No black hole bomb for D-dimensional nonextremal Reissner-Nordström black holes against charged massive scalar perturbation

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The superradiant stability of asymptotically flat D-dimensional nonextremal Reissner-Nordström black holes under charged massive scalar perturbation is analytically studied. In previous works, it was proved that there are no black hole bombs for five- and six-dimensional nonextremal Reissner-Nordström black holes against charged massive scalar perturbation. In this work, we extend the previous discussions to the D-dimensional case ($D \geq 7$) and find that the same conclusion holds in arbitrary higher dimensional case.

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I. INTRODUCTION

Black holes are important objects in both theoretical and observational physics. Linear (in)stability analysis of black holes plays an important role in many topics, such as the (in)stability of black hole solutions, the black hole ring-down phase after binary merger and astrophysics [1–3]. When a charged rotating black hole is scattered off by a charged bosonic field, the electromagnetic or (and) the rotational energy of the black hole may be extracted by the external field under certain conditions. This phenomenon is called superradiance [4–6]. In order to trigger the superradiance, the angular frequency ω of the scattered field should satisfy the following superradiance condition:

$$\omega < \mathrm{m}\Omega_H + e\Phi_H, \tag{1}$$

where e and m are the charge and azimuthal number of the bosonic wave mode, Ω_H is the angular velocity of the black hole horizon and Φ_H is the electromagnetic potential of the black hole horizon. The superradiant scattering was studied a long time ago [7–12] and has broad applications in various areas of physics (for a comprehensive review, see Ref. [4]). If there is a "mirror" between the black hole horizon and spatial infinity, the amplified perturbation will be scattered back and forth between the "mirror" and black hole horizon, and this will lead to the superradiant instability of the system. This is the so-called black hole bomb mechanism [13–16].

The superradiant (in)stability of four-dimensional rotating Kerr black holes under massive scalar or vector perturbation has been studied in [17–31]. Rotating or charged black holes with certain asymptotically curved space are proved to be superradiantly unstable under massless or massive bosonic perturbation [32–43], where the asymptotically curved geometries provide natural mirror-like boundary conditions. The four-dimensional asymptotically flat extremal or nonextremal Reissner-Nordström (RN) black hole has proved superradiantly stable against charged massive scalar perturbation [44–49]. The argument is that the two conditions for the possible superradiant instability of the system, (i) existence of a trapping potential well outside the black hole horizon and (ii) superradiant amplification of the trapped modes, cannot be satisfied simultaneously in the RN black hole and scalar perturbation system [44,46].

The linear stability of higher dimensional black holes has also been studied in the literature (for an incomplete list, see Refs. [50–60]). In Ref. [52], the asymptotically flat RN black holes in $D = 5,6,..,11$ are shown to be stable by studying the time-domain evolution of the massless scalar perturbation with a numerical method. In Ref. [53], the authors provided numerical evidence that asymptotically flat extremal RN black holes are stable for arbitrary D under massless perturbation.

Recently, an analytical method based on the *Descartes' rule of signs* has been used to study the superradiant stability of higher dimensional RN black holes under charged massive scalar perturbation [61–63]. It proved that there is no black hole bombs for five- and six-dimensional (non)extremal RN black holes under charged massive scalar perturbations and the system is superradiantly stable. It is also found that the above conclusion

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still holds for the D-dimensional extremal RN black hole case [64].

In this work, we will use the above mentioned analytical method to study the superradiant stability of D-dimensional ($D \geq 7$) nonextremal RN black hole against charged massive scalar perturbation. We show that there is no potential well for the effective potential experienced by the scalar perturbation. The conditions for the possible black hole bomb cannot be satisfied simultaneously, so there is no black hole bomb for the D-dimensional nonextremal RN black hole and charged massive scalar perturbation system.

The paper is organized as follows: In Sec. II, we present the description of the model and the asymptotic analysis of boundary conditions. In Sec. III, the effective potential of the radial equation of motion is given, and the asymptotic behaviors of the effective potential at the horizon and spatial infinity are discussed. In Sec. IV, we give a brief description of the proof of our main result that there is no potential well outside the black hole horizon for the superradiant bound modes. The details of the proof are in the Appendix. The final section is devoted to the summary.

II. MODEL DESCRIPTION

In this section, we present the model in which we are interested, i.e. a D-dimensional nonextremal RN black hole against charged massive scalar perturbation. The metric of the D-dimensional nonextremal RN black hole [61,62,65] is

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{D-2}^2. \tag{2}$$

The function $f(r)$ reads

$$f(r) = 1 - \frac{2m}{r^{D-3}} + \frac{q^2}{r^{2(D-3)}}, \tag{3}$$

where the parameters m and q are related with the Arnowitt-Deser-Misner mass M and electric charge Q of the RN black hole,

$$m = \frac{8\pi}{(D-2)\mathrm{Vol}(S^{D-2})} M,$$
$$q = \frac{8\pi}{\sqrt{2(D-2)(D-3)}\mathrm{Vol}(S^{D-2})} Q. \tag{4}$$

Here $\mathrm{Vol}(S^{D-2}) = 2\pi^{\frac{D-1}{2}}/\Gamma(\frac{D-1}{2})$ is the volume of unit $(D-2)$ sphere. $d\Omega_{D-2}^2$ is the common line element of a $(D-2)$-dimensional unit sphere S^{D-2} and can be written as

$$d\Omega_{D-2}^2 = d\theta_{D-2}^2 + \sum_{i=1}^{D-3} \prod_{j=i+1}^{D-2} \sin^2(\theta_j) d\theta_i^2, \tag{5}$$

where the ranges of the angular coordinates are taken as $\theta_i \in [0, \pi] (i = 2, \ldots, D-2), \theta_1 \in [0, 2\pi]$. The inner and outer horizons of this RN black hole are

$$r_\pm = \left(m \pm \sqrt{m^2 - q^2}\right)^{1/(D-3)}. \tag{6}$$

The event horizon of the black hole is located at $r_h = r_+$. We introduce two symbols u, v, defined as $u = r_+^{D-3}$, $v = r_-^{D-3}$. It is obvious that we have the following two equalities:

$$u + v = 2m, \qquad uv = q^2. \tag{7}$$

The electromagnetic field outside the black hole horizon is described by the following 1-form vector

$$A = -\sqrt{\frac{D-2}{2(D-3)}} \frac{q}{r^{D-3}} dt = -c_D \frac{q}{r^{D-3}} dt. \tag{8}$$

The equation of motion for a charged massive scalar perturbation in this D-dimensional nonextremal black hole background is governed by the covariant Klein-Gordon equation

$$(D_\nu D^\nu - \mu^2)\phi = 0, \tag{9}$$

where $D_\nu = \nabla_\nu - ieA_\nu$ is the covariant derivative and μ, e are the mass and charge of the scalar field, respectively. Since the RN black hole is stationary, the solution of the above equation with definite angular frequency can be written as

$$\phi(t, r, \theta_i) = e^{-i\omega t} R(r) \Theta(\theta_i). \tag{10}$$

The angular eigenfunctions $\Theta(\theta_i)$ are $(D-2)$-dimensional scalar spherical harmonics and the corresponding eigenvalues are given by $-l(l+D-3), (l = 0, 1, 2, ..)$ [66–70].

The radial equation of motion is described by

$$\Delta \frac{d}{dr}\left(\Delta \frac{dR}{dr}\right) + UR = 0, \tag{11}$$

where

$$\Delta = r^{D-2} f(r),$$
$$U = (\omega + eA_t)^2 r^{2(D-2)} - l(l+D-3) r^{D-4} \Delta - \mu^2 r^{D-2} \Delta. \tag{12}$$

The physical boundary conditions needed here are purely ingoing wave at the outer horizon and exceptionally decaying wave at spatial infinity, which means we discuss the quasibound states. Thus, the asymptotic solution of the radial equation (11) at the outer horizon is

$$\lim_{r\to r_h} R(r) \sim (r-r_h)^{-i\sigma}, \quad \sigma = \frac{r_h^{D-2}(\omega-\omega_c)}{(D-3)(r_h^{D-3}-r_-^{D-3})}, \tag{13}$$

where

$$\omega_c = e\Phi_h = c_D \frac{eq}{r_h^{D-3}} \tag{14}$$

is the critical superradiance frequency and Φ_h is the electric potential of the outer horizon of the RN black hole.

