Central-surface-densities correlation in general MOND theories

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It is shown that the foundational axioms of modified Newtonian dynamics (MOND) alone predict a strong correlation between a bulk measure of the baryonic surface density Σ_B and the corresponding dynamical one Σ_D of an isolated, self-gravitating object, such as a galaxy. The correlation is encapsulated by its high- and low-Σ_B behaviors. For $\Sigma_B \gg \Sigma_M \equiv a_0/2\pi G$ (Σ_M being the critical MOND surface density) one has $\Sigma_D \approx \Sigma_B$. More specifically, their difference—which would be interpreted as the contribution of dark matter—is $\Sigma_p = \Sigma_D - \Sigma_B \sim \Sigma_M \ll \Sigma_B$. In the deep-MOND limit, $\Sigma_B \ll \Sigma_M$, one has $\Sigma_D \sim (\Sigma_M \Sigma_B)^{1/2}$. This is a primary prediction of MOND, shared by all theories that embody its basic tenets. Sharper correlations, even strict algebraic relations, $\Sigma_D(\Sigma_B)$, are predicted in specific MOND theories, for specific classes of mass distribution—e.g., pure disks or spherical systems—and for specific definitions of the surface densities. I proceed to discuss such tighter correlations for the central surface densities of axisymmetric galactic systems, Σ_B^0 and Σ_D^0. Past work has demonstrated such relations for pure disks in the aquadratic Lagrangian and quasilinear MOND theories. Here I consider them in broader classes of MOND theories. For most observed systems, Σ_D^0 cannot be determined directly at present, but, in many cases, a good proxy for it is the acceleration integral $\mathcal{G} \equiv \int_0^\infty g_r d\ln r$, where g_r is the radial acceleration along the reflection-symmetry plane of a system, such as a disk galaxy. $\mathcal{G}$ can be determined directly from the rotation curve. I discuss the extent to which $\mathcal{G}$ is a good proxy for Σ_D^0 and how the relation between them depends on system geometry, from pure disks, through disk-plus-bulge ones, to quasispherical systems.

I. INTRODUCTION

The modified Newtonian dynamics (MOND) program[1] starts with the following basic tenets, which a MOND theory is required to obey,[2,3]: (1) It introduces a new constant, a_0, with the dimensions of acceleration. (2) A MOND theory should converge to standard dynamics in the formal limit $a_0 \to 0$, namely, when all system attributes with the dimensions of acceleration become much larger than a_0. (3) In the opposite, deep-MOND limit (hereafter DML)—which is approached, formally, by taking $a_0 \to \infty$, while $G \to 0$, so that $\mathcal{A}_0 \equiv G a_0$ remains fixed—the theory should become spacetime scale invariant. (See Refs. [8,9] for details of this requirement and many of its consequences.)

One also requires that the theory does not involve dimensionless constants that differ much from unity. This implies, for example, that the transition from the DML to standard dynamics occurs not only around a_0, but also within a range of order a_0 around this value.

The specific MOND theories studied to date, which embody the above tenets, were shown to predict a strong correlation between the characteristic surface density of a baryonic body and that of the putative—but fictitious, or "phantom," according to MOND—dark-matter halo. For example, for an axisymmetric galactic system—such as a disk galaxy—they predicts a strong correlation between the central surface density—i.e., the column density along the symmetry axis—of the baryonic body Σ_B^0 and that of the phantom halo Σ_P^0.

The essence of this central-surface-densities correlation (hereafter CSDC) is captured by its two limits, according to whether Σ_B^0 is smaller or larger than the critical MOND surface density

$$\Sigma_M \equiv \frac{a_0}{2\pi G} = 138(a_0/1.2 \times 10^{-8} \text{ cm s}^{-2})M_\odot \text{ pc}^{-2}. \quad (1)$$

For "high-acceleration" systems with $\Sigma_B^0 \gg \Sigma_M$, it is predicted that $\Sigma_P^0 \sim \Sigma_M$. For systems in the DML, with $\Sigma_B^0 \ll \Sigma_M$, it is predicted that

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$$\Sigma_P^0 = \eta(\Sigma_M \Sigma_B^0)^{1/2}, \qquad (2)$$

with $\eta = O(1)$.[4] The exact interpolation between the two limits, and the exact value of η, depends on the specific MOND theory and generally may depend also on the specific mass distribution.[5]

The CSDC is sometimes plotted in terms of the total "dynamical" central surface density, $\Sigma_D^0 = \Sigma_B^0 + \Sigma_P^0$, which is predicted to converge to Σ_B^0 for $\Sigma_B^0 \gg \Sigma_M$ and to Σ_P^0 in the DML.

As shown in Sec. IV, the MOND basic tenets alone—without resort to a specific theory—predict such a correlation between the baryonic and dynamical bulk surface densities, such as the average one, or the surface density at the half-mass radius. This makes such a correlation a primary MOND prediction.

Instead, however, we concentrate here on the correlations between *central* surface densities of axisymmetric systems, as it emerges in specific MOND theories, for the following reasons: (1) The correlation between bulk surface densities, while it is more general, and not theory dependent in following from only the basic tenets, is not as tightly pinpointed as correlations between central surface densities in specific theories. For example, the aquadratic Lagrangian (AQUAL) [10] and the quasi-linear MOND (QUMOND) [11] theories predict for pure thin disks, an exact, algebraic relation (as opposed to a mere correlation) between Σ_B^0 and Σ_P^0 (or Σ_D^0), which gives $\eta = 2$ [12]. (2) The basic tenets, dealing only with the two limits, are capable of informing us only on the asymptotic branches of a CSDC, while specific theories should predict its full extent. (3) Σ_D^0 is much easier to estimate from observations than bulk, dynamical surface densities—using, e.g., rotation-curve data (see Sec. II A 1). (4) Σ_D^0 lends itself more easily to calculation in a given theory because it requires the dynamical density only along the symmetry axis, not everywhere in the field.

Past observational studies relevant to the issue [13–16]—which I shall describe in more detail in Sec. III—confirm various aspects of the MOND prediction as elucidated in Refs. [12,13,17–19].

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[4]It is often stated that, because of the spacetime scale invariance of the DML, G and a_0 cannot appear separately in DML relations, only the product $\mathcal{A}_0 = Ga_0$; this appears not to be the case in Eq. (2) since $\Sigma_M \propto a_0/G$. Note, however, that for the statement to hold it is required that all system attributes in the relation are normalized such that they have scaling dimensions that match their $[t][\ell][m]$ dimensions. This is the case for Σ_B^0, but not for Σ_P^0, which has dimensions of $m\ell^{-2}$, but scaling dimensions -1, not -2. If, instead, we write the relation for $\Sigma_P^{0*} \equiv G\Sigma_P^0$, which has dimensions ℓt^{-2}, indeed only $\mathcal{A}_0$ appears in the relation.

[5]Detailed predictions find η between 2 for disks and ≈ 4 for spheres.

All the relevant predictions to date were derived either based on the naive algebraic relation between the MOND and Newtonian accelerations (from Ref. [1]) or as they follow in the two workhorse, modified-gravity (hereafter MG) MOND theories, AQUAL and QUMOND.

My main aim in the present paper is to consider the correlation as predicted in more general classes of MOND theories. In particular, I will discuss the value of the coefficient η in the different theories.

A secondary aim is to assess the procedure that we need to use when estimating Σ_D^0 from the observed rotation curve of a galaxy.

In Sec. II, I give the definitions of the various central surface densities and describe ways to determine them observationally, in particular, using rotation-curve data. I defer the discussion of some history of the subject to Sec. III. In Sec. IV, I show how a correlation between baryonic and dynamical surface densities follows from the basic tenets of MOND. In Sec. V, I describe the classes of more general modified-gravity MOND theories that are used in Sec. VI to study the CSDC. Section VII is a summary.

II. DEFINITIONS AND PROCEDURES

In modified-gravity theories of nonrelativistic MOND, the acceleration of test particles in a gravitational field of a density $\rho(\mathbf{r})$ is given by a gradient of the MOND potential $\phi(\mathbf{r})$,

$$\ddot{\mathbf{r}} = \mathbf{g} = -\vec{\nabla}\phi. \qquad (3)$$

Unlike the Newtonian potential $\phi_N(\mathbf{r})$, which is the solution of the Poisson equation $\Delta\phi_N = 4\pi G\rho$ and gives rise to the Newtonian acceleration field $\mathbf{g}_N$, ϕ solves some MOND equation that incorporates the basic tenets of MOND.

The predicted MOND acceleration field $\mathbf{g}$ should be identified with the *observed* acceleration field. So, one can define a dynamical density distribution

$$\rho_D \equiv -(4\pi G)^{-1}\vec{\nabla} \cdot \mathbf{g}. \qquad (4)$$

This auxiliary density would be required in Newtonian dynamics to reproduce the MOND gravitational field via the Poisson equation. The difference,

$$\rho_P(\mathbf{r}) = \rho_D(\mathbf{r}) - \rho(\mathbf{r}), \qquad (5)$$

is called the phantom density [20], which in the context of a Newtonian interpretation of the observed (MOND) acceleration field has to be attributed to the putative dark matter.

Some MOND results can be described, for convenience, in terms of this phantom density. Some examples of such a device can be found in Refs. [18,20,21].

Here, we are interested in the "central surface densities" of the baryons, of the phantom mass, and the total, dynamical one, $\Sigma_B^0, \Sigma_P^0, \Sigma_D^0$, respectively, of an isolated galactic system.[6] This concept is relevant for an axisymmetric object—such as an idealized disk galaxy or an axisymmetric pressure-supported galaxy—and is defined as the column density of the respective component along the symmetry axis, taken as the z axis. It is called so since it is the projected central surface density when the system is viewed along the axis. We consider systems that are also reflection symmetric about a plane perpendicular to the z axis, taken as the x-y plane, which is many times a good approximation. Then

$$\Sigma_B^0 = 2\int_0^\infty \rho_B^0(z)dz = -\frac{1}{2\pi G}\int_0^\infty \vec{\nabla}\cdot\mathbf{g}_N(z)dz,$$
$$\Sigma_D^0 = 2\int_0^\infty \rho_D^0(z)dz = -\frac{1}{2\pi G}\int_0^\infty \vec{\nabla}\cdot\mathbf{g}(z)dz, \qquad (6)$$

where superscript "0" on ρ indicates density on the z axis.

A. Determining the central surface densities

Σ_B^0 is directly measured—at least in principle and not always easily—as the deprojected central surface density of the baryons, with contributions from stars and gas, etc. Σ_D^0 includes the contribution of phantom matter (by MOND) or of dark matter (according to this paradigm). This cannot be detected (in MOND) and has not been detected (as dark matter); so, to measure Σ_D^0 we would need to map the acceleration field in some cylindrical volume around the symmetry axis, so $\vec{\nabla}\cdot\mathbf{g}$ can be determined there.

In spherical systems, it is enough to measure the radial acceleration profile, since we then have from Eq. (6)

$$\Sigma_D^0 = \frac{1}{2\pi G}\int_0^\infty\left(\frac{2g_r}{r}+\frac{dg_r}{dr}\right)dr = \frac{1}{2\pi G}\int_0^\infty\frac{2g_r}{r}dr = \frac{\kappa}{2\pi G}\mathcal{G},$$
$$(7)$$

where

$$\mathcal{G}\equiv\int_0^\infty g_r d\ln r, \qquad (8)$$

and $\kappa = 2$. Here and below I take inward accelerations g_r to be positive. The "acceleration integral" $\mathcal{G}$, a certain average of the radial acceleration, plays an important role in the present context.

In deriving Eq. (7), I used $g_r(\infty) = 0$, and I also assumed that $g_r(0) = 0$, otherwise the integral diverges at least logarithmically at the origin.[7]

$\mathcal{G}$ can be measured with various degrees of accuracy for quasispherical systems, such as elliptical galaxies or galaxy clusters, using methods detailed in the literature, such as hot gas hydrostatics or weak lensing. Such systems tend, however, to have $\Sigma_B^0 \gg \Sigma_M$ and they would probe the "less interesting" region of high accelerations.

