] (there, $q = 0$, but this does not matter in the context under discussion).

From the other hand, if the proper time required for crossing the horizon $\tau \to \infty$, such arguments do not work. This happens if the horizon is extremal. Then, a particle can move in the outward direction from the immediate vicinity of the horizon. But, in doing so, it must have $X = O(N)$ (see Ref. [37] for details). Returning to our issue, we see from (142) that both $X_0 = O(N)$ and $X_2 = O(N)$.

## XIII. EFFICIENCY, PROXIMITY TO HORIZON, AND TYPE OF SCENARIO

Usually, when considering decay, it is assumed that it happens in the turning point of all particles. In doing so, it is often stated that the efficiency reaches its maximum if decay occurs near the horizon (see, e.g. the review [23]). Meanwhile, these statements are not quite accurate and require some essential reservations. Also, they apply to scenario TP3 but, in general, not to all other ones. As this concerns the important aspects of process near a black hole, this needs more careful discussion that is given below. Let us consider different types of scenarios in this context case by case.

### A. Scenario I

It is shown above that, according to (60) this scenario requires $X_0 \leq \frac{N_d b_2}{2m_2}$ in the point of decay $r_d$. This means that for a usual (not fine-tuned) particle that has $X_0 \neq 0$ on the horizon separated from zero, the horizon limit cannot be taken at all, provided $m_2$ is a massive particle. If it has almost vanishing mass, $m_2 = \mu_2 N_d$ with $\mu_2 \neq 0$, then the situation changes since (60) gives us

$$X_0 \leq \frac{b_2}{2\mu_2}. \tag{153}$$

If this criterion is fulfilled, then the horizon limit is indeed possible for a usual particle. Otherwise, the fine-tuning of particle 0 is required.

Let, for simplicity, particle 0 be neutral. Then, if

$$q_2 > q_2^* = \frac{\sqrt{M^2 - Q^2}}{E_0 Q}\sqrt{d}r_+, \tag{154}$$

the maximum is indeed achieved on the horizon (see the Appendix for details). At the same time, condition (117) should be fulfilled as well. On the horizon it reduces to (A6).

However, if $q_2 < q_2^*$, then the horizon corresponds not to the maximum but to a local minimum of $E_2$.

If $m_2 = 0$ exactly, then the horizon limit is also possible, but according to (74) this entails a quite strong condition $X_2 = O(N_d^2)$.

### B. Scenario II

Here, there are no difficulties with decay near the horizon since inequality (70) can be satisfied easily. However, in this scenario both particles fall in a black hole, so this option ceases to be "profitable."

### C. Scenario B

Now, it follows from (142) that for any $q_2Q > 0, \frac{\partial E_2}{\partial r} < 0$, so maximum is indeed achieved on the horizon. However, another difficulty comes into play here. A usual particle cannot start its motion near the horizon in the outward direction—see Sec. XII A above, Sec. I A in [13,37]. This is allowed for fine-tuned or near-fine-tuned particles only, with $X_2 = O(N_d)$. The same reasonings apply to scenario TP3.

### D. Scenario TP2

According to (94),

$$E_2 = \frac{q_2Q}{r} + m_2N. \tag{155}$$

Let us, for simplicity, consider the case of the extremal black hole, so $N = 1 - \frac{r_+}{r}$, $Q = M = r_+$. Then,

$$\frac{\partial E_2}{\partial r} = \frac{r_+}{r^2}(m_2 - q_2). \tag{156}$$

Thus the biggest value of $E_2$ is achieved on the horizon under the condition $m_2 < q_2$ only.

### E. Scenario TP0

According to (105),

$$E_2 = \frac{q_2Q}{r} + \frac{b_2}{2m_0}N. \tag{157}$$

Then, for the extremal black hole we have

$$\frac{\partial E_2}{\partial r} = \frac{r_+}{r^2}\left(\frac{b_2}{2m_0} - q_2\right). \tag{158}$$

The horizon corresponds to the biggest value of $E_2$ for $q_2 > \frac{b_2}{2m_0}$ only.

We see that one should be very careful making the statement about maximum of efficiency. This necessarily includes indication of scenario and reservation about relation between parameters.

## XIV. DIFFERENT SCENARIOS: COMPARISON OF PROPERTIES

Now, it is convenient to summarize the main features of all scenarios in Table I.

We included in it scenarios in which particles 1 and 2 are ejected along the trajectory of particle 0. However, this is done with one exception in a degenerate case. In scenario TP0 not only $P_0 = 0$ but also $L_0 = 0$ — see Sec. IX B. Then, all the components of the three-momentum vanish, so particle 0 is in rest in this point and there is no tangent vector to the trajectory.

