Kantowski-Sachs and Bianchi III dynamics in $f(Q)$ gravity

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We explore the phase-space of homogeneous and anisotropic spacetimes within symmetric teleparallel $f(Q)$ gravity. Specifically, we consider the Kantowski-Sachs and locally rotational Bianchi III geometries to describe the physical space. By analyzing the phase-space, we reconstruct the cosmological history dictated by $f(Q)$ gravity and comment about the theory's viability. Our findings suggest that the free parameters of the connection must be constrained to eliminate nonlinear terms in the field equations. Consequently, new stationary points emerge, rendering the theory cosmologically viable. We identify the existence of anisotropic accelerated universes, which may correspond to the preinflationary epoch.

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I. INTRODUCTION

Even though the flat Friedmann-Lemaître-Robertson-Walker (FLRW) Λ cold dark matter (ΛCDM) model passes most of the observational tests with flying colors, recent cosmological data suggest a tension between early and late Universe measurements of the expansion rate, i.e. the Hubble parameter. Specifically, cosmic microwave background (CMB) data from Planck collaboration in their 2018 release[1] [1], propagated to today using flat ΛCDM suggest a value of the Hubble parameter at $H_0 = 67.4 \pm 0.5$ km/s/Mpc. On the other hand, Cepheid variable stars are used as standard candles to measure the distances of galaxies with the distance ladder method, inferring a value of $H_0 = 74 \pm 1.4$ km/s/Mpc (see SH0ES [2]). The statistical significance between these two measurements is at 5σ. In addition, a nontrivial tension is related to the amplitude of density fluctuations at low redshifts when compared to the one predicted by CMB. The amplitude of these fluctuations, also known as linear matter perturbations, is often defined by the value of the linear matter overdensity field in spheres with a radius $8h^{-1}$ Mpc. This value is called σ_8. The parameter S_8 is defined as $\sigma_8(\Omega_m/0.3)^\alpha$, where Ω_m is the fractional energy density of nonrelativistic matter and α is selected to minimize the correlation between S_8 and Ω_m. Several weak lensing surveys such as DES [3], HSC [4], and Heymans *et al.* [5], have measured S_8, but they have encountered different levels of disagreement with the value inferred from measurements by the Planck satellite. Cosmic shear data tend to recover slightly lower values of S_8 than the CMB. The Kilo-Degree Survey collaboration (KiDS) has reported the most significant disagreement so far with a significance of around 3σ [5–9]. These are only some of the issues that the concordance cosmological model needs to address.

A possible alternative to solve these tensions is to consider extensions beyond ΛCDM; see Ref. [10] for a review. However, incomprehension between the SNIa absolute magnitude and the Cepheid-based distance ladder instead of an exotic late- or early-time physics could be the reason for the tensions [11].

Because of the above, even though inflation is widely considered to be the most plausible explanation for the homogeneity and isotropy of the observable Universe,

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[1]Currently available online in https://wiki.cosmos.esa.int/planck-legacy-archive/index.php/Cosmological_Parameters (accessed on September 22nd, 2023).

among others, most of the studies base their analysis on the fact that the Universe should be flat FLRW and, based on that, they study the evolution of perturbations. However, it could be the case that inflation does occur, but the Universe evolves toward homogeneity and isotropy starting from a more complicated metric. There have been attempts to consider an entirely arbitrary metric, meaning both inhomogeneous and anisotropic [12]; however, the calculations become cumbersome. In this paper, we focus on homogeneous and anisotropic cosmology to extract analytical information. This class of geometries [13] exhibits exciting cosmological features in the inflationary and postinflationary epochs [14]. Nine spatially homogeneous, but generally anisotropic, Bianchi exist based on the real three-dimensional Lie algebra classification. In these spacetimes, three-dimensional hypersurfaces are defined by the orbits of three isometries. An essential characteristic of the Bianchi models is that the physical variables depend only on time, which means that the field equations are a system of ordinary differential equations [15,16]. In recent years, the class of anisotropic geometries has gained much interest because of anisotropic anomalies in the CMB and large-scale structure (LSS) data. The origin of asymmetry and other measures of statistical anisotropy on the large scales of the Universe is a long-standing open question in cosmology. "Planck legacy" temperature anisotropy data [17] show strong evidence of violating the cosmological principle in its isotropic aspect [18,19].

The family of spatially homogeneous Bianchi cosmologies includes as subclasses many important gravitational models, such as the mixmaster Universe or the isotropic FLRW spacetimes [20–23]. As expected, the latter comes as a limit of Bianchi models where the anisotropy vanishes. Indeed, the flat, the open, and the closed FLRW geometries are related to the Bianchi I, V and IX spacetimes respectively [24]. In general, the Bianchi spacetimes are defined by three scale factors [23]; however, the locally rotational spacetimes (LRS) admit an extra fourth isometry, and the LRS Bianchi line elements admit two independent scale factors. It is interesting to mention that the LRS Bianchi IX spacetime is related to the Kantowski-Sachs geometry [25].

Symmetric teleparallel general relativity (STGR) [26,27] is a geometric theory of gravity, equivalent to general relativity. The theory is described by a metric tensor $g_{\mu\nu}(x^\lambda)$ and a symmetric flat connection $\Gamma^\kappa_{\lambda\nu}(x^\lambda)$, different from that of the Levi-Civita connection. It is assumed that the connection $\Gamma^\kappa_{\lambda\nu}(x^\lambda)$ admits the same symmetries with the metric tensor $g_{\mu\nu}(x^\lambda)$ and the corresponding Riemann tensor is assumed to have zero components (flat connection). Since the connection is considered to be symmetric, it means that its torsion also vanishes. The nonmetricity tensor plays an important role in symmetric teleparallel geometries, since it is the fundamental geometric object used to define the gravitational action integral [27] and thus describe gravitational interactions.

To explain the observed cosmic acceleration [28–31] in the context of symmetric teleparallel theory in a natural way, it has been proposed the employ of nonlinear terms of the nonmetricity scalar, Q, in the gravitational action, leading to the symmetric teleparallel $f(Q)$-theory [32,33]. The same approach has been considered before in the case of $f(R)$ [34] and $f(T)$ theories of gravity [35], where the Ricci scalar R of the Levi-Civita connection and the torsion scalar T of the teleparallel connection [36] are considered to define the gravitational theory. For a comprehensive review on teleparallel theories of gravity see Ref. [37]. The novelty of the $f(X)$-theories [38,39], where X is a geometric scalar, is that new terms which are introduced in the modified field equations drive the dynamics to describe the expansion of the universe in a geometric way, without the addition of any exotic form of matter/energy [40].

Preliminary cosmological studies in $f(Q)$-theory have shown that it is a potential geometric dark energy candidate which can challenge the concordance model in cosmology [41–43]. Numerous studies show that $f(Q)$-theory can reproduce various cosmological scenarios [44–47]; in [48], the authors determined the criteria for the existence of scaling cosmological solutions; the detailed phase-space analysis was presented in [49] and it indicates that $f(Q)$-theory can be used to describe not only late-time but also early-time acceleration phases of the Universe. Similar results are presented in [50]. The Hamiltonian analysis of $f(Q)$-theory was studied in [51–53], while quantum cosmology was investigated in [54,55]. For extensions of $f(Q)$-theory we refer the reader to [56–59] and other cosmological applications are discussed in [60–62] and references therein. A recent review on the topic is [63].

Because the connection in $f(Q)$-theory is flat and symmetric, it comes naturally that there exists a coordinate system where all the components of the connection can vanish; as a result, the covariant derivatives are reduced to partial derivatives. This characteristic coordinate system is known as the coincident gauge [33].