1. Disk galaxies—using rotation curves

Our main concern here is with disk galaxies because they offer the most extensive and most accurate sample for studying the CSDC. As we shall see below, the predictions made by modified-gravity MOND theories concern the phantom and dynamical column densities along the symmetry axis. However, except perhaps for the Milky Way,[8] where we can, in principle, use test particles away from the plane to map the potential field (using, e.g., tidal streams; see, for example, Ref. [22]), we can only measure in a relatively robust way, at present, the acceleration field in the plane of the disk, using rotation curves: $g_r(r) = V^2(r)/r$.

In the framework of the dark-matter paradigm, one can approximate the halo by a spherical model, determine its structural parameters (e.g., its characteristic density and size), and then calculate its Σ_P^0. Or, one can determine Σ_P^0 from Eq. (7) with $\kappa = 2$, using the excess contribution to the acceleration integral, over that of the baryons.

In MOND, however, the phantom body of a disk galaxy is made of a phantom disk and a phantom bulge [23]. We thus need a relation similar to (7) for more general mass distributions with our assumed symmetry.

Relation (7) for a spherical system can, in fact, be generalized to spheroids of arbitrary axes ratio q, having a density distribution (in cylindrical coordinates)

$$\rho(r,z) = \hat{\rho}\left(r^2+\frac{z^2}{q^2}\right). \qquad (9)$$

I show in Appendix A 1 that if Σ_D^0 is the column density of such a spheroid along the symmetry axis, and $g_r(r)$ is the acceleration in the x-y, symmetry plane, then relation (7) holds with $\kappa = 1 + q$. For a thin planar disk ($q = 0$), we have $\kappa = 1$.

Since any surface-density distribution in a thin disk can be written as a sum of infinitely flattened homoeoids (see, e.g., Sec. 2.6-1 of Ref. [24]), relation (7) with $\kappa = 1$ holds for any thin disk. This reproduces, as a special case, the relation found in Toomre [25] for thin disks.

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[6]When the system is not isolated, in that it is falling in some external field, the internal dynamics is modified due to the MOND external-field effect.

[7]The acceleration does not vanish at the origin if the density diverges at least as fast as $1/r$ there, in which case Σ_D^0 indeed diverges.

[8]The Milky Way also has $\Sigma_B^0 \gg \Sigma_M$.

As another example of mass models with controllable degree of flatness, for Miyamoto-Nagai models, having a potential field

$$\phi(r, z) = -\frac{MG}{\{r^2 + [a + (z^2 + b^2)^{1/2}]^2\}^{1/2}}, \quad (10)$$

I derive in Appendix A 2

$$\kappa = (1 + \lambda)^2 \int_0^1 \frac{x^3(3 + \lambda x)dx}{(1 + \lambda x)^3 \sqrt{1 - x^2}}, \quad (11)$$

where $\lambda = a/b$. As λ goes from 0 (Plummer sphere) to ∞ (Kuzmin disk), κ goes, monotonically, from 2 to 1.

Relations like Eq. (7) are based on the linear Poisson equation, and their two sides are additive. It can thus be combined linearly for any number of aligned components with our symmetry; e.g., a disk-plus-sphere mass (whether real or phantom).

To avoid possible confusion, note that when we discuss such relations in the context of MOND (which is nonlinear) they should always be understood to apply not to the baryonic surface density of the system, but to the dynamical density distribution predicted by MOND—baryonic plus phantom. Hence, in the present context, such surface densities are designated with a subscript "D." Thus, such relations are not additive with respect to the baryonic components of the mass distribution. Given a baryonic distribution, we first have to determine the MOND, dynamical distribution, identify different components of it, and then apply such relations.

For a system made of several components with different κ values, the acceleration integral and the central surface density are each the sum of contributions of the different components $\mathcal{G} = \sum_i \mathcal{G}_i$, $\Sigma_D^0 = \sum_i \Sigma_{Di}^0$. They are related as in Eq. (7), with an effective coefficient $\bar{\kappa}$, which, however, depends on the relative contributions of the components,

$$\bar{\kappa} = \mathcal{G}^{-1} \sum_i \kappa_i \mathcal{G}_i = \Sigma_D^0 \left[\sum_i \frac{\Sigma_{Di}^0}{\kappa_i} \right]^{-1}. \quad (12)$$

To determine $\bar{\kappa}$ from the first expression in Eq. (12) we would need to know the individual contributions to the full rotation curve. The second expression is more useful, as it requires only knowledge of the separate contributions to the central surface density, which is easier to estimate. For example, a mass distribution that is a combination of oblate spheroids (with $0 \leq q_i \leq 1$) has $1 \leq \bar{\kappa} \leq 2$.

When testing MOND predictions, the contributions of the different phantom components to Σ_P^0, and hence to Σ_D^0, are predicted. So if we can estimate the κ values of the components, we can get a good approximation of $\bar{\kappa}$ from

the second equality in Eq. (12). For example, AQUAL and QUMOND predict that in the DML, where $\Sigma_B^0 \ll \Sigma_M$, $\Sigma_D^0 \approx \Sigma_P^0$ picks up equal contributions from a phantom disk, for which $\kappa \approx 1$, and from an oblate spheroid, for which κ should be between 1 and 2. Then, from Eq. (12), $\bar{\kappa}$ should be between 1 and 4/3. For example, the exact value of $\bar{\kappa}$ predicted by AQUAL and QUMOND for a deep-MOND Kuzmin disk is $\bar{\kappa} = 4/\pi$ [12] (see also Sec. VI B 1 below).[9]

Such accuracy in estimating $\bar{\kappa}$ is good enough for testing the MOND predictions, since, at present, $\mathcal{G}$ and Σ_B^0 themselves are not determined to better accuracy and because the span in Σ_B^0 studied in existing galaxy samples—of several orders of magnitude—is much larger than this uncertainty factor in the predicted $\Sigma_B^0/\mathcal{G}$.[10]

By employing Eq. (7), we turn a MOND prediction of a $\Sigma_B^0 - \Sigma_P^0$ or $\Sigma_B^0 - \Sigma_D^0$ correlation to a $\Sigma_B^0 - \mathcal{G}$ one, with some increase of the scatter due to the uncertainty in $\bar{\kappa}$. The great advantage of the latter is that it lends itself to testing, since both quantities in it can be measured, unlike Σ_P^0 or Σ_D^0 which are very difficult to measure directly.

Note that this prediction of the $\Sigma_B^0 - \mathcal{G}$ relation does not follow from and is independent of the MOND prediction of rotation curves. The latter requires a knowledge of the full baryonic mass distribution and predicts the full rotation curve. The former uses just one attribute of the mass distribution Σ_B^0 and predicts from it one global attribute of the rotation curve. We see already in the spherical case that two systems can have the same baryon central surface density, but different density distributions, and hence different rotation curves, but they are predicted to have approximately the same $\mathcal{G}$.

Examples of other MOND predictions of global rotation-curve characteristics from global baryonic properties are: (1) The mass-asymptotic-speed relation $MGa_0 = V_\infty^4$, which predicts the constant asymptotic rotational speed V_∞ of isolated bodies from their total baryonic mass M [1]. (2) A prediction of the mass-weighted, root-mean-square rotational speed of an isolated galaxy that is fully in the deep-MOND regime from its total mass: $(2/3)(MGa_0)^{1/2} = 2\pi M^{-1} \int_0^\infty r\Sigma(r)V^2(r)dr$ [26].

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[9]For such a Kuzmin disk, AQUAL and QUMOND predict analytic expression for Σ_D^0 and $\mathcal{G}$; see Sec. III A.

[10]It should be realized that such values of $\kappa \sim 1-2$ apply for most observed galaxy mass distributions of a disk plus an oblate-to-spherical bulge. But for general mass distributions (with our symmetry) there is no tight relation between Σ_D^0 and $\mathcal{G}$, and the justification of using $\kappa \sim 1$ has to be checked individually. As extreme counterexamples, for a very long cylinder of constant density, the column density along the symmetry axis goes to infinity as the length of the cylinder does, but $\mathcal{G}$ remains finite; thus in this case $\kappa \to \infty$. This is also true of a prolate spheroid with $q \gg 1$, for which $\kappa = 1 + q$. On the other hand, for a hollow, thin cylindrical shell, for which $\Sigma_D^0 = 0$, $\mathcal{G}$ is finite and for infinite length becomes $2\pi G\Sigma_s$, where Σ_s is the surface density of the thin shell; so $\kappa = 0$.

III. WHAT HAS BEEN DONE TO DATE

From the MOND vantage point, the first statement relevant to our discussion was made when it was pointed out in Ref. [17] that, according to MOND, the gravitational acceleration produced by the fictitious dark matter cannot much exceed a_0. This can be stated, alternatively, as "according to MOND, the central surface density of the phantom halo cannot much exceed Σ_M." This follows from relations such as Eq. (7), which shows that the characteristic surface density in units of Σ_M is some average over the acceleration in units of a_0. Reference [17] based their derivation on either AQUAL or the algebraic formulation, whereby $\mu(g/a_0)\mathbf{g} = \mathbf{g}_N$.

Induced by the prediction in Ref. [17], Ref. [13] then used a sample of disk galaxies and fitted their rotation curves with baryons plus a cored-isothermal-sphere, dark-matter halo. It showed (in their Fig. 4) that the product of the "halo" central density and its core radius is bounded by and concentrated strongly near a value of $\approx 10^2 M_\odot$ pc$^{-2} \approx \Sigma_M$. This translates into the maximum acceleration of the model halo being somewhat below a_0, in agreement with the prediction of Ref. [17]. The product itself is approximately the central surface density of the halo as defined in Eq. (6). So the findings of Ref. [13] are tantamount to the distribution of halo central surface densities of their galaxy sample, being narrowly peaked near $\approx \Sigma_M$. This sample was, however, lacking galaxies deep in the MOND limit.

Other studies have found similar results, without noting that these vindicate a MOND prediction. For example, like Refs. [13,14] fitted rotation curves of a sample of disk galaxies including a cored-isothermal-sphere halo model, while Ref. [15] fitted another set of rotation curves with Burkert model halos (which is also characterized by a central density and a core radius). Both found, for their samples, a distribution of the halo central surface densities that is strongly peaked at a value of $\approx 140 M_\odot$ pc $\approx \Sigma_M$.

On the theoretical side, using the algebraic relation $\mu(g/a_0)\mathbf{g} = \mathbf{g}_N$, Ref. [18] calculated the phantom surface-density profiles—in a plane perpendicular to the axisymmetry axis—for a point mass, for two equal masses of different separations, and for a thin rod of various lengths [in units of their MOND radius, $r_M = (MG/a_0)^{1/2}$]; all these for different choices of $\mu(x)$. The corresponding values of Σ_P^0 for all these models can be read from the figures in Ref. [18] (given there in units of $a_0/G = 2\pi\Sigma_M$).

All the above results concern only the high-surface-density end of the CSDC, predicted to give $\Sigma_P^0 = \gamma\Sigma_M$, with $\gamma = O(1)$.

In Ref. [19], I considered the problem in some generality, using AQUAL, with the observational results of Refs. [13–15] in mind. As noted above, those studies found a quasiuniversal value of $\Sigma_P^0 \approx \Sigma_M$ for the galaxy samples they employed, as MOND predicts for high-acceleration systems (see Sec. IV A) below. However, I noted that MOND does not predict a universal value of Σ_P^0.

This is the case only for high-surface-density systems, but for low-surface-density ones, one predicts a correlation of the form $\Sigma_P^0 \propto (\Sigma_M \Sigma_b)^{1/2}$, where Σ_b is some measure of the average surface density (not the central surface density, which I use in the present paper). The proportionality factor was estimated to be of the order of a few and to depend on the exact form of the interpolating function and on the exact mass distribution. The apparent quasiuniversality was attributed to the lack of truly low-surface-density galaxies in the sample studied. It was shown, with a few examples of such galaxies, that their values of Σ_P^0 indeed fall much below Σ_M, and are consistent with the MOND predictions for the low-surface-density regime.

By far the most extensive observational study of the correlation was described in Ref. [16] (see also Ref. [27] for a study with a much smaller sample, discussed in some detail in Ref. [16]). It shows the correlation between two proxies of the baryonic and dynamical central surface densities for 135 disk galaxies in the Spitzer photometry and accurate rotation curves (SPARC) sample. The proxy for the baryonic central surface density was the stellar value, which may underestimate the baryonic value somewhat because of the contribution of the galactic gas.[11] The proxy for the dynamical central surface density was the acceleration integral, using Eq. (7), with $\kappa = 1$ which, however, is appropriate only for pure disks.