The asymptotic solution of the radial equation (11) at the spatial infinity is

$$\lim_{r\to+\infty} R(r) \sim r_h^{-\frac{D-2}{2}} e^{-\sqrt{\omega^2-\mu^2}r}. \tag{15}$$

The superradiance condition and the bound state condition are, respectively,

$$\mathrm{Re}(\omega) < e\Phi_h = c_D \frac{eq}{r_+^{D-3}}, \tag{16}$$

$$\mathrm{Re}(\omega) < \mu. \tag{17}$$

III. EFFECTIVE POTENTIAL AND ITS ASYMPTOTIC BEHAVIORS

In order to analyze the superradiant stability of the RN black hole and scalar perturbation system, we define a new radial function $\psi = \Delta^{1/2}R$, then the radial equation of motion (11) can be written as a Schrodinger-like equation

$$\frac{d^2\psi}{dr^2} + (\omega^2 - V)\psi = 0, \tag{18}$$

where V is the effective potential, which is the main object we will discuss. The explicit expression for the effective potential V is $V = \omega^2 + \frac{B_1}{A_1}$ and

$$A_1 = 4r^2(r^{2D-6} - 2mr^{D-3} + q^2)^2, \tag{19}$$

$$\begin{aligned} B_1 = {} & 4(\mu^2-\omega^2)r^{4D-10} + (2l+D-2)(2l+D-4)r^{4D-12} - 8(m\mu^2 - c_D eq\omega)r^{3D-7} \\ & - 4m(2\lambda_l + (D-4)(D-2))r^{3D-9} + 4q^2(\mu^2 - c_D^2 e^2)r^{2D-4} \\ & - 2(2m^2 - q^2(2\lambda_l + 3(D-4)(D-2)+2))r^{2D-6} - 4mq^2(D-4)(D-2)r^{D-3} + q^4(D-4)(D-2), \end{aligned} \tag{20}$$

where $\lambda_l = l(l+D-3)$.

A. Asymptotic behaviors of the effective potential

Defining $D_1 = D^2 - 6D + 8 = (D-2)(D-4)$, the derivative of effective potential V reads

$$V'(r) = -\frac{C(r)}{2r^3(r^{2(D-3)} - 2mr^{D-3} + q^2)^3}, \tag{21}$$

$$\begin{aligned} C(r) = {} & a_6 r^{6(D-3)} + a_5' r^{5D-13} + a_5 r^{5(D-3)} + a_4' r^{4D-10} + a_4 r^{4(D-3)} \\ & + a_3' r^{3D-7} + a_3 r^{3(D-3)} + a_2' r^{2D-4} + a_2 r^{2(D-3)} + a_1 r^{D-3} + a_0, \end{aligned} \tag{22}$$

where

$$\begin{aligned} & a_6 = 4\lambda_l + D_1,\ a_5' = 4(D-3)(c_D eq\omega + m\mu^2 - 2m\omega^2), \\ & a_5 = 2m[(2D-14)\lambda_l - 3D_1], \\ & a_4' = -4(D-3)[2m^2\mu^2 - 2c_D meq\omega + q^2(\mu^2 - 2\omega^2 + c_D^2 e^2)], \\ & a_4 = -[4m^2((2D-10)\lambda_l + (D-2)(D^2-9D+21)) + q^2((4D-20)\lambda_l - (D-2)(4D^2-21D+24))], \\ & a_3' = 12(D-3)(m\mu^2 - c_D eq\omega)q^2, \qquad a_2' = 4(D-3)(c_D^2 e^2 - \mu^2)q^4, \\ & a_3 = [8m^3 + 4mq^2((3D-13)\lambda_l - 5(D-2)(D-4) - 2)], \\ & a_2 = (D-4)q^2[(4D^2-12D+12)m^2 - (4\lambda_l + 4D^2 - 27D + 42)q^2], \\ & a_1 = -6(D-2)(D-4)mq^4, \qquad a_0 = (D-2)(D-4)q^6. \end{aligned} \tag{23}$$

When $D \geq 7$, the effective potential V and its derivative $V'(r)$ have the following asymptotic behaviors,

$$r \to r_h, \quad V \to -\infty,$$
$$r \to +\infty, \quad V \to \mu^2,$$
$$r \to +\infty, \quad V'(r) \to -\frac{(2l+D-2)(2l+D-4)}{2r^3} < 0. \tag{24}$$

From the above asymptotic behaviors, we conclude that the effective potential $V(r)$ has at least one maximum outside the event horizon r_h and there is no potential well near the spatial infinity. In the next section, we will prove that in fact there is no potential well between the event horizon and spatial infinity for superradiant bound states by analyzing the derivative of the effective potential V.

IV. FURTHER ANALYSIS OF THE EFFECTIVE POTENTIAL V

In this section, we show that there is only one extreme for the effective potential outside the RN black hole horizon by analyzing the real roots of the derivative of the effective potential. Explicitly, it is shown that only one real root exists for the following equation:

$$V'(r) = 0, \tag{25}$$

when $r > r_h$. Since we are interested in the real roots of $V'(r)$, we just consider the numerator of V', i.e. $C(r)$, which is a polynomial of r.

Making a change of variable from r to $z = r - r_h$, $C(r)$ is changed to a polynomial of z, $C(z + r_h)$, which can be expanded as

$$C(z + r_h) = \sum_{i=0}^{6D-18} b_i z^i. \tag{26}$$

After the change of variables, a real root for $C(r) = 0$ with $r > r_h$ corresponds to a positive real root for $C(z + r_h) = 0$ with $z > 0$. In order to analyze the number of positive real roots for $C(z + r_h) = 0$, we will use a method based on *Descartes' rule of signs*, i.e. we will consider the sign changes of the sequence of the following coefficients:

$$b_{6D-18}, b_{6D-19}, b_{6D-20}, \ldots, b_2, b_1, b_0. \tag{27}$$

Remember that $u = r_+^{D-3}, v = r_-^{D-3}$ and they satisfy $u + v = 2m, uv = q^2$. u, v will be often used in the later discussion. The constant term in $C(z + r_h)$ is

$$b_0 = a_0 + a_1 r_h^{D-3} + a_2 r_h^{2D-6} + a_2' r_h^{2D-4} + a_3 r_h^{3D-9} + a_3' r_h^{3D-7} + a_4 r_h^{4D-12} + a_4' r_h^{4D-10} + a_5 r_h^{5D-15} + a_5' r_h^{5D-13} + a_6 r_h^{6D-18}. \tag{28}$$

Plugging Eq. (23) into the above equation, we can obtain

$$b_0 = -(D-3)u^3(u-v) \times [(D-3)^2(u-v)^2 + 4r_h^2(c_D eq - u\omega)^2] < 0. \tag{29}$$

It is easy to see that $a_6 = D_1 + 4\lambda_l > 0$. Considering the two conditions, Eqs. (16) and (17), it is also easy to prove

$$a_5' = 4(D-3)(c_D eq\omega + m\mu^2 - 2m\omega^2) > 0. \tag{30}$$

So we can immediately obtain that

$$\begin{aligned} b_{6D-18} &= a_6 > 0, \\ b_{6D-19} &= a_6 C_{6D-18}^{6D-19} r_h > 0, \\ &\ldots, \\ b_{5D-12} &= a_6 C_{6D-18}^{5D-12} r_h^{D-6} > 0, \\ b_{5D-13} &= a_6 C_{6D-18}^{5D-13} r_h^{D-5} + a_5' > 0, \\ b_{5D-14} &= a_6 C_{6D-18}^{5D-14} r_h^{D-4} + a_5' C_{5D-13}^{5D-14} r_h > 0. \end{aligned} \tag{31}$$

Let us see the coefficient b_k $(4D-9 \leq k \leq 5D-15)$, which is

$$b_k = a_6 C_{6D-18}^k r_h^{6D-18-k} + a_5' C_{5D-13}^k r_h^{5D-13-k} + a_5 C_{5D-15}^k r_h^{5D-15-k}. \tag{32}$$

The term involving a_5' is positive. Next, we prove the sum of the left two terms are also positive. This sum can be written as

$$\begin{aligned} &a_6 C_{6D-18}^k r_h^{6D-18-k} + a_5 C_{5D-15}^k r_h^{5D-15-k} \\ &= C_{5D-15}^k r_h^{5D-15-k}\left(a_6 u \frac{C_{6D-18}^k}{C_{5D-15}^k} + a_5\right), \\ &= C_{5D-15}^k r_h^{5D-15-k}\left[D_1\left(-3(u+v) + u\frac{C_{6D-18}^k}{C_{5D-15}^k}\right) + \lambda_l\left(2(D-7)(u+v) + 4u\frac{C_{6D-18}^k}{C_{5D-15}^k}\right)\right]. \end{aligned} \tag{33}$$

The λ_l term in the sum is obviously positive when $D \geq 7$. Since

$$\frac{C_{6D-18}^k}{C_{5D-15}^k} = \frac{6D-18}{5D-15}\frac{6D-19}{5D-16}\cdots\frac{6D-18-k+1}{5D-15-k+1} > \left(\frac{6}{5}\right)^k > \left(\frac{6}{5}\right)^{19} > 31 \tag{34}$$

and $31u > 6u > 3(u+v)$, the D_1 term in the above sum is also positive. So we have

$$b_k > 0, \qquad (4D-9 \leq k \leq 5D-15). \tag{35}$$

The signs of the other coefficients are not so easy to prove as the above. Then, we will compare the signs of two adjacent coefficients. For example, since $b_{4D-9} > 0$, when $b_{4D-10} > 0$, we have

$$\mathrm{sign}(b_{4D-9}) = \mathrm{sign}(b_{4D-10}),$$

and when $b_{4D-10} < 0$, we have

$$\mathrm{sign}(b_{4D-9}) > \mathrm{sign}(b_{4D-10}).$$

Thus, we have

$$\mathrm{sign}(b_{p+1}) \geq \mathrm{sign}(b_p), \qquad (p = 4D-10). \tag{36}$$

In fact, it is hard to find certain features of sign relations between adjacent coefficients by directly computing their difference. However, it is found that, after properly normalizing the coefficients with positive factors, we can obtain a feature of the sign relations of the coefficients. With the results in Eqs. (36), (A8), (B8), (B12), (C12), (C20), (D10), (D21), (E6), (E11), (F9), (F7), (G6) and (G7), we prove that

$$\mathrm{sign}(b_{p+1}) \geq \mathrm{sign}(b_p), \qquad (1 \leq p \leq 4D-10). \tag{37}$$

