Calabi-Yau periods for black hole scattering in classical general relativity

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The high-precision description of black hole scattering in classical general relativity using the post-Minkowskian (PM) expansion requires the evaluation of single-scale Feynman integrals at increasing loop orders. Up to 4PM, the scattering angle and the impulse are expressible in terms of polylogarithmic functions and Calabi-Yau (CY) twofold periods. As in QFT, periods of higher dimensional CY n-folds are expected at higher PM order. We find at 5PM in the dissipative leading order self-force sector (5PM-1SF) that the only nonpolylogarithmic functions are the K3 periods encountered before and the ones of a new hypergeometric CY threefold. In the 5PM-2SF sector further CY twofold and threefold periods appear. Griffiths transversality of the CY period motives allows one to transform the differential equations for the master integrals into ϵ-factorized form and to solve them in terms of a well-controlled function space, as we demonstrate in the 5PM-1SF sector.

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I. INTRODUCTION

Today's gravitational wave detectors have observed more than 100 mergers of binary black holes (BHs) or neutron stars (NSs) systems [1–3]. These observations enable studies of fundamental questions in gravitational, astro-, nuclear, and fundamental physics. With the upcoming third generation of gravitational wave detectors [4–6] the need for highest precision predictions of the emitted gravitational waveforms from theory has arisen [7]. To achieve this, perturbative analytical and numerical approaches are being followed: post-Newtonian [8–10], post-Minkowskian (PM) [11–15], self-force [16–19] expansion, as well as numerical relativity [20–22]. In a synergistic fashion, the import of perturbative quantum field theory (QFT) technology has considerably extended our knowledge—in particular, in the PM expansion that is closest to the considerations in particle physics.

In the PM approach, one naturally considers the *scattering* of BHs or NSs [23–27], which are modeled as massive point particles that interact gravitationally in the logic of effective field theory, due to the scale separation between the objects' intrinsic sizes (Schwarzschild or neutron star radius) and their separation ($Gm \ll |b|$) [28]. Using this effective worldline approach, the two-body scattering observables—the impulse (change of momentum), the spin kick, and the far field waveform—have been computed up to high orders in the PM expansion, including spin and tidal effects [29–38]. Complementary QFT approaches based on scattering amplitudes have reached similar precision in the PM expansion [39–42].

In these computations, the advanced toolbox of multiloop Feynman integrals needs to be applied: generation of the integrand, tensor reduction to scalar Feynman integrals, and the systematical reduction to a set of master integrals (MIs) by advanced integration by parts (IBP) algorithms. The present state of the art is at the 4PM (G^4) or three-loop level [34–38,40–42]. The appearing Feynman integrals go beyond the polylogarithmic case at the 4PM order, where quadratic combinations of elliptic integrals appear, in a form that identifies them with periods of a one-parameter K3 family, i.e. a Calabi-Yau (CY) twofold, parametrized by $x = \gamma - \sqrt{\gamma^2 - 1}$. Here is $\gamma = v_1 \cdot v_2$ with the incoming velocities v_i of the BHs.

Families of CY n-fold period motives (CYPM) and their extensions describe typically higher-loop parametric Feynman integrals in their leading ϵ-order in dimensional regularization. These special functions are generalizations of extensions of algebraic or elliptic functions appearing already at low-loop order which can be seen as CY n-fold periods for $n = 0, 1$. See [43–46] for reviews of CY n-fold families and their mixed Hodge structure in this context.

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The best studied all loop series with systematic CY n-fold period identification are the $2d$ $(n+1)$-loop banana graphs [45,47–49], and the $2d$ n-loop fishnet integrals [50]. The corresponding geometries can be realized as singular double covers of a Fano base B branched at two times the canonical class of the base $2K_B$ which shows that they are CY [50]. In this paper, we find similar CY n-folds in the BH scattering problem. They appear in the same realization from the Baikov representation of the Feynman integrals. As in [49] in some cases we find a better smooth realization as complete intersection CY. The transcendental functions that determine the x dependence of the impulse and scattering angle in the BH scattering problem are characterized by CYPM of the corresponding families of CY n-folds. The latter describe the solutions of the periods as determined by the flatness of the Gauss-Manin (GM) connection with additional structures such as Griffiths transversality (GT), integrality, and modularity inherited from the CY geometry [44,45]. The geometrical GM connection is derived from IBP relations for a suitable basis of MIs or alternatively obtained from expansions of special Baikov integrals. The mathematical properties of the CYPM are necessary to calculate the physical quantities. In particular, we use GT as an essential feature to bring the full IBP differential equations into ϵ-factorized form which is convenient to systematically solve them up to the required ϵ-order.

II. WORLDLINE QUANTUM FIELD THEORY

A highly efficient tool to address the PM expansion of the BH or NS scattering problem is the worldline effective field theory approach [29]. The spinless compact objects are modeled as point particles. The action takes the compact form

$$S = -\sum_{i=1}^{2} m_i \int \mathrm{d}\tau \left[\frac{1}{2} g_{\mu\nu}\dot{x}_i^\mu \dot{x}_i^\nu\right] + S_{\mathrm{EH}} \tag{1}$$

using proper time gauge $\dot{x}_i^2 = 1$. The bulk Einstein-Hilbert action S_{EH} includes a de Donder gauge-fixing term, and we employ dimensional regularization with $d = 4 - 2\epsilon$. In the worldline quantum field theory (WQFT) approach [29,30] the fields are expanded about their noninteracting background configurations

$$x_i^\mu = b_i^\mu + v_i^\mu \tau + z_i^\mu, \qquad g_{\mu\nu} = \eta_{\mu\nu} + \sqrt{32\pi G}\, h_{\mu\nu} \tag{2}$$

with the worldline deflections $z_i^\mu(\tau)$ and graviton field $h_{\mu\nu}(x)$. The background data is given by the impact parameter $b^\mu = b_2^\mu - b_1^\mu$ and the incoming velocities v_1, v_2. The fields z_i^μ and $h_{\mu\nu}$ are integrated out in the path integral. One needs to use retarded propagators as a consequence of the Schwinger-Keldysh in-in formalism [32,51]. The WQFT interactions contain the standard bulk graviton vertices as well as worldline vertices coupling a single graviton to worldline deflections [30,36]. The WQFT tree-level one-point functions $\langle z_i^\mu(\tau)\rangle$ solve the classical equations of motion [52]—trivializing the classical limit. As a consequence, the impulse of the (say) first BH or NS, Δp_1^μ, follows from the tree-level one point function $\Delta p_1^\mu = \lim_{\omega\to 0}\omega^2\langle z_1^\mu(\omega)\rangle$ that is evaluated in the PM expansion. As the worldline vertices only conserve the total inflowing energy—opposed to full four-momentum conservation for the bulk graviton vertices—the WQFT tree-level one-point function gives rise to loop-level Feynman integrals whose order grows with the PM order: The nth PM order yields $(n-1)$-loop integrals (plus a trivial Fourier transform over the momentum transfer q).

III. THE IMPULSE IN PM EXPANSION

The PM expanded impulse, $\Delta p_1^\mu = \sum_{n=1}^{\infty} G^n \Delta p^{(n)\mu}$, may be further subdivided into contributions of different self-force (SF) sectors according to the scaling in the masses m_1 and m_2. Concretely, we have at 5PM order

$$\begin{aligned}\Delta p_i^{(5)\mu} = m_1 m_2 \Big(& m_1^4 \Delta p_{0\mathrm{SF}}^{(5)\mu} + m_1^3 m_2 \Delta p_{1\mathrm{SF}}^{(5)\mu} \\ & + m_1^2 m_2^2 \Delta p_{2\mathrm{SF}}^{(5)\mu} + m_1 m_2^3 \Delta \bar{p}_{1\mathrm{SF}}^{(5)\mu} + m_2^4 \Delta \bar{p}_{0\mathrm{SF}}^{(5)\mu}\Big).\end{aligned} \tag{3}$$

In fact, the 0SF contributions $\Delta p_{0\mathrm{SF}}^{(5)\mu}$ and $\Delta \bar{p}_{0\mathrm{SF}}^{(5)\mu}$ are linked to geodesic motion and are, in principle, known to all orders in G [53]. The self-force expansion is a complementary perturbative expansion going beyond geodesic motion in the mass ratio $m_1/m_2 \ll 1$. Importantly, the SF order grows in steps of two in the PM order, e.g. the first 1SF term appears at 3PM and the first 2SF at 5PM order. In this paper, we determine the nonpolylogarithmic function space of the 1SF terms up to the 5PM order. Moreover, we comment on the situation in the 2SF sector. The impulse is a four-vector and will be expressed as a linear combination of the four-vectors b^μ as well as v_1^μ and v_2^μ.

IV. INTEGRAL FAMILY

The generation of the WQFT integrand has been described at 3PM and 4PM orders in [15,31,32,36,37,54]. It employs recursive diagrammatic techniques and tensor reduction for the generation of the WQFT integrands. In the 4PM case the emerging integral family is comprised of 12 propagators and three worldline delta functions [36]. At the 5PM-1SF order, this automated integrand generation is, in principle, identical and poses no technical problems [55] from which the scalar integral families may be read off.