If one wishes to learn from the correlation about dark-matter halos, then a value of κ nearer 2 would be more appropriate in the low-Σ_B^0 region where the spheroidal halo dominates. However, as we saw in Sec. II A 1, when testing the MOND prediction, a value of κ between 1 and 4/3 is more in order; in this case the study of Ref. [16] underestimates Σ_D^0 only a little.

While Ref. [16] plotted a measure of Σ_D^0, MOND makes the finer prediction, that of Σ_P^0, which is harder to separate from Σ_D^0 in the high-surface-density regime, because there Σ_P^0 contributes very little to Σ_D^0. It is, however, important to test this prediction separately from the grosser prediction of $\Sigma_D^0 \approx \Sigma_B^0$.[12]

Next came a derivation of a universal $\Sigma_B^0 - \Sigma_D^0$ algebraic relation for pure baryonic disks predicted by AQUAL and QUMOND, which I describe next.

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[11]In most cases, gas is extended and, even when important globally, contributes little at the center; however, there are gas-dominated galaxies where the gas contribute materially to Σ_B^0.

[12]In a related study, Ref. [28] used the SPARC sample with fits for the separate contributions of the baryons and the phantom components to the rotation curves to plot, separately, the phantom acceleration vs the baryonic one. They find, indeed, that the phantom acceleration achieves a maximum at a value $\sim a_0$ as predicted in Ref. [17].

A. Predictions of AQUAL and QUMOND for pure disks

In Ref. [12] I proved an exact result that holds in AQUAL and QUMOND, stating that the central dynamical surface density of a pure disk is a unique function of the baryonic central surface density,

$$\Sigma_D^0 = \Sigma_M \mathcal{S}(\Sigma_B^0/\Sigma_M), \qquad \mathcal{S}(Y) = \int_0^Y \nu(Y')dY', \quad (13)$$

and $\nu(Y)$ is the interpolating function of the respective theory.[13] The contributions to Σ_D^0 of the disk (baryonic plus phantom) $\Sigma_D^{0,\text{disk}}$ and the bulge (all phantom) $\Sigma_D^{0,\text{sph}}$ are

$$\Sigma_D^{0,\text{disk}} = \Sigma_M \nu(Y)Y, \quad \Sigma_D^{0,\text{sph}} = \Sigma_M \int_0^Y [\nu(Y') - \nu(Y)]dY',$$

$$Y = \Sigma_B^0/\Sigma_M. \qquad (14)$$

ν has the asymptotes $\nu(Y) \to Y^{-1/2}$ for $Y \ll 1$, and $\nu(Y) \to 1$ for $Y \gg 1$.

In the high-central-surface-density limit $\Sigma_B^0 \gg \Sigma_M$, the integral picks up most of the contribution from $y \gg 1$, so $\mathcal{S}(y) \to y$ in this limit, hence, $\Sigma_D^0 \to \Sigma_B^0$. In the opposite limit, the integrand is well approximated by $Y^{-1/2}$, and we have $\Sigma_D^0 \to 2(\Sigma_M \Sigma_B^0)^{1/2}$. It is also quite interesting to consider the finer prediction for the separate contribution Σ_P^0 of the phantom halo. Since $\Sigma_B^0 = \Sigma_M \int_0^{\Sigma_B^0/\Sigma_M} dy$, we have

$$\Sigma_P^0 = \Sigma_D^0 - \Sigma_B^0 = \Sigma_M \int_0^{\Sigma_B^0/\Sigma_M} [\nu(Y) - 1]dY. \quad (15)$$

Solar System constraints imply that $\nu(Y)$ has to approach 1 fast as $Y \gg 1$. So the asymptotic value of Σ_P^0 is $\Sigma_M \int_0^\infty [\nu(Y) - 1]dY$, which should converge fast. The integral is of order unity. For example, for the choice of $\nu(Y) = Y^{-1/2}$ for $Y \leq 1$ and $\nu(Y) = 1$ for $Y \geq 1$, we get $\Sigma_P^0 \to 2\Sigma_M$.

In the DML branch of the relation ($\Sigma_B^0 \ll \Sigma_M$), we have $\Sigma_D^0 = 2(\Sigma_M \Sigma_B^0)^{1/2} \gg \Sigma_B^0$. We see from Eq. (14) that in this limit half of the contribution to Σ_D^0 comes from $\Sigma_D^{0,\text{disk}}$ and half from $\Sigma_D^{0,\text{sph}}$.

Are these exact and universal (disk-independent) relations in AQUAL and QUMOND predicted in a wider range of MG theories? It turns out that they are not quite, as we shall see below.

IV. PREDICTIONS OF THE BASIC TENETS ALONE

Now that we have the machinery in place and before we consider more precise and detailed predictions of the CSDC, I show how the basic tenets of MOND predict a correlation between baryonic and dynamical surface densities. This makes the correlation in its rough form a primary prediction of MOND. Predictions of exact details of the correlation, such as the values of the numerical coefficients that appear, do depend, however, on the specific theory, on the details of the mass distribution, and on the concrete definition of the surface densities.

The basic tenets of MOND inform us in detail only on the two limits of the theory, but not on the exact way in which the interpolation between them occurs in various theories and for different phenomena. As regards the transition itself, because a_0 is the only new dimensioned constant, and we assume that no dimensionless parameters appear that differ much from unity, the transition is predicted to occur around a_0, in acceleration, and within a range of that order; viz. the asymptotic behaviors are already good approximations below a_0/q and above qa_0, for $q \gtrsim 1$. As a result, we can hope to derive from the basic tenets alone only the asymptotic limits of a correlation, but be assured that these limits apply—each in its own regime—almost everywhere except for a region of order Σ_M around $\Sigma_B = \Sigma_M$.

For the arguments here to hold, we need to apply them to baryonic surface-density attributes Σ_B that are a good measure of the overall or average Newtonian acceleration of the system, such as the average surface density, because then Σ_B/Σ_M is a good indicator of whether we are in the MOND or the Newtonian regime.

A. Newtonian limit

A system of mass M and characteristic size R, with $\Sigma_B \sim M/\pi R^2 \gg \Sigma_M$, is well contained within its MOND radius $r_M = (MG/a_0)^{1/2}$; namely, $R \ll r_M$. Whichever way we measure accelerations and whether we are dealing with MG or modified-inertia (MI) theories, the accelerations within $\sim r_M$ are predicted to be Newtonian.[14] Hence, the phantom density picks up contributions only from $r \gtrsim r_M$. In this region we already have approximate spherical symmetry and the average, or central, phantom surface density $\Sigma_p \sim g(r_M)/2\pi G \sim \Sigma_M$, as can be seen, e.g., from Eq. (7).

B. Deep-MOND limit

Consider a density distribution $\rho(\mathbf{r})$ such that the system is fully in the deep-MOND regime. Generate from it a two-parameter family of density distributions

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[13]The interpolating function $\mu(X)$ that appears in AQUAL is represented here by $\nu(Y)$ defined such that if $Y = X\mu(X)$, then $X = Y\nu(Y)$.

[14]In MI theories we have to assume that the orbits of the test-particle probes do not take them far outside r_M.

$$\rho_\lambda^\alpha(\mathbf{r}) = \alpha\lambda^{-3}\rho(\lambda^{-1}\mathbf{r}), \qquad (16)$$

whereby all sizes are stretched by a factor λ and all masses are multiplied by a factor α. Scale invariance of the DML and dimensional arguments imply [3,8] that all kinematic accelerations **a** (which are the same as the MOND gravitational accelerations **g**) that appear in solutions of the theory for ρ, scale to $\alpha^{1/2}\lambda^{-1}\mathbf{a}$ and $\alpha^{1/2}\lambda^{-1}\mathbf{g}$, respectively, in the solutions for ρ_λ^α.[15] Masses scale as $m \to \alpha m$; dynamical densities scale as $\rho_D(\mathbf{r}) \to \alpha^{1/2}\lambda^{-2}\rho_D(\lambda^{-1}\mathbf{r})$ (not as $\alpha\lambda^{-3}$ like ρ_λ^α itself). All dynamical surface densities scale as $\Sigma_D \to \alpha^{1/2}\lambda^{-1}\Sigma_D$ (at the appropriately scaled positions). On the other hand, any baryonic surface density scales as $\Sigma_B \to \alpha\lambda^{-2}\Sigma_B$. Thus, with any definition of some Σ_D and Σ_B, the dimensionless, mass-scaling-invariant, and mass-invariant quantity (again, calculated at the scaled positions)

$$\eta(\lambda, \alpha) = \frac{\Sigma_D}{(\Sigma_M\Sigma_B)^{1/2}} \qquad (17)$$

is, in fact, independent of λ or α, as long as λ is not as small or α is not as large, as to bring the system out of the DML.

To recapitulate this important result: It follows from the basic tenets of MOND alone that, in any given MOND theory that embody them, the DML value of η is constant within any such two-parameter family of galaxies. For example, all thin exponential (pure) disk galaxies in the DML, or all DML galaxies described by a Plummer model, etc., should have the same value of η.

This fact holds for any choice of Σ_B and Σ_M, even if they are unrelated, for example, the dynamical central surface density and the baryonic surface density at the radius containing 90% of the projected mass. However, with arbitrary choices of the two surface densities, η is not a useful ratio to consider, because it could be very large or very small and could depend very strongly on the family ρ_λ^α.

A similar statement is correct if we replace Σ_D in the nominator of η by any measure of the MOND (measured) accelerations, such as $\mathcal{G}$ (or $\mathcal{G}/2\pi G$ if we want the ratio to remain dimensionless) because they scale in the same way as Σ_D under the scaling of Eq. (16).

To proceed with the argument and produce a significant correlation between the two quantities, encompassing all density distributions, with η of order unity, we want to choose the two surface densities so that they are physically related; i.e., the dynamical and baryonic analogs of each other. In particular, we choose their definition such that in the Newtonian limit Σ_D goes to Σ_B—for instance, they can be the central surface densities or the average surface densities $\langle\Sigma_D\rangle$ and $\langle\Sigma_B\rangle$. In this way we can also bring to bear the second MOND tenet, that of the "correspondence principle" in the Newtonian limit.

Given the generating density distribution $\rho(\mathbf{r})$, there is a demarcation line in the $\lambda - \alpha$ plane, $\alpha \propto \lambda^2$, where $\Sigma_B = \Sigma_M$. Much above the line, the corresponding system is Newtonian, and much below it, it is in the DML. In the former region, $\eta \to (\Sigma_B/\Sigma_M)^{1/2}$, since we chose the definitions of Σ_B and Σ_D so that $\Sigma_D \to \Sigma_B$ in this limit. And, as we saw, in the latter region η is some constant (for the family). The essence of our argument is that the dimensionless η cannot make a big jump when going from one side to the other. So, the constant, DML value cannot be very different from the extrapolation of the Newtonian η values to the demarcation line. The latter extrapolate to $\eta = 1$ on the dividing line[16]; so the DML constant value must be $\eta = O(1)$. This follows from the tenet saying that the two limiting expressions of η hold approximately to "factor of order unity," down to the transition.

Since this has to hold for all choices of $\rho(\mathbf{r})$, we have, generally, $\eta = O(1)$ in the DML for all bounded mass distributions.

This establishes a general CSDC. Despite the impression that such a predicted CSDC can be rather loose, with large scatter, one has to note that the span of Σ_B values to which we apply it in practice can be several orders of magnitude; so the predicted correlation would still be quite strong.

The arguments here and in Sec. IV A assume that the ratio of the baryonic surface density we employ to Σ_M is a good measure of whether we are in the DML or the Newtonian regime. This would be the case for global measures of the surface density, such as $\langle\Sigma_B\rangle$, or the surface density at the half-mass radius. Beyond these general arguments that use essentially only the basic tenets, but predict a correlation with possibly large scatter, we would like, if possible, to get more refined predictions, at a cost of being, perhaps, less general, in being theory dependent, and dependent on the exact definitions of the surface densities involved. This leads us to consider the central surface densities in the correlation because, as I said above, Σ_D^0 is both easier to measure and to calculate in a given theory.

