Bañados-Silk-West phenomenon for near-fine-tuned particles with an external force: General classification of scenarios

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If two particles moving toward a black hole collide in the vicinity of the horizon, the energy $E_{\mathrm{c.m.}}$ in the center-of-mass frame can grow indefinitely if one of the particles is fine-tuned. This is the Bañados-Silk-West (BSW) effect. One of the objections against this effect is that, for some types of horizon, fine-tuned particles cannot reach the horizon. However, this difficulty can be overcome if, instead of exact fine-tuning, one particle is nearly fine-tuned, with the value of small detuning being adjusted to the distance to the horizon. Such particles are called near-fine-tuned. We give classification of such particles and describe possible high-energy scenarios of collision in which they participate. We analyze the ranges of possible motion for each type of particle and determine under which condition such particles can reach the horizon. We analyze collision energy $E_{\mathrm{c.m.}}$ and determine under which conditions it may grow indefinitely. We also take into consideration the forces acting on particles and find when the BSW effect with nearly fine-tuned particles is possible with finite forces. We demonstrate that the BSW effect with particles under discussion is consistent with the principle of kinematic censorship. According to this principle, $E_{\mathrm{c.m.}}$ cannot be literally infinite in any event of collision (if no singularity is present), although it can be made as large as one likes.

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I. INTRODUCTION

At present, high-energy particle collisions near black holes (and, more generally, collisions in a strong gravitational field) remain a hot topic. This is mainly due to findings of Bañados *et al.*, who noticed that if two particles move toward an extremal black hole and collide near its horizon, under certain conditions the energy in the center-of-mass frame $E_{\mathrm{c.m.}}$ becomes unbounded [1]. This happens if one of the colliding particles has to be critical (meaning that its radial velocity has to vanish on the horizon). This is what is called the Bañados-Silk-West (BSW) effect. It also revived interest in earlier works on this subject [2,3]. The aforementioned conditions imply that one of colliding particles has fine-tuned parameters (say, a special relation between the energy and angular momentum).

A number of objections were pushed forward against this effect [4–7]. Their meaning can be reduced to the statements that there are some factors that bound the energy of collision $E_{\mathrm{c.m.}}$. However, now it is clear that such a kind of objection does not abolish the BSW effect. Moreover, according to the principle of kinematic censorship, it is impossible to have literally infinite $E_{\mathrm{c.m.}}$ in each event of collision. Instead, this quantity remains finite, but can be made as large as one likes [8]. Therefore, the aforementioned objections simply put limits of validity of the BSW effect, but do not abolish it.

Moreover, it turned out that the requirement of having an extremal horizon is not necessary for the BSW effect. One of factors that prevents infinite $E_{\mathrm{c.m.}}$ is correlation between the type of a trajectory of a fine-tuned particle and a type of a horizon. For example, the critical particle cannot reach the horizon of a nonextremal black hole. However, if one somewhat relaxes the condition of criticality and replaces it with near criticality with a certain relationship between detuning and proximity to the horizon, the BSW effect becomes possible [9] (quite recently the BSW effect near nonextremal black holes was discussed in [10]).

Another objection against the BSW effect is related to backreaction of radiation on a particle. Meanwhile, it was shown that the BSW effect survives under the action of a force for extremal [11] and nonextremal horizons [12].

A large number of particular results for the BSW effect invokes the necessity of constructing the most general scheme that would encompass all possible cases. In our previous paper [13], such a scheme with full classification of scenarios leading to the BSW effect was developed for collisions in which fine-tuned particles participate. In the present paper, we developed a corresponding scheme for near-fine-tuned particles, thus essentially generalizing the observation made in [9]. In doing so, we also take into account a force acting on particles, so, in general, they are not free falling. An important reservation to be mentioned is that we consider motion within the equatorial plane of rotating axially symmetric black holes. Another reservation is that we work in the test particle approximation, neglecting backreaction of particles on the metric.

The paper is organized as follows. In Sec. II, we give a general setup for motion of particles in axially symmetric spacetimes. In Sec. III, we introduce different types of near-fine-tuned particles and analyze possible ranges of their motion. In Sec. IV, we focus on kinematical properties of near-fine-tuned particles for different ranges of their motion, which becomes important in Sec. V, where we investigate behavior of energy in the center-of-mass frame of two colliding particles. In Sec. VI, we give general expressions for an acceleration experienced by near-fine-tuned particles. This becomes useful in Secs. VII–IX, where we analyze near-horizon behavior of acceleration for different ranges of motion of near-fine-tuned particles. In Sec. X, we briefly formulate the results we obtained in previous sections. In Sec. XI, we check the validity of the aforementioned principle of kinematic censorship. Section XII is devoted to a possibility of varying of ranges of particle motion under the action of the external force. In Sec. XIII, we summarize corresponding results of our work.

II. GENERAL SETUP

In this work, we are going to analyze the properties of the BSW phenomenon for near-fine-tuned particles. At first, we need to define what we mean by this term. We are investigating the motion of particles in the background of a rotating black hole which is described in the generalized Boyer-Lindquist coordinates (t, r, θ, φ) by the metric

$$ds^2 = -N^2 dt^2 + g_{\varphi\varphi}(dt - \omega d\varphi)^2 + \frac{dr^2}{A} + g_{\theta\theta} d\theta^2, \tag{1}$$

where all metric coefficients do not depend on t and φ. The horizon is located at $r = r_h$ where $A(r_h) = N(r_h) = 0$. Near the horizon, we utilize a general expansion for the functions N^2, A, and ω,

$$N^2 = \kappa_p v^p + o(v^p), \qquad A = A_q v^q + o(v^q), \tag{2}$$

$$\omega = \omega_H + \omega_k v^k + o(v^k), \tag{3}$$

where q, p, and k are numbers that characterize the rate of a change of the metric functions near the horizon, and $v = r - r_h$.

Now, let us investigate the motion of a particle in such a space-time. If a particle is freely moving, the space-time symmetries with respect to ∂_t and ∂_φ impose conservation of the corresponding components of the four-momentum: $mu_t = -E$, $mu_\varphi = L$. We assume the symmetry with respect to the equatorial plane. In what follows, we restrict ourselves by equatorial motion. This allows us to write the four-velocity of a free-falling particle in the following form:

$$u^\mu = \left(\frac{\mathcal{X}}{N^2}, \sigma\frac{\sqrt{A}}{N}P, 0, \frac{\mathcal{L}}{g_{\varphi\varphi}} + \frac{\omega\mathcal{X}}{N^2}\right), \tag{4}$$

where $\sigma = \pm 1$, $\mathcal{X} = \epsilon - \omega\mathcal{L}$, $\epsilon = E/m$, $\mathcal{L} = L/m$, and P is given by

$$P = \sqrt{\mathcal{X}^2 - N^2\left(1 + \frac{\mathcal{L}^2}{g_{\varphi\varphi}}\right)}. \tag{5}$$

Now, let a particle move nonfreely. In the case of an external force acting on the particles, the quantities ε and $\mathcal{L}$ are obviously not conserved. However, despite this fact, we can still use the expression (4), but with general functions $\mathcal{X}(r)$ and $\mathcal{L}(r)$, provided forces do not depend on time and angle φ. If, additionally, the metric is spherically symmetric and acceleration is pure radial, it is convenient to redefine $\mathcal{X} = \mathcal{E} \pm \int^r dr' a(r')$, where $a(r')$ is the absolute value of acceleration and $\mathcal{E}$ is a constant [14]. However, in a general case, one is led to solving equations of motion without such a substitution.

Near the horizon, for a fine-tuned particle we can use the Taylor expansion,

$$\mathcal{X} = X_s v^s + o(v^s), \qquad \mathcal{L} = L_H + L_b v^b + o(v^b). \tag{6}$$

Here, the fact that a particle is fine-tuned manifests itself in that $\mathcal{X} = 0$ on the horizon where $v = 0$. [Counterparts of it for near-fine-tuned particles will be considered below—see (10)].

In further analysis, we will also require expressions for the tetrad components of the four-velocity. To this end, let us introduce the corresponding tetrad,

$$e^{(0)}_\mu = N(1, 0, 0, 0), \qquad e^{(1)}_\mu = \left(0, \frac{1}{\sqrt{A}}, 0, 0\right), \tag{7}$$

$$e^{(2)}_\mu = \sqrt{g_{\theta\theta}}(0, 0, 1, 0), \qquad e^{(3)}_\mu = \sqrt{g_{\varphi\varphi}}(-\omega, 0, 0, 1). \tag{8}$$

The tetrad components of four-velocity read

$$u^{(a)} = \left(\frac{\mathcal{X}}{N}, \frac{\sigma}{N}P, 0, \frac{\mathcal{L}}{\sqrt{g_{\varphi\varphi}}}\right). \tag{9}$$

III. CLASSIFICATION OF DIFFERENT NEAR-FINE-TUNED PARTICLES

In this section, we are going to define and classify different near-fine-tuned particles that will be useful for further analysis. The definition of near-fine-tuned particles generalizing that introduced in previous works (see [9]) is as follows: a particle is called near-fine-tuned if $\mathcal{X}$ near the horizon has the Taylor expansion

$$\mathcal{X} = \delta + X_s v^s + o(v^s), \quad s > 0, \tag{10}$$

where $\delta \ll 1$ is a dimensionless parameter. Account for small but nonzero δ enabled to find a version of the BSW effect even for nonextremal black holes, contrary to some misconceptions (see [9] for details). On the other hand, the concept of near-fine-tuned particles plays an important role in the analysis of full scenarios of collision including behavior of debris. In particular, it was shown for charged particles in the extremal Reissner-Nordström background that a particle that can escape to infinity must be near critical [15]. However, in what follows we restrict ourselves to consideration of neutral particles. A combined account for rotation and electric charge is a separate complicated problem that can give rise to qualitatively new possibilities and deserves separate attention (for the example of the Kerr-Newman black hole, see [16]).

It is important to note that a particle for which the expansion for $\mathcal{X}$ starts with a constant is generally called usual (and we will see several analogies with usual particles in our further analysis). However, we are going to show that the case $\delta \ll 1$ requires a distinct analysis.

Before we proceed further, we have to note that in our analysis we will also require that the time coordinate during the motion of the particle has to increase (this is the so-called forward-in-time condition). To this end, the time component of the four-velocity has to be positive: $u^t > 0$. From the expression (4), one sees that this requires $\mathcal{X} \geq 0$. In further analysis, we will require this condition to hold for all particles under consideration.

To analyze the behavior of the four-velocity, let us introduce a classification of different types of particles based on different values of s (see Table I).

As we will show, these particles will correspond to generalization, for nonzero δ, of subcritical, critical, and ultracritical particles introduced in Ref. [13]. The only new type of particle is the near-overcritical one that does not have any analog for $\delta = 0$. But before doing this, let us consider an expression for the radial component of the four-velocity. As we will show, it depends strongly on a type of particle that will justify the necessity of introduced classification. For this purpose and for further investigation, let us consider the quantity P near the horizon. First of all, we note that exactly on the horizon $P = |\delta|$ [this becomes obvious if one substitutes Eqs. (10) and (2) into (5) and takes the limit $v \to 0$]. However, as one moves away from the horizon, the quantity P starts to differ from the value $P = |\delta|$. Depending on the parameters of a particle, P may either decrease or increase. In the first case, at some radial distance v_t, where the index "t" stands for "turning point," P becomes zero: $P(v_t) = 0$ (hereafter, "distance" means "coordinate distance"). In the second case, there are no turning points. However, we can still define effective coordinate distances v_e, where the index "e" stands for "effective," at which P changes by values comparable to the value of P on the horizon. Formally, we can define effective distances to be such that $P(v_e) - P(0) \sim \delta$. As we will show in further analysis, the physical properties of the collisional process depend strongly on the point at which this process takes place and the relationship between it and v_t (or v_e).

TABLE I. Table showing classification of different near-fine-tuned particles.

Condition	Type of particle with nonzero $\delta \ll 1$	Abbreviation
$s < p/2$	Near subcritical	NSC
$s = p/2$	Near critical	NC
$s = p/2$ and (24)	Near ultracritical	NUC
$s > p/2$	Near overcritical	NOC

Now, let us find under which conditions P imposes the existence of roots and in what regions a particle may move. Using the expression for P [see (5)], we have

$$P = \sqrt{\mathcal{X}^2 - N^2\left(1 + \frac{\mathcal{L}^2}{g_{\varphi\varphi}}\right)}. \tag{11}$$

The reality of this expression is defined by the condition

$$\mathcal{X}^2 \geq N^2\left(1 + \frac{\mathcal{L}^2}{g_{\varphi\varphi}}\right). \tag{12}$$

As we impose the forward-in-time condition, $\mathcal{X} > 0$. Also, by definition N^2 is non-negative. Thus, we can formally take the "square root" of (12) and get

$$\mathcal{X} \geq N\sqrt{1 + \frac{\mathcal{L}^2}{g_{\varphi\varphi}}}. \tag{13}$$

Our task is to find such ranges of radial coordinate in which this condition holds. To do this, let us at first solve equation $\mathcal{X} = N\sqrt{1 + \frac{\mathcal{L}^2}{g_{\varphi\varphi}}}$. Substituting (10) and (2) one gets

$$\delta + X_s v_t^s + o(v_t^s) = \sqrt{\kappa_p\left(1 + \frac{\mathcal{L}^2}{g_{\varphi\varphi}}\right)} v_t^{p/2} + o(v_t^{p/2}). \tag{14}$$

We cannot find general expressions for all roots, but we can determine if there are any new roots that were absent in the case of zero δ. To analyze the corresponding solutions, we consider different types of particles separately.

A. Near-subcritical particles ($s < p/2$)

In this case, it is obvious that the term v^s is dominant over the term $v^{p/2}$ that gives us $P \approx |\mathcal{X}|$. Therefore, if the forward-in-time condition is satisfied, the quantity P is real. Thus, the entire analysis of regions of motion for near-subcritical particles is restricted to an analysis of the positivity of $\mathcal{X}$. To analyze regions where $\mathcal{X} \geq 0$, we first need to find where $\mathcal{X} = 0$. By substituting (10), we have

$$\delta + X_s v_t^s = o(v_t^s), \tag{15}$$

that gives

$$v_t = \left(-\frac{\delta}{X_s}\right)^{1/s} + o(\delta^{1/s}). \tag{16}$$

Note that this solution is only possible if $\delta > 0$ and $X_s < 0$ or if $\delta < 0$ and $X_s > 0$. In the first case, it can be easily seen that $\mathcal{X}$ is non-negative only in the range $[0, v_t]$, thus the particle can only move from the horizon to v_t, and this region is not connected to infinity. On the other hand, in the second case, $\mathcal{X}$ is non-negative only in the range $[v_t, \infty]$. In this section, when we write ∞, it means that we cannot find other roots that limit the motion of the particle. This implies that there are no roots generated by a nonzero δ that restrict the motion of a particle (although some of them may exist at distances greater than v_t, their existence is not defined by δ).

However, if the conditions $\delta > 0$ and $X_s < 0$, or $\delta < 0$ and $X_s > 0$, do not hold, then there is no δ-related root and a particle may move in the range $[0, \infty]$. The forward-in-time condition for the absence of roots holds only in the case when $\delta > 0$ and $X_s > 0$ (while for the case $\delta < 0$ and $X_s < 0$, it does not hold). As mentioned above, in the case of the absence of roots, we can only define effective distances at which P changes on the order of δ. For near-subcritical particles, where $s < p/2$, we have $P \approx \mathcal{X} \approx \delta + X_s v^s$. By comparing the two terms in the expansion of P, we get

$$v_e \sim \delta^{1/s}. \tag{17}$$

To summarize, we have (hereafter, subscript "c" denotes the collision point)

$$\text{if particle is NSC } (s < p/2),\ v_c \in \begin{cases} [0, v_t] \text{ where } v_t = (-\frac{\delta}{X_s})^{1/s} \text{ if } \delta > 0 \text{ and } X_s < 0, \\ [v_t, \infty] \text{ where } v_t = (-\frac{\delta}{X_s})^{1/s} \text{ if } \delta < 0 \text{ and } X_s > 0, \\ [0, \infty] \text{ if } \delta > 0 \text{ and } X_s > 0, \text{ in this case } v_e \sim \delta^{1/s}. \end{cases} \tag{18}$$

In this case, the ranges of motion were limited only by the forward-in-time condition. Therefore, (18) describes all cases when the forward-in-time condition holds, for general expressions for P. This fact will be used in further analysis.