Together with the results that $b_0 < 0$ and $b_k > 0$ $(4D-9 \leq k \leq 6D-18)$, we find that the sign change for the sequence $(b_{6D-18}, b_{6D-19}, b_{6D-20}, \ldots, b_2, b_1, b_0)$ is always 1. According to Descartes' rule of signs, there is at most one positive real root to the equation $C(z+r_h) = 0$; i.e. there is at most one extreme for the effective potential outside the horizon r_h. And, we already know that there is at least one maximum for the effective potential from the asymptotic analysis of the effective potential. Thus, there is only a potential barrier outside the event horizon and no potential well exists for the superradiant bound modes.

V. SUMMARY

In this work, superradiant stability of D-dimensional $(D \geq 7)$ nonextremal RN black hole under charged massive scalar perturbation is studied analytically. Based on the asymptotic analysis of the effective potential $V(r)$ experienced by the scalar perturbation, it is known that there is at least one maximum for the effective potential outside the black hole event horizon. Then we analyze the numerator of the derivative of the effective potential $C(z+r_h)$, which is a polynomial of $z = r - r_h$. We find that

$$b_0 < 0, \qquad b_k > 0 (4D-9 \leq k \leq 6D-18). \tag{38}$$

We also prove that

$$\mathrm{sign}(b_{p+1}) \geq \mathrm{sign}(b_p) \quad (1 \leq p \leq 4D-10). \tag{39}$$

So the sign change in the following sequence of the real coefficients of $C(z+r_h)$,

$$b_{6D-18}, b_{6D-19}, \ldots, b_{p+1}, b_p, \ldots, b_1, b_0, \tag{40}$$

is always 1. Then according to Descartes' rule of signs, we know there is at most 1 positive root for the equation $C(z+r_h) = 0$ [i.e. $V'(r) = 0$ when $r > r_h$]. Thus, there is only one maximum for the effective potential and there is no potential well outside the horizon for the superradiant bound modes. Namely, there is no black hole bomb for D-dimensional nonextremal RN black hole against charged massive scalar perturbation.

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APPENDIX A: $\mathrm{sign}(b_{p+1}) \geq \mathrm{sign}(b_p)$, $0 < p \leq D-4$

A frequently used identity in the proof is

$$C_n^{m+1} = \frac{n-m}{m+1} C_n^m. \tag{A1}$$

When $0 < p \leq D-3$, the coefficient b_p of z^p in Eq. (26) is

$$\begin{aligned} b_p &= a_6 C^p_{6D-18} r_h^{6D-18-p} + a_5 C^p_{5D-15} r_h^{5D-15-p} + a_4 C^p_{4D-12} r_h^{4D-12-p} + a_3 C^p_{3D-9} r_h^{3D-9-p} + a_2 C^p_{2D-6} r_h^{2D-6-p} + a_1 C^p_{D-3} r_h^{D-3-p} \\ &\quad + a'_5 C^p_{5D-13} r_h^{5D-13-p} + a'_4 C^p_{4D-10} r_h^{4D-10-p} + a'_3 C^p_{3D-7} r_h^{3D-7-p} + a'_2 C^p_{2D-4} r_h^{2D-4-p}, \\ &= r_h^{-p} (a_6 u^6 C^p_{6D-18} + a_5 u^5 C^p_{5D-15} + a_4 u^4 C^p_{4D-12} + a_3 u^3 C^p_{3D-9} + a_2 u^2 C^p_{2D-6} + a_1 u C^p_{D-3}) \\ &\quad + r_h^{-p+2} u^2 (a'_5 u^3 C^p_{5D-13} + a'_4 u^2 C^p_{4D-10} + a'_3 u C^p_{3D-7} + a'_2 C^p_{2D-4}), \end{aligned} \tag{A2}$$

$$= r_h^{-p} \sum_{i=1}^{6} a_i u^i C^p_{i(D-3)} + r_h^{-p+2} u^2 \sum_{j=2}^{5} a'_j u^{j-2} C^p_{j(D-3)+2}, \tag{A3}$$

$$= r_h^{-p} \check{A}_p + r_h^{-p+2} u^2 \bar{B}_p. \tag{A4}$$

The normalized coefficient is defined as

$$b'_p = \frac{r_h^p}{\bar{f}_p} b_p = \frac{\check{A}_p}{\bar{f}_p} + r_h^2 u^2 \frac{\bar{B}_p}{\bar{f}_p} = \frac{\check{A}_p^L}{\bar{f}_p}\lambda_l + \frac{\check{A}_p^R}{\bar{f}_p} + r_h^2 u^2 \left(\frac{\bar{B}_p^M}{\bar{f}_p}(\mu^2 - \omega^2) + \frac{\bar{B}_p^R}{\bar{f}_p}\right), \tag{A5}$$

where $\check{A}_p^L$ is the coefficient of λ_l in $\check{A}_p$, $\check{A}_p^R$ denotes the remaining terms in $\check{A}_p$, $\bar{B}_p^M$ is the coefficient of $(\mu^2 - \omega^2)$ in $\bar{B}_p$, $\bar{B}_p^R$ denotes the remaining terms in $\bar{B}_p$ and

$$\begin{aligned}
\bar{f}_p &= u(u+v)C_{5D-13}^p + (u^2+v^2)C_{4D-10}^p - 3v(u+v)C_{3D-7}^p + 2v^2 C_{2D-4}^p,\\
\check{A}_p^L &= 2u^4(2u^2 C_{6D-18}^p + (D-7)u(u+v)C_{5D-15}^p - (D-5)(u^2+4uv+v^2)C_{4D-12}^p\\
&\quad + (3D-13)v(u+v)C_{3D-9}^p - (2D-8)v^2 C_{2D-6}^p),\\
\check{A}_p^R &= u^3((D-2)(D-4)u^3 C_{6D-18}^p - 3(D-2)(D-4)u^2(u+v)C_{5D-15}^p\\
&\quad + u((4D^3 - 29D^2 + 66D - 48)uv - (D^3 - 11D^2 + 39D - 42)(u+v)^2)C_{4D-12}^p\\
&\quad + (u+v)((u+v)^2 - 2(5D^2 - 30D + 42)uv)C_{3D-9}^p\\
&\quad + (D-4)v((D^2 - 3D + 3)(u+v)^2 - (4D^2 - 27D + 42)uv)C_{2D-6}^p\\
&\quad - 3(D-2)(D-4)(u+v)v^2 C_{D-3}^p),\\
\bar{B}_p^M &= 2u^2(D-3)(C_{5D-13}^p u(u+v) - C_{4D-10}^p(u^2+4uv+v^2) + 3C_{3D-7}^p(u+v)v - 2C_{2D-4}^p v^2),\\
\bar{B}_p^R &= 2(D-3)u(-u\bar{f}_p\omega^2 + 2c_D equ(C_{5D-13}^p u + C_{4D-10}^p(u+v) - 3C_{3D-7}^p v)\omega\\
&\quad - 2c_D^2 e^2 q^2 (C_{4D-10}^p u - C_{2D-4}^p v)).
\end{aligned} \tag{A6}$$

Now we prove the normalization factor is positive, i.e.

$$\bar{f}_p > 0 (p \geq 1). \tag{A7}$$

When $p = 1$,

$$\bar{f}_1 = (9D-23)u^2 - 4(D-2)uv - (D-3)v^2 > (9D - 23 - 4(D-2))uv - (D-3)v^2 > (4D-12)v^2 > 0.$$

When $p \geq 2$,

$$\begin{aligned}
\bar{f}_p &= u(u+v)C_{5D-13}^p + (u^2+v^2)C_{4D-10}^p - 3v(u+v)C_{3D-7}^p + 2v^2 C_{2D-4}^p\\
&> \left(\frac{C_{5D-13}^p + C_{4D-10}^p}{C_{3D-7}^p} - 3\right) v(u+v)C_{3D-7}^p > 1.2v(u+v)C_{3D-7}^p > 0.
\end{aligned}$$

Thus, we finish the proof of Eq. (A7).