In many disk galaxies, Σ_B^0 is close to a global measure of the surface density (e.g., for exponential disks), and for these galaxies the above arguments apply to the central surface densities, as well. However, in principle, Σ_B^0 may be quite different from the average surface density of the galaxy, and hence its ratio to Σ_M is not a good

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[15]To see this, note that the scale invariance of the DML implies that only the combination $\mathcal{A}_0 = Ga_0$ can appear in a DML expression for the MOND accelerations. A change of units must leave any valid expression valid. Change the units of mass such that the values of masses $m \to \alpha m$, the length units such that $\ell \to \lambda\ell$, and the units of time so that $t \to \lambda\alpha^{-1/4}t$. Since the only dimensioned constant of the theory, $\mathcal{A}_0$, remains unchanged under this scaling, the scaled solutions satisfy the DML equations with the correct value of $\mathcal{A}_0$. The kinematic accelerations then indeed scale as $\alpha^{1/2}\lambda^{-1}$.

[16]In specific examples that we have the actual value of η on the dividing line is, in fact, somewhat larger than 1, and the DML values of η vary roughly between 2 for disks and ~4 for spheres.

MOND-vs-Newtonian-dynamics indicator. For example, consider a disk galaxy with a concentric, circular hole around its center, whereby $\Sigma_B^0 = 0$. This remains so all along the above λ, α sequence; so the above argument does not work in this case. Nonetheless, the exact relation (13) holds for this case in AQUAL and QUMOND, and it dictates that $\Sigma_D^0 = 0$, as well, for all values of λ and α.[17] This is a result that we could not have derived based on the above general argument, but requires reference to the specific theory.

With these caveats and subtleties in mind, I now proceed to discuss the predictions of more concrete theories.

V. GENERALIZED THEORIES

In this section, I describe the classes and subclasses of theories that I will use to generate predictions of a CSDC. I shall consider in some detail only MG theories, with some comments on the not-less-promising class of MI theories. The reason for this preference it twofold. First, we do have concrete, full-fledged, and self-consistent examples of MG MOND theories, while we are still lacking such well-developed MI theories. Second, as I explain in Sec. V C below, the concept of phantom density, and hence of a dynamical surface density, is not well defined in MI theories.

Nevertheless, we do have some general results for a certain class of MI theories, which I discuss briefly in Sec. V C.

A. General class of modified-gravity MOND theories

The class of MG MOND theories—more general than AQUAL and QUMOND—that I consider here in some more detail are governed by a Lagrangian density

$$\mathcal{L} = \mathcal{L}_G + \rho\left(\frac{1}{2}\mathbf{v}^2 - \phi\right), \qquad (18)$$

whose gravitational part is

$$\mathcal{L}_G = -\frac{a_0^2}{4\pi G}F\left(\frac{\vec{\nabla}\phi}{a_0}, \frac{\vec{\nabla}\psi_1}{a_0}, \frac{\vec{\nabla}\psi_2}{a_0}, \ldots\right). \qquad (19)$$

(Note that only the first derivatives of the potentials appear.) They are a subclass of the more general theories described in detail in Ref. [29].

Of the scalar potentials $\phi, \psi_1, \psi_2, \ldots$, which are the gravitational degrees of freedom, only ϕ couples directly to the matter density. Variation over the matter degrees of freedom gives the equation of motion $\ddot{\mathbf{r}} = -\vec{\nabla}\phi$, identifying ϕ as the MOND potential, which dictates accelerations of test particles. F is a dimensionless function that depends on the potentials through the scalar products $\vec{\nabla}\phi \cdot \vec{\nabla}\psi_i$ and $\vec{\nabla}\psi_i \cdot \vec{\nabla}\psi_j$, for $j \geq i$.

The field equations are

$$a_0^2\vec{\nabla} \cdot \left(\frac{\partial F}{\partial\vec{\nabla}\phi}\right) = 4\pi G\rho = -\vec{\nabla} \cdot \mathbf{g}_N, \qquad \vec{\nabla} \cdot \left(\frac{\partial F}{\partial\vec{\nabla}\psi_i}\right) = 0. \qquad (20)$$

A useful property of this general class of theories, which I employ below, is that for one-dimensional systems, with spherical, cylindrical, or plane symmetry, the MOND acceleration $\mathbf{g}$ is a universal function of the Newtonian acceleration $\mathbf{g}_N$—a function unique to the theory, but independent of the specific one-dimensional symmetry or the specific mass distribution. We thus have

$$\mathbf{g} = \nu(g_N/a_0)\mathbf{g}_N, \qquad \mathbf{g}_N = -\frac{GM(r)\mathbf{r}}{r^3}. \qquad (21)$$

To see this, apply Gauss's theorem to the field equations in Eq. (20) for constant-r surfaces, where r is the coordinate that underlies the one-dimensional symmetry. All the acceleration-vector arguments of F are collinear and have only an r component; so they can be replaced by $\vec{\nabla}\phi \to g = |d\phi/dr|$, $\vec{\nabla}\psi_i \to g_i = |d\psi_i/dr|$, and all scalar products can be replaced by products of the g's. F then becomes a function of $\chi \equiv g/a_0$ and $\chi_i \equiv g_i/a_0$, $F = \hat{F}(\chi, \chi_1, \chi_2, \ldots)$, and the field equations become algebraic equations between χ, χ_i,

$$\frac{\partial\hat{F}}{\partial\chi}(\chi, \chi_1, \chi_2, \ldots) = \frac{g_N}{a_0}, \qquad (22)$$

$$\frac{\partial\hat{F}}{\partial\chi_i}(\chi, \chi_1, \chi_2, \ldots) = 0. \qquad (23)$$

Equation (23) are used to solve for all the g_i/a_0 as functions of g/a_0, substituted in Eq. (22), which is then inverted to give Eq. (21). The possibility to make all these inversions is a necessary requirement from F, for the theory to predict gravitational fields uniquely.[18]

Equation (21) defines the "one-dimensional interpolating function" $\nu(Y)$.

After some general results that follow for all the theories in the class, I shall concentrate, for more detailed calculations, on a yet smaller subclass—the tripotential

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[17]In this case, there is a region of negative phantom density along the z axis [20], which results in Σ_P^0 vanishing. There is also a region around the center where the acceleration integral $\mathcal{G}$ picks negative contributions (the radial accelerations point outward). The Newtonian rotation curve would then give $\mathcal{G} = 0$. But the observed (MOND) rotation curve will generally give $\mathcal{G} \neq 0$. $\mathcal{G}$ is then not useful for determining Σ_D^0, since in this case the effective $\bar{\kappa} = 0$ in Eq. (12) is very different from unity.

[18]In AQUAL, for example, this is the requirement of ellipticity of the field equations, which requires $x\mu(x)$ to be monotonic.

MOND (TRIMOND) theories discussed in Ref. [30]. While TRIMOND theories—with AQUAL and QUMOND as special cases—form a subclass, with a Lagrangian density of the type (19), the generalizations of QUMOND presented in Ref. [31] are examples of MG theories that are not of this type, because their Lagrangians depend also on higher derivatives of the auxiliary potentials. These latter theories do not predict the algebraic relation (21) for one-dimensional systems.

Finally, as shown in Ref. [29], all the theories in this class predict the DML virial relation for a self-gravitating, isolated system of (pointlike) masses m_p,

$$\sum_p \mathbf{r}_p \cdot \mathbf{K}_p = -(2/3)(Ga_0)^{1/2}\left[\left(\sum_p m_p\right)^{3/2} - \sum_p m_p^{3/2}\right],$$
(24)

where $\mathbf{r}_p$ are the positions of the masses, and $\mathbf{K}_p$ are the forces they are subject to. A special case is the deep-MOND two-body force for arbitrary masses,

$$\mathrm{K}(m_1, m_2, \ell) = \frac{2}{3}\frac{(a_0 G)^{1/2}}{\ell}[(m_1 + m_2)^{3/2} - m_1^{3/2} - m_2^{3/2}]$$
(25)

(ℓ is the distance between the masses). Another important corollary is the DML mass-velocity-dispersion relation

$$\sigma^2 = \frac{2}{3}(MGa_0)^{1/2}\left[1 - \sum_p (m_p/M)^{3/2}\right],$$
(26)

where $\sigma^2 = M^{-1}\sum_p m_p \mathbf{v}_p^2$ (velocities are in the rest frame of the center of mass), and $M = \sum_p m_p$ is the total mass. This reduces to $\sigma^4 = \frac{4}{9}MGa_0$ for systems made of many masses, with $m_p \ll M$.

In particular, all these theories predict the same value of the Q parameter, defined in Ref. [26], which was proposed as a possible discriminator between MOND theories.

After considering in Sec. VI some general prediction of this class of theories, I derive more specific ones for the TRIMOND subclass, which I now recap briefly.

B. TRIMOND recap

Following is a summary of the aspects and equations of TRIMOND from Ref. [30] that we shall be needing here.

The TRIMOND Lagrangian density is

$$\mathcal{L} = -\frac{1}{8\pi G}[2\vec{\nabla}\phi \cdot \vec{\nabla}\psi - a_0^2\mathcal{F}(x, y, z)] + \rho\left(\frac{1}{2}\mathbf{v}^2 - \phi\right),$$
(27)

where

$x \equiv (\vec{\nabla}\psi)^2/a_0^2, \quad y \equiv (\vec{\nabla}\varphi)^2/a_0^2, \quad z = 2\vec{\nabla}\psi \cdot \vec{\nabla}\varphi/a_0^2,$
(28)

and $\mathcal{F}$ is a dimensionless function satisfying the basic tenets of MOND.

This is a special case of the Lagrangian density (19), special in two important regards: (a) It involves only three gravitational potentials. (b) The MOND potential ϕ couples directly to only one of the auxiliary potentials ψ and that linearly. This results in ψ being the Newtonian field and is straightforwardly solved for. As a result, unlike the general case, TRIMOND theories do not require solving coupled equations.

Acceleration of test bodies is given by

$$\ddot{\mathbf{r}}_i = -\vec{\nabla}\phi(\mathbf{r}_i),$$
(29)

where ϕ is the MOND gravitational potential. The first of Eq. (20) becomes here

$$\Delta\psi = 4\pi G\rho,$$
(30)

and the two additional field equations are

$$\vec{\nabla} \cdot (\mathcal{F}_y \vec{\nabla}\varphi) + \vec{\nabla} \cdot (\mathcal{F}_z \vec{\nabla}\psi) = 0,$$
(31)

$$\Delta\phi = \vec{\nabla} \cdot (\mathcal{F}_x \vec{\nabla}\psi) + \vec{\nabla} \cdot (\mathcal{F}_z \vec{\nabla}\varphi) \equiv 4\pi G\rho_D,$$
(32)

where ρ_D is the dynamical density (baryonic plus phantom).

ψ thus equals the Newtonian potential and is solved for from Eq. (30). It is then substituted in Eq. (31), which becomes a nonlinear, second order equation in φ. Solving for it and substituting both in the right-hand side of Eq. (32), we get a Poisson equation for ϕ.

AQUAL is a special case with $\mathcal{F}_x = 0, \mathcal{F}_z = \epsilon$, a constant. QUMOND is a special case gotten for the choice $\mathcal{F}_y = \mathcal{F}_z = 0$.

As a subclass of the above class of MG theories, for systems with one-dimensional symmetry, TRIMOND theories predict the algebraic relation (21) between the MOND and Newtonian accelerations. In such cases, the field equations (31) and (32) can be stripped of the divergences and become algebraic (putting, here, $a_0 = 1$),

$$\mathcal{F}_y(g_N^2, g_\varphi^2, 2g_N g_\varphi)g_\varphi + \mathcal{F}_z(g_N^2, g_\varphi^2, 2g_N g_\varphi)g_N = 0,$$
(33)

from which g_φ can be solved for as a function of g_N. $\mathcal{F}$ has to be such that the solution for g_φ exists and is unique; the ellipticity of Eq. (31) should ensure this. Equation (32) gives, in the same vein,

$$g = \mathcal{F}_x(g_N^2, g_\varphi^2, 2g_N g_\varphi)g_N + \mathcal{F}_z(g_N^2, g_\varphi^2, 2g_N g_\varphi)g_\varphi.$$
(34)

Substituting the g_φ values gotten from Eq. (33), we then get an equation of the form (21).