We also have to make some reservations about ranges of motion of particles. Up to now, we have not considered directions of their motion: ranges where $\mathcal{X} > 0$ and P is real do not depend on whether the particle is ingoing or outgoing. However, it is obvious that in application to the real problem this becomes important. If the particle has the finite proper time to achieve the horizon from some finite distance, then its motion cannot be reversed, and, in application to our problem, this describes only the ingoing particle. (It can be outgoing if one considers a white hole, but we will not focus on this possibility.) If the proper time diverges near the horizon, a particle may move not only toward a horizon but also from its small vicinity in the outward direction, so our analysis is applicable both to ingoing and outgoing particles. Analysis of the proper time is done in Appendix B, where we analyze behavior of proper time for different types of particles. One can see that for different types of particles the behavior of the proper time is different. Meanwhile, we still can conclude that, if $q < 2$, any particle has finite proper time, and in this case our current analysis is applicable only for ingoing particles.

B. Near-critical particles ($s = p/2$)

In this case v^s and $v^{p/2}$ terms are of the same order and Eq. (14) simplifies [we denote $A_{p/2} = \sqrt{\kappa_p(1 + \frac{\mathcal{L}^2}{g_{\varphi\varphi}})}$],

$$\delta \approx (A_{p/2} - X_{p/2}) v_t^{p/2}. \tag{19}$$

Solving this, we have

$$v_t \approx \left(\frac{\delta}{A_{p/2} - X_{p/2}}\right)^{2/p}. \tag{20}$$

This solution is possible only in two cases: if $\delta > 0$ and $A_{p/2} > X_{p/2}$, or if $\delta < 0$ and $A_{p/2} < X_{p/2}$. In the first case, P is positive in the range $[0, v_t]$, while in the second case it is positive in the range $[v_t, \infty]$. In other cases, there is no solution and the particle may move in the range $[0, \infty]$. In this case, we can only define effective distances at which P changes on the values of the order of δ. To find these distances, we write the expression for P in the case of NC particles,

$$P \approx \sqrt{\delta^2 + 2\delta X_{p/2} v^{p/2} + (X_{p/2}^2 - A_{p/2}^2) v^p}. \tag{21}$$

Generally, we need to determine at which distances each term in the expression for P is compatible with δ^2 and choose the dominant solution in δ. We will not provide a general analysis and will just present a result: the effective distance v_e

is such that all terms are of the same order. In this case,

$$v_e \sim \delta^{2/p}. \tag{22}$$

We need to determine whether the forward-in-time condition holds in regions of motion for the NC particle. First, let us consider the case where $\delta > 0$ and $A_{p/2} > X_{p/2}$. Then, the root (20) is closer to the horizon than (16) (this is true because $A_{p/2} > 0$). Therefore, in the entire range $[0, v_t]$, the forward-in-time condition holds.

Next, let us consider the case where $\delta < 0$ and $A_{p/2} < X_{p/2}$. In this case, the root (20) is further from the horizon than (16). Therefore, in the entire range $[v_t, \infty]$, the forward-in-time condition holds.

The remaining cases are (i) $\delta > 0$ and $A_{p/2} < X_{p/2}$ or (ii) $\delta < 0$ and $A_{p/2} > X_{p/2}$. In case (i), the forward-in-time condition holds according to the third case in (18). In case (ii), it only holds for $X_{p/2} > 0$ and for $v \in [(-\frac{\delta}{X_{p/2}})^{2/p}, \infty]$. In all other cases, motion is forbidden.

In summary, generalizing all the above facts, we have

$$\text{if particle is NC}\,(s = p/2),\ v_c \in \begin{cases} [0, v_t] \text{ where } v_t \text{ is in (20) and } \delta > 0 \text{ and } A_{p/2} > X_{p/2}, \\ [v_t, \infty] \text{ where } v_t \text{ is in (20) and } \delta < 0 \text{ and } A_{p/2} < X_{p/2}, \\ [0, \infty] \text{ if } \delta > 0 \text{ and } A_{p/2} < X_{p/2}, \text{ in this case } v_e \sim \delta^{2/p}, \\ \left[(-\frac{\delta}{X_{p/2}})^{2/p}, 0\right] \text{ if } \delta < 0 \text{ and } A_{p/2} > X_{p/2} > 0, v_e \sim \delta^{2/p}. \end{cases} \tag{23}$$

C. Near-ultracritical particles ($s = p/2$ and special condition)

It may appear that coefficients in expansions for P may be such that several first terms in it cancel. This happens in a case when

$$(\mathcal{X} - \delta)^2 - N^2\left(1 + \frac{\mathcal{L}^2}{g_{\varphi\varphi}}\right) = (X_s v^s + o(v^s))^2 - \left(\kappa_p\left(1 + \frac{L_H^2}{g_{\varphi H}}\right)v^p + o(v^c)\right) \tag{24}$$

$$= \frac{\kappa_p}{A_p}(u^r)_c^2 v^{2c+p-q} + o(v^{2c+p-q}), \tag{25}$$

where $(u^r)_c$ and $c > q/2$ are some constants. It is important to note that this cancellation can only occur if $s = p/2$. Additionally, this condition does not involve δ and is the same as Eq. (28) from [13], which is a defining property for ultracritical particles when $\delta = 0$. The unusual notation and choice of parameters $(u^r)_c$ and c were made to simplify the expression for the four-velocity in the case of $\delta = 0$. In this case,

$$u^r = \frac{\sqrt{A}}{N} P \approx (u^r)_c v^c. \tag{26}$$

However, if $\delta \neq 0$, we get from an equation for P,

$$P(v_e) \approx \sqrt{(\delta + X_{p/2} v_e^{p/2})^2 - \left(\kappa_p\left(1 + \frac{L_H^2}{g_{\varphi H}}\right)v_e^p\right)}$$

$$= \sqrt{\delta^2 + 2\delta(X_{p/2} v_e^{p/2}) + (X_{p/2} v_e^{p/2})^2 - \left(\kappa_p\left(1 + \frac{L_H^2}{g_{\varphi H}}\right)v_e^p\right)}. \tag{27}$$

Using (24) we can write

$$P(v_e) \approx \sqrt{\delta^2 + 2\delta X_{p/2} v^{p/2} + \frac{\kappa_p}{A_p}(u^r)_c^2 v^{2c+p-q}} = 0. \tag{28}$$

Generally, there may be three roots: the first is obtained by comparison of the first and second terms in (28) and is given by

$$v_t \approx \left(-\frac{\delta}{2X_{p/2}}\right)^{2/p}. \tag{29}$$

One can easily check that this is the only possible root in (28). This root exists in the same cases as the root for a near-subcritical particle: it exists if $\delta > 0$ and $X_s < 0$, or if $\delta < 0$ and $X_s > 0$. From (28) we can see that in both these cases a particle can only move in the range $[0, v_t]$. As for all parameters, (29) is closer to the horizon than (16), and we see that the forward-in-time condition holds only if $\delta > 0$ and $X_s < 0$. (In the case where $\delta < 0$ and $X_s > 0$, the forward-in-time condition holds for $v \in [(-\frac{\delta}{X_s})^{1/s}, \infty]$, while P is real only for $v \in [0, (-\frac{\delta}{2X_s})^{1/s}]$. These regions obviously do not intersect.)

The only remaining cases are $\delta > 0$ and $X_s > 0$, or $\delta < 0$ and $X_s < 0$. In the first case, both P and $\mathcal{X}$ are positive for all positions of the particle, thus there are no turning points. In the second case, $\mathcal{X}$ is negative, which makes this case impossible. Therefore, we are left with only the case $\delta > 0$ and $X_s > 0$, for which we only need to find the effective distances at which P changes on the values of the order δ. Analyzing (28), we see that the dominant behavior can be obtained by comparing the first and second terms, which give us

$$v_e \sim \delta^{2/p}. \tag{30}$$

Summarizing, we have

$$\text{if particle is NUC}\,(s < p/2),\, v_c \in \begin{cases} [0, v_t] \text{ where } v_t = (-\frac{\delta}{2X_s})^{1/s} \text{ if } \delta > 0 \text{ and } X_s < 0, \\ [0, \infty] \text{ if } \delta > 0 \text{ and } X_s > 0, \text{ in this case } v_e \sim \delta^{2/p}. \end{cases} \tag{31}$$

D. Near-overcritical particles ($s > p/2$)

For such particles the $v^{p/2}$ term is dominant over v^s and (14) becomes

$$\delta = A_{p/2} v_t^{p/2} + o(v_t^{p/2}), \tag{32}$$

where $A_{p/2} = \sqrt{\kappa_p (1 + \frac{\mathcal{L}^2}{g_{\varphi\varphi}})}$. Solving this equation, we have

$$v_t \approx \left(\frac{\delta^2}{A_{p/2}^2}\right)^{1/p} + o(\delta^{2/p}). \tag{33}$$

Note that this root of equation $P = 0$ exists for all δ independent of the forward-in-time condition. In this case, P is given by $\sqrt{\delta^2 - A_{p/2}^2 v^p}$ in dominant orders, and solving the equation $P = 0$ yields (33). Additionally, one may observe that P^2 is non-negative for $[0, v_t]$ regardless of the sign of δ. Now, let us determine the conditions under which the forward-in-time condition holds in this range. At first, we note that (33) is $\sim\delta^{2/p}$, while the root of the equation $\mathcal{X} = 0$, similar to the case of near-subcritical particles (16), is $\sim\delta^{1/s}$. Since for near-overcritical particles $s > p/2$, (33) is closer to the horizon than (16) [it is of lower order in δ than (16)]. Thus, in the case of $\delta > 0$ and $\mathcal{X}_s < 0$ [see the first case in (18)], the forward-in-time condition holds throughout the range $[0, (\frac{\delta^2}{A_{p/2}^2})^{1/p}]$. If $\delta < 0$ and $\mathcal{X}_s > 0$ [see the second case in (18)], the regions of reality of P and positivity of $\mathcal{X}$ do not intersect, so motion in this case is forbidden. The last case is $\delta > 0$ and $X_s > 0$ [see the third case in (18)]. In this case, the forward-in-time condition holds for all points, and thus the positivity of $\mathcal{X}$ does not bound regions of particle motion.

In summary, we have

$$\text{if particle is NOC}\,(s = p/2), \qquad v_c \in [0, v_t]$$
$$\text{where } v_t \approx \left(\frac{\delta^2}{A_{p/2}^2}\right)^{1/p} \quad \text{if } \delta > 0. \tag{34}$$

This concludes the analysis of different regions of motion for different types of particles. It is easy to see that the exact expressions for turning points v_t or effective distances v_e are different for each type of particle. However, for near-critical, near-ultracritical, and near-overcritical particles, they are of the same order in δ [see (22), (30), and (34)]. Therefore, in this sense, they may appear to be indistinguishable. To understand the reason why this classification is still necessary, let us consider the radial component of the four-velocity in the limit $\delta \to 0$. In this case, $P \approx \sqrt{(X_s v^s)^2 - \left(\kappa_p \left(1 + \frac{L_H^2}{g_{\varphi H}}\right) v^p\right)}$. By using the fact that, for near-subcritical particles $s < p/2$ (and thus the v^{2s} term is dominant), for near-critical particles $s = p/2$ (and thus the v^{2s} term and the v^p term are of the same order), and for near-ultracritical particles $s = p/2$ and condition (24) holds, we obtain for $|u^r| = \frac{\sqrt{A}}{N} P$,

$$\text{near-subcritical particle: } |u^r| = \sqrt{\frac{A_q}{\kappa_p}} X_s v^{\frac{q-p}{2}+s}, \tag{35}$$

near-critical particle:

$$|u^r| = \sqrt{\frac{A_q}{\kappa_p}} \sqrt{X_{p/2}^2 - \kappa_p \left(1 + \frac{L_H^2}{g_{\varphi H}}\right)} v^{\frac{q}{2}}, \tag{36}$$

$$\text{near-ultracritical particle: } |u^r| = (u^r)_c v^c. \tag{37}$$

In all three cases, in the limit $\delta \to 0$ we obtain correspondingly subcritical, critical, and ultracritical particles, as introduced in [13]. As one can see, the behavior of the four-velocity in these cases is different that justifies the necessity of distinguishing between near-critical, near-ultracritical, and near-overcritical particles. In our further analysis, we will also observe that other physical quantities are different for these particle types. The only exceptional case is near-overcritical particles that were not considered in [13]. This is because when we take the limit $\delta \to 0$ in this case, P becomes complex near the horizon. Because of this fact, overcritical particles with $\delta = 0$ cannot reach the horizon and cannot participate in the "pure" BSW phenomenon that was the focus of our investigation in [13]. However, nonzero δ allows such particles to reach the horizon and thus they are considered in our work.

IV. DIFFERENT SCALES OF PARAMETERS

Now, we are going to make a next step and discuss the interplay of different parameters in our problem. In the pure BSW phenomenon, there is only one small parameter: the point of collision $v_c \ll r_h$. However, in the case of near-fine-tuned particles, a new parameter δ appears, where $\delta \ll 1$. To analyze the different properties of near-fine-tuned particles, we have to specify the scales of two parameters: v_c and v_e (or v_t if it exists). We have four different cases (see below) which can be described by different relations between v_c and v_t or v_e (in each of these cases, the conditions $\frac{v_c}{r_h} \ll 1$ and $\delta \ll 1$ hold).

Before proceeding further, we must comment on the situation when $v_c < v_t$. This means that a particle cannot arrive at the point of collision from infinity since it would bounce back in the turning point. We assume that it appears between the horizon and a turning point due to a special initial condition and do not specify their nature (say, a particle can appear there due to quantum creation, etc.). In doing so, the interval in which the scenario develops is very tiny [for nonextremal black holes explicit expressions for it can be found in Eq. (18) of [9] for the Kerr metric and in Eq. (18) of [17] for a more general case]. Nonetheless, the BSW effect can indeed exist.

A. First case: $v_c \gg v_e$ (or v_t)

In this case, the point of collision is much further from the horizon than $v_{e,t}$ (hereafter, the notation $v_{e,t}$ means v_e or v_t if it exists). This effectively means that we can take $\delta = 0$ while keeping $\frac{v_c}{r_h}$ terms in all the quantities in which we are interested. To see why this is so, let us consider the expression for P,

$$P \approx \sqrt{(\delta + X_s v_c^s)^2 - \left(\kappa_p \left(1 + \frac{L_H^2}{g_{\varphi H}}\right) v_c^p\right)}. \quad (38)$$

Using that $v_{e,t} \sim \delta^{\max(1/s, 2/p)}$ [see (17), (22), (30), and (34)], or, inversely, $\delta \sim v_{e,t}^{\min(s,p/2)}$, we can substitute this to the expression for (38) and get

$$P \sim \sqrt{(v_{e,t}^{\min(s,p/2)} + X_s v_c^s)^2 - \left(\kappa_p \left(1 + \frac{L_H^2}{g_{\varphi H}}\right) v_c^p\right)}. \quad (39)$$

We see that the ratio $v_e^{\min(s,p/2)}/X_s v_c^s$ tends to zero because of the condition $v_c \gg v_e$. This means that the $v_{e,t}^{\min(s,p/2)}$ term is negligible and dominant terms in (38) contain v_c without δ. Thus, we have

$$P \approx \sqrt{(X_s v_c^s)^2 - \left(\kappa_p \left(1 + \frac{L_H^2}{g_{\varphi H}}\right) v_c^p\right)} + O(\delta) = \quad (40)$$

$$\approx \sqrt{(\mathcal{X} - \delta)^2 - N^2 \left(1 + \frac{\mathcal{L}^2}{g_{\varphi\varphi}}\right)} + O(\delta). \quad (41)$$

One may easily see that the dominant term does not involve δ. Thus, in this case, the dominant behavior can be obtained by taking the $\delta \to 0$ limit. This will correspond to a pure BSW phenomenon, which was completely analyzed in [13]. In this case, the properties of near-fine-tuned particles are similar to corresponding fine-tuned particles (for example, near-subcritical particles correspond to subcritical particles and so on).