In the following four subsections, we will prove $b'_{p+1} > b'_p$, $(0 < p \leq D-4)$, and obtain

$$\mathrm{sign}(b_{p+1}) \geq \mathrm{sign}(b_p), \qquad (0 < p \leq D-4). \tag{A8}$$

This will be achieved by, respectively, proving

$$\frac{\check{A}_{p+1}^L}{\bar{f}_{p+1}} > \frac{\check{A}_p^L}{\bar{f}_p}, \qquad \frac{\check{A}_{p+1}^R}{\bar{f}_{p+1}} > \frac{\check{A}_p^R}{\bar{f}_p}, \qquad \frac{\bar{B}_{p+1}^M}{\bar{f}_{p+1}} > \frac{\bar{B}_p^M}{\bar{f}_p}, \qquad \frac{\bar{B}_{p+1}^R}{\bar{f}_{p+1}} > \frac{\bar{B}_p^R}{\bar{f}_p}. \tag{A9}$$

1. Proof of $\frac{\check{A}^L_{p+1}}{\bar{f}_{p+1}} > \frac{\check{A}^L_p}{\bar{f}_p}$

We first prove

$$\check{A}^L_p > 0 (p \geq 1). \tag{A10}$$

When $p = 1, 2, 3, 4$, the explicit expressions of $\check{A}^L_p$ are

$$\begin{aligned}
\check{A}^L_1 &= 2(D-3)^2 u^4 (u-v)^2, \qquad \check{A}^L_2 = 3(D-3)^2 u^4 (u-v)((3D-8)u-(D-2)v), \\
\check{A}^L_3 &= \frac{(D-3)^2 u^4}{3}((440+D(-333+61D))u^2 - 10(25+D(-24+5D))uv + (-82+D(21+D))v^2), \\
\check{A}^L_4 &= \frac{(D-3)^2 u^4}{12}((-7000+D(8274+D(-3083+369D)))u^2 \\
&\quad - 4(370+D(228+D(-232+39D)))uv + (3728+D(-2682+(619-45D)D))v^2).
\end{aligned} \tag{A11}$$

Since $u > v > 0$, it is easy to prove directly they are positive when $D \geq 7$. When $p \geq 5$, it is easy to see that the sum of the C^p_{3D-9} term and the C^p_{2D-6} term in $\check{A}^L_p$ are positive, i.e.

$$(3D-13)v(u+v)C^p_{3D-9} - (2D-8)v^2 C^p_{2D-6} > 0. \tag{A12}$$

Since $2u^2 > u(u+v)$ and $u^2 + 4uv + v^2 < 3u(u+v)$, for other terms in $\check{A}^L_p$, we have

$$\begin{aligned}
&2u^2 C^p_{6D-18} + (D-7)u(u+v)C^p_{5D-15} - (D-5)(u^2+4uv+v^2)C^p_{4D-12} \\
&> \left(\frac{C^p_{6D-18} + (D-7)C^p_{5D-15}}{C^p_{4D-12}} - 3(D-5)\right) u(u+v)C^p_{4D-12} \\
&> (6/4)^5 + (5/4)^5(D-7) - 3(D-5) > 0.05D + 1.23 > 0.
\end{aligned} \tag{A13}$$

Thus, $\check{A}^L_p > 0$ when $p \geq 1$.

Now we prove

$$\frac{\check{A}^L_{p+1}}{\bar{f}_{p+1}} > \frac{\check{A}^L_p}{\bar{f}_p} (p \geq 1). \tag{A14}$$

For $p = 1$, since $u > v > 0$ and $D \geq 7$ it is easy to see

$$\frac{\check{A}^L_2}{\bar{f}_2} - \frac{\check{A}^L_1}{\bar{f}_1} = \frac{4(D-3)^2}{\bar{f}_2 \bar{f}_1} u(u-v)[(5D-13)((2D-5)u^2 - (D-1)uv) + (D-2)(D-1)v^2] > 0. \tag{A15}$$

For $p = 2$,

$$\frac{\check{A}^L_3}{\bar{f}_3} - \frac{\check{A}^L_2}{\bar{f}_2} = \frac{(D-3)^2}{2\bar{f}_3 \bar{f}_2}(\alpha_0 u^4 + \alpha_1 u^3 v + \alpha_2 u^2 v^2 + \alpha_3 u v^3 + \alpha_4 v^4), \tag{A16}$$

where

$$\begin{aligned}
\alpha_0 &= 400D^4 - 4137D^3 + 15730D^2 - 25977D + 15640, \\
\alpha_1 &= -150D^4 + 762D^3 + 954D^2 - 9888D + 12714, \\
\alpha_2 &= -36D^4 + 888D^3 - 5736D^2 + 14214D - 12138, \\
\alpha_3 &= 2D^4 - 6D^3 - 166D^2 + 924D - 1258, \\
\alpha_4 &= -27D^3 + 234D^2 - 657D + 594.
\end{aligned} \tag{A17}$$

Since $u > v > 0$,

$$\begin{aligned}&\alpha_0 u^4 + \alpha_1 u^3 v + \alpha_2 u^2 v^2 + \alpha_3 u v^3 + \alpha_4 v^4 \\ &\quad > (\alpha_0 + \alpha_1) u^3 v + \alpha_2 u^2 v^2 + \alpha_3 u v^3 + \alpha_4 v^4 \\ &\quad > (\alpha_0 + \alpha_1 + \alpha_2) u^2 v^2 + \alpha_3 u v^3 + \alpha_4 v^4 \\ &\quad > (\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3) u v^3 + \alpha_4 v^4 \\ &\quad > (\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) v^4 > 0. \end{aligned} \tag{A18}$$

In the above inequalities, we use the facts that $\alpha_0 > 0$, $\alpha_0 + \alpha_1 > 0$, $\alpha_0 + \alpha_1 + \alpha_2 > 0$, $\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 > 0$, $\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 > 0$ for $D \geq 7$, which can be checked directly. So we have

$$\frac{\check{A}_3^L}{\bar{f}_3} - \frac{\check{A}_2^L}{\bar{f}_2} > 0. \tag{A19}$$

In the same way, we can also explicitly prove that

$$\frac{\check{A}_{p+1}^L}{\bar{f}_{p+1}} - \frac{\check{A}_p^L}{\bar{f}_p} > 0 \quad (p = 3, 4, 5, 6). \tag{A20}$$

When $p \geq 7$, we first rewrite $\frac{\check{A}_p^L}{\bar{f}_p}$ as $\frac{\check{A}_p^L}{\bar{f}_p} = \frac{\check{A}_p^L / C_{5D-13}^p}{\bar{f}_p / C_{5D-13}^p}$. Then, using the Eq. (A1), we have

$$\begin{aligned}\frac{\bar{f}_p}{C_{5D-13}^p} - \frac{\bar{f}_{p+1}}{C_{5D-13}^{p+1}} &= \frac{D-3}{(5D-13-p) C_{5D-13}^p} (C_{4D-10}^p (u^2 + v^2) - 6 C_{3D-7}^p (u+v) v + 6 C_{2D-4}^p v^2) \\ &> \frac{D-3}{(5D-13-p) C_{5D-13}^p} ((C_{4D-10}^p - 6 C_{3D-7}^p)(u+v) v + 6 C_{2D-4}^p v^2) > 0, \end{aligned} \tag{A21}$$

where we use the fact $\frac{C_{4D-10}^p}{C_{3D-7}^p} > 7.4$ for $p \geq 7$ in the last inequality. And, we also have

$$\begin{aligned}\frac{\check{A}_{p+1}^L}{C_{5D-13}^{p+1}} - \frac{\check{A}_p^L}{C_{5D-13}^p} &= \frac{2u^4}{(5D-13-p) C_{5D-13}^p} (\beta_0 u^2 + \beta_1 u v + \beta_2 v^2), \\ &> \frac{2u^4}{(5D-13-p) C_{5D-13}^p} ((\beta_0 + \beta_1) u v + \beta_2 v^2) > 0, \end{aligned} \tag{A22}$$

where

$$\begin{aligned}\beta_0 &= C_{4D-12}^p (D-5)(D-1) + 2 C_{6D-18}^p (D-5) - 2 C_{5D-15}^p (D-7), \\ \beta_1 &= -2 C_{3D-9}^p (D-2)(3D-13) + 4 C_{4D-12}^p (D-5)(D-1) - 2 C_{5D-15}^p (D-7), \\ \beta_2 &= 2 C_{2D-6}^p (D-4)(3D-7) - 2 C_{3D-9}^p (D-2)(3D-13) + C_{4D-12}^p (D-5)(D-1). \end{aligned} \tag{A23}$$

In the above inequalities, we use the facts that $\beta_0 > 0$, $\beta_2 > 0$, $\beta_0 + \beta_1 > 0$ since $\frac{C_{4D-12}^p}{C_{3D-9}^p} > 7.4$ and $\frac{C_{6D-18}^p}{C_{5D-15}^p} > 3.5$ for $p \geq 7$. With the results in Eqs. (A21), (A22), we conclude that

$$\frac{\check{A}_{p+1}^L}{\bar{f}_{p+1}} = \frac{\check{A}_{p+1}^L / C_{5D-13}^{p+1}}{\bar{f}_{p+1} / C_{5D-13}^{p+1}} > \frac{\check{A}_p^L / C_{5D-13}^p}{\bar{f}_p / C_{5D-13}^p} = \frac{\check{A}_p^L}{\bar{f}_p} \quad (p \geq 7). \tag{A24}$$

Thus, we finish the proof of Eq. (A14) with the results in Eqs. (A15), (A19), (A20) and (A24).