We encounter an integral family comprised of 18 propagators and four worldline delta functions taking the form

$$I^{(\sigma_1,\sigma_2,\ldots,\sigma_7)}_{n_1,n_2,\ldots,n_{22}} = \int_{\ell_1,\ell_2,\ell_3,\ell_4} \frac{\delta^{(n_1-1)}(\ell_1\cdot v_2)\delta^{(n_2-1)}(\ell_2\cdot v_2)\delta^{(n_3-1)}(\ell_3\cdot v_2)\delta^{(n_4-1)}(\ell_4\cdot v_1)}{D_1^{n_5}D_2^{n_6}\cdots D_{18}^{n_{22}}}, \tag{4a}$$

where $\delta^{(n)}(x)$ denotes the nth derivative of the worldline delta function. The propagators ($k=1,2,3$ and $j=1,2,3,4$) are given by

$$D_k = \ell_k\cdot v_1 + \sigma_k i0^+, \qquad D_4 = \ell_4\cdot v_2 + \sigma_4 i0^+, \qquad D_{4+k} = (\ell_k-\ell_4)^2 + \sigma_{4+k}\mathrm{sgn}(\ell_k^0-\ell_4^0)i0^+,$$
$$D_8 = (\ell_1-\ell_2)^2, \qquad D_9 = (\ell_1-\ell_3)^2, \qquad D_{10} = (\ell_2-\ell_3)^2, \qquad D_{10+j} = \ell_j^2, \qquad D_{14+j} = (\ell_j+q)^2. \tag{4b}$$

In principle, all graviton propagators carry a retarded $i0$ prescription. Due to the delta functions only three propagators can go on-shell. The sign of the $i0$ prescription is defined by $\sigma_i = \pm 1$. The 5PM-1SF integral family splits into two branches: even (b-type) or odd (v-type) in parity ($v \to -v$), determined by the number of worldline propagators and derivatives of the delta functions, i.e. the parity of the first eight indices $\{n_1, \ldots n_8\}$. These two integral branches couple in the final result to the vectors b^μ and v_i^μ respectively. The integrals of b- and v-type can have very different function spaces. Notice that all v-type integrals without a worldline propagator vanish, when using Feynman propagators, due to symmetries ($l_i \to -l_i, q \to -q$). Therefore, they only contribute to the dissipative part of the impulse.

The usage of retarded propagators makes the even integrals purely real and the odd integrals pseudoreal. The integrals are effectively one-scale integrals depending on the parameter $x \in [0,1]$. The self-force order determines the indices of the velocities in the delta functions. At nPM-mSF order, we have $(n-1-m)$-loop momenta contracted with v_1 and m momenta contracted with v_2 in the delta functions. At 0SF order, the γ dependence becomes trivial. The complexity of the integration problem increases with every self-force order.

V. IBP REDUCTION AND CHOICE OF BASIS

To study the function space of the PM integrals one necessary step is to derive the differential equations of the involved MIs. The MIs as well as their differential equations are derived from IBP relations using the program KIRA [56–58]. To simplify this task we only reduced the derivatives of the MIs neglecting the full integrand reduction.

It is convenient to group the MIs into a large vector $\underline{I}(x,\epsilon)$. Here, we order MIs from lower to higher sectors, i.e. we start with the subsectors. With this convention we find that the GM equation takes the form $(\mathrm{d} - M(x,\epsilon))\underline{I}(x,\epsilon) = 0$, such that the connection matrix $M(x,\epsilon)$ factorizes into sectors and is of lower block triangular form. As discussed before, we split the integrals into even and odd parity and look at them separately. The diagonal blocks correspond to the maximal cuts of the system [59,60]. They essentially determine the class of transcendental functions appearing in a given sector.

To systematically solve the GM equation up to a given order in ϵ it is useful to go to an ϵ-factorized differential equation [61]. For this one has to construct a rotation into a new basis $\underline{J}(x,\epsilon) = T(x,\epsilon)\underline{I}(x,\epsilon)$ such that the ϵ dependence is factored out in the new connection matrix

$$0 = (\mathrm{d} - \epsilon A(x))\underline{J}(x,\epsilon),$$
$$\epsilon A(x) = (T(x,\epsilon)M(x,\epsilon) + \mathrm{d}T(x,\epsilon))T(x,\epsilon)^{-1}. \tag{5}$$

The complexity of calculating the rotation $T(x,\epsilon)$ strongly depends on the initial choice of MIs. Essential criteria for a good selection of MIs are the absence of powerlike singularities, the absence of general polynomials $p(x,\epsilon)$ in the denominators of $M(x,\epsilon)$, and the manifestation of the different appearing minimally coupled systems, i.e. geometries, in the problem. For our case, this means that the choice of MIs gives rise either to projective spaces with marked points (polylogarithms), CY manifolds or possible additional residues on these geometries [62]. For GM systems related to CY n-folds, the main ingredient then is to use GT to construct linear combinations of MIs involving the CY period integrals as coefficients such that their leading singularities satisfy unipotent differential equations [63]. Practically, this means that one has to split the Wronskian matrix of fundamental solutions $W(x)$ into a semisimple and unipotent part, i.e. $W(x) = W(x)^{\mathrm{ss}}W(x)^{\mathrm{u}}$. After removing the semi-simple part from the initial MIs and further total derivatives one arrives at an ϵ-factorized differential equation. In some cases, additional new transcendental functions being iterated integrals of the CY period integrals have to be introduced during these steps. For more details, we refer to [62,63].

The final ϵ-factorized connection matrix $A(x)$ consists of rational functions and iterated CY period integrals. We can see now clearly that the higher ϵ-orders just give further iterated integrals of these kernels. Also the contributions from subsectors do not change the function space.

VI. CY IN THE SKY

The rank $n+1$ GM connection $(\mathrm{d}-A^M(x))\underline{\Pi}(x)=0$ of a one-parameter period motive of a CY n-fold M appears as subconnection in (5), i.e. as a block in $M(x,0)$. It can be equivalently written as a linear Picard-Fuchs differential operator (PF op) of order $n+1$, i.e. $\mathcal{L}^{(n+1)}=\partial_x^{n+1}+\sum_{i=0}^{n}a_i(x)\partial_x^i$ where $a_i(x)$ is a rational function in x. The solutions $\underline{\Pi}(x)$ to $\mathcal{L}^{(n+1)}\underline{\Pi}(x)=0$ are then the periods $\Pi_k=\int_{C_n^k}\Omega$ of M, with Ω the holomorphic $(n,0)$-form and the n-cycles C_n^k can be chosen to be in $H_n(M,\mathbb{Z})$.

In order to describe a period motive associated to an n-dimensional algebraic variety, $\mathcal{L}^{(n+1)}$ has to have only regular singular points of maximal unipotency n in its moduli space $\mathcal{M}_x$. For $\mathcal{L}^{(n+1)}$ to be also a CY operator, GT requires that $\mathcal{L}^{(n+1)}$ has to be self-adjoint, i.e.

$$\mathcal{L}^{*(n+1)}c(x)=(-1)^{n+1}c(x)\mathcal{L}^{(n+1)}. \tag{6}$$

Here, $\mathcal{L}^{*(n+1)}=\sum_{i=0}^{n+1}(-\partial_x)^i a_i(x)$ is the adjoint operator and the rational function $c(x)$ is determined by $\partial_x c(x)/c(x)=2a_n(x)/(n+1)$ up to a multiplicative constant. CY motives are defined by CY differential operators, see [64] for $n=3$, which have in addition a point of maximal unipotent monodromy (MUM point). At this point they have an integral mirror map and integral BPS expansions [65–67]. The latter are trivial for K3 surfaces, and in this case, the period domain is a symmetric domain. Moreover, the periods are modular functions of congruent subgroups of $\mathrm{SL}(2,\mathbb{Z})$, which is related to the fact that one-parameter K3 period integrals are symmetric squares of elliptic integrals [48,68–70]. For integral BPS expansion in three- and fourfolds see [65,67,71,72]. The integral structures are consequences of the integral monodromy respresentation of $O(\Sigma,\mathbb{Z})$ for n even and $\mathrm{Sp}(2n+2,\mathbb{Z})$ for n odd which also encode the topological type of M. Here Σ is the intersection form on $H_n(M,\mathbb{Z})$.

To identify a given graph Γ in the 5PM expansion having m different MIs in its sector with a CY n-fold, it is essential that the corresponding MIs are chosen such that at $\epsilon=0$ the candidate minimally coupled CY block is decoupled from other contributions which can be additional residues, polylogarithmic contaminations, or additional nontrivial CY manifolds [62,63]. This means that at $\epsilon=0$ the connection form of this sector splits as

$$M_\Gamma(x,0)=\begin{pmatrix} A^{M_\Gamma}(x) & 0 \\ C(x) & D(x)\end{pmatrix}, \tag{7}$$

where the two matrices $C(x)$, $D(x)$ describe the additional structures in the sector. Then, the CY can be identified directly from A^{M_Γ} or via the equivalent PF op. Alternatively, a Baikov representation associated to Γ can be expanded in x by performing a suitable residuum calculation to high order, which corresponds to choose $C_n=T^n$ to obtain a torus integral $\int_{T^n}\Omega$ in M_Γ. An ansatz for $\mathcal{L}^{(n+1)}(x)$ can, in practice, be uniquely fixed by the requirement $\mathcal{L}^{(n+1)}(x)\int_{T^n}\Omega=0$ and can be compared with (7).