The Newtonian limit for $a_0 \to 0$ requires that in this limit $\mathcal{F}_z \to \epsilon$, $\mathcal{F}_y \to \omega$, and $\mathcal{F}_x \to \beta$, all constants, with

$$\beta - \epsilon^2/\omega = 1. \tag{35}$$

Scale invariance in the DML (discussed in detail in Ref. [30]) requires that there is a scaling dimension α of φ, such that in the DML $\mathcal{F}$ goes to some $\mathcal{F}_D(x, y, z)$ that has the property

$$\mathcal{F}_D(\lambda^{-4}x, \lambda^{2\alpha-2}y, \lambda^{\alpha-3}z) = \lambda^{-3}\mathcal{F}_D(x, y, z). \tag{36}$$

It is then necessary and sufficient that the DML of $\mathcal{F}$ is of the form

$$\begin{aligned} \mathcal{F}_D(x, y, z) &= x^{3/4}\mathcal{F}_D[1, yx^{(\alpha-1)/2}, zx^{(\alpha-3)/4}] \\ &\equiv x^{3/4}\bar{\mathcal{F}}_D[yx^{(\alpha-1)/2}, zx^{(\alpha-3)/4}]. \end{aligned} \tag{37}$$

Namely, the DML of TRIMOND requires one to specify a function of two variables, compared with no such freedom in the DML of AQUAL and QUMOND (see Sec. VI C 2 for an explicit example).

C. Modified-inertia theories

MI MOND theories are defined in the following restricted sense [32–35]: The force field (gravitational in our applications) is assumed to be unmodified and is thus Newtonian, with the potential ϕ_N. However, the equation of motion of a test mass in this field is not the Newtonian $\ddot{\mathbf{r}} = \mathbf{g}_N = -\vec{\nabla}\phi_N$, but has the form

$$\mathbf{A}[\mathbf{r}(t), a_0] = -\vec{\nabla}\phi_N, \tag{38}$$

where $\mathbf{A}$—having the dimensions of acceleration—is a functional of the body's trajectory $\mathbf{r}(t)$ [as opposed to a function of the local value of $\mathbf{r}(t)$]. Reference [35] is a review of such theories.

In such theories, a MOND acceleration field is not defined. Different particles can have different accelerations ($\ddot{\mathbf{r}}$) at the same position in the field, depending on details of their trajectories, as explained in detail in Ref. [35].[19,20] Thus the notions of dynamical density and phantom density do not exist, generally, and there is no well-defined sense in speaking of the phantom central surface density.

However, in practice, any procedure that results in some dynamical surface density, whether direct or indirect (e.g., using the acceleration integral $\mathcal{G}$ as a proxy), is based on some measurements of accelerations. As explained in Sec. IV, the general predictions of the basic tenets of MOND then apply to MI theories as well.

Even beyond such a general prediction, we may be able to make more specific predictions in some specific situations. A prominent case in point is the use of rotation curves to determine the dynamical and phantom central surface densities using the procedure described in Sec. II. This procedure uses the measured accelerations on exactly circular orbits, by measuring the radius r and orbital velocity $V(r)$ of these trajectories. Now, for such trajectories, the acceleration functional $\mathbf{A}[\mathbf{r}(t), a_0]$ in Eq. (38) must, in fact, be a *function* of r and V, and so, on dimensional grounds, must be of the form

$$\mathbf{A}[\mathbf{r}(t), a_0] = \frac{V^2}{r}\mu\left(\frac{V^2}{ra_0}\right), \tag{39}$$

where $\mu(x)$ is universal for the theory.[21] Thus, for circular orbits[22] in an axisymmetric body, such modified-inertia theories predict an algebraic relation between the MOND centrifugal acceleration g_c and the Newtonian, gravitational acceleration g_N, $g_c\mu(g_c/a_0) = g_N$, which can be inverted to read

$$g_c = g_N\nu(g_N/a_0), \tag{40}$$

identical with Eq. (21), which is valid for spherical systems in all the modified-gravity theories considered above.

Thus, for such spherical systems, the modified-inertia and the class of modified-gravity theories predict the same distribution of dynamical density [for the same choice of $\nu(Y)$], provided we use rotation-curve data for the determination.

For example, for the deep-MOND polytropes discussed in Ref. [36], the modified-inertia theories predict the same values of the coefficient η in Eq. (2) as the modified-gravity theories.

For disk galaxies, relation (40) still holds for exact circular orbits in the galactic plane, but it does not hold for MG theories. Nevertheless, numerical calculations tell us that AQUAL and QUMOND predict rotation curves for typical galaxy models, such as an exponential disk, that do not differ much from those gotten from this relation [37,38]. This leads us to expect that, for such models,

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[19]This is similar to what happens, e.g., in special relativity, where different electrons have different accelerations at the same position in an electric field, depending on their velocity vector.
[20]When there is a conserved momentum $\mathbf{p}$ in such a theory, $\dot{\mathbf{p}}$ is the same for all particles at the same position and does define a field; but this quantity is not proportional to $\ddot{\mathbf{r}}$, which is what we measure directly.

[21]In Ref. [32], I showed how $\mu(x)$ is derived from the action underlying the theory, when there is one.
[22]In practice, the test particles in disks of galaxies have orbits that depart from exact circular, such as by motions perpendicular to the disk. This causes the predicted velocities to be somewhat smaller than given by Eq. (39). This is particularly important in the inner parts. See discussion of this in Ref. [34].

the $\mathcal{G}$ integral predicted by MG and MI for the same pure-disk distribution would not differ by more than a factor ~ 2 (when a bulge is present, the difference is even smaller). So, if one uses $\mathcal{G}$ as proxy for Σ_D^0, then AQUAL and QUMOND and MI would predict similar Σ_D^0 up to a factor ~ 2 in the DML (nearer the Newtonian regime the differences are smaller).[23]

For example, Kuzmin disks, which I discuss in more detail in Sec. VI B, lend themselves to analytic derivation of the rotation curve in MI and in the class of theories described in Sec. V A. The latter give results that coincide with those of AQUAL and QUMOND for a Kuzmin disk, for which we have in the DML [37],

$$g_{MG} = (V_\infty^2/h)u/(1+u^2), \qquad (41)$$

where $u = r/h$, and h is the scale length of the disk. While

$$g_{MI} = (V_\infty^2/h)u^{1/2}/(1+u^2)^{3/4}. \qquad (42)$$

These give $\mathcal{G}_{MG} = (\pi/2)(V_\infty^2/h)$, while $\mathcal{G}_{MI} = (V_\infty^2/h)$ $\int_0^{\pi/2}(\sin\theta)^{-1/2}d\theta = 2.62(V_\infty^2/h)$, some 70% higher.

VI. PREDICTIONS OF THE CORRELATION

Numerical calculations would be required to derive the predictions concerning the CSDC for general disk-plus-bulge systems in the general theories of Sec. V A. Here, I give some analytic results for this class of theories for spherical systems and for a class of disk-plus-bulge mass models built on the same principle as the Kuzmin disk. I also discuss some results for general pure-disk systems.

A. Spherical systems

Consider first spherical systems. The algebraic relation (21) between the MOND and Newtonian accelerations then holds in the general class of MG theories described above, and we have from Eq. (7)

$$\frac{\Sigma_B^0}{\Sigma_M} = 2\int_0^\infty Y(r)d\ln r, \qquad \frac{\Sigma_D^0}{\Sigma_M} = 2\int_0^\infty \nu[Y(r)]Y(r)d\ln r,$$
$$\frac{\Sigma_P^0}{\Sigma_M} = 2\int_0^\infty \{\nu[Y(r)] - 1\}Y(r)d\ln r, \qquad (43)$$

where $Y(r) = g_N(r)/a_0$.

Another useful expression for Σ_D^0 in terms of the baryonic density distribution is

$$\Sigma_D^0 = -\frac{1}{2\pi G}\int_0^\infty \vec{\nabla}\cdot\mathbf{g}\,dr = -\frac{1}{2\pi G}\int_0^\infty \vec{\nabla}$$
$$\cdot [\nu(g_N/a_0)\mathbf{g}_N]dr = 2\int_0^\infty \nu[Y(r)]\rho_s$$
$$\times (r)dr - \Sigma_M\int_0^\infty (\mathbf{g}_N/a_0)\cdot\vec{\nabla}\nu(g_N/a_0)dr. \qquad (44)$$

The second integral can be shown to vanish [with our assumption that $g_N(0) = 0$], leaving us with

$$\Sigma_D^0 = 2\int_0^\infty \nu[Y(r)]\rho_s(r)dr. \qquad (45)$$

In the high-acceleration limit, $\Sigma_B^0 \gg \Sigma_M$, contribution to the last integral in Eq. (43) comes from outside the mass, so we can put $Y(r) = MG/a_0 r^2$, where M is the total mass. Changing variables to Y and noting that $d\ln r = -d\ln Y/2$, we have in the limit

$$\frac{\Sigma_P^0}{\Sigma_M} = \int_0^\infty [\nu(Y) - 1]dY = O(1) \qquad (46)$$

the same expression we found below Eq. (15) for pure disks in AQUAL/QUMOND.

This result is universal—i.e., holds for any bounded mass—for all the theories in the class, since the phantom density picks up contribution only far from the mass, where we can take the field as spherical.

In the opposite limit, $\Sigma_B^0 \ll \Sigma_M$, we have $y \ll 1$ at all radii; so, in the integral in the second of Eq. (43) we can put $\nu(Y) = Y^{-1/2}$. We then have $\Sigma_D^0 \to \eta(\Sigma_M\Sigma_B^0)^{1/2}$, with the coefficient

$$\eta = \sqrt{2}\frac{\int_0^\infty Y^{1/2}(r)d\ln r}{[\int_0^\infty Y(r)d\ln r]^{1/2}} = \sqrt{2}\frac{\int_0^\infty M^{1/2}(r)r^{-2}dr}{[\int_0^\infty M(r)r^{-3}dr]^{1/2}}. \qquad (47)$$

We see that η depends somewhat on the exact mass distribution. It is independent, however, of the normalization and of the overall length scale of the mass distribution, both of which drop out from expression (47), as long as they do not take us outside the DML. This follows, generally, from basic MOND tenets, as we saw in Sec. IV B.

For spherical systems, η is generally of order of a few. For a homogeneous sphere $\eta = 2\sqrt{3} \approx 3.46$. In Ref. [36], I calculated η for DML polytropes. Figure 11 there shows that, for isotropic-velocity-dispersion polytropes of all orders (which include isothermal spheres as a limiting case), η varies only by a little: $2\sqrt{3} \le \eta \le 3.81$.[24] The values of η are also shown for anisotropic DML polytropes, for different values of the anisotropy parameter (Fig. 16 there). For Plummer spheres, one has $\eta \approx 3.71$.

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[23]We are speaking here of MI-MG similarity in predicting the $\Sigma_B^0 - \mathcal{G}$ correlation, not the $\Sigma_B^0 - \Sigma_D^0$ correlation, which AQUAL and QUMOND predict exactly.

[24]The largest value of $\eta = \sqrt{8\pi/\sqrt{3}} \approx 3.81$ is calculated for a DML isothermal sphere.

We see that spheres have η values 3–5, while disks tend to have $\eta \approx 2$.

B. Kuzmin-inspired disk-plus-bulge systems

I shall discuss the CSDC for pure disks in Sec. VI C. Disk-plus-bulge systems are more difficult to treat analytically.

We can, however, derive some exact results in the class of theories described in Sec. V A for a family of disk-plus-bulge mass distributions. Such exact results are important anchors for some of our general discussion, even if they are derived only for toy examples. The results for these mass models are the same in all the theories in the class and are expressed only in terms of their one-dimensional interpolating function $\nu(Y)$.

These models are constructed based on an idea that generalizes the construction of the Kuzmin disk, as described in Ref. [37].

I use these galaxy models here to test how well the acceleration integral $\mathcal{G}$ actually measures the dynamical central surface density Σ_D^0 and to examine the CSDC that our MG theories predict for them.