2. Proof of $\frac{\check{A}_{p+1}^R}{\bar{f}_{p+1}} > \frac{\check{A}_p^R}{\bar{f}_p}$

In this subsection, we prove

$$\frac{\check{A}_{p+1}^R}{\bar{f}_{p+1}} - \frac{\check{A}_p^R}{\bar{f}_p} > 0 \quad (p \geq 1). \tag{A25}$$

We rewrite $\bar{f}_p$ as

$$\bar{f}_p = y_5 + y_4 - y_3 + y_2, \tag{A26}$$

where

$$\begin{aligned} y_5 &= u(u+v) C_{5D-13}^p, & y_4 &= (u^2 + v^2) C_{4D-10}^p, \\ y_3 &= 3v(u+v) C_{3D-7}^p, & y_2 &= 2 v^2 C_{2D-4}^p, \end{aligned} \tag{A27}$$

and rewrite $\check{A}_p^R$ as

$$\check{A}_p^R = u^3 \sum_{i=1}^{6} g_i C_{i(D-3)}^p, \tag{A28}$$

where

$$g_6 = (D-2)(D-4)u^3, \qquad g_5 = -3(D-2)(D-4)u^2(u+v),$$
$$g_4 = -(D-2)^3 u(u-v)^2 + F_2(u+v)^2 u - F_1 u^2 v,$$
$$g_3 = (u-v)^2(u+v) - 10(D-2)(D-4)(u+v)uv,$$
$$g_2 = (D-2)^2(D-4)v(u-v)^2 + (D-4)v(G_1 uv + G_2(u+v)^2),$$
$$g_1 = -3(D-2)(D-4)(u+v)v^2, \tag{A29}$$

and

$$F_1 = 5D^2 - 18D + 16, \qquad F_2 = 5D^2 - 27D + 34,$$
$$G_1 = 11D - 26, \qquad G_2 = D - 1. \tag{A30}$$

Now consider g_1 term and let us prove

$$\frac{u^3 g_1 C_{D-3}^{p+1}}{\bar{f}_{p+1}} > \frac{u^3 g_1 C_{D-3}^{p}}{\bar{f}_p}, \tag{A31}$$

i.e.

$$\frac{C_{D-3}^{p+1}}{\bar{f}_{p+1}} < \frac{C_{D-3}^{p}}{\bar{f}_p} \Leftrightarrow (D-3-p)\bar{f}_p < (p+1)\bar{f}_{p+1}$$
$$\Leftrightarrow (4D-10)y_5 + (3D-7)y_4 - 2(D-2)y_3 + (D-1)y_2 > 0. \tag{A32}$$

Since $y_5/y_3 > 1.5/3$, $y_4/y_3 > 1.2/3$, the above inequality holds.

Similarly, let us consider the $(D-2)(D-4)$ term in g_3, and prove

$$u^3 \frac{-10(D-2)(D-4)(u+v)uvC_{3D-9}^{p+1}}{\bar{f}_{p+1}} > u^3 \frac{-10(D-2)(D-4)(u+v)uvC_{3D-9}^{p}}{\bar{f}_p}, \tag{A33}$$

i.e.

$$\frac{C_{3D-9}^{p+1}}{\bar{f}_{p+1}} < \frac{C_{3D-9}^{p}}{\bar{f}_p} \Leftrightarrow (3D-9-p)\bar{f}_p < (p+1)\bar{f}_{p+1}$$
$$\Leftrightarrow (2D-4)y_5 + (D-1)y_4 - 2y_3 - (D-5)y_2 > 0. \tag{A34}$$

Since $y_5/y_3 > 1.5/3$, $y_4/y_2 > 1$, the above inequality holds. Then, let us consider the F_1 term in g_4 and prove

$$u^3 \frac{-F_1 u^2 v C_{4D-12}^{p+1}}{\bar{f}_{p+1}} > u^3 \frac{-F_1 u^2 v C_{4D-12}^{p}}{\bar{f}_p}, \tag{A35}$$

i.e.

$$\frac{C_{4D-12}^{p+1}}{\bar{f}_{p+1}} < \frac{C_{4D-12}^{p}}{\bar{f}_p} \Leftrightarrow (4D-12-p)\bar{f}_p < (p+1)\bar{f}_{p+1}$$
$$\Leftrightarrow (D-1)y_5 + 2y_4 + (D-5)y_3 - 2(D-4)y_2 > 0. \tag{A36}$$

Since $y_3/y_2 > 3, y_5 > y_2$, the above inequality holds.

Next, consider the sum of the $(u-v)^2$ terms in g_4, g_2 and for $p \geq 3$ let us prove

$$u^3 \frac{-(D-2)^3 u(u-v)^2 C_{4D-12}^{p+1} + (D-2)^2(D-4)v(u-v)^2 C_{2D-6}^{p+1}}{\bar{f}_{p+1}}$$
$$> u^3 \frac{-(D-2)^3 u(u-v)^2 C_{4D-12}^{p} + (D-2)^2(D-4)v(u-v)^2 C_{2D-6}^{p}}{\bar{f}_p}, \tag{A37}$$

i.e.

$$\frac{-(D-2)uC_{4D-12}^{p+1} + (D-4)vC_{2D-6}^{p+1}}{\bar{f}_{p+1}} > \frac{-(D-2)uC_{4D-12}^{p} + (D-4)vC_{2D-6}^{p}}{\bar{f}_p}$$
$$\Leftrightarrow C_{4D-12}^{p}[-2(D-4)y_2 + (D-5)y_3 + 2y_4 + (D-1)y_5]$$
$$> C_{2D-6}^{p}[2y_2 - (D-1)y_3 + (2D-4)y_4 + (3D-7)y_5]. \tag{A38}$$

Since $y_3/y_2 > 3$, the sum of y_4, y_3, y_2 terms on the left-hand side is positive and the sum of y_3, y_2 terms on the right-hand side is negative. When $p \geq 3$, $C^p_{4D-12}/C^p_{2D-6} > 8$ and the y_5 term on the left-hand side is greater than the sum of y_4, y_5 terms on the right-hand side. Thus, we finish the proof of Eq. (A37).

Finally, let us consider all other terms in $\breve{A}^R_p$, which is

$$S_p = u^3(g_6 C^p_{6D-18} + g_5 C^p_{5D-15} + F_2(u+v)^2 u C^p_{4D-12} \\ +(u-v)^2(u+v)C^p_{3D-9} + (D-4)v(G_1 uv + G_2(u+v)^2)C^p_{2D-6}). \quad (A39)$$

The second line of S_p is obviously positive. Then we prove the first line of S_p is positive for $p \geq 7$ in the following:

$$g_6 C^p_{6D-18} + g_5 C^p_{5D-15} + F_2(u+v)^2 u C^p_{4D-12} > 0 \\ \Leftrightarrow (D-2)(D-4)C^p_{6D-18}u^2 - 3(D-2)(D-4)C^p_{5D-15}u(u+v) + F_2 C^p_{4D-12}(u+v)^2 > 0. \quad (A40)$$

It is known that $u + v = 2m$. To simplify the expression in following proof, hereafter, we take the mass of the black hole to be $1/2$, i.e. $u + v = 1$. This simplification just leads to a difference of an overall positive factor, and does not change the positivity or negativity of a homogeneous expression of u, v. Replacing u with $1 - v$, the above inequality is equivalent to

$$\eta_2 v^2 + \eta_1 v + \eta_0 > 0, \quad (A41)$$

where

$$\eta_2 = (D-2)(D-4)C^p_{6D-18}, \\ \eta_1 = (D-2)(D-4)(3C^p_{5D-15} - 2C^p_{6D-18}), \\ \eta_0 = (D-2)(D-4)(C^p_{6D-18} - 3C^p_{5D-15}) + F_2 C^p_{4D-12}. \quad (A42)$$

When $p \geq 7$, $C^p_{6D-18}/C^p_{5D-15} > 3$, so $\eta_0 > 0$. The quadratic function of v,$\eta_2 v^2 + \eta_1 v + \eta_0$, opens upwards. The symmetric axis satisfies

$$v_s = \frac{1}{2}\frac{2C^p_{6D-18} - 3C^p_{5D-15}}{C^p_{6D-18}} > \frac{1}{2} \quad \text{for } p \geq 7. \quad (A43)$$

The lower bound of $\eta_2 v^2 + \eta_1 v + \eta_0$ is at $v = \frac{1}{2}$, i.e.

$$\frac{1}{4}((D-2)(D-4)C^p_{6D-18} + 4F_2 C^p_{4D-12} \\ - 6(D-2)(D-4)C^p_{5D-15}). \quad (A44)$$