In all cases, we find the higher rank period motives of M_Γ fulfilling the conditions above to be CY motives. They determine the corresponding higher transcendental functions, encoding the observables of the black hole scattering process, which are CY periods $\underline{\Pi}(x)$ and their extensions as we will explicitly exemplify below.

VII. THE 5PM-1SF SECTOR

We have found five different graphs (K3.i, $i=1,\ldots,5$ in Fig. 1) related to a K3 surface. To make this identification, a suitable choice of MIs is given by the corner integrals listed in Table I. Here, each corner integral corresponds to a single graph and is supplemented by additional MIs being the derivatives of the corner one. In this regard, the graphs K3.1-2 consist of four MIs whereas the graphs K3.3-5 consist of three MIs.

For these choices of MIs, the corresponding GM systems can be determined from IBP relations. It is interesting to observe, that all K3 operators appearing in all five graphs at $\epsilon=0$ are related to the same K3 operator also appearing

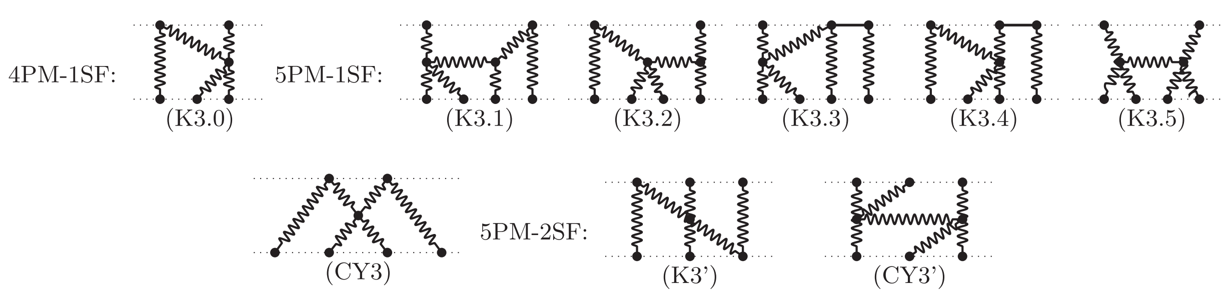

FIG. 1. The graphs up to 5PM whose sectors are associated to CY manifolds. Here, only either the even or odd parity integrals give rise to the CY period integrals. The top line corresponds to the same K3 surface, cf. Table I. Dotted lines represent δ-functions, solid lines the linear propagators $D_1,\ldots,D_4$, and wiggly lines the graviton propagators $D_5,\ldots,D_{18}$.

TABLE I. The master integrals of the K3 sectors.

MI	n_i	Parity
$I_{\mathrm{K3},1}$	1 1 1 1 0 0 0 −1 0 1 1 1 0 1 1 0 0 1 0 0 1 0	Odd
$I_{\mathrm{K3},2}$	1 1 1 1 0 0 0 −1 1 1 0 1 0 1 1 0 0 0 0 0 1 1	Odd
$I_{\mathrm{K3},3}$	1 1 1 1 0 0 1 0 0 1 0 1 0 1 1 0 0 1 0 0 1 0	Odd
$I_{\mathrm{K3},4}$	1 1 1 1 0 0 1 0 1 0 1 1 0 1 1 0 0 0 0 0 1 0	Odd
$I_{\mathrm{K3},5}$	1 1 1 1 0 0 0 0 0 1 0 1 0 1 1 0 0 1 0 0 1 1	Even

in the 4PM-1SF sector (graph K3.0). This operator is conveniently expressed as

$$\mathcal{L}_1^{(3)} = (2\theta - 1)^3 + z^2(2\theta + 1)^3 - 4z\theta(4\theta^2 + 1) \quad (8)$$

with $\theta = z\frac{\mathrm{d}}{\mathrm{d}z}$, $z = x^2$. Self-adjointness of $\mathcal{L}_1^{(3)}$ is guaranteed through $c(z) = 4/(z(1-z)^2)$ from (6), where the 4 has been determined from the intersection Σ of the K3. Monodromy properties can be read off from the corresponding Riemann $\mathcal{P}$-Symbol in Eq. (A1). GT and the representation theory of the monodromy groups [48,68,69] imply that $\mathcal{L}^{(3)} = \mathrm{Sym}^2(\mathcal{L}^{(2)})$, where $\mathcal{L}^{(2)}$ is the PF op of an elliptic curve. In our case, it is the Legendre curve $Y = X(X-1)(X-z)$ with monodromy group $\Gamma_0(4)$ and $\mathcal{L}_1^{(2)} = \theta^2 - z(\theta + \frac{1}{2})^2$. The K3 geometry is then the twisted product of the latter given by

$$Y^2 = X(X-1)(X-z)Z(Z-1)(Z-1/z). \quad (9)$$

Its symmetry makes it immediately clear that the same solution structure appears at $w = 1/z = 1/x^2$. This symmetry is inherited from the physical parametrization $\gamma = (x + x^{-1})/2$ and must occur in all geometries. Since elliptic curves cannot exhibit this symmetry in their moduli space $\mathcal{M}_z$ the occurrence of CY motives in the PM approximation starts with $n = 2$, i.e. K3 surfaces. We can bring all sectors corresponding to the five graphs K3.1-5 in Fig. 1 into ϵ-form using the INITIAL algorithm [73,74]. The corner integrals in Table I serve up to normalization as initial integrals for the INITIAL algorithm.

Besides the K3 surface there is only one other CY n-fold appearing in the 5PM-1SF sector. The graph inducing this (first) CY three-fold is depicted in Fig. 1 (CY3). While its even subsector is polylogarithmic (as reported in [75]) the odd subsector is not and will contribute to the *dissipative* part of the 5PM-1SF impulse. This CY threefold sector is built up of six MIs. The first four MIs, which are the corner integral I_1 in Table II, and its three derivatives, describe the CY threefold part. The MIs I_5, I_6 are instead additional residues on that CY. The corresponding fourth-order CY operator $\mathcal{L}_1^{(4)}$ is of hypergeometric type and is given by

$$\mathcal{L}_1^{(4)} = \theta^4 - 2^8 z\left(\theta + \frac{1}{2}\right)^4 \quad (10)$$

TABLE II. The master integrals of the CY3 sector.

MI	n_i	Parity
I_1	1 1 1 2 0 0 0 0 1 0 1 1 0 1 1 0 0 0 0 0 1 0	Odd
I_5	1 1 2 1 0 0 0 0 1 0 1 1 0 1 1 0 0 0 0 0 1 0	Odd
I_6	1 1 2 2 0 0 0 −1 1 0 1 1 0 1 1 0 0 0 0 0 1 0	Odd

in the variable $z = 2^{-8}x^4$ and after normalizing I_1 by x with $c(z) = 16/(z^3(1 - 2^8 z))$ from (6). $\mathcal{L}_1^{(4)}$ is a Hadamard product $\mathcal{L}_1^{(2)} * \mathcal{L}_1^{(2)}$ of the Legendre operator, see [76] for Hadamard constructions. The corresponding smooth CY threefold one-parameter complex family $z = (2\psi)^{-8}$ can be defined as resolution of four symmetric quadrics

$$x_j^2 + y_j^2 - 2\psi x_{j+1} y_{j+1} = 0, j \in \mathbb{Z}/4\mathbb{Z} \quad (11)$$

in the homogeneous coordinates x_i, y_j, $j = 0, \ldots, 3$ of $\mathbb{P}^7$ [77]. This hypergeometric CY threefold motive appeared in the study of mirror symmetry in [78].

To derive the ϵ-factorization of this CY block, we first have to split the matrix of fundamental solutions $W(x) = (\partial^j \varpi_i)_{0\le i,j\le 3}$ into its semi-simple and unipotent part (for more details see the appendix). The unipotent part satisfies

$$(\mathrm{d} - A^{\mathrm{u}}(x))W^{\mathrm{u}}(x) = 0, \quad A^{\mathrm{u}}(x) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & Y_1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (12)$$

and $A^{\mathrm{u}}(x)$ is nilpotent, i.e. $(A^{\mathrm{u}}(x))^4 = 0$. From the CY perspective it is meaningful to use the rescaled variable $\mathrm{x} = \frac{x}{4}$ in which the integral expansions are manifest. After removing the semisimple part from the initial MIs, we have to introduce four new transcendental functions, which are iterated integrals of CY periods, to obtain the ϵ-form. The two simplest ones are given by

$$G_1(\mathrm{x}) = -\int_0^{\mathrm{x}} \frac{24576\mathrm{x}'(1 + 256\mathrm{x}'^4)}{(1 - 256\mathrm{x}'^4)^2} \frac{\varpi_0(\mathrm{x}')^2}{\alpha_1(\mathrm{x}')} \mathrm{d}\mathrm{x}',$$
$$G_3(\mathrm{x}) = \int_0^{\mathrm{x}} \frac{\mathrm{x}'}{1 - 256\mathrm{x}'^4} \frac{G_1(\mathrm{x}')\alpha_1(\mathrm{x}')^2}{\varpi_0(\mathrm{x}')^2} \mathrm{d}\mathrm{x}'. \quad (13)$$