Different connections, in general, affect the dynamics in a given nonlinear $f(Q)$-theory. Consider for example the case of a spatially flat FLRW metric expressed in a certain coordinate system (Cartesian or spherical). It is found that there exist three distinct choices of connections which share the symmetries of the spacetime [64,65]. One of them is dynamically irrelevant, i.e. its components can vanish through a pure time transformation. The other two, on the other hand, involve functions of time which are to be calculated by solving the relative equations of motion for the connection for a given $f(Q)$ theory. Thus, if one uses one of the latter two connections, additional degrees of freedom are introduced. We will see this happening also in the Kantowski-Sachs/Bianchi type III model which we discuss here, where we shall introduce additional scalar degrees of freedom to incorporate the dynamics owed to the connection [33,66,67]. Although the coincident gauge can always be recovered for a given geometry, it cannot always be used blindly, when a specific assumption for the spacetime metric has been made. See for instance the discussion in [68].

The structure of the paper is as follows. In Sec. II we briefly discuss the basic properties and definitions of symmetric teleparallel general relativity and of the symmetric teleparallel $f(Q)$-theory of gravity. We discuss previous results for the spatially flat and isotropic universe in Sec. III. Homogeneous and isotropic locally rotational spacetimes with nonzero spatial curvature in symmetric teleparallel $f(Q)$-theory are introduced in Sec. IV. We give emphasis in the Kantowski-Sachs and Bianchi III geometries and we present the gravitational field equations in the case of vacuum for a nonlinear function $f(Q)$. Section V includes the main results of this analysis where we present a detailed analysis of the phase-space for the anisotropic cosmological model. From our analysis it follows that for specific values of the free parameters which define the connection the theory can provide the limit of general relativity (GR) and there exist a plethora of asymptotic solutions which can describe anisotropic inflationary solutions. However, in the generic case of the connection these solutions are lost. Thus the cosmological viability of the theory constraints the free parameters of the connection as it follows from the analysis of the asymptotics. Finally, in Sec. VI we discuss our results.

II. SYMMETRIC TELEPARALLEL GEOMETRY AND GRAVITY

In teleparallel theories, parallelism at a distance is achieved by the vanishing of the curvature of the connection, i.e. $R^{\alpha}{}_{\mu\beta\nu} = 0$, which makes the connection become integrable and thus it can be expressed as

$$\Gamma^{\alpha}{}_{\mu\nu} = (\Lambda^{-1})^{\alpha}{}_{\lambda}\partial_{\mu}\Lambda^{\lambda}{}_{\nu}, \tag{1}$$

where $\Lambda \in GL(4, \mathbf{R})$. In addition, symmetric means that the torsion of the connection vanishes, i.e. $T^{\alpha}{}_{\mu\nu} = 2\Gamma^{\alpha}{}_{[\mu\nu]} = 0$, in which case Λ can be written as $\Lambda^{\alpha}{}_{\beta} = \partial_{\beta}\xi^{\alpha}$, with ξ^{α} being an arbitrary coordinate system. This leads to the symmetric teleparallel connection that can be expressed as

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{\partial x^{\alpha}}{\partial \xi^{\lambda}}\partial_{\mu}\partial_{\nu}\xi^{\lambda}. \tag{2}$$

Since ξ^{α} is arbitrary, we can always find a coordinate system in which the connection vanishes by performing a diffeomorphism; this is called the coincident gauge.

Notice that, the theory of gravity could be perfectly formulated just by the metric tensor, $g_{\mu\nu}$, with the kinetic term in the action being $\partial_{\alpha}g_{\mu\nu}$. However, this would not have the same symmetries as general relativity (GR); it would violate diffeomorphism (Diff) invariance. In order to resolve that, we can employ the above ξ's as Stückelberg fields and the Diff symmetry will be restored.

According to the above, the only nontrivial geometric object in a symmetric teleparallel geometry is the nonmetricity tensor, expressed as

$$\begin{aligned} Q_{\alpha\mu\nu} &= \partial_{\alpha}g_{\mu\nu} - \Gamma^{\lambda}{}_{\alpha\mu}g_{\lambda\nu} - \Gamma^{\lambda}{}_{\alpha\nu}g_{\lambda\mu} \\ &= \partial_{\alpha}g_{\mu\nu} - 2\frac{\partial x^{\sigma}}{\partial\xi^{\lambda}}\partial_{\alpha}\partial_{(\mu}\xi^{\lambda}g_{\nu)\sigma}. \end{aligned} \tag{3}$$

This object transforms clearly covariantly and thus any theory formulated with it will be automatically Diff invariant.

A. Symmetric teleparallel equivalent of general relativity

By defining the two independent traces $Q_{\mu} = Q_{\mu\nu}{}^{\nu}$ and $\tilde{Q}^{\nu}{}_{\mu\nu}$, the nonmetricity scalar is defined as

$$Q = Q_{\alpha\mu\nu}P^{\alpha\mu\nu}, \tag{4}$$

where

$$P^{\alpha}{}_{\mu\nu} = -\frac{1}{4}Q^{\alpha}{}_{\mu\nu} + \frac{1}{2}Q_{(\mu}{}^{\alpha}{}_{\nu)} + \frac{1}{4}g_{\mu\nu}(Q^{\alpha} - \tilde{Q}^{\alpha}) - \frac{1}{4}\delta^{\alpha}{}_{(\mu}Q_{\nu)}, \tag{5}$$

with $\delta\alpha_{\mu}$ being the 4-dimensional Kronecker delta and the brackets denote symmetrization $2A_{(\mu\nu)} = A_{\mu\nu} + A_{\nu\mu}$.

As mentioned above, the curvature of the symmetric teleparallel connection (2) is zero. However, the curvature calculated from the Levi-Civita connection, $\mathring{R}^{\alpha}{}_{\mu\beta\nu}$, is not. The relation between the nonmetricity scalar Q and the Ricci scalar $\mathring{R}$ is given by

$$\mathring{R} = Q + \mathring{\nabla}_{\alpha}(Q^{\alpha} - \tilde{Q}^{\alpha}). \tag{6}$$

By taking the functional integral of the above, the last term will act as a boundary term and thus contribute nothing at the dynamics of the theory. This means that GR and STGR are two dynamically equivalent theories since

$$\mathcal{S}_{\mathrm{GR}} = \int \mathrm{d}^4x\sqrt{-g}\mathring{R} \sim \mathcal{S}_{\mathrm{STGR}} = \int \mathrm{d}^4x\sqrt{-g}Q. \tag{7}$$

What is more, any prediction of GR should be predicted by STGR as well and any solution in GR should have an analog solution in STGR. The only thing that changes is the geometric interpretation of gravitational interactions.

B. $f(Q)$ theory

In the same spirit as with $f(\mathring{R})$ theories, we can generalize the nonmetricity scalar (4) with a general function of it, so that the new action will read

$$\mathcal{S}_{f(Q)} = \int \mathrm{d}^4x\sqrt{-g}f(Q). \tag{8}$$

Since the action contains nonlinear terms in Q, not only the two theories are no longer equivalent, but also solutions, like the nonflat FLRW one, which worked fine in the coincident gauge STGR, are no longer solutions in $f(Q)$.