We can write a unifying formula for the energy of particle 2. If it moves in the outward direction after decay of particle 0, then it can, in principle, escape to infinity and is potentially subject to the PP. Otherwise, it falls in a black hole:

$$E_2 = q_2\varphi + X_0\frac{b_2}{2m_0^2} + \delta\frac{\sqrt{d}\sqrt{X_0^2 - m_0^2N^2}}{2m_0^2}. \tag{159}$$

If immediately after decay particle 2 moves in the outward/inward direction, we use shortening "out"/"in." If it is in the radial turning point, then we write "0." In scenario TP0 the factor $\sqrt{X_0^2 - m_0^2N^2} = 0$, so $\delta$ is irrelevant.

The full trajectory of particle 2 is model dependent and cannot be found without specifying the metric. The efficiency $\eta = \frac{E_2}{E_0}$. In the fourth column we indicate whether or not some low bound $v_1 \geq (v_1)_{\min} > 0$ is required for the PP to occur. In scenario II particle 2 falls in a black hole, so the low bound on $v_1$ is pure formal since it has nothing to do with the PP. The general feature consists in that there is such a bound for $\delta = +1$ and it is absent if $\delta = -1$.

In principle, a combined scenario is also possible. If in a point of decay both particles move in the inward direction (scenario II) but after bouncing back in the turning point particle 2 changes direction and moves outwardly. In particular, this can happen in the near-horizon region of the extremal black hole with $X_2 = O(N)$ [16,17].

TABLE I. Classification and main features of scenarios.

| Scenario | Particle 0 | Particle 2 | $\delta$ | $(v_1)_{\min}$ mandatory | $X_2$ near horizon |
|---|---|---|---|---|---|
| I | In | Out | $-1$ | No | $O(N)$ for $m_2 \neq 0$, $O(N^2)$ for $m_2 = 0$ |
| II | In | In | $+1$ | Yes | $O(1)$ |
| TP2 | In | 0 | $-1$ | No | $O(N)$ |
| TP0 | 0 | Out | | No | $O(N)$ |
| TP3 | 0 | 0 | $\pm 1$ | No if $\delta = -1$, yes if $\delta = +1$ | $O(N)$ |
| B | Out | Out | $+1$ | No | $O(N)$ |

In all cases, formally $E_2 \to \infty$ when $q_2 \to \infty$. However, in realistic situations $q_2$ is bounded [16,17].

## XV. DISCREPANCY WITH TWO PREVIOUS WORKS ON THE SUBJECT

The Penrose process with charged particles was also considered in [27] where an additional assumption that the black hole charge $Q$ is small was made. Our results do not agree. The authors of [27], according to their Eqs. (34) and (35), obtained that the outgoing particle after decay near the horizon has a finite nonzero energy. Meanwhile, it follows from our formulas that in this case the energy of an escaping particle (if we neglect $Q$) tends to zero. If we take the charge $Q$ into account, in a similar way the difference $X_2 = E_2 - q_2 \frac{Q}{r_+}$ tends to zero. Actually, the subtlety in the issue under discussion consists in the necessity to take into account correlation between dynamics and kinematics. This means that in scenario I (where particle 2 escapes) the sign before the radical in (52) comes with minus, not plus. Instead, one can consider scenario II with finite $X_1$ and $X_2$ where one of particles has the sign plus but this particle falls in a black hole and does not escape. These features can be seen in Table I.

One more attempt of considering the PP for charged particles was made in [38] with a magnetic field $B$ taken into account. In our view, the results for efficiency $\eta = \frac{E_2}{E_0}$ (in our notations) described by Eqs. (31) and (32) of Ref. [38] are incorrect. In the neutral case $q = 0$, $Q = 0$, and $B = 0$ they give $\eta = 0$ instead of the Schwarzschild value $\eta_{\mathrm{Sch}}$. Therefore, for the RN metric, it also does not reproduce the formulas like (110) of the present paper.

The reason of discrepancy can be explained as follows. The results (31), (32) of [38] are based on their Eq. (30) that contains the angular velocity of a particle. Therefore, account for angular momentum is necessary. For a decay, say, in the turning point of radial motion for all three particles, the correct values are given by our Eqs. (114) and (115). Meanwhile, in [38] all angular momenta are put to be zero. This leads to contradiction.