The x expansions of α_1 and all G_i functions are given in the Eqs. (A5) and (B1), respectively. This allows one to construct the full ϵ-factorized connection matrix in the CY sector by using the special properties of the CY geometry, in particular, GT and the period integrals. The new transcendental functions are given as power series which can be easily analytically continued to the whole complex plane. As an important observation, the new transcendental functions have all integer coefficient expansions in x,

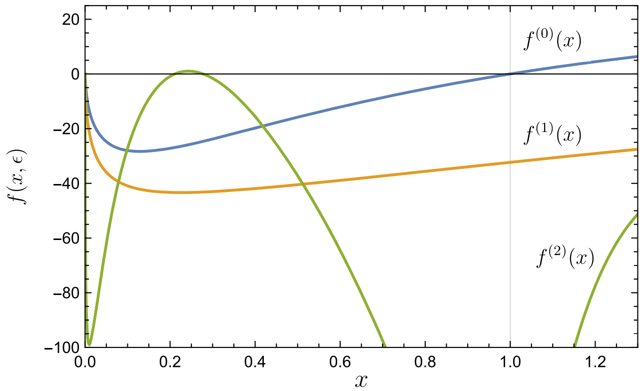

FIG. 2. Plot of ϵ-orders of $f(x,\epsilon)$ in original variable x.

similarly to the novel transcendental functions appearing in [62,63,79–81] which for K3 surfaces are related to magnetic modular forms [63,82,83]. This gives us full analytic control over the function space in the 5PM-1SF sector [84]. In Appendix B, we provide the ϵ-factorized differential equation for the CY threefold in expanded form [see Eqs. (B2) and (B3)] together with a plot (see Fig. 2) showing explicitly the higher ϵ-orders.

VIII. OUTLOOK TO 5PM-2SF

This sector contains many more and new CY n-folds compared to the 1SF sector. For instance, for the graph K3' in Fig. 1 we find among the nine MIs a K3 block that leads with $z = x^2$, removing an apparent singularity at $z = 1$ and normalizing the corresponding MIs by $z/(1-z)$ to the famous Apèry operator

$$\mathcal{L}_A^{(3)} = \theta^3 + z^2(\theta+1)^3 - z(2\theta+1)(17\theta^2+17\theta+5) \tag{14}$$

with $c(z) = \frac{\kappa}{z^2(z^2-34z+1)}$ from (6), that was used in [85] to prove the irrationality of $\zeta(3)$, see [86] for a review. The smooth K3 of Picard rank 19 was described in [87] as the resolution of the affine equation

$$1-(1-XY)Z - zXYZ(1-X)(1-Y)(1-Z) = 0. \tag{15}$$

In fact, the corresponding Baikov integral representation and the evaluation of ϖ_0 near $z=0$ by performing the T^2 integral can be also found in [87].

In recent work [88] a singular Baikov integral representation for the graph CY3' (see Fig. 1) was given. It can be viewed as a double cover of $\mathbb{P}^3$ branched at

$$\begin{aligned} P = {} & t^2(WX+Y^2)^2(W+Z)^2(X+Z)^2 \\ & + 2^6(1+t)(WXY)^2 Z(W+X+Z) \end{aligned} \tag{16}$$

with $t = x^2 - 1$. The corresponding geometry $U^2 = P$ is highly singular and was not resolved in [88] to a CY unlike (15) in [87]. Also the differential operator given in [88] with an apparent singularity is not of CY type [64]. It lacks integral BPS expansions at the MUM point $t = 0$. With $z = -\frac{t^2}{2^{12}(1+t)}$, we lift the apparent singularity and in Eq. (A2) we show that the transformed operator

$$\begin{aligned} \mathcal{L}_2^{(4)} = {} & \theta^4 - 2^{30}z^3\left(\theta+\frac{1}{2}\right)^4 \\ & - 2^4 z(192\theta^4+128\theta^3+112\theta^2+48\theta+7) \\ & + 2^{14}z^2(192\theta^4+256\theta^3+208\theta^2+64\theta+7) \end{aligned} \tag{17}$$

corresponds to a one-parameter CY family with topological data $\chi = 80$, $\kappa = 4$, and $\gamma = \kappa(6m-5)$, which fixes the topological type according to [89]. Using integral BPS expansion we can also relate it to a Hadamard construction.

IX. CONCLUSIONS

In this work, we have shown that the function space describing the radiated momentum of a scattering encounter of two BHs involves CY threefolds starting at the 5PM order. We have completely resolved the nonpolylogarithmic function space at the 5PM-1SF order by ϵ-factorizing the differential equations for the MIs in these sectors. In addition, an exemplary outlook into two CY sectors at 2SF was given. We established that all CY n-fold periods appearing so far are either symmetric or Hadamard products of elliptic functions. Clearly, the appearance of these functions indicates that the use of advanced mathematics is necessary to address the classical two-body problem in general relativity (in the PM or SF expansions). While the Newtonian problem is famously linked to elliptic integrals, it is fascinating to see that the general relativistic problem leads to their natural generalizations in terms of CY periods.

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APPENDIX A: FURTHER PROPERTIES OF THE CY OPERATORS

In this appendix, we list more properties of the CY manifolds appearing at 5PM. We start with the Riemann $\mathcal{P}$-symbols of the CY operators in the 5PM-1SF sector

$$\mathcal{P}_{\mathcal{L}_1^{(3)}}\left\{\begin{array}{ccc} 0 & 1 & \infty \\ \hline \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{array}\right\}, \qquad \mathcal{P}_{\mathcal{L}_1^{(4)}}\left\{\begin{array}{ccc} 0 & \frac{1}{2^8} & \infty \\ \hline 0 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 2 & \frac{1}{2} \end{array}\right\}, \tag{A1}$$

and in the 2SF sector

$$\mathcal{P}_{\mathcal{L}_B^{(3)}}\left\{\begin{array}{cccc} 0 & p & \frac{1}{p} & \infty \\ \hline 0 & 0 & 0 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 1 & 1 & 1 \end{array}\right\}, \qquad \mathcal{P}_{\mathcal{L}_2^{(4)}}\left\{\begin{array}{ccc} 0 & \frac{1}{2^{10}} & \infty \\ \hline 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{3}{2} & \frac{1}{2} \\ 0 & 2 & \frac{1}{2} \end{array}\right\}, \tag{A2}$$

where p, p^{-1} are the two roots of the quadratic equation $u^2 - 34u + 1 = 0$. A Riemann $\mathcal{P}$-symbol $\mathcal{P}_{\mathcal{L}^{(r)}}$ records the local exponents γ_i, $i = 1, \ldots, r$ of the solutions to a rth-order operator $\mathcal{L}^{(r)}(z)$, below the regular singular points or the apparent singularities in the moduli space parametrized by z. If all γ_i are equal, z is a MUM point.

At a MUM point a Frobenius basis with local exponent γ_1 is given by $\underline{\tilde{\Pi}} = (\varpi_i, i = 0, \ldots, n)$ with $\varpi_j(z) = z^{\gamma_1} \sum_{k=0}^{n} \frac{1}{k!} \log(z)^k f_{n-k}(z)$, where the power series are normalized like $f_0(0) = 1$, $f_{i>0}(0) = 0$. Then, we get, e.g. for a threefold, an integral symplectic basis [49,77,90] by

$$\underline{\Pi} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2\pi i} & 0 & 0 \\ \frac{c_2}{24} & \frac{\sigma}{2\pi i} & \frac{\kappa}{4\pi^2} & 0 \\ \frac{\chi\zeta(3)}{2(2\pi i)^3} & \frac{c_2}{24(2\pi i)} & 0 & \frac{\kappa}{(2\pi i)^3} \end{pmatrix} \underline{\tilde{\Pi}}. \tag{A3}$$

Introducing the mirror map $q(z) = \exp(2\pi i t(z))$ with $t(z) = \varpi_1/(2\pi i \varpi_0)$, we can write $\underline{\Pi} = \varpi_0(1, t, \partial_t \mathcal{F}(t), 2\mathcal{F}(t) - t\partial_t \mathcal{F}(t))$ in terms of a prepotential $\mathcal{F} = -\frac{\kappa t^3}{6} - \frac{\sigma t^2}{2} + \frac{c_2}{24} t + \sum_{d=0}^{\infty} n_0^d \mathrm{Li}_3(q^d)$ with $n_0^0 = \chi/2$ and κ, c_2, χ, which are the topological data of the mirror to M. For general CY n-folds the analog of the transformation in (A3) is obtained by the $\hat{\Gamma}$-class formalism as explained in [49]. The main invariant of CY n-fold motives are the mirror maps $t_i(z_j)$ and the triple couplings in quantum cohomology, also known as Yukawa coupling in the string compactification context, as they encode all enumerative invariants. For the one-parameter CY threefold case, there is only one triple coupling, namely $c_{ttt}(t) = \frac{c(z)}{\varpi_0^2} (\frac{\partial z}{\partial t})^3 = \partial_t^3 \mathcal{F}$. The last equality is a consequence of GT. With coordinates $x^2 = z$ and naive normalization $\kappa = 1$, they appear as Y_k-invariants and structure series α_k in [91]. For our CY (11), we find explicitly

$$\begin{aligned} \alpha_1 &= \frac{1}{(\theta t)(\mathrm{x}^2)} = 1 - 64\mathrm{x}^4 - 5952\mathrm{x}^8 + \mathcal{O}(\mathrm{x}^{12}), \\ Y_1 &= \frac{c_{ttt}(t(\mathrm{x}^2))}{16} = 1 + 32\mathrm{x}^4 + 6944\mathrm{x}^8 + \mathcal{O}(\mathrm{x}^{12}). \end{aligned} \tag{A4}$$

The recursive definition of the Y_k, $k = 0, 1, \ldots, n-2$ and α_k, $k = 0, 1, \ldots, n$ uses the differential operators $\mathcal{N}_0 = 1$, $\mathcal{N}_1 = \frac{\theta}{\varpi_0}$, and $\mathcal{N}_{k+1} = \theta \frac{1}{\mathcal{N}_k(\varpi_k)}$ and defines $\alpha_k = \mathcal{N}_k(\varpi_k)^{-1}$ and $Y_k = \frac{\alpha_1}{\alpha_{k+1}}$. It is useful to provide the triple coupling in quantum cohomology of CY n-folds as in [66] and bring the one-parameter GM in a standard form [66,71,72,91]. The theory for multiparameter GM was worked out in [92], see [44] for review.