To construct the baryonic models, start with a spherical density distribution $\rho_s(R)$ with accumulated mass $M(R)$. For the models, the Newtonian potential above the x-y symmetry plane is that of our spherical mass centered on the z axis at $z = -h$. The potential below the symmetry plane is a reflection of the one above. The Newtonian potential satisfies the Poisson equation outside the symmetry plane, with the (baryonic) density

$$\rho_B(r, z) = \rho_s(R), \qquad R = [r^2 + (z + h)^2]^{1/2} \qquad (48)$$

as a source (using cylindrical coordinates r, z); this density distribution constitutes our model baryonic bulge. The Newtonian acceleration field outside the disk is

$$\mathbf{g}_N = -\frac{GM(R)}{R^3}(r, z + h). \qquad (49)$$

The z component of $\mathbf{g}_N$ just above the disk is

$$g_{Nz}^+(r) = -\frac{GM(\bar{R})h}{\bar{R}^3}, \qquad \bar{R} = (r^2 + h^2)^{1/2}, \qquad (50)$$

and just below the disk $g_{Nz}^-(r) = -g_{Nz}^+(r)$. This jump implies that, for the Poisson equation to be satisfied everywhere, the mass distribution has to be supplemented with a thin disk of (baryonic in our context) surface density

$$\Sigma_B^{\text{disk}}(r) = -\frac{1}{2\pi G}g_{Nz}^+(r) = \frac{hM(\bar{R})}{2\pi \bar{R}^3}. \qquad (51)$$

If $\rho_s(R)$ is nonincreasing, then so is $\Sigma_B^{\text{disk}}(r)$. The resulting mass distribution is a disk and a bulge made of the two caps

of the mass distribution $M(R)$. Since we want the bulge mass to be finite, $\Sigma_B^{\text{disk}}(r)$ goes asymptotically to the Kuzmin surface density $\Sigma_B^{\text{disk}}(r) \propto \bar{R}^{-3}$. For example, for a sphere of radius R_0 and constant density ρ_0, we get a disk of constant surface density $2h\rho_0/3$ up to $r = (R_0^2 - h^2)^{1/2}$ and a Kuzmin disk beyond this, plus a constant-density lenslike bulge.[25] The Kuzmin disk is gotten when the whole spherical mass is below the plane; namely, when $M(R > h)$ is constant (e.g., for a point mass).

Despite the complexity of the MG theories described in Sec. V A—they yield a system of many nonlinear, coupled equations—and their generality, the MOND field equations in Eq. (20) can be solved analytically for the above mass models.

The mass distribution and the Newtonian field above the plane are spherical with respect to the center at $r = 0, z = -h$; so, as shown in Sec. V A, there is a solution of the full MOND equations—the one-dimensional solution—whereby $\vec{\nabla}\phi$ and the $\vec{\nabla}\psi_i$ are all aligned with $\vec{\nabla}\phi_N$. This solves the MOND equations above the symmetry plane with ρ_B from Eq. (48) as source, and its reflection solves them below the plane. All the vector fields being radial from some point, they have a vanishing curl. Thus, this MOND solution solves the field equations (20) with the divergences stripped. Namely,

$$a_0^2 \frac{\partial F}{\partial \vec{\nabla}\phi} = \vec{\nabla}\phi_N, \qquad \frac{\partial F}{\partial \vec{\nabla}\psi_i} = 0, \qquad (52)$$

which hold above and below the plane. But this means that this solution also satisfies all the jump conditions across the disk, dictated by Σ_B^{disk} from Eq. (51),[26] since the Newtonian solution does. This establishes it as the MOND solution of the problem.[27]

In particular, the MOND acceleration field above the disk is

$$\mathbf{g} = -\nu\left[\frac{GM(R)}{a_0R^2}\right]\frac{GM(R)}{R^3}(r, z + h), \qquad (53)$$

with $\nu(Y)$ the one-dimensional interpolating function of the theory. The parallel component of $\mathbf{g}$ is continuous across the disk. The jump in the perpendicular MOND field is interpreted as a dynamical surface density (baryonic + phantom)

[25]If we start with the center of the sphere above the plane, we end up with an hourglass-shaped bulge.
[26]The jump conditions from Eq. (20) are that the perpendicular component of $\partial F/\partial \vec{\nabla}\psi_i$ has to be continuous and that of $a_0^2(\partial F/\partial \vec{\nabla}\phi)$ has to be $2\pi G\Sigma_B^{\text{disk}}$.
[27]For this solution $\vec{\nabla}\psi_i = \nu_i(|\vec{\nabla}\phi_N|/a_0)\vec{\nabla}\phi_N$, and in particular, $\vec{\nabla}\phi = \nu(|\vec{\nabla}\phi_N|/a_0)\vec{\nabla}\phi_N$, with $\nu(Y)$ the one-dimensional interpolating function of the theory.

$$\Sigma_D^{\mathrm{disk}}(r) = \nu\left[\frac{GM(\bar{R})}{a_0\bar{R}^2}\right]\Sigma_B^{\mathrm{disk}}(r). \tag{54}$$

The baryonic central surface density is

$$\Sigma_B^0 = 2\int_h^\infty \rho_s(\bar{z})d\bar{z} + \frac{M(h)}{2\pi h^2} = \frac{1}{\pi}\int_h^\infty \frac{M(\bar{z})}{\bar{z}^3}d\bar{z}, \tag{55}$$

where the second equality is gotten by writing $\rho_s(\bar{z}) = M'(\bar{z})/4\pi\bar{z}^2$ and integrating by parts.

The dynamical (MOND) central surface density picks up a contribution $\Sigma_D^{0,\mathrm{disk}}$ from the disk (baryonic plus phantom) and one $\Sigma_D^{0,\mathrm{sph}}$ from the bulge, with

$$\Sigma_D^{0,\mathrm{disk}} = \nu\left[\frac{GM(h)}{a_0h^2}\right]\frac{M(h)}{2\pi h^2}, \tag{56}$$

$$\Sigma_D^{0,\mathrm{sph}} = -\frac{1}{2\pi G}\int_0^\infty \vec{\nabla}\cdot\left\{\nu\left[\frac{GM(z+h)}{a_0(z+h)^2}\right]\mathbf{g}_N\right\}dz, \tag{57}$$

from which

$$\Sigma_D^{0,\mathrm{sph}} = 2\int_0^\infty \nu\left[\frac{GM(z+h)}{a_0(z+h)^2}\right]\rho_s(z+h)dz + \frac{1}{2\pi G}$$
$$\times \int_0^\infty g_N\frac{d\nu}{dz}dz \tag{58}$$

($\mathbf{g}_N$ points to the origin along the z axis, and $g_N = |\mathbf{g}_N|$), and after some manipulations (change of variables from z to $Y = g_N/a_0$ and integration by parts, with $\bar{z} = z+h$)

$$\Sigma_D^0 = \Sigma_D^{0,\mathrm{disk}} + \Sigma_D^{0,\mathrm{sph}} = 2\int_h^\infty \nu\left[\frac{GM(\bar{z})}{a_0\bar{z}^2}\right]\rho_s(\bar{z})d\bar{z}$$
$$+ \Sigma_M\int_0^{g_N^+(0)/a_0}\nu(Y)dY. \tag{59}$$

However, in the last integral, $Y = g_N(0,z)/a_0 = GM(\bar{z})/a_0\bar{z}^2$. So, changing variables from Y to $\bar{z}$,

$$dY = d\bar{z}\left[\frac{-2GM(\bar{z})}{a_0\bar{z}^3} + \frac{4\pi G\rho_s(\bar{z})}{a_0}\right]. \tag{60}$$

Putting all the above together we get

$$\Sigma_D^0 = \frac{1}{\pi}\int_h^\infty \nu\left[\frac{GM(\bar{z})}{a_0\bar{z}^2}\right]\frac{M(\bar{z})}{\bar{z}^3}d\bar{z}. \tag{61}$$

A pure Kuzmin disk is gotten when $M(\bar{z}\geq h) = M_k$ is constant. Then, change variables in the integral to $Y = GM_K/a_0\bar{z}^2$ to obtain

$$\Sigma_D^0 = \Sigma_M\int_0^{\Sigma_B^0/\Sigma_M}\nu(Y)dY = \Sigma_M\mathcal{S}(\Sigma_B^0/\Sigma_M). \tag{62}$$

This is the universal expression predicted by AQUAL/QUMOND; here we see that it is predicted for Kuzmin disks in the general class of MG theories.

The radial MOND acceleration in the plane of the disk is

$$g_r(r) = g(\bar{R})\frac{r}{\bar{R}} = \nu\left[\frac{GM(\bar{R})}{a_0\bar{R}^2}\right]\frac{GM(\bar{R})r}{\bar{R}^3}, \tag{63}$$

from which we get the acceleration integral

$$\mathcal{G} = \int_0^\infty \frac{g(\bar{R})}{\bar{R}}dr = \int_h^\infty \frac{g(\bar{R})}{(\bar{R}^2-h^2)^{1/2}}d\bar{R}. \tag{64}$$

For completeness, we can also calculate the value of $\mathcal{G}$ for these models, with the MI expression, whereby the accelerations are determined from rotational speeds of exactly circular trajectories. In this case, the MOND radial acceleration is related to the Newtonian radial acceleration by $g_r^{\mathrm{MI}} = \nu(g_r^N/a_0)g_r^N$ instead of $g_r^{\mathrm{MG}} = \nu(g^N/a_0)g_r^N$ (the difference being that, in the former, the r component appears in the argument of ν instead of the full Newtonian acceleration in the latter). We are now equipped to examine the correlations we are after.

1. The $\Sigma_D^0 - \mathcal{G}$ correlation

Using expression (61) for Σ_D^0 and expression (64) for $\mathcal{G}$, we get

$$\kappa = \frac{2\pi G\Sigma_D^0}{\mathcal{G}} = 2\int_h^\infty \frac{g(\bar{R})d\bar{R}}{\bar{R}}\left[\int_h^\infty \frac{g(\bar{R})d\bar{R}}{(\bar{R}^2-h^2)^{1/2}}\right]^{-1}. \tag{65}$$

The two integrals differ only by the replacement of one power of $\bar{R}$ in one by $(\bar{R}^2-h^2)^{1/2}$ in the other. For $h=0$, for which the model is spherical, we get $\kappa = 2$ as expected. For $h > 0$, the second integral is larger than the first and $k < 2$.

For a Kuzmin disk, for which $M(\bar{R} > h) = M_K$ constant, we have

$$\kappa = 2\int_h^\infty \nu\left[\frac{GM_K}{a_0\bar{R}^2}\right]\frac{d\bar{R}}{\bar{R}^3}\left\{\int_h^\infty \nu\left[\frac{GM_K}{a_0\bar{R}^2}\right]\frac{d\bar{R}}{\bar{R}^2(\bar{R}^2-h^2)^{1/2}}\right\}^{-1}. \tag{66}$$

In the Newtonian limit, $\nu \equiv 1$, one gets $\kappa = 1$, as expected from Toomre's result. However, in MOND, the dynamical mass acquires a phantom bulge, so we expect κ to be between 1 and 2. Indeed, the ν factor in the integrands increases with increasing values of $\bar{R}$. This causes the value of κ to increase over its value for $\nu = 1$. For example, in the DML, where we can put $\nu(Y) = Y^{-1/2}$, we get from expression (66) $\kappa = 4/\pi$.

2. The $\Sigma_D^0 - \Sigma_B^0$ correlation

We have several expressions for the Σ_D^0 and the corresponding ones for Σ_B^0, gotten from them by putting $\nu(Y) \equiv 1$. Such pairs can be used to asses the level of the CSDC in our class of theories. All these theories (including AQUAL and QUMOND) give the same results for the Kuzmin-like, disk-plus-bulge mass models, when expressed in terms of the one-dimensional interpolating function $\nu(Y)$.

Take, for example, the pair Σ_B^0 from Eq. (55) (second equality) and Σ_D^0 from Eq. (61). As expected, there is no universal functional relation between the two, as there is for AQUAL and QUMOND for pure disks. We already saw that for a spherical mass the correlation does depend on the mass distribution.