Alternatively, one may consider a scenario in which all particles move radially with $L_0 = L_1 = L_2 = 0$. However, in this case Eqs. (19), (20) of [38] from which (31) and (32) were derived, loose their sense since (19) is obtained for motion along a circle. Again, we obtain contradiction.

## XVI. RELATION TO THE BSW EFFECT

Up to now, we discussed the standard Penrose process based on particle decay. Meanwhile, the aforementioned properties are applicable to the collisional version of the PP as well, when particles 1 and 2 collide to produce particles 3 and 4. This is due to the fact that particles 1 and can be considered as a combined one with characteristics obeying (10)–(13) and $m_0$ equal to the energy $E_{\mathrm{c.m.}}$ in the center of mass frame [39]. The BSW process and properties of debris were considered before in somewhat different approach in which consideration was restricted from the very beginning to the immediate vicinity of the horizon and approximate formulas were used [40]. Now, it is instructive to compare it with the present approach when one starts from exact formulas from the very beginning.

We are mainly interested in the situation when collisions lead to the BSW effect. To make comparison possible, we assume that the RN black hole is extremal and consider pure radial motion like in [16]. For the extremal RN black hole, the Coulomb potential $\varphi = 1 - N$. Let particle 1 be critical. By definition, this means that on the horizon $X = 0$. Then, in this setting, $X_1 = E_1 N$. We want to show that our exact formulas in the limit when $N_c \to 0$ turn into the results of [16]. (Here subscript "c" denotes the point of collision.) It is sufficient to trace correspondence with Eq. (26) of [16] which is the key point of the analysis there.

According to the BSW effect [8] and its electric counterpart [41], the energy $E_{\mathrm{c.m.}}$ in the center of mass of two colliding particles grows unbounded if (i) one of particles (say, 1) is critical, (ii) collision happens near the horizon, when $N \to 0$. In our context,

$$m_0^2 \approx \frac{\beta}{N}, \tag{160}$$

where $\beta$ is a constant [15] equal to

$$\beta = 2(X_2)_c A, \quad A = E_1 - \sqrt{E_1^2 - m_1^2}. \tag{161}$$

This formula can be obtained in the near-horizon limit directly from (123), if it is applied to particles 1 and 2. As a result,

$$\sqrt{X_0^2 - m_0^2 N^2} \approx X_0 - \frac{\beta N}{2X_0}, \tag{162}$$

where $X_0 = X_2 + E_1 N \approx X_2$ near the horizon.

Particle 4 falls in a black hole, particle 3 is a near-critical particle. Similarly to [16], we define

$$q_3 = E_3(1 + \delta), \tag{163}$$

where

$$\delta = C N_c \ll 1, \tag{164}$$

subscript "c" means the point of collision, $C$ is a constant.

Then, $X_3$ and $X_4$ are given by our formulas in which 1 should be replaced by 4 and 2 should be replaced by 3. We have

$$X_3 = E_3 - q_3 + q_3 N = E_3(N - C N_c), \tag{165}$$

$$X_3(N_c) = E_3 N_c (1 - C), \tag{166}$$

$$P_3(N_c) = N_c \sqrt{E_3^2 (1 - C)^2 - \beta}. \tag{167}$$

From the other hand, it follows from (126) with $\gamma_{01}$ replaced by $\gamma \equiv \gamma_{12}$ that

$$\gamma m_1 m_2 \approx \frac{(X_2)_c (E_1 - \sqrt{E_1^2 - m_1^2})}{N_c}, \tag{168}$$

$$b_3 = m_0^2 + m_3^2 - m_4^2, \tag{169}$$

$$d = b_3^2 - 4m_0^2 m_3^2. \tag{170}$$

Taking into account (52) for $N_c \ll 1$, we obtain

$$E_3(1 - C) = \frac{m_3^2}{2A} + \frac{1}{2}A. \tag{171}$$

This agrees completely with Eq. (26) [16], so further analysis from the aforementioned paper applies. See also [40]. If particle 1 moves from the vicinity of the horizon in the outward direction (so-called Schnittman process [42]), consideration runs along the same lines.

## XVII. SUMMARY AND CONCLUSIONS

Thus we considered the Penrose process for motion of all three particles within the same plane. We relied on exact formulas for characteristics of daughter particles in terms of a parent one. In doing so, we gave full classification of possible scenarios and derive the bounds on possible values of angular momenta of daughter particles. We selected those scenarios for which ejection occurs along the trajectory of a parent particle. Significance of scenarios of such a type consists in that we obtain maximum (minimum) value of energy. This is confirmed by direct computation of extrema of the energy of corresponding particles with respect to the angular momentum of one of them.