B. Second case: $v_c \sim v_e$ (or v_t)

In this case, the point of collision is in the same scale of distances as $v_{e,t}$. Let us obtain expressions for $P(v_c)$ and $\mathcal{X}(v_c)$ in the leading order for different types of particles.

(i) Near-subcritical particle:

For such particles, the terms δ^2, v^{2s}, and δv^s are dominant in the expansion for P^2 [see (14)]. This indicates that $\mathcal{X}$ terms prevail over N^2 in the expansion of P [see (5)]. Therefore, in this case, we have

$$P = \sqrt{\mathcal{X}^2 - N^2\left(1 + \frac{\mathcal{L}^2}{g_{\varphi\varphi}}\right)} \approx \mathcal{X} - \frac{N^2}{2\mathcal{X}}\left(1 + \frac{\mathcal{L}^2}{g_{\varphi\varphi}}\right). \quad (42)$$

In this expression, we have also included higher-order terms because, as we will demonstrate below in the analysis of the energy of the collision, the dominant terms will cancel each other out, and the energy will be determined by higher-order corrections.

If there is no turning point [that occurs only when $\delta > 0$ and $X_s > 0$, see the third condition in (18)], the dominant term in P (specifically $\mathcal{X}$) at $v_c \sim v_e$ takes the following form:

$$P \approx \mathcal{X} = \delta + X_s v_c^s + o(\delta). \quad (43)$$

Using the approximation $v_c \sim v_e \sim \delta^{1/s}$, we observe that $P \sim \mathcal{X} \sim \delta$. However, if a turning point exists, the situation becomes more complex. In this case, we can invert (16) and obtain $\delta \approx -X_s v_t^s$. By substituting this into the expression for $\mathcal{X}$, we have

$$\mathcal{X} \approx \delta + X_s v_c^s \approx X_s(v_c^s - v_t^s). \tag{44}$$

One can easily see that, since $v_c \sim v_e$, P is of the order δ. However, if v_c approaches v_t (this means that the collision point reaches the turning point), the quantity $\mathcal{X}$ tends to zero. To describe the small difference that arises in this case, we make the assumption

$$\delta = -X_s v_c^s + B_r v_c^r \tag{45}$$

[in fact, we could assume that difference $v_c - v_t$ is some small parameter, but further analysis will be simpler if we consider a more concrete example according to (45)]. In this case,

$$\mathcal{X} \approx B_r v_c^r, \tag{46}$$

$$P \approx B_r v_c^r - \frac{A_{p/2}^2}{2B_r} v_c^{p-r}. \tag{47}$$

From the expression for $\mathcal{X}$, we can observe that the forward-in-time condition holds only if $B_r > 0$. On the other hand, from the expression for P, we can see that the expansion (42) holds only if $s < r < p/2$. It may seem strange that this special case needs to be considered. However, as we will demonstrate in the analysis of the energy of the collision, the behavior of energy becomes quite special when v_c approaches v_t.

In further analysis, we will need to know the behavior of the quantity $\sqrt{\mathcal{X}^2 - N^2}$. Considering that in this case $s < p/2$ and thus the $\mathcal{X}$ term is dominant, we have

$$\sqrt{\mathcal{X}^2 - N^2} \approx \mathcal{X} - \frac{N^2}{2\mathcal{X}}. \tag{48}$$

As one can see, the behavior of this quantity is similar to the behavior of P.

(ii) Near-critical particle:

For these particles, the situation is more complicated because all terms are comparable, and we have (in the main order of δ)

$$P \approx \sqrt{(\delta + X_{p/2} v_c^{p/2})^2 - A_{p/2}^2 v_c^p}. \tag{49}$$

If there is no turning point, we take into account that $v_c \sim v_e \sim \delta^{2/p}$ [see (22)] and we see that $P \sim \delta$. In this case, we also have

$$\mathcal{X} \approx \delta + X_{p/2} v_c^{p/2} \sim \delta. \tag{50}$$

For $\sqrt{\mathcal{X}^2 - N^2}$, we can observe that both $\mathcal{X}^2$ and N^2 are of the same order, and we have $\sqrt{\mathcal{X}^2 - N^2} \sim \delta$.

If there are roots of equation $P = 0$ [that occurs if $\delta > 0$ and $A_{p/2} > X_{p/2}$ or $\delta < 0$ and $A_{p/2} < X_{p/2}$, see (23)], the expression for P becomes [by inverting (20) that gives us $\delta = (A_{p/2} - X_{p/2}) v_t^{p/2}$]

$$P \approx \sqrt{\delta^2 + 2\delta X_{p/2} v_c^{p/2} + (X_{p/2}^2 - A_{p/2}^2) v_c^p} \approx \tag{51}$$

$$\sqrt{(A_{p/2} - X_{p/2})^2 v_t^p + 2(A_{p/2} - X_{p/2}) X_{p/2} v_t^{p/2} v_c^{p/2} + (X_{p/2}^2 - A_{p/2}^2) v_c^p}. \tag{52}$$

Note that if $v_c \sim v_t \sim \delta^{2/p}$, we have $P \sim \delta$. However, if $v_c \to v_t$ one can check that $P \to 0$. In this case, we assume

$$\delta = (A_{p/2} - X_{p/2}) v_c^{p/2} + B_r v_c^r, \tag{53}$$

where $r > p/2$.

Substituting this to P we have

$$P \approx \sqrt{(\delta + X_{p/2} v_c^{p/2})^2 - A_{p/2}^2 v_c^p} = \sqrt{(A_{p/2} v_c^{p/2} + B_r v_c^r)^2 - A_{p/2}^2 v_c^p} \approx \tag{54}$$

$$\approx \sqrt{2A_{p/2} B_r v_c^{p/2+r} + B_r^2 v_c^{2r}} \approx \sqrt{2A_{p/2} B_r} v_c^{p/4+r/2}. \tag{55}$$

This will not have any special consequences in the behavior of the energy of collision, but will influence the behavior of acceleration.

(iii) Near-ultracritical particle:

In this case, we can use (24) and (30) that gives us

$$P \approx \sqrt{\delta^2 + 2\delta X_{p/2} v_c^{p/2}}. \tag{56}$$

If there is no turning point, using (30) we have $v_c \sim v_e \sim \delta^{2/p}$. Substituting this to (56) we see that $P \sim \delta$. In this case, we also have

$$\mathcal{X} \approx \delta + X_{p/2} v_c^{p/2} \sim \delta. \tag{57}$$

The same holds for $\sqrt{\mathcal{X}^2 - N^2}$,

$$\sqrt{\mathcal{X}^2 - N^2} \approx \sqrt{(\delta + X_{p/2} v_c^{p/2})^2 - (\kappa_p v_c^p + o(v_c^p))} \sim \delta. \tag{58}$$

If a turning point exists, inverting (31) we have $\delta = -2X_s v_t^s$. Substituting this in the expression for P we have

$$P \approx 2X_{p/2} \sqrt{v_t^{p/2}(v_t^{p/2} - v_c^{p/2})}. \tag{59}$$

If $v_c \sim v_t$ we see that $P \sim \delta$. While if $v_c \to v_t$, $P \to 0$. In this case, let us write

$$\delta = -2X_{p/2} v_c^{p/2} + B_r v_c^r, \tag{60}$$

where $r > p/2$.

Substituting this in the expression for P we have

$$P \approx \sqrt{(-2X_{p/2} v_c^{p/2} + B_r v_c^r)^2 + 2(-2X_{p/2} v_c^{p/2} + B_r v_c^r) X_{p/2} v_c^{p/2}} \approx \tag{61}$$

$$\approx \sqrt{2X_{p/2} B_r v_c^{p/2+r} + B_r^2 v_c^{2r}} \approx \sqrt{2X_{p/2} B_r} v_c^{p/4+r/2}. \tag{62}$$

However, we will see that in this case the energy of the collision does not change drastically depending on v_c.

(iv) Near-overcritical particle:

In this case, only δ^2 and v_c^p terms are dominant [see (34)] and we get

$$P \approx \sqrt{\delta^2 - A_{p/2}^2 v_c^p}. \tag{63}$$

Now, the turning point always exists. Inverting (33) we have $\delta \approx A_{p/2} v_t^{p/2}$. Substituting this in P, we obtain

$$P \approx \sqrt{A_{p/2}^2 (v_t^p - v_c^p)}. \tag{64}$$

If $v_c \sim v_t$ we see that $P \sim \delta$. While if $v_c \to v_t$, $P \to 0$. In this case, let us write

$$\delta = A_{p/2} v_c^{p/2} + K_r v_c^r, \tag{65}$$

where $r > p/2$. Substituting this in the expression for P, we have

$$P \approx \sqrt{\delta^2 - A_{p/2}^2 v_c^p}$$

$$\approx \sqrt{(A_{p/2} v_c^{p/2} + K_r v_c^r)^2 - A_{p/2}^2 v_c^p} \approx \tag{66}$$

$$\approx \sqrt{2A_{p/2} B_r v_c^{p/2+r} + B_r^2 v_c^{2r}} \approx \sqrt{2A_{p/2} B_r} v_c^{p/4+r/2}. \tag{67}$$

However, we will see that in this case the energy of the collision does not change drastically depending on v_c.

C. Third case: $v_c \ll v_e$

Now, the point of collision is much closer to the horizon than v_e. This case can be obtained simply by taking the limit $v_c \to 0$ while keeping terms with δ. To see this, we have to use the fact that $v_{e,t} \sim \delta^{\max(1/s,2/p)}$ [see (17), (22), (30), and (34)] or, inversely, $\delta \sim v_{e,t}^{\min(s,p/2)}$. We can substitute this into the expression for (38) and get

$$P \sim \sqrt{(v_{e,t}^{\min(s,p/2)} + X_s v_c^s)^2 - \left(\kappa_p \left(1 + \frac{L_H^2}{g_{\varphi H}}\right) v_c^p\right)}. \tag{68}$$

First of all, we note that the ratio of the second and first terms in P is $\frac{v_c^s}{v_{e,t}^{\min(s,p/2)}}$. Since we assume $v_c \ll v_e$, the second term is much less than the first one. Additionally, we observe that the third term is much less than the first term because $\frac{v_c^p}{v_{e,t}^{2\min(s,p/2)}} \ll 1$ due to $v_c \ll v_e$. Therefore, the first term is dominant and we can write

$$P = \sqrt{\mathcal{X}^2 - N^2\left(1 + \frac{\mathcal{L}^2}{g_{\varphi\varphi}}\right)} \approx \mathcal{X} - \frac{N^2}{2\mathcal{X}}\left(1 + \frac{\mathcal{L}^2}{g_{\varphi\varphi}}\right). \quad (69)$$

We keep here higher-order corrections because they will be important for analysis of energy of collision. Also we note that in this case

$$\sqrt{\mathcal{X}^2 - N^2} \approx \mathcal{X} - \frac{N^2}{2\mathcal{X}}. \quad (70)$$

Generally speaking, this corresponds to usual particles because all entries of the point of collision in the expression for P are much less than the value of $\mathcal{X}$ on the horizon.

The main idea of this section is that there are four cases of possible interplay between the small parameters v_c and δ (actually, we saw that the classification is mainly defined by the relations between v_c and δ raised to some power). The first case occurs when $v_{e,t} \ll v_c$ (but $v_c \ll r_h$ still holds) that, as we showed, corresponds to the pure BSW phenomenon, which is not of interest in this work. The second case is $v_{e,t} \sim v_c$, for which, as we showed, P, $\mathcal{X}$, and $\sqrt{\mathcal{X}^2 - N^2}$ are $\sim\delta$ (exact expressions can be found in the corresponding parts of the text). In the case $v_c \ll v_{e,t}$, all v_c terms in the expressions for P, $\mathcal{X}$, and $\sqrt{\mathcal{X}^2 - N^2}$ are negligible that correspond to the case of usual particles that have already been investigated.

D. Fourth case: $v_c \to v_t$

The last case is possible if the particles impose the existence of turning points and when $v_c \to v_t$. In this case, depending on the type of particle, we assume that δ is given by (45), (53), (60), or (65). As we showed, in these cases P is either $\sim v_c^r$ (for near-subcritical particles) or $\sim v_c^{p/4+r/2}$ (for near-critical, near-ultracritical, or near-overcritical particles).

To summarize, new scenarios of particle motion related to the nonzero δ can only be obtained if $v_{e,t} \sim v_c$ or $v_c \to v_t$. All other ranges of the v coordinate correspond to already investigated cases [13].

V. ENERGY OF COLLISION

A. General relations

As we mentioned above, we are mainly interested in the possibility of the BSW phenomenon that is related to an unbounded growth of energy in the center-of-mass frame of two colliding particles. This energy is given by

$$\begin{aligned} E^2_{\text{c.m.}} &= -(m_1 u_{1\mu} + m_2 u_{2\mu})(m_1 u_1^\mu + m_2 u_2^\mu) \\ &= m_1^2 + m_2^2 - 2 m_1 m_2 u_1^\mu u_{2\mu}, \end{aligned} \quad (71)$$

where $\gamma = -u_{1\mu} u^{2\mu}$ is the Lorentz γ factor of relative motion. Substituting the expression for the four-velocity (4), we have

$$\gamma = \frac{\mathcal{X}_1 \mathcal{X}_2 - P_1 P_2}{N^2} - \frac{\mathcal{L}_1 \mathcal{L}_2}{g_{\varphi\varphi}}. \quad (72)$$

Hereafter, we assume that both particles move toward the horizon, so $\sigma_1 = \sigma_2 = -1$. The second term in (72) is regular, so we are interested, when the first one is unbounded.

Let us discuss all possible cases of particle collision depending on types of particles. Cases of collision between fine-tuned particles with usual or other fine-tuned particles have already been discussed in [13]. Thus, we are left with a discussion of the collision between near-fine-tuned particles with fine-tuned or usual particles, as well as the cases where both particles participating in the collision are near-fine-tuned.

B. First particle is fine-tuned (or usual), second is near-fine-tuned

Let us start with the analysis of the case in which one particle (let us call this particle 1) is fine-tuned (or usual) and particle 2 is near-fine-tuned. Before we proceed further, let us remind the reader of several properties of fine-tuned particles. Different types of particles are defined through their expansion of $\mathcal{X}$. Generally, $\mathcal{X}$ for fine-tuned particles has an expansion in the form

$$\mathcal{X} = X_s v^s + o(v^s), \quad (73)$$

where for usual particles $s = 0$, for subcritical $0 < s < p/2$, for critical $s = p/2$, for ultracritical $s = p/2$, and the condition (24) has to hold. In further analysis, we use the abbreviations "U" for usual particle, "SC" for subcritical, "C" for critical, and "UC" for ultracritical.

For usual and subcritical particles, as $\frac{N^2}{\mathcal{X}} \to 0$ as $v \to 0$, we can expand the function P and obtain

$$P = \mathcal{X} - \frac{N^2}{2\mathcal{X}}\left(\frac{\mathcal{L}^2}{g_{\varphi\varphi}} + 1\right) + \cdots = \mathcal{X} + O(v^{p-s}). \quad (74)$$

For critical particles N^2 and $\mathcal{X}$ are of the same order, so we have

$$P = P_{p/2} v^{p/2} + \cdots, \quad (75)$$

where $P_{p/2} = \sqrt{X_{p/2}^2 - \kappa_p\left(\frac{L_H^2}{g_{\varphi H}} + 1\right)}$.

For ultracritical particles

$$P = P_{c+(p-q)/2} v^{c+\frac{p-q}{2}} + \cdots, \quad (76)$$

where $c > q/2$ and $P_{c+(p-q)/2} = \sqrt{\frac{\kappa_p}{A_q}}(u^r)_c$ [note that these expansions correspond to the ones obtained by taking the limit $\delta \to 0$ for near-fine-tuned particles, see (35)–(37)].

Now, let us discuss the properties of particle 2. As concluded at the end of Sec. IV, the only interesting cases are those when for the second particle either $v_c \sim v_{e,t}$ or $v_c \to v_t$, so we only need to consider these cases.