As regards the asymptotes, the Newtonian limit takes the form that it always does, $\Sigma_P^0 = \Sigma_D^0 - \Sigma_B^0 \to \alpha \Sigma_M$, with α depending only on the exact form of $\nu(Y)$, as given in Eq. (46). In the DML, we get the value of $\eta \equiv \Sigma_D^0/(\Sigma_M \Sigma_B^0)^{1/2}$ from Eqs. (55) and (61),

$$\eta = \sqrt{2} \frac{\int_h^\infty \bar{z}^{-2} M^{1/2}(\bar{z}) d\bar{z}}{[\int_h^\infty \bar{z}^{-3} M(\bar{z}) d\bar{z}]^{1/2}}. \tag{67}$$

As predicted by the basic tenets, we see that η does not depend on the normalization of M, nor on the characteristic scale lengths of the mass distribution, as long as we scale them and h by the same factor. For $M(R)$ constant beyond $R = h$, we have the Kuzmin pure disk, and expression (67) gives $\eta = 2$. For $h = 0$ we get the result for the spherical case, Eq. (47). For AQUAL/QUMOND this is the value that applies to all pure disks; here we derived it for a Kuzmin disk in the general class of theories.

C. The CSDC for pure disks

We saw that AQUAL and QUMOND predict a universal (i.e., disk independent) $\Sigma_B^0 - \Sigma_D^0$ relation for pure disks, the relation hinging only on the ν interpolating function of these theories. We also saw that the more general class of MG theories predict the same relation, but only for Kuzmin disks, in which the one-dimensional interpolating function of the specific theory appears in the same way. A Kuzmin disk is a mass model whose gravitational field is describable, in some sense, in terms of a one-dimensional mass model, so perhaps this is not so surprising. But, in the general class of theories, it cannot be expected that a similar strict algebraic relation holds for general pure disks and, in particular, that only the one-dimensional interpolating function will define a CSDC for such disks. What can we, nevertheless, say about such a CSDC?

One general result that applies to all the theories in the class discussed in Sec. V A is that there appears a phantom disk, whose local surface density Σ_P^{disk} depends only on the local baryonic disk surface density Σ_B^{disk} according to

$$\Sigma_P^{\text{disk}} = \left[\nu\left(\frac{\Sigma_B^{\text{disk}}}{\Sigma_M}\right) - 1\right]\Sigma_B^{\text{disk}}, \tag{68}$$

where $\nu(Y)$ is the one-dimensional interpolating function of the theory, which is derived from Eqs. (22) and (23). This results from the fact that the jump conditions of the MOND and the Newtonian fields across a thin disk are related by the one-dimensional behavior of the theory. Relation (68) holds, in particular, for the relation between the central surface densities $\Sigma_P^{0,\text{disk}}$ and Σ_B^{disk}.[28,29]

There is no baryonic mass outside the disk, so there we have the contribution to the dynamical, central surface density from a phantom bulge

$$\Sigma_D^0(\text{out}) = \frac{1}{2\pi G} \int_{0^+}^\infty \Delta\phi \, d\zeta, \tag{69}$$

where ϕ is the MOND potential gotten by solving Eq. (20), and the integration is along the z axis from $\zeta = 0^+$ just outside the disk. (Here and hereafter, I use ζ for the z coordinate to avoid confusion with the above use of z as a scalar variable in the TRIMOND Lagrangian.)

The formulation of the general Lagrangian is not specific enough for me to derive further general results, relevant to the CSDC; so I will confine myself to the subclass of TRIMOND theories, where the MOND potential enters the gravitational Lagrangian in a specific manner, as explained in Sec. V B.

1. The pure-disk CSDC in TRIMOND

In TRIMOND theories, ϕ in Eq. (69) is to be determined from the TRIMOND field equations (31) and (32). Given that the Newtonian field ψ satisfies $\Delta\psi = 0$ outside the disk, we can get from Eq. (31),

$$\Delta\varphi = -(\vec{\nabla}\psi \cdot \vec{\nabla}\mathcal{F}_z + \vec{\nabla}\varphi \cdot \vec{\nabla}\mathcal{F}_y)/\mathcal{F}_y. \tag{70}$$

Substituting in Eq. (32) we get an expression for $\Delta\phi$,

$$\Delta\phi = \vec{\nabla}\psi \cdot \vec{\nabla}\mathcal{F}_x + \vec{\nabla}\varphi \cdot \vec{\nabla}\mathcal{F}_z - \frac{\mathcal{F}_z}{\mathcal{F}_y}$$
$$\times (\vec{\nabla}\psi \cdot \vec{\nabla}\mathcal{F}_z + \vec{\nabla}\varphi \cdot \vec{\nabla}\mathcal{F}_y). \tag{71}$$

In the integrand in Eq. (69) $\Delta\phi$ is needed only on the ζ axis, where, from the symmetry, all the gradients are along the axis, so we can write

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[28]Such a phantom disk also occurs when there is a bulge present.

[29]Note that this is a secondary prediction of MOND that holds in this class of MG theories, but it does not follow from the basic tenets. In Ref. [34], I showed that MI theories, for example, can make rather different predictions here. In particular, the notion of phantom disk is not defined in such theories.

$$\Sigma_D^0(\text{out}) = \frac{1}{2\pi G}\int_{0^+}^\infty d\zeta \left(\frac{d\mathcal{F}_x}{d\zeta} - \frac{\mathcal{F}_z}{\mathcal{F}_y}\frac{d\mathcal{F}_z}{d\zeta}\right) g_N$$
$$+ \left(\frac{d\mathcal{F}_z}{d\zeta} - \frac{\mathcal{F}_z}{\mathcal{F}_y}\frac{d\mathcal{F}_y}{d\zeta}\right) g_\varphi. \tag{72}$$

This expression is useful because the integrand now depends only on the accelerations on the ζ axis, since all the second derivatives have disappeared. On the ζ axis, $\mathcal{F}_x$, $\mathcal{F}_y$, $\mathcal{F}_z$ are also functions of only g_N and g_φ.

Near the disk and far from it—more specifically, at ζ values small and large compared with the central, characteristic size of the disk—the gravitational field becomes one-dimensional, so the algebraic relations (33) and (34) hold. Had this been the case all along the ζ axis, we would have gotten the same relation (involving the one-dimensional interpolating function of the theory) between Σ_D^0 and Σ_B^0 as we have for Kuzmin disks, where, indeed, the field outside the disk is one dimensional. Thus, the deviation from such a relation, and the dependence on the specific disk model, result from the contribution of the intermediate region on the ζ axis.

My aim here is, indeed, to demonstrate in more detail why, in general, the relation between Σ_D^0 and Σ_B^0 for pure disks is not universal, as in the special cases of AQUAL and QUMOND. For this, it is enough to further confine ourselves to TRIMOND theories in which $\mathcal{F}_z = \epsilon$ is a constant (AQUAL and QUMOND are such cases). Equation (72) then reduces to

$$\Sigma_D^0(\text{out}) = \frac{1}{2\pi G}\int_{0^+}^\infty \left(\frac{d\mathcal{F}_x}{d\zeta} g_N - \frac{\epsilon}{\mathcal{F}_y}\frac{d\mathcal{F}_y}{d\zeta} g_\varphi\right) d\zeta. \tag{73}$$

The field configuration near the disk (e.g., at 0^+) is one dimensional and the three accelerations are related there by the one-dimensional relations (33) and (34) reduced to our special case,

$$\mathcal{F}_y^+ g_\varphi^+ + \epsilon g_N^+ = 0, \tag{74}$$

$$g^+ = \mathcal{F}_x^+ g_N^+ + \epsilon g_\varphi^+. \tag{75}$$

Since the only quantities that enter the integrand in Eq. (73) are the acceleration values on the ζ axis, *for a given disk*, we can think of one quantity in the integrand as a function of any other. For example, along the ζ axis, $\mathcal{F}_x(\zeta)$ can be treated as a function $\nu_1(Y)$, where $Y \equiv g_N$ (in this section, again, we put $a_0 = 1$ for convenience). Strictly speaking, this is only so if $Y(z)$ is monotonic from its value g_N^+ to 0 at infinity, because then $Y(\zeta)$ can be inverted and $\zeta(Y)$ gives $\mathcal{F}[\zeta(Y)]$. There are disks for which this is not so; e.g., for a disk with a hole at the center, g_N vanishes at the origin and at infinity. In this case we need to make the treatment piecewise. But here, for the demonstration, we deal with disks for which inversion is possible.

$\nu_1[Y(\zeta)]$ is not universal for the given theory, like the one-dimensional interpolating function, depending not only on the disk model, but also on the value of Σ_B^0 relative to Σ_M; it is defined only on the ζ axis and only for argument values between 0 and g_N^+.

Write, then, the integral of the first term in Eq. (73),

$$\int_{0^+}^\infty \left[\frac{d}{d\zeta}[Y\nu_1(Y)] - \nu_1(Y)\frac{dY}{d\zeta}\right] d\zeta$$
$$= -[Y\nu_1(Y)]^{0^+} + \int_{Y(\infty)}^{Y(0^+)} \nu_1(Y)dY$$
$$= -g_N^+ \nu_1(g_N^+) + \int_0^{g_N^+} \nu_1(Y)dY. \tag{76}$$

In a similar vein, on the ζ axis, we can think of $\mathcal{F}_y$ as a function of $g_\varphi(\zeta)$: $F_y(\zeta) = \mu[g_\varphi(\zeta)]$. Then, if we denote

$$Y = -\epsilon^{-1}X\mu(X) \tag{77}$$

and invert this relation, then $\nu_2(Y)$ is defined such that

$$X = \epsilon^{-1}Y\nu_2(Y). \tag{78}$$

Take the X derivative of Eq. (77) to get

$$-\frac{\epsilon}{\mu}\frac{dY}{dX} - 1 = \frac{X\mu'(X)}{\mu}. \tag{79}$$

But, from Eqs. (77) and (78) we have

$$1/\mu = -\epsilon^{-2}\nu_2, \tag{80}$$

and the second integrand in Eq. (73) is $(-\epsilon X/\mu)\mu'(dX/d\zeta)$. Then, the second integral can be written as

$$-\epsilon X(0^+) + \int_0^{g_N^+} \nu_2(Y)dY, \tag{81}$$

where we used the fact that at 0^+ the one-dimensional relations hold; so, for example, $Y(0^+) = -\epsilon^{-1}X\mu(X)(0^+) = g_N^+$ [from Eq. (74)]. All in all, we get

$$\Sigma_D^0(\text{out}) = \frac{1}{2\pi G}\left\{-\mathcal{F}_x^+ g_N^+ - \epsilon X^+ + \int_0^{g_N^+}[\nu_1(Y)\right.$$
$$\left. + \nu_2(Y)]dY\right\}. \tag{82}$$

From Eq. (75) the first two terms give $-(2\pi G)^{-1}g^+ = -\Sigma_D^0(\text{disk})$, which cancels with the contribution of the disk to Σ_D^0, to give (restoring a_0)

$$\Sigma_D^0 = \Sigma_M \int_0^{\Sigma_B^0/\Sigma_M} \nu^*(Y)dY \equiv \Sigma_M \mathcal{S}^*(\Sigma_B^0/\Sigma_M), \tag{83}$$

where $\nu^*(Y) = \nu_1(Y) + \nu_2(Y)$. This is similar in form to the universal AQUAL/QUMOND relation (13), but here $\mathcal{S}^*$ is not universal. In fact, being too dependent on the model, $\nu^*(Y)$ is not very useful for calculations, as it is not known beforehand for a given theory and model. I use this concept here only to better bring out the differences from AQUAL and QUMOND.

Indeed, in the above derivation, we encounter, in a sense, a combination of AQUAL and QUMOND. Remember that QUMOND is a special case with $\epsilon = 0$, so $\nu_2 = 0$, and $\nu^* = \nu_1$ is then the interpolating function of QUMOND. For AQUAL, $\mathcal{F}_x \equiv 0$; so, $\nu_1 = 0$, and $\nu^* = \nu_2$ is the AQUAL, ν-type interpolating function.

Also, ν^* here has the same asymptotic limits as ν. For $Y \gg 1$, the statement around Eq. (35), on the Newtonian limit of TRIMOND, tells us that in this limit $\nu_1 = \mathcal{F}_x \to \beta$, and from Eq. (80) and the definition of $\mu = \mathcal{F}_y$, $\nu_2 \to -\epsilon^2/\omega$. So, by Eq. (35), $\nu_1 + \nu_2 \to 1$.

2. The deep-MOND limit of TRIMOND theories with $\mathcal{F}_z$ constant

I now describe in some detail the form that the DML equations take for this subclass, also as an opportunity to see how the DML in TRIMOND theories can differ from that in AQUAL/QUMOND, the Lagrangian of which involves a function of a single variable. The DML in TRIMOND, whose Lagrangian involves generally a function of three variables, was discussed in Ref. [30].