So in this case, we have both the metric and the connection (2) as fundamental variables. Varying the action (8) with respect to the metric, we get

$$\frac{2}{\sqrt{-g}}\nabla_\lambda\left(\sqrt{-g}f'(Q)P^\lambda{}_{\mu\nu}\right) - \frac{1}{2}f(Q)g_{\mu\nu} + f'(Q)\left(P_{\mu\rho\sigma}Q_\nu{}^{\rho\sigma} - 2Q_{\rho\sigma\mu}P^{\rho\sigma}{}_\nu\right) = 0, \tag{9}$$

where the prime denotes differentiation with respect to the argument, i.e. $f'(Q) = f_{,Q}$. Respectively, varying the action with respect to the connection[2] we get

$$\nabla_\mu\nabla_\nu(\sqrt{-g}f'(Q)P^{\mu\nu}{}_\sigma) = 0. \tag{10}$$

In the coincident gauge, the latter is satisfied identically.

III. FLAT FLRW COSMOLOGY IN $f(Q)$ GRAVITY

Consider a homogeneous and isotropic flat FLRW metric of the form

$$\mathrm{d}s^2 = -N(t)^2\mathrm{d}t^2 + a(t)^2\left(\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\right), \tag{11}$$

with N being the lapse function and a the scale factor. By forcing the symmetries of this metric, i.e. three rotations and three translations, to an arbitrary connection, we find that the connection depends on five unknown functions (out of 64 independent components). Once we impose the symmetric teleparallel constraints, i.e. vanishing curvature and torsion of the connection, we end up with three unknown functions $\{C_1, C_2, C_3\}$ and three constrain equations

$$C_1C_3 - C_3^2 - \dot{C}_3 = 0, \tag{12}$$

$$C_1C_2 - C_2C_3 + \dot{C}_2 = 0, \tag{13}$$

$$C_2C_3 = 0. \tag{14}$$

[2]Obviously, instead of the connection $\Gamma^\alpha{}_{\mu\nu}$, one could vary the action with respect to the arbitrary ξ's; the equations of motion would be the same.

Because of the last of the above, we have three cases

(i) Case I: $C_2 = 0 = C_3$ where $\Gamma^t{}_{tt} = \gamma$.
(ii) Case II: $C_2 = 0$ and $C_3 \neq 0$ where $\Gamma^t{}_{tt} = \gamma + \frac{\dot{\gamma}}{\gamma}$ and $\Gamma^r{}_{tr} = \Gamma^r{}_{rt} = \Gamma^\theta{}_{t\theta} = \Gamma^\theta{}_{\theta t} = \Gamma^\phi{}_{t\phi} = \Gamma^\phi{}_{\phi t} = \gamma$.
(iii) Case III: $C_2 \neq 0$ and $C_3 = 0$ where $\Gamma^t{}_{tt} = -\frac{\dot{\gamma}}{\gamma}$ and $\Gamma^t{}_{rr} = \gamma$, $\Gamma^t{}_{\theta\theta} = \gamma r^2$, $\Gamma^t{}_{\phi\phi} = \gamma r^2\sin^2\theta$.

The rest of the components of the connection are the same as the Levi-Civita one for the three-dimensional flat space, i.e.

$$\Gamma^r{}_{\theta\theta} = -r, \quad \Gamma^r{}_{\phi\phi} = -r\sin^2\theta, \quad \Gamma^\theta{}_{\phi\phi} = -\sin\theta\cos\theta,$$
$$\Gamma^\phi{}_{\theta\phi} = \Gamma^\phi{}_{\phi\theta} = \cot\theta, \quad \Gamma^\theta{}_{r\theta} = \Gamma^\theta{}_{\theta r} = \Gamma^\phi{}_{r\phi} = \Gamma^\phi{}_{\phi r} = r^{-1}. \tag{15}$$

Summarizing, a connection which is spatially flat, homogeneous, isotropic, torsionless and with no curvature can be parametrized in the above three distinct ways, which could lead to interesting phenomenology in cosmology.

IV. KANTOWSKI-SACHS AND BIANCHI III GEOMETRY

We proceed our study by considering the anisotropic cosmological models with line element

$$ds^2 = -N(t)^2dt^2 + a^2(t)\left(e^{2b(t)}dx^2 + e^{-b(t)}(dy^2 + S^2(y)dz^2)\right), \tag{16}$$

with $S(y) = \sin y$ or $S(y) = \sinh y$.

For $S(y) = \sin y$ the line element (16) describes the Kantowski-Sachs space, while for $S(y) = \sinh y$ the line element (16) corresponds to the locally rotational (LRS) Bianchi III geometry. The scale factor $a(t)$ describes the size of the universe; that is, the volume of the three-dimensional space is defined as $V = a^3$. Moreover, $b(t)$ is the anisotropic parameter.

Indeed, spacetime (16) admits a four-dimensional Lie algebra consisted by the vector fields

$$\xi_1 = \partial_z, \quad \xi_2 = \cos z\partial_y - \frac{S'(y)}{S(y)}\sin z\partial_z$$
$$\xi_3 = \sin z\partial_y + \frac{S'(y)}{S(y)}\cos z\partial_z \quad \text{and} \quad \xi_4 = \partial_x.$$

In this study we consider the symmetric and flat connection $\Gamma^\mu_{\kappa\nu}$ with nonzero components [67]

$$\Gamma^{t}{}_{tt}=-\frac{1}{\gamma_2}[\dot{\gamma}_2+c_1\gamma_1(2-c_2\gamma_1)+k],\qquad \Gamma^{t}{}_{tx}=c_1(1-c_2\gamma_1),\qquad \Gamma^{t}{}_{xx}=c_1c_2\gamma_2,$$
$$\Gamma^{t}{}_{yy}=\gamma_2,\qquad \Gamma^{t}{}_{zz}=\gamma_2S(y)^2,\qquad \Gamma^{x}{}_{tt}=\frac{1}{\gamma_2^2}[\gamma_1(k+c_1\gamma_1)(c_2\gamma_1-1)-\gamma_2\dot{\gamma}_1]$$
$$\Gamma^{x}{}_{tx}=-\frac{c_2\gamma_1}{\gamma_2}(k+c_1\gamma_1),\qquad \Gamma^{x}{}_{xx}=c_1+c_2k+c_1c_2\gamma_1,\qquad \Gamma^{x}{}_{yy}=\gamma_1,\qquad \Gamma^{x}{}_{zz}=\gamma_1S(y)^2,$$
$$\Gamma^{y}{}_{ty}=\Gamma^{z}{}_{tz}=-\frac{k+c_1\gamma_1}{\gamma_2},\qquad \Gamma^{y}{}_{xy}=\Gamma^{z}{}_{xz}=c_1,\qquad \Gamma^{y}{}_{zz}=-S(y)S'(y),\qquad \Gamma^{z}{}_{yz}=\frac{S'(y)}{S(y)},\tag{17}$$

with $\gamma_1=-\frac{1}{c_2}-\frac{k}{c_1}$. This choice for the γ_1 is needed in order to eliminate the nondiagonal term produced in the field equations [67]. This value of γ_1 is selected in order the nondiagonal terms of the field equations to be zero. If we do not make this choice, then the resulting field equations imply that either $f(Q)$ is linear (theory equivalent to general relativity) or that $Q=\text{const.}$ (theory equivalent to general relativity plus a cosmological constant). Hence, the choice for γ_1 is unique if we want to study a modification of General Relativity with an actual effect on the dynamics. The field equations for arbitrary γ_1 are presented in the Appendix.