Next, consider the collision of two particles. If the first particle is usual or subcritical, we substitute (74) and (73) to (72) and get

$$\gamma \approx \frac{X_{s_1}^{(1)} v_c^{s_1} [\mathcal{X}_2 - P_2]}{N^2} + \frac{P_2}{2X_{s_1}^{(1)} v_c^{s_1}} \left(\frac{L_{H1}^2}{g_{\varphi\varphi}} + 1\right)$$
$$\approx \frac{X_s^{(1)}}{\kappa_p} \frac{[\mathcal{X}_2 - P_2]}{v_c^{p-s_1}} + \frac{\left(\frac{L_{H1}^2}{g_{\varphi\varphi}} + 1\right)}{2X_{s_1}^{(1)}} \frac{P_2}{v_c^{s_1}}, \tag{77}$$

where upper index (1) means quantities related to the first particle (fine-tuned). We will postpone analysis of this complicated expression to the next subsections.

If the first particle is critical, then we substitute (75) and (73) to (72) and have

$$\gamma \approx \frac{1}{\kappa_p} \frac{X_{p/2}^{(1)} \mathcal{X}_2 - P_{p/2}^{(1)} P_2}{v_c^{p/2}}. \tag{78}$$

If the first particle is ultracritical,

$$\gamma \approx \frac{1}{\kappa_p} \frac{X_{s_1}^{(1)} \mathcal{X}_2}{v_c^{p/2}} \tag{79}$$

(note that in this case the term with P_1 is absent because P_1 is of higher order in v_c than $\mathcal{X}_1$).

1. Near-subcritical particles

Now, let us analyze the behavior of the γ factor concerning various types of the second particle. We begin by considering a scenario where particle 2 is near-subcritical. The initial case for analysis involves the situation where particle 1 is usual or subcritical. Before delving into the analysis of the behavior of γ, it is important to note that in this case [using (42)]

$$\mathcal{X}_2 - P_2 = \frac{N^2}{2\mathcal{X}_2} \left(1 + \frac{\mathcal{L}_2^2}{g_{\varphi\varphi}}\right). \tag{80}$$

Substituting this in (77) we have

$$\gamma \approx \frac{X_{s_1}^{(1)} v_c^{s_1}}{2\mathcal{X}_2} \left(1 + \frac{L_{H2}^2}{g_{\varphi\varphi}}\right) + \frac{\left(\frac{L_{H1}^2}{g_{\varphi\varphi}} + 1\right)}{2X_{s_1}^{(1)}} \frac{\mathcal{X}_2}{v_c^{s_1}}. \tag{81}$$

Now, let us analyze the various possible cases. As demonstrated in Sec. IV, the behavior of $\mathcal{X}_2$ depends on whether $v_c \sim v_{e,t}$ or $v_c \to v_t$. If $v_c \sim v_{e,t}$, we can use (17) [or (18) if a turning point exists] and get $v_c \sim \delta^{1/s_2}$ (or, inverting, $\delta \sim v_c^{s_2}$). Furthermore, using the relation $\mathcal{X}_2 \sim \delta$ when $v_c \sim v_{e,t}$, we obtain two terms in the expression for γ: the first one is proportional to $\sim v_c^{s_1 - s_2}$, while the second term is proportional to $\sim v_c^{s_2 - s_1}$. The dominant term is determined by the smaller degree in v_c between these two. Combining these terms, we can find that $\gamma \sim v_c^{-|s_2 - s_1|}$.

If the turning point exists [this, as one can see from (18), happens if $\delta^{(2)} > 0$ and $X_s^{(2)} < 0$ or if $\delta^{(2)} < 0$ and $X_s^{(2)} > 0$] and if $v_c \to v_t$, it follows from (46) that $\mathcal{X}_2 \approx B_{r_2}^{(2)} v_c^{r_2}$. Substituting this in (77) we get

$$\gamma \approx \frac{X_{s_1}^{(1)} v_c^{s_1}}{2B_{r_2}^{(2)} v_c^{r_2}} \left(1 + \frac{L_{H2}^2}{g_{\varphi\varphi}}\right) + \frac{\left(\frac{L_{H1}^2}{g_{\varphi\varphi}} + 1\right)}{2X_{s_1}^{(1)}} \frac{B_{r_2}^{(2)} v_c^{r_2}}{v_c^{s_1}}. \tag{82}$$

As one can see, there are two terms: the first is $\sim v_c^{s_1 - r_2}$, while the second is $\sim v_c^{r_2 - s_1}$. Since the divergence of the γ factor is defined by the dominant term, we can combine these two terms and write $\gamma \sim v_c^{-|s_1 - r_2|}$.

All these expressions can be formulated briefly under this condition.

$$\text{if the first particle is U or SC and the second is NSC, then} \begin{cases} \gamma \sim v_c^{-|s_2 - s_1|} & \text{if } v_c \sim v_{e,t}, \\ \gamma \sim v_c^{-|s_1 - r_2|} & \text{if } v_c \to v_t [\text{see (45)}]. \end{cases} \tag{83}$$

In the case where the first particle is critical [see (78)] or ultracritical [see (79)] and if $v_c \sim v_{e,t}$ we can use that $\mathcal{X}_2 \sim \delta$ [see (43) and discussion after it] and get $\gamma \sim \frac{\delta}{v_c^{p/2}}$. Now, $v_c \sim \delta^{1/s_2}$ [see (17) and (16)]. Inverting it, we have $\delta \sim v_c^{s_2}$, so we can write $\gamma \sim v_c^{s_2 - p/2}$.

If the turning point exists [this, as one can see from (18), happens if $\delta^{(2)} > 0$ and $X_s^{(2)} < 0$ or if $\delta^{(2)} < 0$ and $X_s^{(2)} > 0$] and if $v_c \to v_e$, we can use (46) and write $\mathcal{X}_2 \approx B_{r_2}^{(2)} v_c^{r_2}$. If particle 1 is critical, we can use (78) and write

$$\gamma \approx \frac{1}{\kappa_p} \frac{(X_{p/2}^{(1)} - P_{p/2}^{(1)}) B_{r_2}^{(2)} v_c^{r_2}}{v_c^{p/2}}. \tag{84}$$

From this expression, we see easily that $\gamma \sim v_c^{r_2 - p/2}$. The same holds for the case when the first particle is ultracritical [to see this one has to substitute (46) to (79)]. To summarize, we have

$$\text{if the first particle is C or UC and the second is NSC, then} \begin{cases} \gamma \sim v_c^{s_2-\frac{p}{2}} & \text{if } v_c \sim v_e, \\ \gamma \sim v_c^{r_2-\frac{p}{2}} & \text{if } v_c \to v_t [\text{see (45)}]. \end{cases} \tag{85}$$

2. Near-critical, near-ultracritical, and near-overcritical particles

Now let us consider the cases when the second particle is near-critical, near-ultracritical, or near-overcritical. Before we proceed further, let us consider $\mathcal{X}_2 - P_2$ for different types of particles. We start with a case of near-critical particles. Using (49) and (50) we have

$$\mathcal{X}_2 - P_2 \approx (\delta + X_{p/2} v_c^{p/2}) - \sqrt{(\delta + X_{p/2} v_c^{p/2})^2 - \kappa_p \left(1 + \frac{L_H^2}{g_{\varphi H}}\right) v_c^p}. \tag{86}$$

If $v_c \sim v_{e,t}$, corresponding terms do not cancel each other and, using (22), we have $\mathcal{X}_2 - P_2 \sim \delta$ (note that this also holds for the case $v_c \to v_t$ because in this limit $P_2 \to 0$ while $\mathcal{X}_2$ remains $\sim\delta$).

In the case of near-overcritical particles, the situation is somehow similar. Using (63) we have

$$\mathcal{X}_2 - P_2 \approx \delta - \sqrt{\delta^2 - \kappa_p \left(1 + \frac{L_H^2}{g_{\varphi H}}\right) v_c^p}. \tag{87}$$

If $v_c \sim v_{e,t}$ (and $v_c \to v_t$) corresponding terms do not cancel each other and, using (34), we have $\mathcal{X}_2 - P_2 \sim \delta$.

In the case of near-ultracritical particles, the situation is also similar. Using (56) and (57), we get

$$\mathcal{X}_2 - P_2 \approx (\delta + X_{p/2} v_c^{p/2}) - \sqrt{\delta^2 + 2\delta X_{p/2} v_c^{p/2}}. \tag{88}$$

If $v_c \sim v_{e,t}$ (and $v_c \to v_t$), the corresponding terms do not cancel each other, and using (30), we have $\mathcal{X}_2 - P_2 \sim \delta$.

All these cases are similar in the sense that if $v_c \sim v_{e,t}$ (or $v_c \to v_t$), then $\mathcal{X}_2 - P_2 \sim \delta$. Using these facts, we are ready to analyze the γ factor. We start with a case when the first particle is usual or subcritical [see (77)]. If $v_c \sim v_{e,t}$, then we have two terms: the first one is $\sim \frac{\delta}{v_c^{p-s_1}}$, while the second one is $\sim \frac{\delta}{v_c^{s_1}}$. Since for usual or subcritical particles, $0 < s_1 < p/2$, the first term is dominant. Using the fact that $v_c \sim v_{e,t}$ and Eqs. (22), (34), or (30), we have $v_c \sim \delta^{2/p}$ or, inversely, $\delta \sim v_c^{p/2}$. Substituting this into (77), we get $\gamma \sim v_c^{-(\frac{p}{2}-s_1)}$. To summarize, we have

$$\text{if the first particle is U or SC and the second is NC, NUC, or NOC, then } \gamma \sim v_c^{-(\frac{p}{2}-s_1)} \quad \text{if } v_c \sim v_{e,t}. \tag{89}$$

If particle 1 is critical [see (78)] of ultracritical [see (79)], $\gamma \sim \frac{\delta}{v_c^{p/2}}$ if $v_c \sim v_{e,t}$. Reverting (22) or (34), we can write that in a case $v_c \sim v_{e,t}$, $\delta \sim v_c^{p/2}$. Substituting this in the expression for the γ factor, we have

$$\text{if first the particle is C or UC and the second is NC, NUC, or NOC, then } \gamma = O(1) \quad \text{if } v_c \sim v_{e,t} \text{ or if } v_c \to v_t. \tag{90}$$

C. First and second particles are near-fine-tuned

First of all, let us formulate which cases we have to consider. As we concluded at the end of Sec. IV, the only new cases are such that for both particles either $v_c \sim v_{e,t}$ or $v_c \to v_t$. Let us consider different subcases.

1. First and second particles are near-subcritical

In this case, we can use (42) for both particles and get

$$\gamma \approx \frac{\mathcal{X}_1}{2\mathcal{X}_2}\left(1 + \frac{\mathcal{L}_2^2}{g_{\varphi\varphi}}\right) + \frac{\mathcal{X}_2}{2\mathcal{X}_1}\left(1 + \frac{\mathcal{L}_1^2}{g_{\varphi\varphi}}\right). \tag{91}$$

Additionally, there are four subcases.

(i) If both particles satisfy $v_c^{(1,2)} \sim v_{e,t}^{(1,2)}$, (43) holds for both particles, and γ is given by two terms: the first term is $\sim \frac{\delta_1}{\delta_2}$ and the second one is $\sim \frac{\delta_2}{\delta_1}$. Using the fact that $\delta_{1,2} \sim v_c^{s_{1,2}}$ for both particles, we can combine both terms and write $\gamma \sim v_c^{-|s_1 - s_2|}$.

(ii) If for the first particle $v_c^{(1)} \sim v_{e,t}^{(1)}$, while for the second particle $v_c^{(2)} \to v_t^{(2)}$, we can use (43) for the first particle and (46) for the second particle. There are two terms: the first one is $\sim \frac{\delta_1}{v_c^{r_2}}$, while the second one is $\sim \frac{v_c^{r_2}}{\delta_1}$. Since for the first particle $\delta_1 \sim v_c^{s_1}$, we can combine these two terms and get $\gamma \sim v_c^{-|s_1 - r_2|}$.

(iii) If for the first particle $v_c^{(1)} \to v_t^{(1)}$, while for the second particle $v_c^{(2)} \sim v_{e,t}^{(2)}$, we can use the expressions from the previous subcase and write $\gamma \sim v_c^{-|r_1 - s_2|}$.

TABLE II. Table showing behavior of γ factor for $v_c \sim v_{e,t}$ in a case when first particle is fine-tuned and second is near-fine-tuned. Here d is defined by relation $\gamma \sim v_c^{-d}$.

	First particle	Second particle	d
1	U or SC	NSC	$\lvert s_1 - s_2\rvert$ or $\lvert s_1 - r_2\rvert$ if $\delta_2 = -X_s^{(2)} v_c^s + B_r^{(2)} v_c^r$
2	C or UC	NSC	$\frac{p}{2} - s_2$ or $\frac{p}{2} - r_2$ if $\delta_2 = -X_s^{(2)} v_c^s + B_r^{(2)} v_c^r$
3	U or SC	NC, NUC, or NOC	$\frac{p}{2} - s_1$
4	C or UC	NC, NUC, or NOC	0

(iv) If for both particles $v_c^{(1,2)} \to v_t^{(1,2)}$, we can use (46) for them. We have two terms: the first one is $\sim \frac{v_c^{r_1}}{v_c^{r_2}}$, while the second one is $\sim \frac{v_c^{r_2}}{v_c^{r_1}}$. We can combine them and write $\gamma \sim v_c^{-\lvert r_1 - r_2\rvert}$.

2. First particle is NSC while the second is NC, NUC, or NOC

For the first particle we can use (42) and write

$$\gamma \approx \frac{\mathcal{X}_1}{\kappa_p} \frac{[\mathcal{X}_2 - P_2]}{v_c^p} + \frac{\left(\frac{L_{H1}^2}{g_{\varphi\varphi}} + 1\right)}{2\mathcal{X}_1} P_2. \tag{92}$$

Here, there are several cases.

(i) For both particles $v_c^{(1,2)} \sim v_{e,t}^{(1,2)}$. In this case, we can use (43) for the first particle and (49), (56), or (63) for the second particle. (All of these cases, in fact, give the same result because $P_2 \sim \mathcal{X}_2 \sim \delta_2$). Substituting this into the expressions for γ, we obtain two terms: the first term is of the order $\sim \frac{\delta_1 \delta_2}{v_c^p}$, while the second one is $\sim \frac{\delta_2}{\delta_1}$. Using the fact that $\delta_1 \sim v_c^{s_1}$ for the first particle and $\delta_2 \sim v_c^{p/2}$ for the second particle, we see that the expression for γ has two terms: the first term is $\sim v_c^{s_1 - p/2}$, and the second term is $\sim v_c^{p/2 - s_1}$. Since the first particle is near-subcritical, it holds that $s_1 < p/2$. From this fact, we can deduce that the first term is dominant, and we have

$$\gamma \approx v_c^{-(p/2 - s_1)}. \tag{93}$$

(ii) For the first particle $v_c^{(1)} \to v_t^{(1)}$, while for the second one $v_c^{(2)} \sim v_{e,t}^{(2)}$. In this case, for the first particle condition (46) holds and the behavior of the γ factor may be obtained by changing $s_1 \to r_1$. Thus, we have

$$\gamma \approx v_c^{-(p/2 - r_1)}. \tag{94}$$

Note that taking limit $v_c^{(2)} \to v_t^{(2)}$ does not change the relations $P_2 \sim \mathcal{X}_2 \sim \delta_2$. So this case does not require special analysis.

3. Both particles are NC, NUC, or NOC

According to Sec. IV B, the relations $\mathcal{X}_{1,2} \sim P_{1,2} \sim \delta_{1,2}$ hold for both particles. Substituting this in the expression for γ (72), one has

$$\gamma \sim \frac{\delta_1 \delta_2}{v_c^p}. \tag{95}$$

Using that for NC, NUC, and NOC particles holds $\delta_{1,2} \sim v_c^{p/2}$, we can deduce that $\gamma = O(1)$.

Now, let us briefly formulate all the aforementioned conditions. To this end, we introduce a parameter d that characterizes how fast γ changes as a function of the collision point v_c: $\gamma \sim v_c^{-d}$. By analyzing (83)–(85), (89), and (90), we obtain the conditions summarized in Tables II and III.

Now, let us discuss the new scenarios that can be obtained. It is important to note that for cases when $v_c \sim v_{e,t}$, the behavior of the γ factor is the same as for corresponding particles with $\delta = 0$. The only difference is

TABLE III. Table showing behavior of γ factor for $v_c \sim v_{e,t}$ in a case when both particles are near-fine-tuned. Here d is defined by relation $\gamma \sim v_c^{-d}$.