It is easy to check the above is positive when $p \geq 7$. Thus, $S_p > 0$ when $p \geq 7$. Then, let us prove

$$\frac{S_{p+1}}{\bar{f}_{p+1}} > \frac{S_p}{\bar{f}_p} \quad \text{for } p \geq 8. \quad (A45)$$

The above inequality is equivalent to

$$\frac{S_{p+1}/C^{p+1}_{5D-13}}{\bar{f}_{p+1}/C^{p+1}_{5D-13}} > \frac{S_p/C^p_{5D-13}}{\bar{f}_p/C^p_{5D-13}}. \quad (A46)$$

Consider the following difference

$$\frac{S_{p+1}}{C^{p+1}_{5D-13}} - \frac{S_p}{C^p_{5D-13}} = \frac{(p+1)S_{p+1} - (5D-13-p)S_p}{(5D-13-p)C^p_{5D-13}}. \quad (A47)$$

When $p \geq 8$, the numerator of the above difference is positive, i.e.

$$(p+1)S_{p+1} - (5D-13-p)S_p = u^3[(D-5)(D-2)(D-4)C^p_{6D-18}u^3 \\ + 6(D-2)(D-4)C^p_{5D-15}(u+v)u^2 - (D-1)F_2(u+v)^2 u C^p_{4D-12} \\ - 2(D-2)(u-v)^2(u+v)C^p_{3D-9} - (3D-7)(D-4)v(G_1 uv + G_2(u+v)^2)C^p_{2D-6}] > 0, \quad (A48)$$

where we use $u > 1/2$, $(D-2)(D-4)C^p_{5D-15}(u+v)u^2 > 2(D-2)(u-v)^2(u+v)C^p_{3D-9}$, $v(G_1 uv + G_2(u+v)^2) < 15u(D-2)/4$, and

$$\frac{(D-5)(D-2)(D-4)C^p_{6D-18}/4 + 5(D-2)(D-4)C^p_{5D-15}/2}{(D-1)F_2 C^p_{4D-12} + 15(3D-7)(D-4)(D-2)C^p_{2D-6}/4} > 1.2 \quad \text{for } p \geq 8. \quad (A49)$$

Then, together with Eq. (A21), we prove Eq. (A46), i.e. Eq. (A45).

With the results in Eqs. (A31), (A33), (A35), (A37) and (A45), we prove that

$$\frac{\check{A}^R_{p+1}}{\bar{f}_{p+1}} - \frac{\check{A}^R_p}{\bar{f}_p} > 0 \quad (p \geq 8). \tag{A50}$$

For $1 \leq p \leq 7$, we can check directly that $\frac{\check{A}^R_{p+1}}{\bar{f}_{p+1}} - \frac{\check{A}^R_p}{\bar{f}_p} > 0$ holds. When $p = 1$, the difference $\frac{\check{A}^R_{p+1}}{\bar{f}_{p+1}} - \frac{\check{A}^R_p}{\bar{f}_p}$ is

$$\begin{aligned} &2(D-3)^3(u-v)^2((D-1)u + (D-3)v) \\ &\quad \times ((2D-5)(5D-13)u^2 \\ &\quad - (D-1)(5D-13)uv + (D-2)(D-1)v^2). \end{aligned} \tag{A51}$$

Since $u > v$, it is obviously positive when $D \geq 7$. When $p = 2$, the difference satisfies

$$\begin{aligned} &\frac{1}{2}(D-3)^3(u-v)^2(k_3u^3 + k_2u^2v + k_1uv^2 + k_0v^3) \\ &> \frac{1}{2}(D-3)^3(u-v)^2((k_3+k_2+k_1)uv^2 + k_0v^3) \\ &> \frac{1}{2}(D-3)^3(u-v)^2(k_3+k_2+k_1+k_0)v^3 > 0, \end{aligned} \tag{A52}$$

where

$$\begin{aligned} k_3 &= D(D(2D(4D(25D-281)+4789)-18799)+15881)-3586, \\ k_2 &= (D(D(2D(D(101D-1523)+8984)-51829)+73117)-40354), \\ k_1 &= -(D-2)(D(2D(4D^2+D-255)+2143)-2525), \\ k_0 &= -(D-3)(D-2)(2D(D(5D-53)+175)-353), \end{aligned} \tag{A53}$$

and we also use the results: $k_3 > 0, k_2 > 0, k_3 + k_2 + k_1 > 0, k_3 + k_2 + k_1 + k_0 > 0$ for $D \geq 7$. When $p = 3$, the similar argument can be used to prove the difference is positive. When $p = 4$, the difference is

$$\frac{1}{1440}(D-3)^4(1-2v)(n_4v^4 + n_3v^3 + n_2v^2 + n_1v + n_0), \tag{A54}$$

where

$$\begin{aligned} n_0 &= 80000D^8 - 1445688D^7 + 9955851D^6 - 27463065D^5 - 19290150D^4 \\ &\quad + 336996048D^3 - 928280156D^2 + 1138695960D - 545407200, \\ n_1 &= -307992D^8 + 5330579D^7 - 33196467D^6 + 58112421D^5 + 317588127D^4 \\ &\quad - 2036657124D^3 + 4908514212D^2 - 5709234656D + 2657836800, \\ n_2 &= 2(180912D^8 - 2705933D^7 + 10350693D^6 + 49629585D^5 - 609429207D^4 \\ &\quad + 2467053018D^3 - 5154998928D^2 + 5589738800D - 2495614080), \\ n_3 &= -8(16460D^8 - 111888D^7 - 1939017D^6 + 30813387D^5 - 187841085D^4 \\ &\quad + 621263463D^3 - 1176724838D^2 + 1203712368D - 517001280), \\ n_4 &= 96(6032D^7 - 129296D^6 + 1179584D^5 - 5934980D^4 + 17778503D^3 \\ &\quad - 31693094D^2 + 31117416D - 12974640). \end{aligned} \tag{A55}$$

Consider the following linear function of v

$$24n_4v + 6n_3. \tag{A56}$$

It is easy to check the above function is negative for $0 < v < \frac{1}{2}$ when $D \geq 7$. Then, the following integral of the above function

$$12n_4v^2 + 6n_3v + 2n_2 \tag{A57}$$

is monotonically decreasing in the domain $0 < v < \frac{1}{2}$ when $D \geq 7$. Its lower bound is at $v = \frac{1}{2}$ and is positive when $D \geq 7$. Then, the following integral of the above function

$$4n_4v^3 + 3n_3v^2 + 2n_2v + n_1 \tag{A58}$$

is monotonically increasing in the domain $0 < v < \frac{1}{2}$ when $D \geq 7$. Its upper bound is at $v = \frac{1}{2}$ and is negative when $D \geq 7$. Then, the following integral of the above function

$$n_4 v^4 + n_3 v^3 + n_2 v^2 + n_1 v + n_0 \tag{A59}$$

is monotonically decreasing in the domain $0 < v < \frac{1}{2}$ when $D \geq 7$. Its lower bound is at $v = \frac{1}{2}$ and is positive when $D \geq 7$. Thus, it is easy to see that the difference in Eq. (A54) is positive, i.e.

$$\frac{\check{A}_5^R}{\bar{f}_5} - \frac{\check{A}_4^R}{\bar{f}_4} > 0. \tag{A60}$$

We can use the same strategy to prove differences of the $p = 5, 6, 7$ cases are also positive. Thus, we finish the proof of Eq. (A25).

3. Proof of $\frac{\bar{B}_{p+1}^M}{\bar{f}_{p+1}} > \frac{\bar{B}_p^M}{\bar{f}_p}$

We first prove

$$\bar{B}_p^M > 0 \quad (p \geq 1). \tag{A61}$$

Since $2u^2(D-3)$ is positive, we just consider the rest factor of $\bar{B}_p^M$, i.e.

$$C_{5D-13}^p u(u+v) - C_{4D-10}^p (u^2 + 4uv + v^2) + 3C_{3D-7}^p (u+v)v - 2C_{2D-4}^p v^2. \tag{A62}$$

Replacing u with $(1-v)$, we obtain

$$2v^2(C_{4D-10}^p - C_{2D-4}^p) + v(3C_{3D-7}^p - 2C_{4D-10}^p - C_{5D-13}^p) + C_{5D-13}^p - C_{4D-10}^p. \tag{A63}$$

The above quadratic function of v has a positive intersection and opens upwards. Its symmetric axis satisfies

$$v_s = \frac{1}{2} \frac{C_{5D-13}^p + 2C_{4D-10}^p - 3C_{3D-7}^p}{2(C_{4D-10}^p - C_{2D-4}^p)} \geq \frac{1}{2}. \tag{A64}$$

Proof of the above inequality. It is easy to see that $C_{5D-13}^p + 2C_{4D-10}^p - 3C_{3D-7}^p > 0$ and $2(C_{4D-10}^p - C_{2D-4}^p) > 0$. Then, their difference is

$$(C_{5D-13}^p + 2C_{4D-10}^p - 3C_{3D-7}^p) - 2(C_{4D-10}^p - C_{2D-4}^p) = C_{5D-13}^p + 2C_{2D-4}^p - 3C_{3D-7}^p. \tag{A65}$$

When $p \geq 3$, $\frac{C_{5D-13}^p}{C_{3D-7}^p} > 3$, the above expression is positive. When $p = 1$, the above expression is zero. When $p = 2$, the above expression equals to $3(D-3)^2$, which is positive.