Furthermore, the k in the previous relations is a constant given by $k=-S''(y)/S(y)$ and it can be equal to $+1$ or -1. For the latter connection and the line element (16) the nonmetricity scalar is derived to be

$$Q=-6H^2+2\frac{k}{a^2}e^b+\frac{3}{2}\frac{\dot{b}^2}{N^2}-\frac{6kH}{\gamma_2N}+\frac{e^b}{a^2}(2+c_1c_2e^{-3b})\left(HN\gamma_2+\dot{\gamma}_2+\frac{\gamma_2\dot{N}}{N}\right)+2\frac{e^b}{a^2}(1-c_1c_2e^{-3b})\gamma_2\dot{b}$$
$$+\frac{3(2c_1+c_2k)^2}{4c_1c_2}\frac{H}{N\gamma_2}-\frac{(c_2k-2c_1)^2}{4c_1c_2N^2\gamma_2}\left(\frac{\dot{N}}{N}+\frac{\dot{\gamma}_2}{\gamma_2}\right),\tag{18}$$

in which H stands for $H=\frac{\dot{a}}{Na}$.

Hence, the gravitational field equations in the vacuum are [67]

tt:

$$f'(Q)\left(3H^2+\frac{k}{a^2}e^b-\frac{3}{4}\frac{\dot{b}^2}{N^2}\right)+\frac{1}{2}(f(Q)-Qf'(Q))+\frac{\dot{Q}}{N}f''(Q)\left[\frac{(c_2k-2c_1)^2}{8c_1c_2N\gamma_2}-\frac{e^bN\gamma_2}{a^2}\left(1+\frac{c_1c_2}{2}e^{-3b}\right)\right]=0,\tag{19}$$

xx:

$$f'(Q)\left(\frac{\ddot{b}}{N^2}+3H\frac{\dot{b}}{N}-\frac{\dot{b}\dot{N}}{N^3}-2\frac{\dot{H}}{N}-3H^2-\frac{ke^b}{a^2}-\frac{3}{4}\frac{\dot{b}^2}{N^2}\right)-\frac{1}{2}(f(Q)-Qf'(Q))$$
$$+\frac{\dot{Q}}{N}f''(Q)\left[\frac{(c_2k-2c_1)^2}{8c_1c_2N\gamma_2}+\frac{e^bN\gamma_2}{a^2}\left(1-\frac{c_1c_2}{2}e^{-3b}\right)+\frac{\dot{b}}{N}-2H\right]=0,\tag{20}$$

yy, *zz*:

$$f'(Q)\left(\frac{\ddot{b}}{N^2}+3H\frac{\dot{b}}{N}-\frac{\dot{b}\dot{N}}{N^3}+4\frac{\dot{H}}{N}+6H^2+\frac{3}{2}\frac{\dot{b}^2}{N^2}\right)+(f(Q)-Qf'(Q))$$
$$+\frac{\dot{Q}}{N}f''(Q)\left[4H+\frac{\dot{b}}{N}-\frac{(c_2k-2c_1)^2}{4c_1c_2N\gamma_2}-\frac{c_1c_2e^{-2b}}{a^2}N\gamma_2\right]=0.\tag{21}$$

In order to write the field equations in a simpler form we introduce the scalar field $\phi=f'(Q)$, the potential function $V(\phi)=(f(Q)-Qf'(Q))$, from where we can write the pointlike Lagrangian [66]

$$L(a,\dot{a},b,\dot{b},\phi,\dot{\phi},\Psi,\dot{\Psi})=\frac{1}{N}\left(\frac{3}{2}a^3\phi\dot{b}^2-6a\phi\dot{a}^2-\frac{a^3(k-2\alpha)^2\dot{\Psi}\dot{\phi}}{4\alpha}\right)-N\left(\frac{ae^{-2b}(2e^{3b}+\beta)\dot{\phi}}{\dot{\Psi}}-2kae^b\phi-a^3V(\phi)\right),\tag{22}$$

with $\gamma_2=\frac{1}{\dot{\Psi}}$, $\alpha=\frac{c_1}{c_2}$ and $\beta=c_1c_2$.

The field equations follow from the variation of the latter Lagrangian with respect to the dynamical variables $\{N, a, b, \phi, \Psi\}$. Specifically, the gravitational field equations are (here and henceforth we impose $N = 1$ for the lapse function)

$$\dot{H} = \frac{12a\dot{a}\,\dot{\phi} + 6\phi\dot{a}^2 - 2ke^b\phi}{12a\phi} + \frac{a(6\alpha\phi\dot{b}^2 + (k-2\alpha)^2\dot{\Psi}(-\dot{\phi}) + 4\alpha V(\phi))}{16\alpha\phi} - \frac{(\beta e^{-2b} + 2e^b)\dot{\phi}}{12a\phi\dot{\Psi}} + \frac{a(6\alpha\phi\dot{b}^2 + (k-2\alpha)^2\dot{\Psi}(-\dot{\phi}) + 4\alpha V(\phi))}{16\alpha\phi}, \tag{23}$$

$$\ddot{b} = -\frac{3\dot{a}\,\dot{b}}{a} + \frac{2e^{-2b}\left(ke^{3b} + \frac{(\beta - e^{3b})\dot{\phi}}{\phi\dot{\Psi}}\right)}{3a^2} - \frac{\dot{b}\,\dot{\phi}}{\phi}, \tag{24}$$

$$\ddot{\phi} = \frac{\dot{\phi}(\dot{a}(40\alpha a^2 e^{2b}(k-2\alpha)^2(2e^{3b}+\beta)\dot{\Psi}^2 - 3a^4e^{4b}(k-2\alpha)^4\dot{\Psi}^4 + 16\alpha^2(2e^{3b}+\beta)^2))}{a(a^2e^{2b}(k-2\alpha)^2\dot{\Psi}^2 - 4\alpha(2e^{3b}+\beta))^2} + \frac{\dot{\phi}(8\alpha a(6\alpha a^2e^{2b}(2e^{3b}+\beta)\dot{b}^2\dot{\Psi} + 4\alpha e^{2b}(2e^{3b}+\beta)\dot{\Psi}(a^2V'(\phi) + 2ke^b)))}{a(a^2e^{2b}(k-2\alpha)^2\dot{\Psi}^2 - 4\alpha(2e^{3b}+\beta))^2} + \frac{\dot{\phi}(8\alpha a((e^{3b}-\beta)\dot{b}(a^2e^{2b}(k-2\alpha)^2\dot{\Psi}^2 + 4\alpha\beta + 8\alpha e^{3b})))}{a(a^2e^{2b}(k-2\alpha)^2\dot{\Psi}^2 - 4\alpha(2e^{3b}+\beta))^2} + \frac{\dot{\phi}(-192\alpha^2 ae^{2b}\dot{a}^2(2e^{3b}+\beta)\dot{\Psi})}{a(a^2e^{2b}(k-2\alpha)^2\dot{\Psi}^2 - 4\alpha(2e^{3b}+\beta))^2}, \tag{25}$$

$$\ddot{\Psi} = \frac{\dot{\Psi}(8\alpha e^{3b}(\dot{a} + a(\dot{b} + k\Psi')) + ae^{2b}\dot{\Psi}(3a(k-2\alpha)^2\dot{a}\,\dot{\Psi} - 24\alpha\dot{a}^2 + 2\alpha a^2(3\dot{b}^2 + 2V'(\phi))))}{4\alpha a(2e^{3b}+\beta) - a^3e^{2b}(k-2\alpha)^2\dot{\Psi}^2} + \frac{\dot{\Psi}(4\alpha\beta(\dot{a} - 2a\dot{b}))}{4\alpha a(2e^{3b}+\beta) - a^3e^{2b}(k-2\alpha)^2\dot{\Psi}^2}. \tag{26}$$

Additionally, the Friedmann equation reads

$$\frac{2ke^b\phi}{a^2} - \frac{\beta e^{-2b}\dot{\phi}}{a^2\dot{\Psi}} - \frac{2e^b\dot{\phi}}{a^2\dot{\Psi}} - \frac{3}{2}\phi\dot{b}^2 + 6H^2\phi + \frac{k^2\dot{\Psi}\,\dot{\phi}}{4\alpha} - k\dot{\Psi}\,\dot{\phi} + \alpha\dot{\Psi}\,\dot{\phi} + V(\phi) = 0. \tag{27}$$

For simplicity, we kept the scale factor a in the field equations, but in the computations, we write everything in terms of the Hubble function, that is $a = e^{\int H dt}$. In the following we consider the power-law potential $V(\phi) = V_0\phi^\lambda$, which correspond to a power-law $f(Q)$ function.