	First particle	Second particle	d
1	NSC	NSC	$\lvert s_1 - s_2\rvert$ $\lvert r_1 - s_2\rvert$ if $\delta_1 = -X_s^{(1)} v_c^s + B_r^{(1)} v_c^r$ $\lvert s_1 - r_2\rvert$ if $\delta_2 = -X_s^{(2)} v_c^s + B_r^{(2)} v_c^r$ $\lvert r_1 - r_2\rvert$ if $\delta_{1,2} = -X_s^{(1,2)} v_c^s + B_r^{(1,2)} v_c^r$
2	NSC	NC, NUC, or NOC	$\frac{p}{2} - s_1$
3	NC, NUC, or NOC	NC, NUC, or NOC	0

the existence of near-overcritical particles, which do not have any analog in the case of $\delta = 0$, and the possibility of collisional processes involving them. As we will show, such particles are very important because they allow for high-energy collisions for nonextremal horizons.

There are also additional differences in cases when $v_c \to v_t$. In these cases, the behavior of the γ factor is described by different expressions. For example, if two near-subcritical particles with $s_1 = s_2$ participate in a collision, according to Table III, $\gamma = O(1)$. However, if we choose δ's for these particles in such a way that $r_1 \neq r_2$, then the γ factor diverges. Thus, in some cases, "fine-tuning" of the δ's makes previously forbidden BSW effects possible.

VI. BEHAVIOR OF ACCELERATION: GENERAL APPROACH

Now we are going to analyze the forces acting on particles of different types. In order to do this, at first we need to answer a question: in which frame do we have to compute acceleration? Since the stationary frame is singular near the horizon, we have to choose a frame that does not have this property. The natural frame for this purpose is the one attached to a particle, known as the free-falling zero angular momentum observer (FZAMO) frame. It is worth noting that, for near-fine-tuned particles, the radial velocity on the horizon is not zero and the particle can cross the horizon, so we have to compute acceleration in the FZAMO frame, unlike fine-tuned particles for which the FZAMO frame is singular and acceleration has to be computed in the nonsingular orbital zero angular momentum observer (OZAMO) frame [11]. General definition of ZAMO and description of its properties is given in [18].

To proceed further, let us use the expressions for acceleration in the tetrad frame that are obtained in Appendix A,

$$a_f^{(t)} = 0, \tag{96}$$

$$a_f^{(r)} = \frac{u^r}{\sqrt{\mathcal{X}^2 - N^2}}(\partial_r \mathcal{X} + L\partial_r \omega), \tag{97}$$

$$a_f^{(\varphi)} = \frac{1}{\sqrt{\mathcal{X}^2 - N^2}} \frac{1}{\sqrt{g_{\varphi\varphi}}} \frac{\sqrt{A}}{N} \times [(\mathcal{X}^2 - N^2)\partial_r \mathcal{L} - \mathcal{L}\mathcal{X}(\partial_r \mathcal{X} + \mathcal{L}\partial_r \omega)], \tag{98}$$

where subscript f means that the corresponding components of acceleration are computed in the FZAMO frame. As one can see, there are only two nonzero acceleration components and we are going to analyze at first $a_f^{(r)}$ to understand the structure of acceleration and which terms are dominant for each case discussed in Sec. IV. To this end, we use expansions (2), (3), and (10) that give us

$$a_f^{(r)} \approx \sigma \frac{P}{\sqrt{\mathcal{X}^2 - N^2}} \sqrt{\frac{A_q}{\kappa_p}} \left(X_s s v_c^{s+\frac{q-p}{2}-1} + L_H \omega_k k v_c^{k+\frac{q-p}{2}-1} \right), \tag{99}$$

where $\sigma = \pm 1$ depending on whether the particle is outgoing or ingoing. Our task is to consider different ranges of v_c and find the behavior of acceleration.

(i) If $v_c \sim v_{e,t}$,

$$a_f^{(r)} \approx (a_f^{(r)})_{m_1} \delta^{m_1}, \qquad a_f^{(\varphi)} \approx (a_f^{(\varphi)})_{m_2} \delta^{m_2}. \tag{100}$$

(ii) If $v_c \ll v_{e,t}$,

$$a_f^{(r)} \approx (a_f^{(r)})_{n_1} v_c^{n_1}, \qquad a_f^{(\varphi)} \approx (a_f^{(\varphi)})_{n_2} v_c^{n_2}. \tag{101}$$

(iii) If there exists a turning point, in the limit $v_c \to v_t$ we have

$$a_f^{(r)} \approx (a_f^{(r)})_{i_1} v_c^{i_1}, \qquad a_f^{(\varphi)} \approx (a_f^{(\varphi)})_{i_2} v_c^{i_2}. \tag{102}$$

Our goal is to obtain relations between $m_1, m_2, n_1, n_2, i_1, i_2$ and the type of a particle.

VII. CASE $v_c \ll v_{e,t}$

As we discussed in the analysis of the γ factor (see Sec. V), the case $v_c \ll v_{e,t}$ corresponds to usual particles. However, in [13], the acceleration for this case was analyzed only qualitatively (the reason for this is that, in [13], the authors were mainly focused on acceleration for fine-tuned particles, for which the FZAMO frame is singular on the horizon). To fill this gap, we are going to analyze acceleration of usual particles in the present work.

A. General analysis of acceleration

We start with the radial component of acceleration. First of all, we refer to the fact that $P \sim \sqrt{\mathcal{X}^2 - N^2} \sim \delta$ for $v_c \ll v_{e,t}$ [to obtain this, one has to use (69) and (70); the dominant term $\mathcal{X}$ in these expressions has the order δ on the horizon]. Using this fact, we see that the prefactor $\frac{P}{\sqrt{\mathcal{X}^2 - N^2}}$ in (99) is $= O(1)$. Thus, we are left with the terms in brackets in (99). One can see that the first term is $\sim v_c^{s+\frac{q-p}{2}-1}$, and the second term is $\sim v_c^{k+\frac{q-p}{2}-1}$. Comparing these terms with (101), we get

$$n_1 = \min(s, k) + \frac{q - p}{2} - 1. \tag{103}$$

However, this expression does not describe the most general case. It may appear that the function $\omega = \omega_H$ is constant (that corresponds to a static metric because there exists a corresponding coordinate transformation $\tilde{\varphi} = \varphi - \omega_H t$ that brings the metric to an explicitly static form). In this case, the second term in (99) is absent and we obtain

$$n_1 = s + \frac{q-p}{2} - 1 \quad \text{if } \omega = \omega_H. \tag{104}$$

There is also a special case when the coefficients in the expansion of $\mathcal{X}$ and $\mathcal{L}$ are such that several terms in the power series (potentially divergent, generally speaking) in the expression for acceleration cancel each other. Full cancellation happens, for example, for a freely falling particle, provided $\mathcal{X} + \omega\mathcal{L} = \epsilon$, where ϵ and $\mathcal{L}$ are constants that give us zero acceleration. Then, $\partial_r(\mathcal{X} + \omega\mathcal{L}) = 0$ exactly. In a more general case, we can consider

$$\epsilon = \mathcal{X} + \omega\mathcal{L} = \epsilon_0 + \text{terms of } v^m \text{ order}, m > k. \tag{105}$$

Then, let us rewrite acceleration (97) as a function of ϵ,

$$a_f^{(r)} = \frac{u^r}{\sqrt{\mathcal{X}^2 - N^2}}(\partial_r\epsilon - \omega\partial_r\mathcal{L}). \tag{106}$$

We have

$$n_1 = \min(m, b) + \frac{q-p}{2} - 1 \text{ in a case of } (105) \tag{107}$$

[the quantity b was defined in (6)]. However, we will not pay much attention to this case in our further analysis.

Now let us invert all aforementioned conditions to obtain s as a function of n_1,

$$s = \begin{cases} n_1 + 1 + \frac{p-q}{2} \text{ if } 0 \le n_1 < k + \frac{q-p}{2} - 1, \\ \text{may be any value } s \ge k \ge 0 \text{ if } n_1 = k + \frac{q-p}{2} - 1, \\ n_1 + 1 + \frac{p-q}{2} \text{ for any } 0 \le n_1 \text{ if } \omega = \omega_H, \\ \text{may be any value in a case of } (105) \left(n_1 = \min(m, b) + \frac{q-p}{2} - 1\right). \end{cases} \tag{108}$$

Now, let us move to an angular component of acceleration. To analyze finiteness of this component, we note that it follows from (97) and (98) and the fact that $u^r = \sigma\frac{\sqrt{A}}{N}P$ that

$$\mathcal{X}\mathcal{L}a_f^{(r)} + \sigma P\sqrt{g_{\varphi\varphi}}a_f^{(\varphi)} = \frac{\sqrt{A}}{N}P\sqrt{\mathcal{X}^2 - N^2}\partial_r\mathcal{L}. \tag{109}$$

As in this section we are considering the near-horizon limit, $\mathcal{X} \approx P \approx \sqrt{\mathcal{X}^2 - N^2} \approx \delta$. Thus, we can write in this limit

$$a_f^{(\varphi)} + \sigma\mathcal{L}a_f^{(r)} = \delta\frac{\sqrt{A}}{N}\partial_r\mathcal{L}. \tag{110}$$

We require that both $a_f^{(\varphi)}$ and $a_f^{(r)}$ be finite. From (110), it follows that if the acceleration components are finite, then the left-hand side of (110) is also finite. Because the equality (110) has to hold, the right-hand side also has to be finite. If $q \ge p$, then the ratio $\frac{\sqrt{A}}{N}$ is finite on the horizon, and the whole right-hand side is finite for any expansion of $\mathcal{L}$. While if $q < p$, then the right-hand side behaves like

$$\frac{\sqrt{A}}{N}\partial_r\mathcal{L} \sim v_c^{b+\frac{q-p}{2}-1}, \tag{111}$$

[here we used (6)]. This quantity is finite only for

$$b \ge 1 + \frac{p-q}{2}. \tag{112}$$

The reversed statement thus can be easily proved by extraction of $a_f^{(\varphi)}$ from (110),

$$a_f^{(\varphi)} = \delta\frac{\sqrt{A}}{N}\partial_r\mathcal{L} - \sigma\mathcal{L}a_f^{(r)}.$$

If (112) holds and $a_f^{(r)}$ is finite, then $a_f^{(\varphi)}$ is also finite. This allows us to state a proposition:

Proposition 1. If in case $v_c \ll v_{e,t}$ for some particle $a_f^{(r)}$ is finite and condition (112) holds, then $a_f^{(\varphi)}$ is also finite and vice versa.

B. Behavior of acceleration for different types of particles

In this subsection, we are going to analyze which types of particles are compatible with finite acceleration.

1. Near-subcritical particles

We start with near-subcritical particles. We would like to remind the reader that the defining condition for them is $0 < s < p/2$. Let us begin with the first solution in (108). By substituting the range of $s: 0 < s < p/2$, we obtain the corresponding range for n_1: $\frac{q-p}{2} - 1 < n_1 < \frac{q-2}{2}$. Now, let us determine how this correlates with the condition for the existence of the first solution in (108): $n_1 \le k + \frac{q-p-2}{2}$. For $k < \frac{p}{2}$, the condition $n_1 \le k + \frac{q-p-2}{2}$ is stronger than $n_1 < \frac{q-2}{2}$ [it is also worth noting that, according to the first solution in (108), $s < k$ in this case]. However, for $k \geq \frac{p}{2}$, the condition $n_1 < \frac{q-2}{2}$ becomes stronger. The lower bound for n_1 remains the same for all positive k. Therefore, we can conclude that, for the first solution in (108),

$$\frac{q-p}{2} - 1 < n_1 \leq k + \frac{q-p}{2} - 1 \quad \text{if } s < k < \frac{p}{2}, \tag{113}$$

$$\frac{q-p}{2} - 1 < n_1 < \frac{q-2}{2} \quad \text{if } k \geq \frac{p}{2}. \tag{114}$$

The second solution in (108) gives any s that is greater than k with n_1 fixed by (104): $n_1 = k + \frac{q-p}{2} - 1$. This value is non-negative if

$$k \geq \frac{p-q}{2} + 1. \tag{115}$$

As we are specifically considering near-subcritical particles with $0 < s < p/2$, this solution is possible only if $0 < k \leq s < p/2$. This condition necessitates that $k < p/2$. By combining this condition with (115), we can deduce that $q > 2$ for this solution to exist. In all other cases, this solution does not exist. To summarize the above findings regarding the existence of the second solution in (108), we have

$$n_1 = k + \frac{q-p}{2} - 1 \quad \text{if } 0 < k \leq s < p/2, \tag{116}$$

$$\text{second solution in (108) is absent if } 0 < s < k < p/2 \text{ or if } k \geq p/2. \tag{117}$$

If the function ω is constant [the third solution in (108)], we obtain the same range within which n_1 can change. It is the first solution in (108): $\frac{q-p}{2} - 1 < n_1 < \frac{q-2}{2}$.

By combining these facts, we can write

$$\text{for near-subcritical particles holds} \begin{cases} \frac{q-p}{2} - 1 < n_1 \leq k + \frac{q-p}{2} - 1 \text{ if } k < \frac{p}{2}, \\ \frac{q-p}{2} - 1 < n_1 < \frac{q-2}{2} \text{ if } k \geq \frac{p}{2} \text{ or if } \omega = \omega_H. \end{cases} \tag{118}$$

2. *Near-critical and near-ultracritical particles*

In these cases, when $s = \frac{p}{2}$ [but in the case of near-ultracritical particles, the additional condition (24) must also hold], despite the difference in P functions, the acceleration for them is the same [because acceleration depends only on s which is the same for the considered cases, see (108)].

If we substitute $s = p/2$ into the first solution in (108), it will give us $n_1 = \frac{q-2}{2}$, so n_1 is non-negative if $q \geq 2$. The first solution exists if $n_1 < k + \frac{q-p}{2} - 1$ that requires $k > \frac{p}{2}$.

The second solution is true for all $s = p/2 \geq k$ and, similar to the case of near-subcritical particles, gives a non-negative n_1 only if $k \geq \frac{p-q}{2} + 1$ (in this case, $n_1 = k + \frac{q-p}{2} - 1$).

If the function ω is constant [the third solution in (108)], we get $n_1 = \frac{q-2}{2}$ without any additional limitations.

We can combine all the aforementioned solutions and write

$$\text{for NC and NUC particles holds} \begin{cases} n_1 = k + \frac{q-p-2}{2} \text{ if } k \leq \frac{p}{2}, \\ n_1 = \frac{q-2}{2} \text{ if } k > \frac{p}{2} \text{ or if } \omega = \omega_H. \end{cases} \tag{119}$$

3. *Near-overcritical particles*

For near-overcritical particles $s > \frac{p}{2}$. The first solution in (108) gives us $n_1 > \frac{q-2}{2}$. As the first solution exists for $n_1 < k + \frac{q-p}{2} - 1$ only, we have

$$\frac{q-2}{2} < n_1 < k + \frac{q-p}{2} - 1 \text{ if } k > \frac{p}{2},$$

$$\text{first solution is impossible if } k \leq \frac{p}{2}. \tag{120}$$

Substituting the inequalities $\frac{q-2}{2} < n_1 < k + \frac{q-p}{2} - 1$ back to (108), one gets that the first solution in (108) requires $p/2 < s < k$.

The second solution in (108) allows all $s \geq k$ and is only possible if $k \geq \frac{p-q}{2} + 1$ (in which case $n_1 = k + \frac{q-p}{2} - 1$).

The third solution simply gives us $\frac{q-2}{2} < n_1$ without any upper limitations.