Since $0 < v < 1/2$, the lower bound of Eq. (A63) is at $v = 1/2$, which is

$$\frac{1}{2}(C_{5D-13}^p - 3C_{4D-10}^p + 3C_{3D-7}^p - C_{2D-4}^p). \tag{A66}$$

When $p = 1, 2, 3, 4, 5$, the values of the above lower bound are, respectively, $\{0, 0, (D-3)^3, (D-3)^3(7D-20)/2, (D-3)^3(5D-16)(5D-14)/4\}$, which are all positive. When $p \geq 6$, $\frac{C_{5D-13}^p}{C_{4D-10}^p} > 3$ and the above lower bound is also positive. So we prove Eq. (A61).

Now we prove

$$\frac{\bar{B}_{p+1}^M}{\bar{f}_{p+1}} - \frac{\bar{B}_p^M}{\bar{f}_p} > 0 \quad (p \geq 1). \tag{A67}$$

For $p = 1, 2, 3$, we calculate the differences directly and prove they are positive. When $p = 1$, we have

$$\begin{aligned} \frac{\bar{B}_2^M}{\bar{f}_2} - \frac{\bar{B}_1^M}{\bar{f}_1} &= \frac{2(D-3)u(u-v)}{\bar{f}_2 \bar{f}_1}((10D^2 - 51D + 65)u^2 + (-5D^2 + 18D - 13)uv + (D^2 - 3D + 2)v^2) \\ &> \frac{2(D-3)u(u-v)}{\bar{f}_2 \bar{f}_1}((5D^2 - 33D + 52)uv + (D^2 - 3D + 2)v^2) > 0, \end{aligned} \tag{A68}$$

where we use the facts that $u > v > 0$ and $10D^2 - 51D + 65 > 0, 5D^2 - 33D + 52 > 0$ for $D \geq 7$. When $p = 2$, we have

$$\begin{aligned} \frac{\bar{B}_3^M}{\bar{f}_3} - \frac{\bar{B}_2^M}{\bar{f}_2} &= \frac{(D-3)u}{3\bar{f}_3 \bar{f}_2}(\gamma_0 u^3 + \gamma_1 u^2 v + \gamma_2 u v^2 + \gamma_3 v^3) > \frac{(D-3)u}{3\bar{f}_3 \bar{f}_2}((\gamma_0 + \gamma_1)u^2 v + \gamma_2 u v^2 + \gamma_3 v^3) \\ &> \frac{(D-3)u}{3\bar{f}_3 \bar{f}_2}((\gamma_0 + \gamma_1 + \gamma_2)u v^2 + \gamma_3 v^3) > 0, \end{aligned} \tag{A69}$$

where

$$\begin{aligned}\gamma_0 &= 200D^4 - 2130D^3 + 8501D^2 - 15069D + 10010,\\ \gamma_1 &= -3(25D^4 - 210D^3 + 612D^2 - 681D + 182),\\ \gamma_2 &= -3(6D^4 - 84D^3 + 401D^2 - 801D + 574),\\ \gamma_3 &= D^4 - 12D^3 + 46D^2 - 69D + 34,\end{aligned} \tag{A70}$$

and we also use the facts that $u > v > 0$ and $\gamma_0 > 0, \gamma_0 + \gamma_1 > 0, \gamma_0 + \gamma_1 + \gamma_2 > 0, \gamma_3 > 0$ for $D \geq 7$. The proof of $p = 3$ case is similar to the proof of $p = 2$ case. When $p \geq 4$, we have

$$\begin{aligned}\frac{\bar{B}^M_{p+1}}{\bar{f}_{p+1}} - \frac{\bar{B}^M_p}{\bar{f}_p} = \frac{2(D-3)u}{(p+1)\bar{f}_{p+1}\bar{f}_p}&(-6C^p_{5D-13}v(u+v)(-C^p_{2D-4}v + C^p_{3D-7}(u+v))\\ &+ C^p_{4D-10}(-8C^p_{2D-4}v^3 + 6C^p_{3D-7}v^2(u+v) + C^p_{5D-13}(u+v)^3)).\end{aligned} \tag{A71}$$

Replacing u with $1 - v$ in the above equation, and since $v < 1/2$, we obtain

$$\begin{aligned}\frac{\bar{B}^M_{p+1}}{\bar{f}_{p+1}} - \frac{\bar{B}^M_p}{\bar{f}_p} &= \frac{2(D-3)u}{(p+1)\bar{f}_{p+1}\bar{f}_p}(C^p_{4D-10}C^p_{5D-13} - 6C^p_{3D-7}C^p_{5D-13}v + 6(C^p_{3D-7}C^p_{4D-10} + C^p_{2D-4}C^p_{5D-13})v^2 - 8C^p_{2D-4}C^p_{4D-10}v^3)\\ &> \frac{2(D-3)u}{(p+1)\bar{f}_{p+1}\bar{f}_p}(C^p_{4D-10}C^p_{5D-13} - 6C^p_{3D-7}C^p_{5D-13}v + 6(C^p_{3D-7}C^p_{4D-10} + C^p_{2D-4}C^p_{5D-13})v^2 - 4C^p_{2D-4}C^p_{4D-10}v^2).\end{aligned} \tag{A72}$$

Since $\frac{2(D-3)u}{(p+1)\bar{f}_{p+1}\bar{f}_p} > 0$ we just consider the rest factor, i.e.

$$\begin{aligned}&C^p_{4D-10}C^p_{5D-13} - 6C^p_{3D-7}C^p_{5D-13}v\\ &\quad + 6(C^p_{3D-7}C^p_{4D-10} + C^p_{2D-4}C^p_{5D-13})v^2 - 4C^p_{2D-4}C^p_{4D-10}v^2.\end{aligned}$$

Taking it as a quadratic function of $v (0 < v < \frac{1}{2})$, it is obvious that this function has a positive intersection and opens upwards. The symmetric axis is at

$$v_s = \frac{1}{2} \cdot \frac{6C^p_{3D-7}C^p_{5D-13}}{6(C^p_{3D-7}C^p_{4D-10} + C^p_{2D-4}C^p_{5D-13}) - 4C^p_{2D-4}C^p_{4D-10}}. \tag{A73}$$

Since $1 - \frac{C^p_{4D-10}}{C^p_{5D-13}} - \frac{C^p_{2D-4}}{C^p_{3D-7}} > 0.3$ for $p \geq 4$ we have $v_s > \frac{1}{2}$. The lower bound of the quadratic function is at $v = \frac{1}{2}$, which is

$$\begin{aligned}&C^p_{4D-10}\left(\frac{3}{2}C^p_{3D-7} - C^p_{2D-4}\right) + \frac{3}{2}C^p_{2D-4}C^p_{5D-13}\\ &\quad + C^p_{5D-13}(C^p_{4D-10} - 3C^p_{3D-7}).\end{aligned}$$

Because $\frac{C^p_{4D-10}}{C^p_{3D-7}} > 3$ for $p \geq 4$, the above lower bound is positive. Then, Eq. (A72) is positive. We finish the proof of Eq. (A67).

4. Proof of $\frac{\bar{B}^R_{p+1}}{\bar{f}_{p+1}} > \frac{\bar{B}^R_p}{\bar{f}_p}$

Now we prove

$$\frac{\bar{B}^R_{p+1}}{\bar{f}_{p+1}} - \frac{\bar{B}^R_p}{\bar{f}_p} > 0 \quad (p \geq 1). \tag{A74}$$

An important observation is that the difference $\frac{\bar{B}^R_{p+1}}{\bar{f}_{p+1}} - \frac{\bar{B}^R_p}{\bar{f}_p}$ is linear in ω. In order to prove the positivity of the difference, we just need to check the $\omega = 0$ and $\omega = \frac{c_D eq}{u}$ cases (remember the superradiance condition $0 < \omega < \frac{c_D eq}{u}$). When $\omega = 0$, the difference is

$$\begin{aligned}\left(\frac{\bar{B}^R_{p+1}}{\bar{f}_{p+1}} - \frac{\bar{B}^R_p}{\bar{f}_p}\right)_{\omega=0} &= \frac{4(D-3)^2u(u+v)c_D^2e^2q^2}{\bar{f}_{p+1}\bar{f}_p}\\ &\quad \times (\delta_0u^2 + \delta_1uv + \delta_2v^2),\end{aligned} \tag{A75}$$

where

$$\begin{aligned}\delta_0 &= C^p_{4D-10}C^p_{5D-13},\\ \delta_1 &= 2(C^p_{3D-7} - C^p_{2D-4})C^p_{4D-10}\\ &\quad + C^p_{3D-7}C^p_{4D-10} - 3C^p_{2D-4}C^p_{5D-13},\\ \delta_2 &= C^p_{2D-4}(3C^p_{3D-7} - 2C^p_{4D-10}).\end{aligned} \tag{A76}$$