V. PHASE-SPACE ANALYSIS

In this section, we set $k = 2\alpha$ in the pointlike Lagrangian (22) and obtain a simplified version of system (23)–(26). This choice corresponds to having $\gamma_1 = 0$ in the components of the connection. The ensuing equations are

$$\dot{H} = -\frac{\alpha e^b}{3a^2} + \frac{\beta e^{-2b}\dot{\phi}}{12a^2\phi\dot{\Psi}} + \frac{e^b\dot{\phi}}{6a^2\phi\dot{\Psi}} - \frac{3}{8}\dot{b}^2 - \frac{H\dot{\phi}}{\phi} - \frac{3}{2}H^2 - \frac{V(\phi)}{4\phi}, \tag{28}$$

$$\ddot{b} = \frac{4\alpha e^b}{3a^2} + \frac{1}{3a^2\phi\dot{\Psi}}(2\beta e^{-2b}\dot{\phi} - 2e^b\dot{\phi}) - 3H\dot{b} - \frac{\dot{b}\,\dot{\phi}}{\phi}, \tag{29}$$

$$\ddot{\phi} = \frac{1}{2e^{3b}+\beta}\left(3a^2e^{2b}\dot{b}^2\dot{\Psi}\,\dot{\phi} + 2a^2e^{2b}\dot{\Psi}\,\dot{\phi}\,V'(\phi) + 2e^{3b}\dot{b}\,\dot{\phi}\right) - \frac{1}{2e^{3b}+\beta}\left(12a^2e^{2b}H^2\dot{\Psi}\,\dot{\phi} + 2\beta\dot{b}\,\dot{\phi}\right) + \frac{1}{2e^{3b}+\beta}\left(2e^{3b}H\dot{\phi} + \beta H\dot{\phi} + 8\alpha e^{3b}\dot{\Psi}\,\dot{\phi}\right), \tag{30}$$

$$\ddot{\Psi} = \frac{1}{2e^{3b}+\beta}\left(\frac{3a^2e^{2b}\dot{b}^2\dot{\Psi}^2}{2} + a^2e^{2b}\dot{\Psi}^2V'(\phi) + 2e^{3b}\dot{b}\,\dot{\Psi}\right) - \frac{1}{2e^{3b}+\beta}\left(6a^2e^{2b}H^2\dot{\Psi}^2 + 2\beta\dot{b}\,\dot{\Psi}\right) + \frac{1}{2e^{3b}+\beta}\left(2e^{3b}H\dot{\Psi} + \beta H\dot{\Psi} + 4\alpha e^{3b}\dot{\Psi}^2\right). \tag{31}$$

In this case, the Friedmann equation becomes

$$\frac{4\alpha e^b\phi}{a^2} - \frac{\beta e^{-2b}\dot{\phi}}{a^2\dot{\Psi}} - \frac{2e^b\dot{\phi}}{a^2\dot{\Psi}} - \frac{3}{2}\phi\dot{b}^2 + 6H^2\phi + V(\phi) = 0. \tag{32}$$

At this point, we define the dimensionless variables

$$\Omega_R = \frac{2\alpha e^{b-2\int Hdt}}{3H^2}, \qquad y^2 = \frac{V(\phi)}{6H^2\phi}, \qquad \Sigma^2 = \frac{\dot{b}^2}{4H^2}, \tag{33}$$

$$x = \frac{\dot{\phi}}{H\phi}, \qquad Z = \frac{H}{\dot{\Psi}}, \qquad w = -\frac{(2e^{3b}+\beta)\dot{\phi}e^{-2(b+\int Hdt)}}{6H^2\phi\dot{\Psi}}, \tag{34}$$

and the new independent variable $\tau = \ln a$, such that $x' = \frac{dx}{d\tau}$.

Using these variables, we can write the modified Friedmann's equation as

$$\Omega_R - \Sigma^2 + w + y^2 + 1 = 0, \tag{35}$$

and solve it for w to reduce the dimension of the dynamical system, that is

$$w = \Sigma^2 - y^2 - \Omega_R - 1. \tag{36}$$

The field equations are reduced to the following system of first order differential equations

$$\Omega_R' = 2\Omega_R(2\Sigma^2 + \Sigma + x + y^2), \tag{37}$$

$$y' = \frac{1}{2}y(4\Sigma^2 + \lambda x + x + 2y^2 + 2),$$
$$\Sigma' = \Sigma'^2(\Sigma+1) + \frac{3}{2}\Omega_R\left(2 - \frac{xZ}{\alpha}\right) + (\Sigma+2)y^2, \tag{38}$$

$$x' = x\left(\frac{x(2\alpha(\Sigma^2+\lambda y^2-1) + \Omega_R(2\alpha+3\Sigma Z))}{\alpha(-\Sigma^2+y^2+\Omega_R+1)} + 2(\Sigma-1)^2 + y^2\right), \tag{39}$$

$$Z' = -\frac{\Omega_R Z(2\alpha(\Sigma-1)^2 + x(2\alpha+3\Sigma Z))}{\alpha(-\Sigma^2+y^2+\Omega_R+1)} + \frac{Z(2\alpha(\Sigma-1)^3(\Sigma+1) - \alpha y^4)}{\alpha(-\Sigma^2+y^2+\ \Omega\mathrm{R}+1)} - \frac{y^2 Z((\Sigma-4)\Sigma + (\lambda+1)x + \Omega_R + 3)}{-\Sigma^2+y^2+\Omega_R+1}. \tag{40}$$

Each stationary point of the latter system describes an asymptotic solution with deceleration parameter

$$q = -1 - \frac{\dot{H}}{H^2} = 2\Sigma^2 + x + y^2. \tag{41}$$

The initial assumption that $k = 2\alpha$ means that the system, the critical points, and their stability depends only on two parameters α and λ. Since $k = \pm 1$, this means that $\alpha = \pm\frac{1}{2}$. The dynamical system has the following equilibrium points $(\Omega_R, y, \Sigma, x, Z)$:

(1) The family $L_1 = (\Omega_{Rc}, 0, -1, -1, -2\alpha)$, where $\Omega_{Rc} \in \mathbb{R}$ is a free parameter. Hence, the asymptotic solution is that of anisotropic or Kantowski-Sachs or Bianchi III universe. On the surface $\Omega_{Rc} = 0$, the Bianchi type I dynamics are recovered. The value of the deceleration parameter is $q(L_1) = 1$, this means that L_1 defines a decelerated solution. The eigenvalues are $\{0, -2, 2, 6, \frac{5-\lambda}{2}\}$. The family is normally hyperbolic the stability is given by the nonzero eigenvalues, L_1 is a saddle. From (41) we calculate $H(t) = \frac{1}{2t}$, that is, $a(t) = a_0\sqrt{t}$ and $\dot{b}^2 = \frac{2}{t^2}$.

(2) The family $L_2 = (\Omega_{Rc}, 0, 1, -3, -\frac{2\alpha}{3})$, where $\Omega_{Rc} \in \mathbb{R}$ is a free parameter. The value of the deceleration parameter is $q(L_2) = 1$, which means it has the same physical properties with point L_1. The eigenvalues are $\{0, -6, -6, 6, -\frac{3}{2}(\lambda-1)\}$, as L_1, L_2 is a saddle.