Now let us join all the aforementioned solutions. Let us consider the case of $k > p/2$. Depending on whether $s < k$

or $s \geq k$, we get different solutions. In the case $s < k$, we have $\frac{q-2}{2} < n_1 < k + \frac{q-p}{2} - 1$ [see (120) and the discussion after it], while in the case $s \geq k$, we have $n_1 = k + \frac{q-p}{2} - 1$. We can join these conditions and write $\frac{q-2}{2} < n_1 \leq k + \frac{q-p}{2} - 1$ for any $k > p/2$, independent of whether s is greater or smaller than k. In the case $k \leq p/2$, the situation is simpler: solution (120) is absent, and we are left only with the second solution in (108): $n_1 = k + \frac{q-p}{2} - 1$. By joining all these cases, we can write

for near-over critical particles holds

$$\times \begin{cases} \frac{q-2}{2} < n_1 \leq k + \frac{q-p}{2} - 1 \text{ if } k > \frac{p}{2}, \\ n_1 = k + \frac{q-p}{2} - 1 \text{ if } k \leq p/2, \\ n_1 > \frac{q-2}{2} \text{ if } \omega = \omega_H. \end{cases} \tag{121}$$

C. Different k regions

To reformulate conditions obtained in previous subsections, first of all we note that the existence of solutions is defined only by different values of k. To simplify analysis further, we introduce different k regions as we did in Sec. VII E in [13],

$$0 < k < \frac{p-q}{2} + 1, \qquad \text{region I},$$
$$\frac{p-q}{2} + 1 \leq k < \frac{p+1-q/2}{2}, \qquad \text{region II},$$
$$\frac{p+1-q/2}{2} \leq k < \frac{p}{2}, \qquad \text{region III},$$
$$k \geq \frac{p}{2}, \qquad \text{region IV}. \tag{122}$$

As was discussed in [13], all these regions exist and do not intersect if $q > 2$. However, if $q \leq 2$, then classification in these cases has to be introduced differently,

$$0 < k < \frac{p-q}{2} + 1, \qquad \text{region I},$$
$$k \geq \frac{p-q}{2} + 1, \qquad \text{region IV}. \tag{123}$$

Let us start our analysis with the stationary metric (where ω is nonconstant). We see that in region I n_1 is negative for any type of particle because in this case it follows from (118), (119), and (121) that, for any type of particle, n_1 is either limited by the value $k + \frac{q-p-2}{2}$ or is equal to it. However, as $k + \frac{q-p-2}{2}$ is negative in region I, n_1 is also negative.

In regions II and III (where $\frac{p-q}{2} + 1 \leq k < \frac{p}{2}$) we see that for near-subcritical particles, we have to choose the first solution in (118) that gives us $\frac{q-p-2}{2} < n_1 \leq k + \frac{q-p-2}{2}$. For near-critical and near-ultracritical particles, we take the first solution in (119). For near-overcritical particles, we take the second solution in (121) that gives us $n_1 = k + \frac{q-p}{2} - 1$ for all these particles.

In region IV (where $k \geq \frac{p}{2}$), for near-subcritical particles, we use the second solution in (118) that gives us $\frac{q-p-2}{2} < n_1 < \frac{q-2}{2}$. To include non-negative values of n_1 in this region, we have to require $q > 2$. Otherwise, accelerations for near-subcritical particles diverge in region IV.

For near-critical and near-ultracritical particles, we use either the first solution in (119) (if $k = p/2$) that entails $n_1 = \frac{q-2}{2}$ or we use the second solution in (119) that leads to the same value $n_1 = \frac{q-2}{2}$. The acceleration in these cases is non-negative only if $q \geq 2$.

TABLE IV. Classification of near-horizon trajectories for different k regions for $q > 2$ (ultraextremal horizon). The fourth solution in (108) is not presented in this table.

	k region	n_1 range	s	Type of trajectory
		Stationary metric		
1	I	For any type of trajectory n_1 is negative (forces diverge)		
2	II and III	$\max(0, \frac{q-p}{2} - 1) < n_1 \leq k + \frac{q-p}{2} - 1$	First and second in (108)	NSC
		$n_1 = k + \frac{q-p}{2} - 1$	Second in (108)	NC and NOC
		$n_1 = k + \frac{q-p}{2} - 1$ and (24)	Second in (108)	NUC
3	IV	$\max(0, \frac{q-p}{2} - 1) < n_1 < \frac{q-2}{2}$	First in (108)	NSC
		$n_1 = \frac{q-2}{2}$	First in (108)	NC
		$n_1 = \frac{q-2}{2}$ and (24)	First in (108)	NUC
		$\frac{q-2}{2} < n_1 \leq k + \frac{q-p}{2} - 1$	First and second in (108)	NOC
		Static metric		
4	$k = 0$	Same results as in IV for stationary metric		NSC, NC, and NUC
		$n_1 > \frac{q-2}{2}$	Third in (108)	NOC

TABLE V. Classification of near-horizon trajectories for different k regions for $q = 2$ (ultraextremal horizon). The fourth solution in (108) is not presented in this table.

	k region	n_1 range	s	Type of trajectory
		Stationary metric		
1	I		For any type of trajectory n_1 is negative (forces diverge)	
2	IV	$n_1 = 0$	First in (108)	NC
		$n_1 = 0$ and (24)	First in (108)	NUC
		$0 \leq n_1 \leq k + \frac{q-p}{2} - 1$	First and second in (108)	NOC
		Static metric		
3	$k = 0$	Same results as in IV for stationary metric		NC and NUC
		$n_1 > 0$	Third in (108)	NOC

TABLE VI. Classification of near-horizon trajectories for different k regions for $q < 2$ (ultraextremal horizon). The fourth solution in (108) is not presented in this table.

	k region	n_1 range	s	Type of trajectory
		Stationary metric		
1	I		For any type of trajectory n_1 is negative (forces diverge)	
2	IV	$0 < n_1 \leq k + \frac{q-p}{2} - 1$	First and second in (108)	NOC
		Static metric		
3	$k = 0$	$0 < n_1$	Third in (108)	NOC

For near-overcritical particles, we use either the first solution in (121) that gives us $\frac{q-2}{2} < n_1 < k + \frac{q-p}{2} - 1$ (this solution is true if $\frac{p}{2} < s < k$), or we use the second solution in (121) that gives us $n_1 = k + \frac{q-p}{2} - 1$ (if $s \geq k$.) This solution in region IV is presented only if $k = \frac{p}{2}$. These two solutions in region IV can be joined that gives us for near-overcritical particles $\frac{q-2}{2} < n_1 \leq k + \frac{q-p}{2} - 1$ (independent of whether s or k is greater). This condition does not have further limitations if $q > 2$.

If $q \leq 2$, the lower bound for n_1 is negative. The upper bound for n_1 (that is the same as in the case $q > 2$) is positive, with the reservation about redefinition of k regions in this case [see (123)].

For a constant ω, we observe that near-subcritical, near-critical, and near-ultracritical particles experience the same accelerations as nonconstant ω in region IV [one can refer to second solutions in (118) and (119)]. The only difference occurs with near-overcritical particles, where in the case of constant ω, $n_1 > \frac{q-2}{2}$ [see third solution (121)].

We summarize all the aforementioned results in Tables IV–VI.

Also note that results obtained in this section include also the case of usual particles [with arbitrary $\delta = O(1)$].

VIII. CASE $v_c \sim v_{e,t}$

As we already mentioned, in this case in the main approximation acceleration reads

$$a_f^{(r)} \approx (a_f^{(r)})_{m_1} v_c^{m_1}, \qquad a_f^{(\varphi)} \approx (a_f^{(\varphi)})_{m_2} v_c^{m_2}. \tag{124}$$

Our task is to find m_1 and m_2 depending on the particle type. We begin with the radial component of acceleration $a_f^{(r)}$. At first, let us discuss the prefactor in the expression for (99). As stated in Sec. IV B, for all types of particles, $P \sim \sqrt{\mathcal{X}^2 - N^2} \sim \delta$ (note that this applies to the $v_c \sim v_{e,t}$ case, but not to the limit $v_c \to v_t$). Thus, the prefactor $\frac{P}{\sqrt{\mathcal{X}^2 - N^2}} = O(1)$ and can be ignored in this analysis.

In the expression for $a_f^{(r)}$ (99), there are two terms dependent on v_c: the first term is $\sim v_c^{s+\frac{q-p}{2}-1}$, and the second term is $\sim v_c^{k+\frac{q-p}{2}-1}$. Therefore, the analysis of the radial component of acceleration is the same as for $v_c \ll v_{e,t}$, and we can use the results obtained in the previous section.

The only differences are related to the angular component of acceleration. In this case, $\mathcal{X} \sim \sqrt{\mathcal{X}^2 - N^2} \sim P \sim \delta$, and we can use Eq. (110),

$$a_f^{(\varphi)} + \sigma \mathcal{L} a_f^{(r)} \sim \delta \frac{\sqrt{A}}{N} \partial_r \mathcal{L}. \tag{125}$$

Relying on this condition, let us determine when $a_f^{(\varphi)}$ is finite. The right-hand side (rhs) of this condition is of the order of $\delta v_c^{b+\frac{q-p}{2}-1}$. It is evident that if condition (112) is satisfied, then the rhs is also finite. Therefore, we can conclude that, for the range $v_c \sim v_{e,t}$, if $a_f^{(r)}$ is finite and condition (112) is met, then $a_f^{(\varphi)}$ is also finite (see Proposition 1).

Considering that the expression for n_1 is the same for both cases $v_c \sim v_{e,t}$ and $v_c \ll v_{e,t}$, we can write

Proposition 2. If acceleration is finite for $v_c \ll v_{e,t}$, it is finite for $v_c \sim v_{e,t}$.

This proposition is very useful because we can use the results already obtained for $v_c \ll v_{e,t}$.

IX. CASE $v_c \to v_t$

If a particle is such that a turning point exists, the quantity P tends to zero when this point is approached. However, for different types of particles, it tends to zero at different rates. Therefore, we will consider each of these cases separately.

A. Near-subcritical particles

For such particles, we use (46)–(48) and see that in the leading order

$$P \approx \sqrt{\mathcal{X}^2 - N^2} \approx \mathcal{X} \approx B_r v_c^r. \tag{126}$$

Thus, prefactor $\frac{P}{\sqrt{\mathcal{X}^2 - N^2}}$ in expression (99) is $= O(1)$. This means that for such particles acceleration for the $v_c \to v_t$ case is given by the same expressions as for the case of $v_c \sim v_t$.

B. Near-critical, near-ultracritical, and near-overcritical particles

All these cases are similar in the sense that in the leading order [see (50), (53), (55), (60), (61), (65), and (66)], we can write

$$\mathcal{X} \sim \sqrt{\mathcal{X}^2 - N^2} \sim v_c^{p/2}, \qquad P \sim v_c^{p/4 + r/2}. \tag{127}$$

Now, let us analyze the radial component of acceleration. It can be observed that the prefactor $\frac{P}{\sqrt{\mathcal{X}^2 - N^2}}$ in expression (99) is approximately $\sim v_c^{(\frac{r}{2} - \frac{p}{4})}$. Note that, by definition, $r > p/2$, so this prefactor does not diverge. The structure of the terms in brackets is the same as in the previously analyzed cases of $v_c \ll v_{e,t}$ and $v_c \sim v_{e,t}$. Therefore, we can conclude that if the acceleration is finite in these two cases, it will also be finite in the case of $v_c \to v_t$.

Next, we analyze the angular component (this analysis is independent of the type of particle). To this end, we will use expression (98),

$$a_f^{(\varphi)} = \frac{1}{\sqrt{\mathcal{X}^2 - N^2}} \frac{1}{\sqrt{g_{\varphi\varphi}}} \frac{\sqrt{A}}{N} \times [(\mathcal{X}^2 - N^2)\partial_r \mathcal{L} - \mathcal{L}\mathcal{X}(\partial_r \mathcal{X} + \mathcal{L}\partial_r \omega)]. \tag{128}$$

Using that in the limit $v_c \to v_t$ quantities $\sqrt{\mathcal{X}^2 - N^2} \sim \mathcal{X}$ are of order of δ [this is true because v_t is defined by condition $\mathcal{X}^2 = (1 + \frac{\mathcal{L}^2}{g_{\varphi\varphi}})N^2$, while $\mathcal{X}$ and N remain $\sim\delta$], and using expansions (2) and (6), we have three terms in (128). The first term is $\sim \delta v_c^{b + \frac{q-p}{2} - 1}$, the second term is $\sim v_c^{s + \frac{q-p}{2} - 1}$, and the third term is $\sim v_c^{k + \frac{q-p}{2} - 1}$. The second and third terms are finite if $\min(s, k) + \frac{q-p}{2} - 1$ is non-negative. Note that this expression is the same as n_1 in the case $v_c \ll v_{e,t}$ given by (103). So, if the radial component of the acceleration is finite in the case $v_c \ll v_{e,t}$, then the second and third terms are also finite. The first term is finite if (112) is satisfied, which is a necessary condition for the finiteness of the angular component of the acceleration in the case $v_c \ll v_{e,t}$. Therefore, we can state the proposition:

Proposition 3. If acceleration is finite for $v_c \ll v_{e,t}$, it is finite for $v_c \to v_t$.

X. RESULTS FOR DIFFERENT TYPES OF HORIZONS

In this section, we formulate briefly the results obtained for different particles near different types of horizons. To this end, we analyze accelerations for $v_c \ll v_{e,t}$, because, as we showed above (see Propositions 1–3), if acceleration is finite in this range, it will be finite in all other ranges.

A. Nonextremal horizons

Nonextremal horizons are such that $q = p = 1$ (for explanation of why we use such a definition and other properties of nonextremal horizons, see [19]). All results corresponding to this case may be found in Table VI. One can see that in this case (unlike that of fine-tuned particles) it is possible to have particles that experience a finite force near the horizon: such particles are near-overcritical ones in range IV (or for static space-times). This correlates with the result obtained in [20] where such a situation was demonstrated explicitly for the Schwarzchild space-time. According to our classification, such particles are near-overcritical ones that, as we already discussed above, do not have a fine-tuned analog.

B. Extremal horizons

Extremal horizons are such that $q = 2$ and $p \geq q$. All results corresponding to this case may be found in Table V. From this table, we see that finite forces act only on near-critical, near-ultracritical, and near-overcritical particles in region IV or in static metric.

C. Ultraextremal horizon

Ultraextremal horizons are such that $q > 2$. All results corresponding to this case may be found in Table IV. One can see that in this case a force is finite for all types of particles if k is in regions II–IV or if space-time is static.

TABLE VII. Classification of cases when forces are finite for different types of horizons and trajectories.

	Type of horizon	Type of trajectory	Region of k
1	Nonextremal	NOC	IV or static
2	Extremal	NC, NUC, and NOC	IV or static
3	Ultraextremal	NSC, NC, NUC, and NOC	II, III, IV, or static

We summarize all the aforementioned results in Table VII.

XI. PARTICLES WITH FINITE PROPER TIME: IS KINEMATIC CENSORSHIP PRESERVED?

In this section, we are going to probe the so-called principle of kinematic censorship (KC). It excludes unphysical situations in which the energy released in a collisional event in a regular system is infinite in a literal sense [8]. Something should prevent such an event. For example, the proper time to the horizon can diverge [5], so collision with infinite $E_{\text{c.m.}}$ never occurs, although it can be made as large as one likes. What happens if the corresponding proper time is finite and $E_{\text{c.m.}}$ is infinite? In Sec. IX of our previous work [13], we showed that for fine-tuned particles this is possible in two cases only: either (1) a force acting on such particles is infinite or (2) the horizon fails to be regular. As the system becomes singular in a geometrical or dynamic sense, the KC does not apply to it and no contradiction arises. Now, we are going to prove that this principle is preserved for near-fine-tuned particles. However, this requires consideration of one more factor: (3) an interval within which motion leading to diverging $E_{\text{c.m.}}$ is allowed shrinks to the point. This makes the scenario degenerate and unphysical.

Let us prove the corresponding theorem.

Theorem 1. If for a near-subcritical, near-critical, or near-ultracritical particle a proper time needed to reach the horizon is finite for all possible relations between v_c and $v_{t,e}$, then at least one of three aforementioned conditions is fulfilled.

Proof. We start with an analysis of near-subcritical, near-critical, and near-ultracritical particles in the $v_c \gg v_{e,t}$ case. As was discussed in Sec. IV A, near-fine-tuned particles in this range behave in the same way as corresponding fine-tuned ones. The corresponding part of the theorem (requiring the horizon's regularity) was proven for such particles in [13].

For cases $v_c \ll v_{e,t}$ and $v_c \sim v_{e,t}$, proof is different. We will conduct it by considering different types of particles separately.