For example, for our first CY threefold (11) the topological data of the mirror are $\chi = -128$, $\kappa = 16$, $c_2 = 64$ ($\sigma = \kappa \bmod 2 = 0$) which are precisely those topological data fixing its topological type [89] and the number of lines $n_0^1 = 512$ in it, which was observed in [78]. The equation (A3) fixes the integral symplectic basis at $z = 0$. The analytic continuation of $\underline{\Pi}$ to $z = \infty$ is given exactly by a Barnes integral representation and to the conifold $z = \frac{1}{2^{10}}$ by the construction of a Kuga-Sato variety. The latter implies that the transition matrix is given in terms of the periods and quasiperiods of weight four Hecke eigenforms of $S_4(\Gamma_0(8))$.

From (9) we can evaluate the T^2 period of the K3 at $x = 0$ as residuum, i.e. $\varpi_0^{(0)} = \int_{T^2} \Omega$ with

$$\begin{aligned} \varpi_0^{(0)} &= \frac{1}{(2\pi i)^2} \oint \oint \frac{\mathrm{d}X \wedge \mathrm{d}Z}{\sqrt{Y}} \\ &= \sqrt{z} \left(\sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} z^k \right)^2 = \frac{4}{\pi^2} \sqrt{z} K(z)^2. \end{aligned} \tag{A5}$$

This is clearly a solution of $\mathcal{L}_1^{(3)}$ at $z = x^2 = 0$ and, in fact, one can obtain $\mathcal{L}_1^{(3)}$ as the third-order differential operator that annihilates it. The other two K3 periods over integral monodromy cycles, which have logarithmic and double logarithmic degenerations at this MUM point, yield precisely those quadratic combinations of elliptic functions that appear as transcendental functions in the 4PM approximation. Similarly, the operator (14) can be written as a symmetric square of the second-order operator $\mathcal{L}_2^{(2)} = \theta^2 + z^2(\theta + \frac{1}{2})^2 - \frac{1}{2}z(\theta^2 + 34\theta + 5)$.

For the second CY threefold defined by $U^2 = P$ with P from the Baikov representation of the graph CY3' in Eq. (16), we can evaluate, as in (A5), the T^3 period integral in affine coordinates

$$\begin{aligned}\varpi_0^{(1)} &= \frac{8}{(2\pi i)^4}\oint_{T^4}\frac{\sqrt{1+t}}{\sqrt{Q}}\frac{\mathrm{d}W}{W}\wedge\frac{\mathrm{d}X}{X}\wedge\frac{\mathrm{d}Y}{Y}\wedge\frac{\mathrm{d}Z}{Z}\\ &= 1 - \frac{7}{2^8}t^2 + \frac{7}{2^8}t^3 - \frac{25711}{2^{20}}t^4 + \cdots. \end{aligned} \tag{A6}$$

Here, $t = x^2 - 1$ and the Laurent polynomial Q is $Q = P/(WXYZ)^2$. We can find an operator annihilating the expansion of $\varpi_0^{(1)}$ from an ansatz $\mathcal{L}(\theta, t)$ of order four and six in θ and t, respectively. This operator $\tilde{\mathcal{L}}^{(4)}$, which is equivalent to the one in [88], has an apparent singularity and lacks the integrality properties of a CY operator. Since it is difficult to resolve the $U^2 = P$ geometry we reconstruct the CY threefold motive from (17) by running the $\hat{\Gamma}$-class argument of [49] backwards. This means we determine an integral symplectic basis by calculating the monodromies around $z = 0$ and $z = \frac{1}{2^{10}}$, and determine thereby the topological data in (A3). By analytic continuation we find that such a choice is unique up to an integer $m \in \mathbb{Z}$ with the monodromies $M_{z=0}$ and $M_{z=\frac{1}{2^{10}}}$ ($M^2_{z=\frac{1}{2^{10}}} = \mathbb{1}$) given by

$$\begin{pmatrix} 1 & 1 & 0 & 0\\ 0 & 1 & 0 & 0\\ 2 & 2m-1 & 1 & -1\\ -4 & -2 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} m & 0 & -1 & 0\\ m^2-1 & 0 & -m & 0\\ 0 & 1-m^2 & 0 & m \end{pmatrix}, \tag{A7}$$

respectively. This not only determines an integral symplectic basis but also restricts the topological data to $\chi = 20\kappa$, $c_2 = \kappa(6m-5)$ with $\kappa = 4$. The genus zero BPS numbers are integral at both MUM points $\{n_0^d\} = \{-640, -27680, -2158729, \ldots, d = 1, 2, 3\ldots\}$ as required [64,91]. They appear[1] also in case 2.33 of [76]. This CY threefold is defined as Hadamard product [76] with CY operator $\mathcal{L}^{(4)}_{\mathrm{Had}} = \theta^4 + 2^{16}w^2\prod_{k=0}^{3}(4\theta+2k+1) - 2^4 z(4\theta+1)(4\theta+3)(32\theta^2+32\theta+13)$. Its periods are related to the periods $\underline{\Pi}(z)$ of (17) by $\underline{\Pi}(z) = 1/\sqrt{1+2^{10}z}\,\underline{\Pi}_{\mathrm{Had}}(z/(1+2^{10}z)^2)$, which means that the CY periods associated to the geometry (16) are also realized in a Hadamard product of two elliptic curves.

[1]We thank Duco van Straten for pointing this out.

APPENDIX B: ϵ-FACTORIZED DIFFERENTIAL EQUATION

Finally, we want to give explicit results for the ϵ-factorized differential equation in the CY threefold sector at 5PM-1SF. We do this as an expansion of the rescaled variable x to obtain directly integer coefficient expansions. The four new transcendental functions G_i for $i = 1, 2, 3, 4$ are given by

$$\begin{aligned} G_1(\mathrm{x}) &= -6144\mathrm{x}^4\big(1 + 432\mathrm{x}^4 + 138784\mathrm{x}^8 + \mathcal{O}(\mathrm{x}^{12})\big),\\ G_2(\mathrm{x}) &= \frac{128}{3}\mathrm{x}^2\big(7 + 2512\mathrm{x}^2 + 29344\mathrm{x}^4 + \mathcal{O}(\mathrm{x}^6)\big),\\ G_3(\mathrm{x}) &= -1536\mathrm{x}^4\big(1 + 264\mathrm{x}^4 + 66432\mathrm{x}^8 + \mathcal{O}(\mathrm{x}^{12})\big),\\ G_4(\mathrm{x}) &= -\frac{64}{3}\mathrm{x}^2\big(7 + 900\mathrm{x}^2 - 1120\mathrm{x}^4 + \mathcal{O}(\mathrm{x}^6)\big). \end{aligned} \tag{B1}$$

Notice, that all $G_i(\mathrm{x})$ have an integer coefficient expansion after a suitable normalization. The functions $G_1(\mathrm{x}), G_3(\mathrm{x})$ have exponents being only multiples of four whereas $G_2(\mathrm{x}), G_4(\mathrm{x})$ have just even exponents. The final ϵ-form of the connection form $\epsilon A_{\mathrm{CY3}}(\mathrm{x})$ in the CY threefold sector at 5PM-1SF is given by

$$A_{\mathrm{CY3}}(\mathrm{x}) = \begin{pmatrix} K_{11}(\mathrm{x}) & K_{12}(\mathrm{x}) & 0 & 0\\ K_{21}(\mathrm{x}) & K_{22}(\mathrm{x}) & K_{23}(\mathrm{x}) & 0\\ K_{31}(\mathrm{x}) & K_{32}(\mathrm{x}) & K_{22}(\mathrm{x}) & K_{12}(\mathrm{x})\\ K_{41}(\mathrm{x}) & K_{31}(\mathrm{x}) & K_{21}(\mathrm{x}) & K_{11}(\mathrm{x}) \end{pmatrix} \tag{B2}$$