(3) $P_1 = (-\frac{2}{3}, 0, 0, 0, 0)$, with eigenvalues $\{-2, 2, -1 + i\sqrt{3}, -1 - i\sqrt{3}, 1\}$. P_1 is a saddle. The value of the deceleration parameter is $q(P_1) = 0$. The asymptotic solution is isotropic and with nonzero spatially curvature. That is the limit of Milne universe, since $\dot{b} = 0$, $a(t) = a_0 t$ and $\Omega_R \neq 0$. It is a solution which provides the limit of GR in the theory.

(4) $P_2 = (2(\sqrt{13}-4), 0, \frac{1}{2}(3-\sqrt{13}), \frac{1}{2}(7\sqrt{13}-25), 0)$. The eigenvalues are

(a) for $\alpha = \frac{1}{2}$, $\{3(3-\sqrt{13}), 3(3-\sqrt{13}), \frac{1}{2}(9-3\sqrt{13} \pm i\sqrt{14(37\sqrt{13}-133)}), \frac{1}{4}(7\sqrt{13}\lambda - 25\lambda - 5\sqrt{13} + 23)\}$,

(b) for $\alpha = -\frac{1}{2}$, $\{3(3-\sqrt{13}), 3(3-\sqrt{13}), \frac{1}{2}(9-3\sqrt{13} \mp i\sqrt{14(37\sqrt{13}-133)}), \frac{1}{4}(7\sqrt{13}\lambda - 25\lambda - 5\sqrt{13} + 23)\}$.

The point is an attractor for $\lambda < \frac{5\sqrt{13}-23}{7\sqrt{13}-25}$ and a saddle in any other case, the stability does not change if $\alpha = \frac{1}{2}$ or $-\frac{1}{2}$. The value of the deceleration parameter is $q(P_2) = \frac{1}{2}(\sqrt{13}-3) \approx 0.3 > 0$, this means that P_2 describes a decelerated solution. Those asymptotic solutions belong to an anisotropic Bianchi III universe because $\Omega_R(P_2) < 0$.

(5) $P_3 = (-2(4+\sqrt{13}), 0, \frac{1}{2}(3+\sqrt{13}), \frac{1}{2}(-25-7\sqrt{13}), 0)$, the eigenvalues are

(a) for $\alpha = \frac{1}{2}$, $\{3(3+\sqrt{13}), 3(3+\sqrt{13}), \frac{1}{2}(9+3\sqrt{13} \mp \sqrt{14(133+37\sqrt{13})}), \frac{1}{4}(-7\sqrt{13}\lambda - 25\lambda + 5\sqrt{13} + 23)\}$,

(b) for $\alpha = -\frac{1}{2}$, $\{3(3+\sqrt{13}), 3(3+\sqrt{13}), \frac{1}{2}(9+3\sqrt{13} \pm \sqrt{14(133+37\sqrt{13})}), \frac{1}{4}(-7\sqrt{13}\lambda - 25\lambda + 5\sqrt{13} + 23)\}$.

The point is a saddle for $\lambda < \frac{23+5\sqrt{13}}{25+7\sqrt{13}}$ or $\lambda > \frac{23+5\sqrt{13}}{25+7\sqrt{13}}$, the stability does not change if $\alpha = \frac{1}{2}$ or $-\frac{1}{2}$. The value of the deceleration parameter is $q(P_2) = \frac{1}{2}(-3-\sqrt{13}) \approx -3.3 < -1$, this means that P_3 describes an accelerated anisotropic universe with negative curvature, that is, a Bianchi III geometry.

(6) $P_4 = (\Omega_R, y, \Sigma, x, z) = (\frac{6\lambda+6}{5-7\lambda}, \frac{\sqrt{6}\sqrt{\lambda(\lambda+20)-17}}{7\lambda-5}, -\frac{2(\lambda-2)}{7\lambda-5}, \frac{18}{5-7\lambda}, 0)$. This point exist for $\lambda \neq 5/7$ and $\lambda \leq -10-3\sqrt{13}$ or $\lambda \geq 3\sqrt{13}-10$. The eigenvalues are $l_1 = -\frac{12(\lambda-2)}{7\lambda-5}$, $l_2 = f_1(\alpha,\lambda)$, $l_3 = f_2(\alpha,\lambda)$, $l_4 = f_3(\alpha,\lambda)$, $l_5 = f_4(\alpha,\lambda)$, where f_i are complicated expressions depending on the parameters. Setting $\alpha = \pm\frac{1}{2}$ slightly changes the eigenvalues but not the stability, see Fig. 1. The point is a saddle. The value of the deceleration parameter is $q(P_4) = \frac{2(\lambda-2)}{7\lambda-5}$, P_4 describes an accelerated solution for $\frac{5}{7} < \lambda < 2$ and a de Sitter solution for $\lambda = 1$. In Fig. 2 we present some numerical

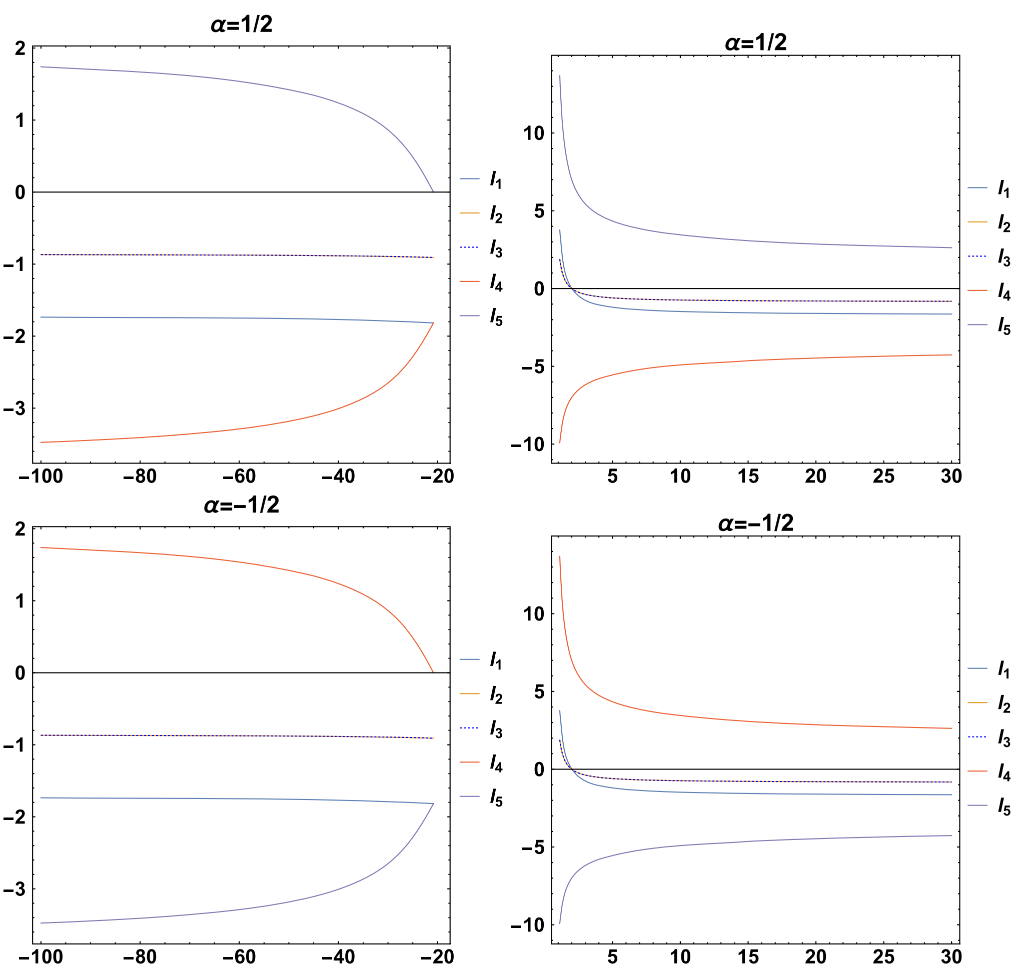

FIG. 1. Real part of the eigenvalues for point P_4 for $\alpha = \pm\frac{1}{2}$ and different ranges for λ where the point exists.