Let us start with near-subcritical particles. As one can see from Appendix B, a proper time for them is finite if $q < p+2$ (in the range $v_c \ll v_{e,t}$) and $s < \frac{p-q}{2}+1$ (in the range $v_c \sim v_{e,t}$). As we require the proper time to be finite for all relations between v_c and $v_{e,t}$, this means that both these conditions have to hold.

Now let us analyze an acceleration for these particles. We will focus only on the acceleration for $v_c \ll v_{e,t}$ because, as we showed above, if acceleration is finite in this range it will be finite in other ranges. Finiteness of the acceleration is defined by the sign of n_1 which is given by (103),

$$n_1 = \min(s,k) + \frac{q-p}{2} - 1. \tag{129}$$

First of all, we note that from defining property of $\min(s,k)$ it follows that $\min(s,k) \le s$, and using condition $s < \frac{p-q}{2}+1$ we have $\min(s,k) < \frac{p-q}{2}+1$. Property $q < p+2$ tells us that $\frac{p-q}{2}-1$ is a positive value, so $n_1 = \min(s,k) - (\frac{p-q}{2}+1)$ is negative, which means that a force diverges.

Now let us consider near-critical and near-ultracritical particles. For them, $s = p/2$. As one can note from Appendix B, a proper time for these particles is finite if $q < 2$. As we mentioned, $\min(s,k) \le s$ that gives $\min(s,k) \le p/2$. Thus, we see that n_1 is given by

$$n_1 = \left(\min(s,k) - \frac{p}{2}\right) + \left(\frac{q}{2} - 1\right). \tag{130}$$

The first bracket is ≤ 0 because $\min(s,k) \le p/2$, the second one is < 0 because $q < 2$. This makes n_1 negative and makes a force divergent.

Now let us consider near-overcritical particles. First of all, we note that according to the result we obtained in Sec. III D, such particles always have a turning point and a particle may move only in the $[0, v_t]$ interval. Thus, the case $v_c \gg v_{e,t}$ is now impossible, so conclusions from the theorem in Sec. IX of [13] simply cannot be applied to them.

Then we focus on the $v_c \ll v_{e,t}$ and $v_c \sim v_{e,t}$ cases. The conditions of finiteness of the proper time for such particles are the same as for near-critical and near-ultracritical particles, namely, $q < p+2$ and $q < 2$ (these conditions, obviously, can be merged to give $q < 2$). However, such particles have $s > p/2$ that causes the main difference between them and the case of near-critical and near-ultracritical particles. Indeed, the expression for n_1 is given by (129). This quantity may be made non-negative if one chooses $s \ge \frac{p-q}{2}+1$ and $k \ge \frac{p-q}{2}+1$ (or if the metric is static). As $q < 2$, the quantity $\frac{p-q}{2}+1 > \frac{p}{2}$. Because of this, we obtain that $k > \frac{p}{2}$ (that corresponds to region IV) or the metric is static.

Thus, for NOC particles, the proper time is finite, and acceleration is also finite. Meanwhile, an infinite $E_{\text{c.m.}}$ is possible for scenarios in which such a particle collides with a subcritical or near-subcritical particle—see Table II (line 3) and Table III (line 2). At first glance, the KC principle is violated. However, this is not so. Let us remind

TABLE VIII. Conditions which have to hold for an acceleration to make different particles possible to achieve infinity.

	$s < k$	$s = k$	$s > k$
NSC	$(a_f^{(r)})_{n_1} < 0$	$(a_f^{(r)})_{n_1} < -\sqrt{\frac{A_q}{\kappa_p}} L_H \omega_k k$	Impossible
NC	$(a_f^{(r)})_{n_1} < -\sqrt{\frac{A_q}{\kappa_p}} s A_{p/2}$	$(a_f^{(r)})_{n_1} < -\sqrt{\frac{A_q}{\kappa_p}} (s A_{p/2} + L_H \omega_k k)$	Impossible
NUC	$(a_f^{(r)})_{n_1} < 0$	$(a_f^{(r)})_{n_1} < -\sqrt{\frac{A_q}{\kappa_p}} L_H \omega_k k$	Impossible
NOC	Impossible	Impossible	Impossible

a reader that, say, for nonextremal black holes, the BSW effect is possible with small but nonzero δ but in such a way that a corresponding near-overcritical particle moves in a very narrow strip. This was shown in [9] for the Kerr metric. For a more general rotating axially symmetric black hole, the corresponding allowed interval of N is proportional to δ according to Eq. (18) of Ref. [17]. In the limit $\delta \to 0$ this interval shrinks to a point and the process of collision (as well as motion of such a particle) loses its sense. The KC is not violated since it cannot simply be applied to a system.

The similar situation happens for NOC particles in a more general case. Indeed, if $\delta \to 0$, the corresponding $v_t \to 0$ according to (33) and the allowed coordinate interval $0 \leq v_c \leq v_t$ shrinks to the point. Moreover, this is valid for the proper distance

$$l = \int_{v_c}^{v_t} \frac{dr}{\sqrt{A}}. \tag{131}$$

Indeed, as in the case under discussion $\sqrt{A} \sim v^{q/2}$ with $q < 2$, the integral converges and, as the limits of integration shrink, so does τ.

This completes the proof of the theorem. ∎

XII. VARYING RANGES OF MOTION OF PARTICLES

Our previous analysis primarily focused on investigating the properties of the collisional process for a given particle. Although we classified all possible cases where the BSW phenomenon is possible with forces acting on particles, there are still several problems with this approach. As is shown in Sec. III, not all particles that potentially participate in the BSW effect can reach freely the collision point from infinity. This means that such particles can only be created in a narrow region disconnected from infinity, either due to a quantum creation process in this region or through multiple scattering [9]. We consider these scenarios to be exotic and put them aside. Instead, we are interested in the question of whether it is possible to change the ranges of motion of a particle by the action of force in such a way that the particle could reach infinity. To answer this question, we need to analyze the force acting on a particle for different types of particles.

A. Near-subcritical particle

We begin our analysis with near-subcritical particles (as we will show later, the situation is similar for other particles). For near-subcritical particles, using Eq. (18), we find that the particle can reach infinity only if $\delta > 0$ and $X_s > 0$. Therefore, our task is to determine the conditions on a force under which we can have $X_s > 0$. To do this, we revisit the expression for the radial component of the acceleration (99),

$$a_f^{(r)} \approx \sigma \frac{P}{\sqrt{\mathcal{X}^2 - N^2}} \sqrt{\frac{A_q}{\kappa_p}} \left(X_s s v_c^{s + \frac{q-p}{2} - 1} + L_H \omega_k k v_c^{k + \frac{q-p}{2} - 1} \right). \tag{132}$$

As we are considering near-subcritical particles, for which $s < p/2$ and $\mathcal{X} \ll N$, we have that $P \approx \sqrt{\mathcal{X}^2 - N^2} \approx |\mathcal{X}|$ (for a proof see Sec. IV B), thus $\frac{P}{\sqrt{\mathcal{X}^2 - N^2}} \approx 1$. This allows us to write

$$a_f^{(r)} \approx \sigma \sqrt{\frac{A_q}{\kappa_p}} \left(X_s s v_c^{s + \frac{q-p}{2} - 1} + L_H \omega_k k v_c^{k + \frac{q-p}{2} - 1} \right). \tag{133}$$

As we have previously discussed, there are two terms here and, depending on whether $s > k$, $s = k$, or $s < k$, we obtain different behavior. Our task is to consider these different cases and determine how the coefficients in the expansions for the acceleration (100) and (101) in different ranges of the coordinate v_c are related to X_s. This will allow us to determine whether we can control the ranges of motion of the particle or not. It is important to note that, in this investigation, we will only analyze the cases $v_c \ll v_{e,t}$ and $v_c \sim v_{e,t}$. The case $v_c \gg v_{e,t}$ will not be investigated here because, as we have already discussed, at such distances, the particle can be considered as fine-tuned (with negligible influence of the parameter δ) and we return to the pure BSW effect.

(i) $s < k$:

In this case, the first term in (133) is dominant and we have $a_f^{(r)} \approx \sigma\sqrt{\frac{A_q}{\kappa_p}} X_s s v_c^{s+\frac{q-p}{2}-1}$. Substituting here (100) and (101) one obtains

$$(a_f^{(r)})_{n_1} = (a_f^{(r)})_{m_1} = \sigma\sqrt{\frac{A_q}{\kappa_p}} s X_s,$$
$$n_1 = m_1 = s + \frac{q-p}{2} - 1. \tag{134}$$

This gives us in this case

$$X_s = \sigma\sqrt{\frac{\kappa_p}{A_q}}(a_f^{(r)})_{m_1} = \sigma\sqrt{\frac{\kappa_p}{A_q}}(a_f^{(r)})_{n_1}. \tag{135}$$

Therefore, we observe that the sign of X_s and thus the possibility of reaching infinity is determined by the sign of the acceleration. For ingoing particles ($\sigma = -1$), it is only possible if the force is negative (attractive). This case also corresponds to static space-times.

(ii) $s = k$:

In this case, both terms in (133) are comparable and we have $a_f^{(r)} \approx \sigma\sqrt{\frac{A_q}{\kappa_p}}(X_s s v_c^{s+\frac{q-p}{2}-1} + L_H \omega_k k v_c^{k+\frac{q-p}{2}-1})$.

Substituting here (100) and (101) one obtains

$$(a_f^{(r)})_{n_1} = (a_f^{(r)})_{m_1} = \sigma\sqrt{\frac{A_q}{\kappa_p}}(s X_s + L_H \omega_k k),$$
$$n_1 = m_1 = k + \frac{q-p}{2} - 1. \tag{136}$$

Reversing it, one gets

$$X_s = \frac{1}{s}\left(\sqrt{\frac{\kappa_p}{A_q}}\frac{(a_f^{(r)})_{n_1}}{\sigma} - L_H \omega_k k\right). \tag{137}$$

One can observe that the sign of this expression can be controlled by the choice of the proper values of acceleration. For example, if the particle is ingoing and we want to have a positive X_s, one must have an acceleration that satisfies the condition $(a_f^{(r)})_{n_1} < -\sqrt{\frac{A_q}{\kappa_p}} L_H \omega_k k$.

(iii) $s > k$:

In this case, the second term in Eq. (133) is dominant, and we have $a_f^{(r)} \approx \sigma\sqrt{\frac{A_q}{\kappa_p}} L_H \omega_k k v_c^{k+\frac{q-p}{2}-1}$. As one can see, in this case, the acceleration is independent of X_s, so X_s cannot be controlled by leading terms in the external force.

Therefore, we observe that the range of motion of a near-subcritical particle can be controlled by force if $s \le k$. For these cases, n_1 is defined by the first and the second solutions in Eq. (108).

B. Near-critical, near-ultracritical, and near-overcritical particles

As far as the behavior of acceleration is concerned, all these cases are similar. Let us start with the case $v_c \ll v_{e,t}$. Then, as is shown in Sec. IV C, for any particle, $P \approx \sqrt{\mathcal{X}^2 - N^2} \approx \delta$ that gives us $\frac{P}{\sqrt{\mathcal{X}^2 - N^2}} \approx 1$. Thus, the expression for acceleration (99) is the same as for the case of a near-subcritical particle (133), so the analysis is the same in those cases. In the case $v_c \sim v_{e,t}$, the quantity $\frac{P}{\sqrt{\mathcal{X}^2 - N^2}}$ differs from 1. Let us consider the Taylor expansion for this quantity,

$$\frac{P}{\sqrt{\mathcal{X}^2 - N^2}} \approx 1 + d_{p/2} v_c^{p/2}, \tag{138}$$

where $d_{p/2}$ is some coefficient. Substituting this to (99) one gets

$$a_f^{(r)} \approx \sigma(1 + d_{p/2} v_c^{p/2})\sqrt{\frac{A_q}{\kappa_p}}\left(X_s s v_c^{s+\frac{q-p}{2}-1} + L_H \omega_k k v_c^{k+\frac{q-p}{2}-1}\right). \tag{139}$$

Comparing this with the expansion (101), we see that the dominant term is obtained by taking $\frac{P}{\sqrt{\mathcal{X}^2 - N^2}} = 1$ that gives us the same expression as in the case of a near-subcritical particle. So, we see that, for all types of particles, the quantity X_s is given by (135) if $s < k$ and by (137) if $s = k$.

We observe that, for all types of particles, X_s is only controllable in the case of $s \le k$ (for any range of point of collision). We have already extensively worked on finding the conditions at which we obtain the first solution (that corresponds to $s < k$) or the second solution (which corresponds to $s \ge k$) in (108). The corresponding results are given in Tables IV–VI. These tables are useful in the sense that, for a given type of particle and a given value of k, we can easily deduce which solution for s we can use among the ones given in (108). If it is the first solution, then X_s is given by (135). However, if it is the second solution in (108) and additionally $s = k$, then X_s is given by (137). If it is the second solution in (108) and $s > k$, then X_s cannot be controlled.

Now, let us discuss how we can manipulate by the particle parameters to make it possible to reach infinity.

If the particle is near ultracritical, it may potentially reach infinity if $\delta > 0$ and $X_s > 0$ [see (31)]. These conditions are the same as for the case of near-subcritical particles that has already been investigated.

If the particle is near overcritical, it is impossible to make any such particle to reach infinity [see (34)].

However, if the particle is near critical, the corresponding particle may reach infinity only if $\delta > 0$ and $A_{p/2} < X_{p/2}$

[see third condition in (23)]. If $s < k$, using (135), this gives us $(a_f^{(r)})_{n_1} < \sigma\sqrt{\frac{A_q}{\kappa_p}}sA_{p/2}$. If $s = k$, using (137), this gives $(a_f^{(r)})_{n_1} < \sigma\sqrt{\frac{A_q}{\kappa_p}}(sA_{p/2} + L_H\omega_k k)$. We can summarize all these cases in Table VIII.

XIII. SUMMARY AND CONCLUSIONS

We have shown that the BSW effect is possible for a quite rich family of configurations that include the combination of a type of horizon, that of a near-fine-tuned particle, and a force. We have presented the classification of different types of near-fine-tuned particles, generalizing the one presented for fine-tuned ones (near subcritical, near critical, and near ultracritical) and adding a new type that is possible only for the case of near-fine-tuned ones: near-overcritical particles (see Table I). Particles of each type differ in the behavior of the components of the four-velocity near the horizon that causes different kinematical properties. Specifically, we have analyzed the allowed ranges of motion for each type of particle (and we have shown that the corresponding ranges of allowed motion are different) and the near-horizon limits of the components of the four-velocity, which also turn out to be different. These results have been used to describe the behavior of energy in the center-of-mass frame of two colliding particles. It is important to note that the investigation of near-fine-tuned particles has opened up a wider variety of different scenarios for particle collision. In this process, one near-fine-tuned particle may participate with a fine-tuned one (or a usual one), or there may be two near-fine-tuned particles. For all of these cases, we have formulated how the energy of collision would behave as the point of collision approaches the horizon and have formulated the conditions that must be met to make the energy divergent (which is the main property of the BSW effect). The corresponding results have been briefly summarized in Tables II and III. Table II essentially generalizes the situation considered earlier for nonextremal black holes [9,17].

Furthermore, we have focused on the dynamic properties of particles participating in the BSW phenomenon. We have analyzed the behavior of the forces acting on such particles and investigated under which conditions these forces are finite. The corresponding results are briefly summarized in Tables IV–VI. In Sec. X and Table VII we indicated which types of particles experience finite force for a given type of horizon. Then we focused on the possibility of the preservation kinematic censorship principle in the case of near-fine-tuned particles, and we showed that this principle holds for them. An additional investigation concerns the possibility of changing the

TABLE IX. Table showing for which particles and which ranges of their motion the BSW phenomenon is possible if forces acting on both particles are finite. This table describes all new possible cases when both particles are near-fine-tuned or one is fine-tuned while the second particle is near-fine-tuned. In near-horizon collisions, near-fine-tuned particles with $v_c \ll v_{t,e}$ behave similar to usual ones. If $v_c \gg v_{t,e}$, near-fined particles are similar to fine-tuned ones. The last column displays the equation number describing a corresponding case.