with the eight different kernels

$$\begin{aligned} K_{11}(\mathrm{x}) &= -\frac{2}{\mathrm{x}} + 128\mathrm{x} + 512\mathrm{x}^3 + 32768\mathrm{x}^5 + \mathcal{O}(\mathrm{x}^7),\\ K_{12}(\mathrm{x}) &= \frac{1}{\mathrm{x}} + 64\mathrm{x}^3 + 10048\mathrm{x}^7 + 1878016\mathrm{x}^{11} + \mathcal{O}(\mathrm{x}^{15}),\\ K_{21}(\mathrm{x}) &= \frac{10}{3\mathrm{x}} + 448\mathrm{x} + 65408\mathrm{x}^3 + 200704\mathrm{x}^5 + \mathcal{O}(\mathrm{x}^7),\\ K_{22}(\mathrm{x}) &= -\frac{2}{\mathrm{x}} + 128\mathrm{x} - 2560\mathrm{x}^3 + 32768\mathrm{x}^5 + \mathcal{O}(\mathrm{x}^7),\\ K_{23}(\mathrm{x}) &= \frac{1}{\mathrm{x}} + 96\mathrm{x}^3 + 19040\mathrm{x}^7 + 4199424\mathrm{x}^{11} + \mathcal{O}(\mathrm{x}^{15}),\\ K_{31}(\mathrm{x}) &= -5376\mathrm{x} - 1208320\mathrm{x}^3 - 10149888\mathrm{x}^5 + \mathcal{O}(\mathrm{x}^7),\\ K_{32}(\mathrm{x}) &= \frac{40}{3\mathrm{x}} + 896\mathrm{x} + \frac{438016\mathrm{x}^3}{3} + 831488\mathrm{x}^5 + \mathcal{O}(\mathrm{x}^7),\\ K_{41}(\mathrm{x}) &= -\frac{476}{9\mathrm{x}} + \frac{8960\mathrm{x}}{3} + \frac{21856640\mathrm{x}^3}{3} + \mathcal{O}(\mathrm{x}^5). \end{aligned} \tag{B3}$$

We can see that on the diagonals $A_{\mathrm{CY3}}(\mathrm{x})$ exhibits a symmetry reducing the number of independent kernels which was also observed for the ϵ-form of the banana integrals [62,63,79–81]. Notice, that this is not a general feature for ϵ-deformed differential equations related to CY operators [63]. With the ϵ-form (B2) we can now systematically compute the ϵ-expansion which by analytic continuation can be made global. To demonstrate this we plot in Fig. 2 the real part of the first three ϵ-orders of the sample function

$$f(\mathrm{x},\epsilon) = f^{(0)}(\mathrm{x}) + \epsilon f^{(1)}(\mathrm{x}) + \epsilon^2 f^{(2)}(\mathrm{x}) + \mathcal{O}(\epsilon^3). \tag{B4}$$

The function f is a linear combination of the maximal cuts of the MIs $I_1, \ldots, I_4$ in the CY3 sector. In particular, we have chosen this linear combination such that $f^{(0)}$ is proportional to the conifold vanishing period of the CY threefold. To be precise we have taken

$$\begin{aligned}
f^{(0)}(\mathrm{x}) &= \frac{2}{3}\mathrm{x}(4\log^3(\mathrm{x}) - \pi^2 \log(\mathrm{x}) - 3\zeta(3)) + \mathcal{O}(\mathrm{x}^2),\\
f^{(1)}(\mathrm{x}) &= -\frac{2}{3}\mathrm{x}(8\log^4(\mathrm{x}) + 7\pi^2\log^2(\mathrm{x}),\\
&\quad - 6\zeta(3)\log(\mathrm{x})) + \mathcal{O}(\mathrm{x}^2),\\
f^{(2)}(\mathrm{x}) &= \frac{2}{27}\mathrm{x}(108\log^5(\mathrm{x}) - (2700 - 119\pi^2)\log^3(\mathrm{x})\\
&\quad - 99\zeta(3)\log^2(\mathrm{x}) + 4050\log(\mathrm{x})) + \mathcal{O}(\mathrm{x}^2).
\end{aligned} \tag{B5}$$

We can see that at the conifold singularity located at $x = 1$ ($\mathrm{x} = \frac{1}{4}$) the different ϵ-orders can be smooth or exhibit a singularity. Nevertheless, the analytic continuation can be done beyond this singularity of the differential equation.

[1] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), Observation of gravitational waves from a binary black hole merger, Phys. Rev. Lett. **116**, 061102 (2016).
[2] B. P. Abbott *et al.* (LIGO Scientific and Virgo Collaborations), GW170817: Observation of gravitational waves from a binary neutron star inspiral, Phys. Rev. Lett. **119**, 161101 (2017).
[3] R. Abbott *et al.* (LIGO Scientific, Virgo, and KAGRA Collaborations), GWTC-3: Compact binary coalescences observed by LIGO and Virgo during the second part of the third observing run, Phys. Rev. X **13**, 041039 (2023).
[4] P. Amaro-Seoane *et al.* (LISA Collaboration), Laser Interferometer Space Antenna, arXiv:1702.00786.
[5] M. Punturo *et al.*, The Einstein Telescope: A third-generation gravitational wave observatory, Classical Quantum Gravity **27**, 194002 (2010).
[6] S. W. Ballmer *et al.*, Snowmass2021 cosmic frontier white paper: Future gravitational-wave detector facilities, in Snowmass 2021 (2022), arXiv:2203.08228.
[7] M. Pürrer and C.-J. Haster, Gravitational waveform accuracy requirements for future ground-based detectors, Phys. Rev. Res. **2**, 023151 (2020).
[8] L. Blanchet, Gravitational radiation from post-Newtonian sources and inspiralling compact binaries, Living Rev. Relativity **17**, 2 (2014).
[9] R. A. Porto, The effective field theorist's approach to gravitational dynamics, Phys. Rep. **633**, 1 (2016).
[10] M. Levi, Effective field theories of post-Newtonian gravity: A comprehensive review, Rep. Prog. Phys. **83**, 075901 (2020).
[11] D. A. Kosower, R. Monteiro, and D. O'Connell, The SAGEX review on scattering amplitudes chapter 14: Classical gravity from scattering amplitudes, J. Phys. A **55**, 443015 (2022).
[12] N. E. J. Bjerrum-Bohr, P. H. Damgaard, L. Plante, and P. Vanhove, The SAGEX review on scattering amplitudes chapter 13: Post-Minkowskian expansion from scattering amplitudes, J. Phys. A **55**, 443014 (2022).
[13] A. Buonanno, M. Khalil, D. O'Connell, R. Roiban, M. P. Solon, and M. Zeng, Snowmass white paper: Gravitational waves and scattering amplitudes, in Snowmass 2021 (2022), arXiv:2204.05194.
[14] P. Di Vecchia, C. Heissenberg, R. Russo, and G. Veneziano, The gravitational eikonal: From particle, string and brane collisions to black-hole encounters, arXiv:2306.16488.
[15] G. U. Jakobsen, Gravitational scattering of compact bodies from worldline quantum field theory, Ph.D. thesis, Humboldt-University Berlin, 2023.
[16] Y. Mino, M. Sasaki, and T. Tanaka, Gravitational radiation reaction to a particle motion, Phys. Rev. D **55**, 3457 (1997).
[17] E. Poisson, A. Pound, and I. Vega, The motion of point particles in curved spacetime, Living Rev. Relativity **14**, 7 (2011).
[18] L. Barack and A. Pound, Self-force and radiation reaction in general relativity, Rep. Prog. Phys. **82**, 016904 (2019).
[19] S. E. Gralla and K. Lobo, Self-force effects in post-Minkowskian scattering, Classical Quantum Gravity **39**, 095001 (2022).
[20] F. Pretorius, Evolution of binary black hole spacetimes, Phys. Rev. Lett. **95**, 121101 (2005).
[21] M. Boyle *et al.*, The SXS Collaboration catalog of binary black hole simulations, Classical Quantum Gravity **36**, 195006 (2019).
[22] T. Damour, F. Guercilena, I. Hinder, S. Hopper, A. Nagar, and L. Rezzolla, Strong-field scattering of two black holes: Numerics versus analytics, Phys. Rev. D **89**, 081503 (2014).
[23] S. J. Kovacs and K. S. Thorne, The generation of gravitational waves. 4. Bremsstrahlung, Astrophys. J. **224**, 62 (1978).

[24] K. Westpfahl and M. Goller, Gravitational scattering of two relativistic particles in postlinear approximation, Lett. Nuovo Cimento **26**, 573 (1979).

[25] L. Bel, T. Damour, N. Deruelle, J. Ibanez, and J. Martin, Poincaré-invariant gravitational field and equations of motion of two pointlike objects: The postlinear approximation of general relativity, Gen. Relativ. Gravit. **13**, 963 (1981).

[26] T. Damour, High-energy gravitational scattering and the general relativistic two-body problem, Phys. Rev. D **97**, 044038 (2018).

[27] S. Hopper, A. Nagar, and P. Rettegno, Strong-field scattering of two spinning black holes: Numerics versus analytics, Phys. Rev. D **107**, 124034 (2023).

[28] W. D. Goldberger and I. Z. Rothstein, An effective field theory of gravity for extended objects, Phys. Rev. D **73**, 104029 (2006).