The Lagrangian of the subclass I consider here involves a function of two variables and has the form

$$\mathcal{F}(x, y, z) = \epsilon z + \bar{\mathcal{F}}(x, y). \tag{84}$$

From the requirement of scale invariance in the DML, expressed as Eq. (36), the DML of $\mathcal{F}$ has to be such that

$$\epsilon z + \bar{\mathcal{F}}(x, y) = \lambda^3 [\epsilon \lambda^{\alpha-3} z + \bar{\mathcal{F}}(\lambda^{-4} x, \lambda^{\alpha-2} y)], \tag{85}$$

for some α (which is the scaling dimension of φ) and for any $\lambda > 0$. This means that we have to take $\alpha = 0$, and the DML of $\bar{\mathcal{F}}$ has to satisfy

$$\bar{\mathcal{F}}(x, y) = \lambda^3 \bar{\mathcal{F}}(\lambda^{-4} x, \lambda^{-2} y). \tag{86}$$

This implies that $\bar{\mathcal{F}}$ is of the form (taking $\lambda = x^{1/4}$)

$$\bar{\mathcal{F}}(x, y) = x^{3/4} F(y/x^{1/2}). \tag{87}$$

Any choice of F would give a scale-invariant DML, pending some consistency requirement, such as existence and uniqueness of solutions. In contradistinction from AQUAL and QUMOND, where the DML of the theory is completely fixed by the requirement of scale invariance, here one still needs to specify a function of one variable to specify the theory's DML (for general TRIMOND theories,

one is left with the choice of a function of two variables). From Eq. (87) follows that, in the DML,

$$\mathcal{F}_x = \frac{3}{4} x^{-1/4} F - \frac{1}{2} y x^{-3/4} F', \quad \mathcal{F}_y = x^{1/4} F', \quad \mathcal{F}_z = \epsilon. \tag{88}$$

The standard choice of the normalization of a_0 is such that, asymptotically far from a mass, we have $g = \sqrt{a_0 g_N}$. This normalization still has to be imposed, as follows: In the spherical case, which is relevant to this, we use the algebraic relation (33), which reads here

$$F'(q) q^{1/2} + \epsilon = 0, \qquad q \equiv y x^{-1/2}. \tag{89}$$

This implies that in a one-dimensional, DML system, $y = q_0 x^{1/2}$, where q_0 is the solution of Eq. (89). Thus, for the theory to be consistent, $F'(q)$ has to be such that this equation has a solution and that the solution is unique.

The second one-dimensional equation (34) then tells us that

$$g = \frac{3}{4} [F(q_0) - 2q_0 F'(q_0)] g_N^{1/2}. \tag{90}$$

We want to choose the still-free additive constant to F, so that in this case $g = g_N^{1/2}$ ($a_0 = 1$).

In summary, in constructing such a theory, we choose F up to an additive constant and a value of ϵ. Then, Eq. (89) determines q_0, and Eq. (90) determines the additive constant.

While, from their definition, $\nu_1(Y)$ and $\nu_2(Y)$, and hence $\nu^*(Y)$, scale as $Y^{-1/2}$ for $Y \to 0$, the proportionality constant depends on the disk model. While in AQUAL/QUMOND, the canonical normalization of a_0 determines the DML normalization $\nu(Y) = Y^{-1/2}$, here it does not determine that of the DML of ν^* and hence of the value of η.

VII. SUMMARY

The CSDC is a correlation between the dynamical central surface density Σ_D^0 of an axisymmetric and plane-symmetric galactic system and the baryonic central surface density Σ_B^0. Various versions of the CSDC emerge from various MOND starting points.

I have emphasized, in the first place, that the notions of dynamical density and dynamical surface density are defined only in modified-gravity theories. In these, the Newtonian potential is modified into a MOND potential, from which one can define a dynamical density: the density that would source the MOND potential in a Poisson equation. In modified-inertia versions of MOND there is no gravitational potential defined, and so no dynamical density is defined. However, since in actual tests one uses proxies for the dynamical surface density, and these may be

well defined in MI theories, the CSDC may emerge in the MI context in some circumstances.

I showed that from MOND's defining tenets alone follows a CSDC in a basic form: The underlying tenets pertain to the behavior of the theory at high and low accelerations, but do not specify the interpolation scheme between these behaviors, and how it takes place in different phenomena.[30] So, they can predict only the high- and low-surface-density ends of the CSDC.

At the high end, $\Sigma_B^0 \gg \Sigma_M$, where Σ_M is the MOND critical surface density, it is predicted that the phantom surface density $\Sigma_P^0 \equiv \Sigma_D^0 - \Sigma_B^0 \to \gamma\Sigma_M$, with $\gamma = O(1)$ that depends on the theory, but not on the system, because in this limit the contribution to Σ_P^0 comes from far outside the system, where it can be thought of as a point mass, whose structure is immaterial here.

At the other end, $\Sigma_B^0 \ll \Sigma_M$, it is predicted that $\Sigma_D^0 = \eta(\Sigma_M\Sigma_B^0)^{1/2}$, with $\eta = O(1)$ that does not depend on the mass or overall size of the system, but that can depend on dimensionless attributes of the system, such as whether we have a sphere, a pure disk, or a combination, or generally on ratios of scale-length attributes of the system.

This makes a rough CSDC, as described above, a primary prediction of MOND that is shared by all MOND theories that embody the basic axioms.

Specific MOND theories can predict more constrained forms of a CSDC, especially if we also consider specific system geometries. For example, the AQUAL and QUMOND theories predict for pure disks a universal (disk-independent), tight algebraic relation $\Sigma_D^0(\Sigma_B^0)$.

As other examples, I considered the detailed predictions of a large class of modified-gravity theories that include AQUAL, QUMOND, and TRIMOND as special cases. They all coincide in their predictions for one-dimensional systems, such as spherical systems or near-disk dynamics. Such predictions then hinge only on the so-called one-dimensional interpolating function—a function of the Newtonian acceleration of the system.

Importantly, though, and in contradistinction from the special cases of AQUAL and QUMOND, these theories do not predict a universal relation for pure disks, with the exception of the Kuzmin disk for which they all make the same prediction of a tight algebraic CSDC. In particular, the value of η that describes the low-Σ_B^0 limit can depend somewhat on the exact disk model and on the exact theory. Such details of the CSDC are thus secondary MOND predictions, as they do not follow from the basic tenets alone.

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[30]The only assumption about the interpolation is that it occurs within a range of order a_0 around a_0, so that the theory does not involve dimensionless constants $\gg 1$ or $\ll 1$.

APPENDIX: RELATION BETWEEN Σ_D^0 AND THE ACCELERATION INTEGRAL $\mathcal{G}$ FOR SOME MASS MODELS

Here I derive the relation between central surface density and the acceleration integral for some mass models. To avoid confusion, note that, because this relation is linear and is derived on the basis of Newtonian dynamics, these models model not the baryonic mass distribution of the galaxy, but its MOND, dynamical mass distribution (baryonic plus phantom). Hence these surface densities are designated with a subscript D. For example, a pure baryonic disk has both a disk and a spheroidal ("bulge") component of dynamical mass, so the Miyamoto-Nagai models below represent this compound distribution.

1. Spheroids

Consider a thin spherical shell of radius r_s and surface density Σ_s, so its mass is $M_s = 4\pi r_s^2 \Sigma_s$. The (Newtonian) acceleration vanishes inside the shell, and outside it is $g(r) = GM_s/r^2$. Its central surface density is $2\Sigma_s$, and the acceleration integral is

$$\mathcal{G} \equiv \int_0^\infty g\, d\ln r = \frac{GM_s}{2r_s^2} = 2\pi G\Sigma_s, \qquad \text{(A1)}$$

so it has $\kappa = 2$, as expected.

Contracting or expanding the shell by a factor q along the z axis, we get a homoeoid thin shell, which still produces zero acceleration inside it. The acceleration along the equator, a distance $r > r_s$ from the center is $g(r) = \Sigma_s f(q, r_s, r)$, but on dimensional grounds it has to be of the form

$$g(r) = 4\pi G\Sigma_s \hat{f}(q, r/r_s). \qquad \text{(A2)}$$

The acceleration integral is thus

$$\mathcal{G} = 4\pi G\Sigma_s \int_{r_s}^\infty \hat{f}(q, r/r_s)\, d\ln r = 4\pi G\Sigma_s$$
$$\times \int_1^\infty \hat{f}(q, x)\, d\ln x \equiv 4\pi G\Sigma_s/\kappa(q). \qquad \text{(A3)}$$

We see then that, for this shell, κ defined in Eq. (7) is $\kappa(q)$, which depends only on q.

Because of the additivity of the central surface density and of $\mathcal{G}$, the same relation holds for any spheroid of r-independent oblateness factor q.

Clearly, $\kappa(q)$ is a decreasing function of q: the smaller q is, the nearer the shell elements are to a point on the external equator and, in addition, the larger the radial component is. So, $\mathcal{G}$ increases with decreasing q, but Σ_D^0 is independent of q, so κ decreases.

To get an explicit expression for $k(q)$, start from Eq. (2-91) in Ref. [24] (somewhat paraphrased), which gives the acceleration in the midplane for an axisymmetric spheroid, of density distribution (in cylindrical coordinates r, z)

$$\rho(r,z) = \hat{\rho}(y), \qquad y \equiv \left(r^2 + \frac{z^2}{q^2}\right)^{1/2}, \qquad \text{(A4)}$$

as

$$g(r) = \frac{4\pi G q}{r} \int_0^r \frac{\hat{\rho}(y) y^2 dy}{\sqrt{r^2 - e^2 y^2}}, \qquad e = \sqrt{1 - q^2}. \qquad \text{(A5)}$$

The acceleration integral is then

$$\mathcal{G} = 4\pi G q \int_0^\infty \frac{dr}{r^2} \int_0^r \frac{\hat{\rho}(y) y^2 dy}{\sqrt{r^2 - e^2 y^2}}, \qquad \text{(A6)}$$

which after some rearrangement and changing the order of the integration gives

$$\mathcal{G} = 4\pi G q \int_0^1 \frac{x dx}{\sqrt{1 - e^2 x^2}} \int_0^\infty \hat{\rho}(u) du = \frac{4\pi G q}{1 + q} \int_0^\infty \hat{\rho}(u) du. \qquad \text{(A7)}$$

The central surface density is

$$\Sigma_D^0 = 2 \int_0^\infty \rho(0, z) dz = 2q \int_0^\infty \hat{\rho}(u) du. \qquad \text{(A8)}$$

Putting the two together we get from the definition of κ, Eq. (7),

$$\kappa(q) = 1 + q. \qquad \text{(A9)}$$

2. Miyamoto-Nagai models

For Miyamoto-Nagai models the potential field is

$$\phi(r, z) = -\frac{MG}{\{r^2 + [a + (z^2 + b^2)^{1/2}]^2\}^{1/2}}. \qquad \text{(A10)}$$

So, in the symmetry plane $g_r(r, 0) = MGr/[r^2 + (a + b)^2]^{3/2}$ yielding

$$\mathcal{G} = \frac{MG}{(a + b)^2}. \qquad \text{(A11)}$$

The source density along the z axis is (e.g., Ref. [24])

$$\rho(0, z) = \left(\frac{Mb^2}{4\pi}\right) \frac{a + 3(z^2 + b^2)^{1/2}}{[a + (z^2 + b^2)^{1/2}]^3 (z^2 + b^2)^{3/2}}, \qquad \text{(A12)}$$

and the expression for Σ_D^0 can be brought to the form

$$\Sigma_D^0 = \left(\frac{M}{2\pi b^2}\right) \int_0^1 \frac{x^3(3 + \lambda x) dx}{(1 + \lambda x)^3 \sqrt{1 - x^2}}, \qquad \text{(A13)}$$

from which

$$\kappa = (1 + \lambda)^2 \int_0^1 \frac{x^3(3 + \lambda x) dx}{(1 + \lambda x)^3 \sqrt{1 - x^2}}, \qquad \text{(A14)}$$

where $\lambda = a/b$. As λ goes from 0 (Plummer sphere) to ∞ (Kuzmin disk), κ goes, monotonically, from 2 to 1.

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