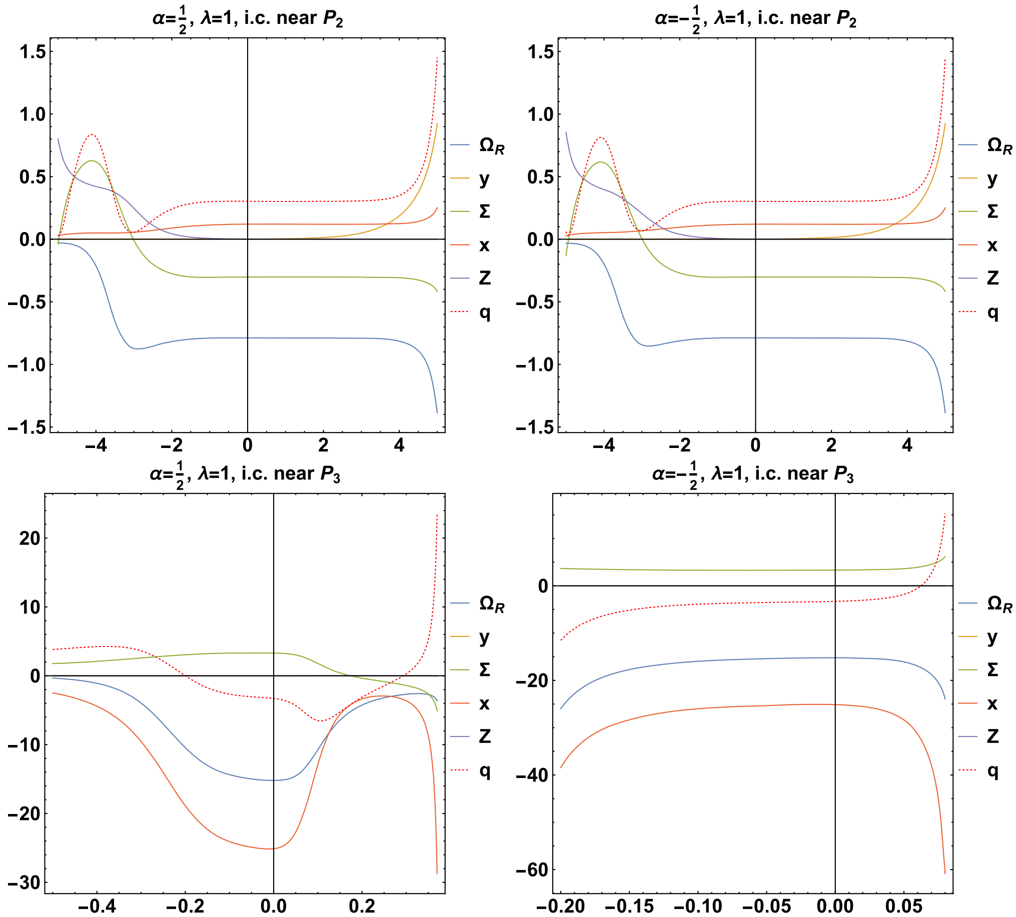

FIG. 2. Evolution of $\Omega_R, y, \Sigma, x, Z$ evaluated at a numerical solutions of system (37)–(40) for $\alpha = \pm\frac{1}{2}$ and $\lambda = 1$ for initial conditions (i.c.) near the points P_2 and P_3 with a displacement of $\epsilon = \frac{1}{1000}$. Also present as a red dotted line is the evolution of the deceleration parameter q evaluated at these points.

solutions of system (37)–(40) for $\alpha = \pm\frac{1}{2}$ and $\lambda = 1$ for initial conditions (i.c.) near the points P_2 and P_3 with a displacement of $\varepsilon = \frac{1}{1000}$. Additionally, depicted as a red dotted line is the evolution of the deceleration parameter evaluated at each solution.

(7) $P_5 = (\Omega_R, y, \Sigma, x, z) = (\frac{6\lambda+6}{5-7\lambda}, \frac{\sqrt{6}\sqrt{\lambda(\lambda+20)-17}}{5-7\lambda}, -\frac{2(\lambda-2)}{7\lambda-5}, \frac{18}{5-7\lambda}, 0)$. The difference with P_4 is a minus sign in the y coordinate, it exists for the same values of P_4 as well. The stability is also the same (saddle) since they share eigenvalues see Fig. 1 for reference. The value of the deceleration parameter is $q(P_5) = \frac{2(\lambda-2)}{7\lambda-5}$, P_5 describes an accelerated solution for $\frac{5}{7} < \lambda < 2$ and a de Sitter solution for $\lambda = 1$.

A. General case

For the general case with α arbitrary, we introduce the dimensionless variables

$$\Omega_R = \frac{k e^{b-2\int H dt}}{3H^2}, \qquad y^2 = \frac{V(\phi)}{6H^2\phi}, \qquad \Sigma^2 = \frac{\dot{b}^2}{4H^2}, \qquad x = \frac{\dot{\phi}}{H\phi}, \qquad z = \frac{\dot{\Psi}}{H}, \qquad w = \frac{\beta\dot{\phi}e^{-2b-2\int H dt}}{6H^2\phi\dot{\Psi}}. \tag{42}$$

In these variables, the Friedmann equation reads

$$\frac{(k-2\alpha)^2 xz}{24\alpha} - \frac{x\Omega_R}{kz} - \Sigma^2 - w + y^2 + \Omega_R + 1 = 0. \tag{43}$$

This means that we can solve (43) for w to obtain the dynamical system

$$\Omega_R' = \frac{x\Omega_R(12\alpha - (k-2\alpha)^2 z)}{6\alpha} + 2\Omega_R(2\Sigma^2 + \Sigma + y^2), \tag{44}$$

$$y' = \frac{xy(6\alpha(\lambda+1) - (k-2\alpha)^2 z)}{12\alpha} + y(2\Sigma^2 + y^2 + 1) \tag{45}$$

$$\Sigma' = -\frac{(k-2\alpha)^2(\Sigma-1)xz}{12\alpha} - \frac{3x\Omega_R}{kz} + 2(\Sigma-1)^2(\Sigma+1) + (\Sigma+2)y^2 + 3\Omega_R, \tag{46}$$

$$\begin{aligned}(-\Sigma^2 + y^2 + \Omega_R + 1)z' &= z\big(-2(\Sigma-1)^3(\Sigma+1) + 2\Omega_R\big((\Sigma-1)^2 + x\big)\big) + z\big(y^2\big((\Sigma-4)\Sigma + (\lambda+1)x \\ &\quad + \Omega_R + 3\big) + y^4\big) - \frac{(k-2\alpha)^2}{12\alpha}xz^2\big(-(\Sigma-1)^2 + y^2 + \Omega_R\big) + \frac{6}{k}\Sigma x\Omega_R,\end{aligned} \tag{47}$$