[29] G. Kälin and R. A. Porto, Post-Minkowskian effective field theory for conservative binary dynamics, J. High Energy Phys. 11 (2020) 106; G. Kälin, Z. Liu, and R. A. Porto, Conservative dynamics of binary systems to third post-Minkowskian order from the effective field theory approach, Phys. Rev. Lett. **125**, 261103 (2020); Conservative tidal effects in compact binary systems to next-to-leading post-Minkowskian order, Phys. Rev. D **102**, 124025 (2020); C. Dlapa, G. Kälin, Z. Liu, and R. A. Porto, Dynamics of binary systems to fourth Post-Minkowskian order from the effective field theory approach, Phys. Lett. B **831**, 137203 (2022); Conservative dynamics of binary systems at fourth post-Minkowskian order in the large-eccentricity expansion, Phys. Rev. Lett. **128**, 161104 (2022); Z. Liu, R. A. Porto, and Z. Yang, Spin effects in the effective field theory approach to post-Minkowskian conservative dynamics, J. High Energy Phys. 06 (2021) 012; S. Mougiakakos, M. M. Riva, and F. Vernizzi, Gravitational bremsstrahlung in the post-Minkowskian effective field theory, Phys. Rev. D **104**, 024041 (2021); M. M. Riva and F. Vernizzi, Radiated momentum in the post-Minkowskian worldline approach via reverse unitarity, J. High Energy Phys. 11 (2021) 228; S. Mougiakakos, M. M. Riva, and F. Vernizzi, Gravitational bremsstrahlung with tidal effects in the post-Minkowskian expansion, Phys. Rev. Lett. **129**, 121101 (2022); M. M. Riva, F. Vernizzi, and L. K. Wong, Gravitational bremsstrahlung from spinning binaries in the post-Minkowskian expansion, Phys. Rev. D **106**, 044013 (2022); G. Mogull, J. Plefka, and J. Steinhoff, Classical black hole scattering from a worldline quantum field theory, J. High Energy Phys. 02 (2021) 048; G. U. Jakobsen, G. Mogull, J. Plefka, and J. Steinhoff, Classical gravitational bremsstrahlung from a worldline quantum field theory, Phys. Rev. Lett. **126**, 201103 (2021); Gravitational bremsstrahlung and hidden supersymmetry of spinning bodies, Phys. Rev. Lett. **128**, 011101 (2022).

[30] G. U. Jakobsen, G. Mogull, J. Plefka, and J. Steinhoff, SUSY in the sky with gravitons, J. High Energy Phys. 01 (2022) 027.

[31] G. U. Jakobsen and G. Mogull, Conservative and radiative dynamics of spinning bodies at third post-Minkowskian order using worldline quantum field theory, Phys. Rev. Lett. **128**, 141102 (2022).

[32] G. U. Jakobsen, G. Mogull, J. Plefka, and B. Sauer, All things retarded: Radiation-reaction in worldline quantum field theory, J. High Energy Phys. 10 (2022) 128.

[33] C. Shi and J. Plefka, Classical double copy of worldline quantum field theory, Phys. Rev. D **105**, 026007 (2022); F. Bastianelli, F. Comberiati, and L. de la Cruz, Light bending from eikonal in worldline quantum field theory, J. High Energy Phys. 02 (2022) 209; F. Comberiati and C. Shi, Classical double copy of spinning worldline quantum field theory, J. High Energy Phys. 04 (2023) 008; T. Wang, Binary dynamics from worldline QFT for scalar QED, Phys. Rev. D **107**, 085011 (2023); M. Ben-Shahar, Scattering of spinning compact objects from a worldline EFT, arXiv:2311.01430; A. Bhattacharyya, D. Ghosh, S. Ghosh, and S. Pal, Observables from classical black hole scattering in scalar-tensor theory of gravity from worldline quantum field theory, arXiv:2401.05492.

[34] C. Dlapa, G. Kälin, Z. Liu, J. Neef, and R. A. Porto, Radiation reaction and gravitational waves at fourth post-Minkowskian order, Phys. Rev. Lett. **130**, 101401 (2023).

[35] C. Dlapa, G. Kälin, Z. Liu, and R. A. Porto, Bootstrapping the relativistic two-body problem, J. High Energy Phys. 08 (2023) 109.

[36] G. U. Jakobsen, G. Mogull, J. Plefka, B. Sauer, and Y. Xu, Conservative scattering of spinning black holes at fourth post-Minkowskian order, Phys. Rev. Lett. **131**, 151401 (2023).

[37] G. U. Jakobsen, G. Mogull, J. Plefka, and B. Sauer, Dissipative scattering of spinning black holes at fourth post-Minkowskian order, Phys. Rev. Lett. **131**, 241402 (2023).

[38] G. U. Jakobsen, G. Mogull, J. Plefka, and B. Sauer, Tidal effects and renormalization at fourth post-Minkowskian order, Phys. Rev. D **109**, L041504 (2024).

[39] D. Neill and I. Z. Rothstein, Classical space-times from the S matrix, Nucl. Phys. **B877**, 177 (2013); A. Luna, I. Nicholson, D. O'Connell, and C. D. White, Inelastic black hole scattering from charged scalar amplitudes, J. High Energy Phys. 03 (2018) 044; D. A. Kosower, B. Maybee, and D. O'Connell, Amplitudes, observables, and classical scattering, J. High Energy Phys. 02 (2019) 137; A. Cristofoli, R. Gonzo, D. A. Kosower, and D. O'Connell, Waveforms from amplitudes, Phys. Rev. D **106**, 056007 (2022); N. E. J. Bjerrum-Bohr, J. F. Donoghue, and P. Vanhove, On-shell techniques and universal results in quantum gravity, J. High Energy Phys. 02 (2014) 111; N. E. J. Bjerrum-Bohr, P. H. Damgaard, G. Festuccia, L. Planté, and P. Vanhove, General relativity from scattering amplitudes, Phys. Rev. Lett. **121**, 171601 (2018); Z. Bern, C. Cheung, R. Roiban, C.-H. Shen, M. P. Solon, and M. Zeng, Scattering amplitudes and the conservative Hamiltonian for binary systems at third post-Minkowskian order, Phys. Rev. Lett. **122**, 201603 (2019); Z. Bern, C. Cheung, R. Roiban, C.-H. Shen, M. P. Solon, and M. Zeng, Black hole binary dynamics from the double copy and effective theory, J. High Energy Phys. 10 (2019) 206; N. E. J. Bjerrum-Bohr, L. Planté, and P. Vanhove, Post-Minkowskian radial action from soft limits and velocity cuts, J. High Energy Phys. 03 (2022) 071; C. Cheung and M. P. Solon, Classical gravitational scattering at $\mathcal{O}(G^3)$ from Feynman diagrams, J. High Energy Phys. 06 (2020) 144; N. E. J. Bjerrum-Bohr,

P. H. Damgaard, L. Planté, and P. Vanhove, The amplitude for classical gravitational scattering at third post-Minkowskian order, J. High Energy Phys. 08 (2021) 172; P. Di Vecchia, C. Heissenberg, R. Russo, and G. Veneziano, Universality of ultra-relativistic gravitational scattering, Phys. Lett. B **811**, 135924 (2020); The eikonal approach to gravitational scattering and radiation at $\mathcal{O}(G^3)$, J. High Energy Phys. 07 (2021) 169; Radiation reaction from soft theorems, Phys. Lett. B **818**, 136379 (2021); Classical gravitational observables from the eikonal operator, Phys. Lett. B **843**, 138049 (2023); C. Heissenberg, Angular momentum loss due to tidal effects in the post-Minkowskian expansion, Phys. Rev. Lett. **131**, 011603 (2023); T. Damour, Radiative contribution to classical gravitational scattering at the third order in G, Phys. Rev. D **102**, 124008 (2020); E. Herrmann, J. Parra-Martinez, M. S. Ruf, and M. Zeng, Radiative classical gravitational observables at $\mathcal{O}(G^3)$ from scattering amplitudes, J. High Energy Phys. 10 (2021) 148; P. H. Damgaard, K. Haddad, and A. Helset, Heavy black hole effective theory, J. High Energy Phys. 11 (2019) 070; P. H. Damgaard, L. Plante, and P. Vanhove, On an exponential representation of the gravitational S-matrix, J. High Energy Phys. 11 (2021) 213; P. H. Damgaard, E. R. Hansen, L. Planté, and P. Vanhove, The relation between KMOC and worldline formalisms for classical gravity, J. High Energy Phys. 09 (2023) 059; R. Aoude, K. Haddad, and A. Helset, On-shell heavy particle effective theories, J. High Energy Phys. 05 (2020) 051; M. Accettulli Huber, A. Brandhuber, S. De Angelis, and G. Travaglini, From amplitudes to gravitational radiation with cubic interactions and tidal effects, Phys. Rev. D **103**, 045015 (2021); A. Brandhuber, G. Chen, G. Travaglini, and C. Wen, Classical gravitational scattering from a gauge-invariant double copy, J. High Energy Phys. 10 (2021) 118.

[40] Z. Bern, J. Parra-Martinez, R. Roiban, M. S. Ruf, C.-H. Shen, M. P. Solon, and M. Zeng, Scattering amplitudes and conservative binary dynamics at $\mathcal{O}(G^4)$, Phys. Rev. Lett. **126**, 171601 (2021).

[41] Z. Bern, J. Parra-Martinez, R. Roiban, M. S. Ruf, C.-H. Shen, M. P. Solon, and M. Zeng, Scattering amplitudes, the Tail effect, and conservative binary dynamics at O(G4), Phys. Rev. Lett. **128**, 161103 (2022).