We formulated the results in two forms. The first one includes the dependence of the outcome on masses of particles. The second form expresses them in terms of velocities and gamma factors of relative motion between a parent particle and daughter ones. The corresponding formulas obtained for a static charged metric are similar to whose obtained in [29] for rotating metrics and neutral particles. Meanwhile, there are some essential differences. For the Penrose process to occur in the Kerr or other rotating metric, there is a lower bound on velocity of a daughter particle in the center of mass frame, this bound being rather high [29]. This creates obstacles for using this process. Meanwhile, in our case, the situation is more diverse and there are scenarios in which the velocity obeys the upper (not lower) bound, so the Penrose process can be realized more easily.

We discussed in detail an important question how the efficiency of the Penrose process depends on a point in which decay occurred. The results depends in general on a scenario and in this sense classification of scenarios developed in our work enables to elucidate this issue in general setting. One may think that this will be useful for analysis of diverse process near astrophysical black holes since we did not restrict ourselves by a separate scenario or special set of date.

Although we discussed decay, the results are applicable to other reactions between particles including the collisional Penrose process in which particles 1 and 2 collide to create new particles 3 and 4. In particular, this can lead to the BSW effect. We traced how the current approach based on exact formulas agrees with the previous one used before for description of the aforementioned effect.

In our consideration, we implied that the metric is the Reissner-Nordström one. However, the approach can be applied to a more general metrics of type (1).

Our goal consisted in the present article not in considering some concrete astrophysical problem but in development of general formalism. We carried out the analysis of possible types of scenarios in the simplest case of processes with charged particles thus developing general approach. This implied consideration of a static metric and electric field. The next step is supposed to be inclusion of rotation, electric charge and a magnetic field altogether into consideration.

We hope that the corresponding formalism will be useful for solving realistic problems in astrophysics. This is supposed to include processes in the accretion discs, extraction of energy from magnetized black holes, properties of ionization of neutral particle falling into a black hole, etc. This needs separate treatment. We hope that model-independent approach developed in our previous work [26] and the present one, will be useful for a diverse set of such problems.

## APPENDIX: DEPENDENCE OF EFFICIENCY ON A POINT

For simplicity, we consider here a neutral particle 0. Then, $q_0 = 0$, $q_1 = -q_2$, and $X_0 = E_0$. After decay the energy $E_2$ (and thus efficiency $\eta = E_2/E_0$) is given by an equation of the type

$$E_2 = q_2 \frac{Q}{r} + \frac{b_2}{2m_0^2} E_0 + \frac{\sqrt{d}}{2m_0^2} \sqrt{E_0^2 - m_0^2 N^2}. \tag{A1}$$

Here, $q_2 > 0$. As $\frac{dN}{dr} > 0$, it follows that $\frac{dE_2}{dr} > 0$ and the maximum is formally achieved for decay on the horizon but with all necessary reservations made in Sec. XIII above. A more involved case arises if

$$E_2 = q_2 \frac{Q}{r} + \frac{b_2}{2m_0^2} E_0 - \frac{\sqrt{d}}{2m_0^2} \sqrt{E_0^2 - m_0^2 N^2}. \tag{A2}$$

Then, direct calculation shows that on the horizon

$$\left(\frac{dE_2}{dr}\right)_+ = \frac{1}{r_+^2}\left(\frac{\sqrt{M^2-Q^2}}{E_0}\sqrt{d} - q_2 Q\right). \quad \text{(A3)}$$

Thus if the biggest value of $E_2$ is required to occur on the horizon, it is necessary that $(\frac{dE_2}{dr})_+ < 0$, whence

$$q_2 > q_2^* = \frac{\sqrt{M^2-Q^2}}{E_0 Q}\sqrt{d}. \quad \text{(A4)}$$

However, if

$$q_2 < q_2^*, \quad \text{(A5)}$$

the horizon corresponds to the smallest value.

Simultaneously, the threshold for the PP (117) gives us on the horizon for $|q_1| = q_2$

$$q_2 \geq \frac{E_0 r_+}{2m_0^2 Q}(b_1 + \sqrt{d}), \quad \text{(A6)}$$

where we have taken into account that now $|q_1| = q_2$. Both inequalities (A5) and (A6) are compatible with each other, provided

$$\frac{E_0^2}{m_0^2} \leq 2\frac{\sqrt{d}\sqrt{M^2-Q^2}}{(b_1+\sqrt{d})r_+}. \quad \text{(A7)}$$


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