$$\begin{aligned}\frac{12\alpha z}{x}(-\Sigma^2 + y^2 + \Omega_R + 1)^2 x' &= -(k-2\alpha)^2 xz^2\big(x(\Sigma^2 + \lambda y^2 + \Omega_R - 1)\big) + (k-2\alpha)^2 xz^2\big((-(\Sigma-6)\Sigma + y^2 + \Omega_R - 5) \\ &\quad \times(-\Sigma^2 + y^2 + \Omega_R + 1)\big) - \frac{6z}{k}\big((k-2\alpha)^2\Sigma x^2\Omega_R\big) - 6z\big(4\alpha x(-\Sigma^2 + y^2 + \Omega_R + 1) \\ &\quad \times(\Sigma^2 + \lambda y^2 + \Omega_R - 1)\big) - \frac{6z}{k}\big(2\alpha k\big(2(\Sigma-1)^2 + y^2\big)(-\Sigma^2 + y^2 + \Omega_R + 1)^2\big) \\ &\quad + \frac{(k-2\alpha)^4}{6\alpha}(\Sigma-1)x^2z^3 - \frac{72\alpha\Sigma}{k}x\Omega_R(-\Sigma^2 + y^2 + \Omega_R + 1),\end{aligned} \tag{48}$$

where once more the prime means a total derivative with respect the independent variable $\tau = \ln(a)$.

The equilibrium points for the latter system are given by the family $(\Omega_R, y, \Sigma, x, z) = (0, 0, \Sigma_c, \frac{24\alpha(\Sigma_c^2-1)}{(k-2\alpha)^2 z_c}, z_c)$ where Σ_c and $z_c \in \mathbb{R}$ are the parameters that define the family. This family is a normally hyperbolic set of equilibrium points and therefore it has two zero eigenvalues. We observe that there are not any asymptotic solutions which can describe anisotropic solutions with nonzero spatial curvature, that is, the limit of GR is not recovered in this case.

We conclude that the general case is not of physical interest, thus we end the discussion here.

VI. CONCLUDING REMARKS

In this study we investigated the asymptotic dynamics for the field equations in symmetric teleparallel $f(Q)$-theory for Kantowski-Sachs and Bianchi type III background geometries. The field equations of $f(Q)$-theory are of second-order where the geometrodynamical degrees of freedom can be attributed to two scalar fields. By using the scalar field description we were able to write a minisuperspace Lagrangian. From the minisuperspace approach we observed that, for specific values of some of the free parameters of theory, some nonlinear terms in the field equations are eliminated.

To understand the overall evolution of physical parameters in the solution space, we determined the stationary points of the phase-space and investigated their stability properties. Employing the Hubble normalization approach, we transformed the field equations into a system of algebraic-differential equations. Each stationary point of this system corresponded to an asymptotic solution, whose stability properties and physical characteristics we thoroughly examined.

We found that for the general form of the symmetric and teleparallel connection provided by the theory, the field equations admit asymptotic solutions describing dynamics similar to that of the Bianchi type I geometry, without recovering the limit of general relativity (GR). However, for specific values of the free parameters, new stationary points emerged, describing the limit of GR and potentially representing anisotropic and accelerated solutions that could describe the pre-inflationary epoch of the universe.

From the results of this work it follows that symmetric teleparallel $f(Q)$-theory can describe anisotropic solutions with acceleration. However, we have considered a power-law function for the $f(Q)$-only and we have not found any future attractor which can describe an accelerating universe. However, for a more general $f(Q)$ function, new stationary points exist. Finally, we demonstrated how the phase-space analysis can be utilized to constrain the free parameters of the connection, in order to ensure the viability of the theory.

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APPENDIX: FIELD EQUATIONS FOR ARBITRARY γ_1

In this Appendix we present the general definition of the nonmetricity scalar Q and the gravitational field equations for arbitrary γ_1 with the lapse function N set to 1 for simplicity.

For the connection of our consideration the nonmetricity scalar reads

$$Q = \frac{e^{-2b}(c_1c_2(\gamma_2(H-2\dot{b})+\dot{\gamma}_2)+2e^{3b}(\gamma_2(\dot{b}+H)+k+\dot{\gamma}_2))}{a^2}+\frac{3}{2}\dot{b}^2-\frac{3H(c_2\gamma_1+2)(c_1\gamma_1+k)}{\gamma_2} - \frac{\dot{\gamma}_1(2c_1c_2\gamma_1+2c_1+c_2k)}{\gamma_2}+\frac{(c_2\gamma_1+2)\dot{\gamma}_2(c_1\gamma_1+k)}{\gamma_2^2}-6H^2. \tag{A1}$$

Furthermore, the gravitational field equations are

tt:

$$0 = \frac{\gamma_2 f'(Q)(-3a^2\dot{b}^2+12a^2H^2+4ke^b)+2a^2\gamma_2(f(Q)-Qf'(Q))}{4a^2\gamma_2} - \frac{\dot{Q}f''(Q)(a^2(c_2\gamma_1+2)(c_1\gamma_1+k)+e^{-2b}\gamma_2^2(2e^{3b}+c_1c_2))}{2a^2\gamma_2} \tag{A2}$$

tx:

$$0 = \dot{Q}f''(Q(2c_1c_2\gamma_1+2c_1+c_2k) \tag{A3}$$

xx:

$$0 = -2a^2e^{2b}(f(Q)-Qf'(Q))-e^{2b}f'(Q)(-12a^2H\dot{b}+a^2(3\dot{b}^2-4\ddot{b})+8a(a\dot{H}+aH^2)+4a^2H^2+4ke^b) + \frac{2\dot{Q}f''(Q)(-a^2e^{2b}(-2\gamma_2\dot{b}+c_1c_2\gamma_1^2+2c_1\gamma_1+c_2k\gamma_1+2k)-4a^2e^{2b}H\gamma_2+\gamma_2^2(2e^{3b}-c_1c_2))}{\gamma_2} \tag{A4}$$

yy, *zz*:

$$0 = -\frac{2\dot{Q}f''(Q)(a^2e^{2b}(\gamma_2\dot{b}+\gamma_1(2c_1+c_2k)+c_1c_2\gamma_1^2+2k)+4a^2e^{2b}H\gamma_2-c_1c_2\gamma_2^2)}{\gamma_2} - e^{2b}f'(Q)(6a^2H\dot{b}+a(a(2\ddot{b}+3\dot{b}^2)+8(a\dot{H}+aH^2))+4a^2H^2)+2a^2e^{2b}(f(Q)-Qf'(Q)) \tag{A5}$$

Finally, the equations of motion for the connection are

$$0 = \frac{e^{-2b}(\gamma_2(2\dot{b}\,\dot{Q}\,(e^{3b}-c_1c_2)f''(Q)+(2e^{3b}+c_1c_2)(f^{(3)}(Q)\dot{Q}^2+\ddot{Q}f''(Q))))}{a^2} + \frac{e^{-2b}(H\gamma_2\dot{Q}(2e^{3b}+c_1c_2)f''(Q)+2\dot{Q}\,\dot{\gamma}_2\,(2e^{3b}+c_1c_2)f''(Q))}{a^2} + \frac{(c_2\gamma_1+2)(c_1\gamma_1+k)(f^{(3)}(Q)\dot{Q}^2+3H\dot{Q}f''(Q)+\ddot{Q}f''(Q))}{\gamma_2} \tag{A6}$$

$$0 = f''(Q)(-\dot{Q}(3H(2c_1c_2\gamma_1+2c_1+c_2k)+2c_1c_2\dot{\gamma}_1)-\ddot{Q}(2c_1c_2\gamma_1+2c_1+c_2k)) - f^{(3)}(Q)\dot{Q}^2(2c_1c_2\gamma_1+2c_1+c_2k) \tag{A7}$$

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