[42] P. H. Damgaard, E. R. Hansen, L. Planté, and P. Vanhove, Classical observables from the exponential representation of the gravitational S-matrix, J. High Energy Phys. 09 (2023) 183.

[43] P. Vanhove, The physics and the mixed Hodge structure of Feynman integrals, Proc. Symp. Pure Math. **88**, 161 (2014).

[44] A. Klemm, The B-model approach to topological string theory on Calabi-Yau n-folds, in *B-Model Gromov-Witten theory*, Trends in Mathematics (Birkhäuser/Springer, Cham, 2018), pp. 79–397.

[45] K. Bönisch, C. Duhr, F. Fischbach, A. Klemm, and C. Nega, Feynman integrals in dimensional regularization and extensions of Calabi-Yau motives, J. High Energy Phys. 09 (2022) 156.

[46] J. L. Bourjaily, A. J. McLeod, M. von Hippel, and M. Wilhelm, Bounded collection of Feynman integral Calabi-Yau geometries, Phys. Rev. Lett. **122**, 031601 (2019).

[47] S. Bloch, M. Kerr, and P. Vanhove, Local mirror symmetry and the sunset Feynman integral, Adv. Theor. Math. Phys. **21**, 1373 (2017).

[48] A. Klemm, C. Nega, and R. Safari, The l-loop banana amplitude from GKZ systems and relative Calabi-Yau periods, J. High Energy Phys. 04 (2020) 088.

[49] K. Bönisch, F. Fischbach, A. Klemm, C. Nega, and R. Safari, Analytic structure of all loop banana integrals, J. High Energy Phys. 05 (2021) 066.

[50] C. Duhr, A. Klemm, F. Loebbert, C. Nega, and F. Porkert, Yangian-invariant fishnet integrals in two dimensions as volumes of Calabi-Yau varieties, Phys. Rev. Lett. **130**, 041602 (2023).

[51] G. Kälin, J. Neef, and R. A. Porto, Radiation-reaction in the effective field theory approach to Post-Minkowskian dynamics, J. High Energy Phys. 01 (2023) 140.

[52] D. G. Boulware and L. S. Brown, Tree graphs and classical fields, Phys. Rev. **172**, 1628 (1968).

[53] P. H. Damgaard, J. Hoogeveen, A. Luna, and J. Vines, Scattering angles in Kerr metrics, Phys. Rev. D **106**, 124030 (2022).

[54] G. U. Jakobsen and G. Mogull, Linear response, Hamiltonian, and radiative spinning two-body dynamics, Phys. Rev. D **107**, 044033 (2023).

[55] M. Driesse, G. U. Jakobsen, G. Mogull, J. Plefka, B. Sauer, and J. Usovitsch, Conservative black hole scattering at fifth post-minkowskian and first self-force order, arXiv:2403.07781.

[56] P. Maierhöfer, J. Usovitsch, and P. Uwer, Kira—A Feynman integral reduction program, Comput. Phys. Commun. **230**, 99 (2018).

[57] J. Klappert, F. Lange, P. Maierhöfer, and J. Usovitsch, Integral reduction with Kira 2.0 and finite field methods, Comput. Phys. Commun. **266**, 108024 (2021).

[58] F. Lange, P. Maierhöfer, and J. Usovitsch, Developments since Kira 2.0, SciPost Phys. Proc. **7**, 017 (2022).

[59] A. Primo and L. Tancredi, On the maximal cut of Feynman integrals and the solution of their differential equations, Nucl. Phys. **B916**, 94 (2017).

[60] A. Primo and L. Tancredi, Maximal cuts and differential equations for Feynman integrals. An application to the three-loop massive banana graph, Nucl. Phys. **B921**, 316 (2017).

[61] J. M. Henn, Multiloop integrals in dimensional regularization made simple, Phys. Rev. Lett. **110**, 251601 (2013).

[62] L. Görges, C. Nega, L. Tancredi, and F. J. Wagner, On a procedure to derive ϵ-factorised differential equations beyond polylogarithms, J. High Energy Phys. 07 (2023) 206.

[63] C. Duhr, S. Maggio, C. Nega, L. Tancredi, and F. J. Wagner (to be published).

[64] D. van Straten, Calabi-Yau operators, in *Uniformization, Riemann-Hilbert Correspondence, Calabi-Yau Manifolds & Picard-Fuchs Equations*, Vol. 42 of Advanced Lectures in Mathematics (ALM) (Internation Press, Somerville, MA, 2018), pp. 401–451.

[65] P. S. Aspinwall and D. R. Morrison, Topological field theory and rational curves, Commun. Math. Phys. **151**, 245 (1993).

[66] B. R. Greene, D. R. Morrison, and M. R. Plesser, Mirror manifolds in higher dimension [MR1357989 (96h:32038)], in *Mirror Symmetry, II*, Vol. 1 of AMS/IP Studies in

Advanced Mathematics (American Mathematical Society, Providence, RI, 1997), pp. 745–791.
[67] A. Klemm and R. Pandharipande, Enumerative geometry of Calabi-Yau 4-folds, Commun. Math. Phys. **281**, 621 (2008).
[68] W. Lerche, D. J. Smit, and N. P. Warner, Differential equations for periods and flat coordinates in two-dimensional topological matter theories, Nucl. Phys. **B372**, 87 (1992).
[69] H. A. Verrill, Root lattices and pencils of varieties, J. Math. Kyoto Univ. **36**, 423 (1996).
[70] J. Broedel, C. Duhr, F. Dulat, R. Marzucca, B. Penante, and L. Tancredi, An analytic solution for the equal-mass banana graph, J. High Energy Phys. 09 (2019) 112.
[71] A. Klemm, B. Lian, S. S. Roan, and S.-T. Yau, Calabi-Yau fourfolds for M theory and F theory compactifications, Nucl. Phys. **B518**, 515 (1998).
[72] P. Mayr, Mirror symmetry, $N = 1$ superpotentials and tensionless strings on Calabi-Yau four folds, Nucl. Phys. **B494**, 489 (1997).
[73] C. Dlapa, J. M. Henn, and F. J. Wagner, An algorithmic approach to finding canonical differential equations for elliptic Feynman integrals, J. High Energy Phys. 08 (2023) 120.
[74] C. Dlapa, J. Henn, and K. Yan, Deriving canonical differential equations for Feynman integrals from a single uniform weight integral, J. High Energy Phys. 05 (2020) 025.
[75] M. Ruf, Towards gravitational scattering at the fifth order in g, in *Talk at Amplitudes 23* (CERN, Geneva, 2023).
[76] G. Almkvist and D. van Straten, Calabi-Yau operators of degree two, Journal of Algebraic Combinatorics **58**, 1203 (2023).
[77] K. Bönisch, A. Klemm, E. Scheidegger, and D. Zagier, D-brane masses at special fibres of hypergeometric families of Calabi-Yau threefolds, modular forms, and periods, arXiv:2203.09426.
[78] A. Libgober and J. Teitelbaum, Lines on Calabi-Yau complete intersections, mirror symmetry, and Picard-Fuchs equations, Int. Math. Res. Not. **1993**, 29 (1993).
[79] S. Pögel, X. Wang, and S. Weinzierl, The three-loop equal-mass banana integral in ε-factorised form with meromorphic modular forms, J. High Energy Phys. 09 (2022) 062.
[80] S. Pögel, X. Wang, and S. Weinzierl, Taming Calabi-Yau Feynman integrals: The four-loop equal-mass banana integral, Phys. Rev. Lett. **130**, 101601 (2023).
[81] S. Pögel, X. Wang, and S. Weinzierl, Bananas of equal mass: Any loop, any order in the dimensional regularisation parameter, J. High Energy Phys. 04 (2023) 117.
[82] D. Broadhurst and W. Zudilin, A magnetic double integral, J. Aust. Math. Soc. **107**, 9 (2019).
[83] V. Paşol and W. Zudilin, Magnetic (quasi-)modular forms, Nagoya mathematical Journal **248**, 849 (2022).
[84] A. Klemm, C. Nega, J. Plefka, B. Sauer *et al.* (to be published).
[85] R. Apéry, Astérisque **61**, 11 (1979).
[86] D. Zagier, The arithmetic and topology of differential equations, in *European Congress of Mathematics* (European Mathematical Society, Zürich, 2018), pp. 717–776.
[87] F. Beukers and C. A. M. Peters, A family of $K3$ surfaces and $\zeta(3)$, J. Reine Angew. Math. **351**, 42 (1984).
[88] H. Frellesvig, R. Morales, and M. Wilhelm, Calabi-Yau meets Gravity: A Calabi-Yau three-fold at fifth post-Minkowskian order, arXiv:2312.11371.
[89] C. T. C. Wall, Classification problems in differential topology. V. On certain 6-manifolds, Inventiones Mathematicae **1**, 355 (1966).
[90] S. Hosono, A. Klemm, S. Theisen, and S.-T. Yau, Mirror symmetry, mirror map and applications to complete intersection Calabi-Yau spaces, Nucl. Phys. **B433**, 501 (1995).
[91] M. Bogner, Algebraic characterization of differential operators of Calabi-Yau type, arXiv:1304.5434.
[92] N. Cabo Bizet, A. Klemm, and D. Vieira Lopes, Landscaping with fluxes and the E8 Yukawa point in F-theory, arXiv:1404.7645.