Type of horizon	First particle's type and range			Second particle's type and range			
Nonextremal	NOC	$v_c \ll v_t^{(1)}$ U	$\delta_1 \gg v_c^{p/2}$	NOC	$v_c \sim v_t^{(2)}$	$\delta_2 \sim v_c^{p/2}$	(89)
Extremal	NC, NUC, NOC	$v_c \ll v_{e,t}^{(1)}$ U	$\delta_1 \gg v_c^{p/2}$	NC, NUC, NOC	$v_c \sim v_{e,t}^{(2)}$	$\delta_2 \sim v_c^{p/2}$	(89)
Ultraextremal	NSC	$v_c \ll v_{e,t}^{(1)}$ U	$\delta_1 \gg v_c^s$	NSC	$v_c \sim v_{e,t}^{(2)}$	$\delta_2 \sim v_c^s$	(83)
	NSC	$v_c \sim v_{e,t}^{(1)}$	$\delta_1 \sim v_c^s$	NSC	$v_c \sim v_{e,t}^{(2)}$	$\delta_2 \sim v_c^s$	(91)
	NSC	$v_c \gg v_{e,t}^{(1)}$ SC	$\delta_1 \ll v_c^s$	NSC	$v_c \sim v_{e,t}^{(2)}$	$\delta_2 \sim v_c^s$	(83)
	NSC	$v_c \ll v_{e,t}^{(1)}$ U	$\delta_1 \gg v_c^s$	NC, NUC, NOC	$v_c \sim v_{e,t}^{(2)}$	$\delta_2 \sim v_c^{p/2}$	(89)
	NSC	$v_c \sim v_{e,t}^{(1)}$	$\delta_1 \sim v_c^s$	NC, NUC, NOC	$v_c \sim v_{e,t}^{(2)}$	$\delta_2 \sim v_c^{p/2}$	(92)
	NSC	$v_c \sim v_{e,t}^{(1)}$	$\delta_1 \sim v_c^s$	NC, NUC	$v_c \gg v_{e,t}^{(2)}$ C, UC	$\delta_2 \ll v_c^{p/2}$	(85)
	NSC	$v_c \gg v_{e,t}^{(1)}$ SC	$\delta_1 \ll v_c^s$	NC, NUC, NOC	$v_c \sim v_{e,t}^{(2)}$	$\delta_2 \sim v_c^{p/2}$	(89)
	NC, NUC, NOC	$v_c \ll v_{e,t}^{(1)}$ U	$\delta_1 \gg v_c^{p/2}$	NC, NUC, NOC	$v_c \sim v_{e,t}^{(2)}$	$\delta_2 \sim v_c^{p/2}$	(89)

TABLE X. Table showing for which particles and which ranges of their motion BSW phenomenon is possible if forces acting on both particles are finite. This table describes all the cases with the BSW effect when usual, fine-tuned, or near-fined particles participate. The last column indicates the line number in Table II from [13] that describes a corresponding case.

Type of horizon	First particle's type and range			Second particle's type and range			
Extremal	NC, NUC, NOC	$v_c \ll v_{e,t}^{(1)}$	$\delta_1 \gg v_c^{p/2}$	NC, NUC	$v_c \gg v_{e,t}^{(2)}$	$\delta_2 \ll v_c^{p/2}$	4
		U					
		U			C, UC		
Ultraextremal	NSC, NC, NUC, NOC	$v_c \ll v_{e,t}^{(1)}$	$\delta_1 \gg v_c^{p/2}$	NSC	$v_c \gg v_{e,t}^{(2)}$	$\delta_2 \ll v_c^{s}$	2
		U					
		U			SC		
	NSC, NC, NUC, NOC	$v_c \gg v_{e,t}^{(1)}$	$\delta_1 \ll v_c^{p/2}$	NSC	$v_c \gg v_{e,t}^{(2)}$	$\delta_2 \ll v_c^{s}$	3
		SC					
		SC			SC		
	NSC	$v_c \ll v_{e,t}^{(1)}$	$\delta_1 \gg v_c^{s}$	NC, NUC	$v_c \gg v_{e,t}^{(2)}$	$\delta_2 \ll v_c^{p/2}$	4
		U					
		U			C, UC		
	NSC	$v_c \gg v_{e,t}^{(1)}$	$\delta_1 \ll v_c^{s}$	NC, NUC	$v_c \gg v_{e,t}^{(2)}$	$\delta_2 \ll v_c^{p/2}$	4
		SC					
		SC			C, UC		

ranges of motion of particles by the action of an external force. This is an important topic in the analysis of the possibility of having the BSW effect and allowing particles falling from infinity to achieve the horizon and participate in this phenomenon. The corresponding results have been briefly summarized in Table VIII.

As the summary of our work, we present Tables IX and X that show all possible cases when the BSW phenomenon is possible for near-fine-tuned and fine-tuned particles experiencing an action of a finite force. These tables, being the union of Table VII (for near-fine-tuned ones) and Table VIII from [13] (for fine-tuned ones) with Tables II and III and Table II from [13] give the final answer to the question related to the possibility of having BSW phenomenon with finite forces for near-fine-tuned and fine-tuned particles. Note an important difference between these tables. In Table X both particles have either $v_c \ll v_{e,t}$ or $v_c \gg v_{e,t}$. This effectively means that such particles behave as usual (in $v_c \ll v_{e,t}$ range) or as fine-tuned ones (in $v_c \gg v_{e,t}$ range) (see Secs. IV A and IV C), so this table effectively describes the possibility of standard BSW phenomenon. However, one can see that, if one of particles is near-fine-tuned and $v_c \sim v_{e,t}$, we obtain new scenarios described in Table IX. The most obvious case is that appearance of near-fine-tuned particles made the BSW phenomenon possible, forces for the nonextremal horizon being finite (that is forbidden for fine-tuned ones). In other words, in Table IX one of particles has properties specific for near-fine-tuned particles that happens for $v_c \sim v_{e,t}$. In Table X any particle behaves either as usual or as a fine-tuned one.

One reservation is in order. In the case of the pure BSW effect NOC particles do not exist. However, we can consider them formally if δ is large enough. For $\delta \gg v_c^{p/2}$ such a particle is indistinguishable from a usual one. This justifies why we mention NOC for particle 1 in the first line of Tables IX and X.

APPENDIX A: COMPUTATION OF ACCELERATION IN FZAMO FRAME

In this investigation, we are dealing with near-fine-tuned particles that have a nonzero (but quite small) P on a horizon. As a result, the radial velocity for such particles is nonzero on a horizon, and to compute the components of acceleration, we need to choose a suitable frame. We will use a free-falling zero angular momentum observer frame, which is attached to a falling particle. This frame is such that the three-velocity of a particle is zero in this frame. To obtain this frame, we start with a stationary tetrad [(7) and (8)]. The three-velocity of a particle in this frame is given by

$$V^{(i)} = -\frac{e_\mu^{(i)} u^\mu}{e_\mu^{(0)} u^\mu} = \left(\frac{\sigma P}{\mathcal{X}}, 0, \frac{\mathcal{L} N}{\mathcal{X}\sqrt{g_{\varphi\varphi}}}\right) = |v|(\cos\psi, 0, \sin\psi), \tag{A1}$$

where

$$|v| = \sqrt{1 - \frac{N^2}{\mathcal{X}^2}}, \qquad \tan\psi = \frac{N\mathcal{L}}{P\sqrt{g_{\varphi\varphi}}}. \tag{A2}$$

The FZAMO frame may be obtained if we perform several transformations to a stationary tetrad:

(i) Rotate frame in such a way that new radial tetrad vector is codirected with direction of three-velocity,

$$\tilde{e}^{(0)}_\mu = e^{(0)}_\mu, \qquad \tilde{e}^{(2)}_\mu = e^{(2)}_\mu, \tag{A3}$$

$$\tilde{e}^{(1)}_\mu = e^{(1)}_\mu \cos\psi + e^{(3)}_\mu \sin\psi,$$
$$\tilde{e}^{(3)}_\mu = e^{(3)}_\mu \cos\psi - e^{(1)}_\mu \sin\psi. \tag{A4}$$

(ii) Perform a boost in a direction of particle's motion,

$$e^{(2)\prime}_\mu = \tilde{e}^{(2)}_\mu, \qquad e^{(3)\prime}_\mu = \tilde{e}^{(3)}_\mu, \tag{A5}$$

$$e^{(0)\prime}_\mu = \gamma(\tilde{e}^{(0)}_\mu - |v|\tilde{e}^{(1)}_\mu), \quad e^{(1)\prime}_\mu = \gamma(\tilde{e}^{(1)}_\mu - |v|\tilde{e}^{(0)}_\mu), \tag{A6}$$

where $\gamma = \frac{1}{\sqrt{1-v^2}} = \frac{\mathcal{X}}{N}$.

After these actions, we obtain tetrad vectors in a form

$$e^{(0)\prime}_\mu = \left(\mathcal{X} + \omega\mathcal{L}, -\frac{P}{\sqrt{A}N}, 0, -\mathcal{L}\right), \tag{A7}$$

$$e^{(1)\prime}_\mu = \frac{1}{\sqrt{\mathcal{X}^2 - N^2}}\left(-\omega\mathcal{L}\mathcal{X} - \mathcal{X}^2 + N^2, \frac{P\mathcal{X}}{\sqrt{A}N}, 0, \mathcal{L}\mathcal{X}\right), \tag{A8}$$

$$e^{(2)\prime}_\mu = \sqrt{g_{\theta\theta}}(0, 0, 1, 0), \tag{A9}$$

$$e^{(3)\prime}_\mu = \frac{1}{\sqrt{\mathcal{X}^2 - N^2}}\left(-\omega P\sqrt{g_{\varphi\varphi}}, -\frac{\mathcal{L}N}{\sqrt{g_{\varphi\varphi}A}}, 0, \sqrt{g_{\varphi\varphi}}P\right). \tag{A10}$$

One can easily check than indeed in this frame corresponding three-velocity,

$$V^{(i)\prime} = -\frac{e^{(i)\prime}_\mu u^\mu}{e^{(0)\prime}_\mu u^\mu}, \tag{A11}$$

is zero.

Acceleration components in this frame may be computed from the definition $a^{(a)}_F = e^{(a)\prime}_\mu a^\mu$. Using expressions for acceleration in an OZAMO frame, given in Sec. VI in [13],

$$a^t = \frac{a^{(t)}_o}{N} = \frac{u^r}{N^2}(\partial_r\mathcal{X} + \mathcal{L}\partial_r\omega), \tag{A12}$$

$$a^r = \sqrt{A}a^{(r)}_o = \mathcal{X}\frac{A}{N^2}\left(\partial_r\mathcal{X} + \mathcal{L}\partial_r\omega - \frac{N^2}{\mathcal{X}}\frac{\mathcal{L}\partial_r\mathcal{L}}{g_{\varphi\varphi}}\right), \tag{A13}$$

$$a^\varphi = \frac{a^{(\varphi)}_o}{\sqrt{g_{\varphi\varphi}}} + \omega a^t = u^r\left(\frac{\partial_r\mathcal{L}}{g_{\varphi\varphi}} + \frac{\omega}{N^2}(\partial_r\mathcal{X} + \mathcal{L}\partial_r\omega)\right), \tag{A14}$$

we have

$$a^{(t)}_F = 0, \tag{A15}$$

$$a^{(r)}_F = \frac{u^r}{\sqrt{\mathcal{X}^2 - N^2}}(\partial_r\mathcal{X} + \mathcal{L}\partial_r\omega), \tag{A16}$$

$$a^{(\varphi)}_F = \frac{1}{\sqrt{\mathcal{X}^2 - N^2}}\frac{\sqrt{A}}{N} \times \left[(\mathcal{X}^2 - N^2)\frac{\partial_r\mathcal{L}}{\sqrt{g_{\varphi\varphi}}} - \frac{\mathcal{L}\mathcal{X}}{\sqrt{g_{\varphi\varphi}}}(\partial_r\mathcal{X} + \mathcal{L}\partial_r\omega)\right]. \tag{A17}$$

APPENDIX B: PROPER TIME

In this appendix, we are going to analyze the proper time of near-fine-tuned particles for different scenarios of particle motion. Our main aim is to find out under which conditions the proper time will be finite and how its behavior for near-fine-tuned particles correlates with corresponding fine-tuned ones. To this end, let us consider separately different ranges of particle motion.

1. $v_c \gg v_{e,t}$

In this case, as was shown in Sec. IV A, behavior of all physical quantities is the same as for fine-tuned particles of the same type. Thus, the proper time given by

$$\tau = \int \frac{dr}{u^r} + C \tag{B1}$$

in this range of coordinates can be found in [13]. Here C is a constant of integration, which we will omit in a further analysis because we are interested in a near-horizon behavior of the proper time, which is independent of C.

2. $v_c \ll v_{e,t}$

In this case, we can refer to Sec. IV C and simply take the limit $v_c \longrightarrow 0$ while keeping terms with δ in expressions for the P and $\mathcal{X}$. Thus, we can write

$$P = \sqrt{\mathcal{X}^2 - \left(1 + \frac{L^2}{g_{\varphi\varphi}}\right)N^2} \approx \delta. \tag{B2}$$

The radial velocity in this case is given by

$$u^r = \frac{\sqrt{A}}{N} P \approx \sqrt{\frac{A_q}{\kappa_p} v_c^{\frac{q-p}{2}}} \delta. \tag{B3}$$

The proper time is given by

$$\tau = \int \frac{dr}{u^r} \approx \sqrt{\frac{\kappa_p}{A_q}} \frac{v_c^{\frac{p-q}{2}+1}}{\delta(\frac{p-q}{2}+1)}. \tag{B4}$$

We see that in this range of coordinates $\tau \sim v_c^{\frac{p-q}{2}+1}$ and this result does not depend on a type of corresponding near-fine-tuned particle. Comparing this with Table I in [13], we see, that the proper time in this range is the same as for usual particles. This is not surprising, because, as was discussed in Sec. IV C, in range $v_c \ll v_{e,t}$ all kinematic properties of all near-fine-tuned particles are the same as for usual ones. The proper time is finite if $q < p + 2$.

3. $v_c \sim v_{e,t}$

In this range of a particle's motion expression for proper time cannot be explicitly integrated. However, we are rather interested in an asymptotical behavior of the proper time than in an exact expression. For this we can use that in $v_c \sim v_t$ range holds $P \sim \delta$ (see Sec. IV B). This allows us to write

$$\tau = \int \frac{dr}{u^r} \approx \sqrt{\frac{\kappa_p}{A_q}} \int \frac{dr}{v^{\frac{q-p}{2}} P} \sim \frac{1}{\delta} \int \frac{dr}{v^{\frac{q-p}{2}}} \sim \frac{v_c^{\frac{p-q}{2}+1}}{\delta}. \tag{B5}$$

Note that this expression depends on both v_c and δ. However, we can relate these quantities as far as we consider the $v_c \sim v_{e,t}$ case. For near-subcritical particles, one can use that $v_{e,t} \sim \delta^{1/s}$ according to (16) and (17) or, inverting $\delta \sim v_c^s$, we have

$$\tau \sim v_c^{\frac{p-q}{2}+1-s}. \tag{B6}$$

As for near-subcritical particles $0 < s < p/2$, we see that proper time behaves as $\tau \sim v_c^{-\alpha}$, where $\frac{q-p-2}{2} < \alpha < \frac{q-2}{2}$. Comparing this result with Table I in [13], we see that the proper time for near-subcritical particles in region $v_c \sim v_{e,t}$ behaves in the same way as for corresponding subcritical particles. The proper time for such particles is finite if $s < \frac{p-q}{2} + 1$.

For near-critical, near-ultracritical, and near-overcritical ones, the situation differs. For them holds $\delta \sim v_c^{p/2}$, which allows us to write

$$\tau \sim v_c^{\frac{2-q}{2}}. \tag{B7}$$

From this we see that $\tau \sim v_c^{-\alpha}$ with $\alpha = \frac{q-2}{2}$. Comparing this with Table I in [13], we see, that in range $v_c \sim v_t$ behavior of the proper time for near-critical particles is the same as for critical ones. However, for near-ultracritical particles, behavior is not the same as for ultracritical ones and is the same as for critical ones (this also concerns near-overcritical ones). The proper time is finite if $q < 2$.

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