For $p = 1$,

$$\delta_0 = 2(2D-5)(5D-13),\quad \delta_1 = -2(D-1)(5D-13),\quad \delta_2 = 2(D-1)(D-2). \tag{A77}$$

Since $u > v > 0$ we have $\delta_0 u^2 + \delta_1 uv + \delta_2 v^2 > (\delta_0+\delta_1)uv + \delta_2 v^2 > 0$. For $p \geq 2$, we have

$$\delta_0 u^2 + \delta_1 uv + \delta_2 v^2 > (\delta_0+\delta_1)uv + \delta_2 v^2 > (\delta_0+\delta_1+\delta_2)v^2 > 0, \tag{A78}$$

where we use the facts that $\frac{C^p_{4D-10}}{C^p_{2D-4}} > 3$ and $\frac{3C^p_{3D-7}}{4C^p_{2D-4}} > 1.5$. So we obtain

$$\left(\frac{\bar{B}^R_{p+1}}{\bar{f}_{p+1}} - \frac{\bar{B}^R_p}{\bar{f}_p}\right)_{\omega=0} > 0. \tag{A79}$$

When $\omega = \frac{c_D e q}{u}$, we find that

$$\left(\frac{\bar{B}^R_{p+1}}{\bar{f}_{p+1}} - \frac{\bar{B}^R_p}{\bar{f}_p}\right)_{\omega=\frac{c_D e q}{u}} = \frac{4(D-3)^2 u(u-v)c_D^2 e^2 q^2}{\bar{f}_{p+1}\bar{f}_p} \times (\delta_0 u^2 + \delta_1 uv + \delta_2 v^2) > 0. \tag{A80}$$

Then we finish the proof of Eq. (A74).

APPENDIX B: $\mathrm{sign}(b_{p+1}) \geq \mathrm{sign}(b_p)$, $D-3 \leq p \leq 2D-7$

When $D-2 \leq p \leq 2D-6$, the coefficient of z^p is

$$\begin{aligned} b_p &= a_6 C^p_{6D-18} r_h^{6D-18-p} + a_5 C^p_{5D-15} r_h^{5D-15-p} + a_4 C^p_{4D-12} r_h^{4D-12-p} + a_3 C^p_{3D-9} r_h^{3D-9-p} + a_2 C^p_{2D-6} r_h^{2D-6-p} \\ &\quad + a'_5 C^p_{5D-13} r_h^{5D-13-p} + a'_4 C^p_{4D-10} r_h^{4D-10-p} + a'_3 C^p_{3D-7} r_h^{3D-7-p} + a'_2 C^p_{2D-4} r_h^{2D-4-p} \\ &= r_h^{-p}(a_6 u^6 C^p_{6D-18} + a_5 u^5 C^p_{5D-15} + a_4 u^4 C^p_{4D-12} + a_3 u^3 C^p_{3D-9} + a_2 u^2 C^p_{2D-6}) \\ &\quad + r_h^{-p+2} u^2 (a'_5 u^3 C^p_{5D-13} + a'_4 u^2 C^p_{4D-10} + a'_3 u C^p_{3D-7} + a'_2 C^p_{2D-4}). \end{aligned} \tag{B1}$$

Define the normalized coefficient as

$$b'_p = \frac{r_h^p}{\bar{f}_p} b_p = \frac{\tilde{A}_p}{\bar{f}_p} + r_h^2 u^2 \frac{\bar{B}_p}{\bar{f}_p} = \frac{\tilde{A}^L_p}{\bar{f}_p}\lambda_l + \frac{\tilde{A}^R_p}{\bar{f}_p} + r_h^2 u^2 \frac{\bar{B}_p}{\bar{f}_p}, \tag{B2}$$

where

$$\begin{aligned} \tilde{A}^L_p &= 2u^4(2u^2 C^p_{6D-18} + (D-7)u(u+v)C^p_{5D-15} - (D-5)(u^2+4uv+v^2)C^p_{4D-12} \\ &\quad + (3D-13)v(u+v)C^p_{3D-9} - (2D-8)v^2 C^p_{2D-6}), \\ \tilde{A}^R_p &= u^3((D-2)(D-4)u^3 C^p_{6D-18} - 3(D-2)(D-4)u^2(u+v)C^p_{5D-15} \\ &\quad + u((4D^3 - 29D^2 + 66D - 48)uv - (D^3 - 11D^2 + 39D - 42)(u+v)^2)C^p_{4D-12} \\ &\quad + (u+v)((u+v)^2 - 2(5D^2 - 30D + 42)uv)C^p_{3D-9} \\ &\quad + (D-4)v((D^2 - 3D + 3)(u+v)^2 - (4D^2 - 27D + 42)uv)C^p_{2D-6}). \end{aligned} \tag{B3}$$

Since $\tilde{A}^L_p = \check{A}^L_p$, according to the proof in Appendix A 1, we have

$$\frac{\tilde{A}^L_{p+1}}{\bar{f}_{p+1}} > \frac{\tilde{A}^L_p}{\bar{f}_p}. \tag{B4}$$

According to the proofs in Appendices A 3 and A 4, we also have

$$r_h^2 u^2 \frac{\bar{B}_{p+1}}{\bar{f}_{p+1}} > r_h^2 u^2 \frac{\bar{B}_p}{\bar{f}_p}. \tag{B5}$$

The difference between $\tilde{A}_p^R$ and $\check{A}_p^R$ is the g_1 term in Eq. (A28). In the proof in Appendix A 2, the g_1 term and other terms are discussed separately, so we have

$$\frac{\tilde{A}_{p+1}^R}{\bar{f}_{p+1}} > \frac{\tilde{A}_p^R}{\bar{f}_p}. \tag{B6}$$

Thus, we prove

$$b'_{p+1} > b'_p, \qquad (D-2 \le p \le 2D-5), \tag{B7}$$

and obtain

$$\mathrm{sign}(b_{p+1}) \ge \mathrm{sign}(b_p), \quad (D-2 \le p \le 2D-5). \tag{B8}$$

When $p = D-3$, we need prove $b'_{D-2} > b'_{D-3}$, i.e.

$$\frac{\tilde{A}_{D-2}}{\bar{f}_{D-2}} + r_h^2 u^2 \frac{\bar{B}_{D-2}}{\bar{f}_{D-2}} > \frac{\check{A}_{D-3}}{\bar{f}_{D-3}} + r_h^2 u^2 \frac{\bar{B}_{D-3}}{\bar{f}_{D-3}}. \tag{B9}$$

Based on the proof of Eq. (B7), it is easy to deduce

$$\frac{\tilde{A}_{D-2}}{\bar{f}_{D-2}} + r_h^2 u^2 \frac{\bar{B}_{D-2}}{\bar{f}_{D-2}} > \frac{\tilde{A}_{D-3}}{\bar{f}_{D-3}} + r_h^2 u^2 \frac{\bar{B}_{D-3}}{\bar{f}_{D-3}}. \tag{B10}$$

According to the definitions of $\check{A}_p$ and $\tilde{A}_p$, we have

$$\check{A}_{D-3} = \tilde{A}_{D-3} - 3(D-2)(D-4)u^3(u+v)v^2 C_{D-3}^p < \tilde{A}_{D-3}, \tag{B11}$$

and then

$$\frac{\tilde{A}_{D-2}}{\bar{f}_{D-2}} + r_h^2 u^2 \frac{\bar{B}_{D-2}}{\bar{f}_{D-2}} > \frac{\check{A}_{D-3}}{\bar{f}_{D-3}} + r_h^2 u^2 \frac{\bar{B}_{D-3}}{\bar{f}_{D-3}}.$$

Thus, we have

$$\mathrm{sign}(b_{p+1}) \ge \mathrm{sign}(b_p), \quad (p = D-3). \tag{B12}$$

APPENDIX C: $\mathbf{sign}(b_{p+1}) \ge \mathbf{sign}(b_p)$, $p = 2D-6, 2D-5$

The coefficient b_p $(p = 2D-4, 2D-5)$ is

$$\begin{aligned} b_p &= a_6 C_{6D-18}^p r_h^{6D-18-p} + a_5 C_{5D-15}^p r_h^{5D-15-p} + a_4 C_{4D-12}^p r_h^{4D-12-p} + a_3 C_{3D-9}^p r_h^{3D-9-p} \\ &\quad + a'_5 C_{5D-13}^p r_h^{5D-13-p} + a'_4 C_{4D-10}^p r_h^{4D-10-p} + a'_3 C_{3D-7}^p r_h^{3D-7-p} + a'_2 C_{2D-4}^p r_h^{2D-4-p}, \\ &= r_h^{-p}(a_6 u^6 C_{6D-18}^p + a_5 u^5 C_{5D-15}^p + a_4 u^4 C_{4D-12}^p + a_3 u^3 C_{3D-9}^p) \\ &\quad + r_h^{-p+2} u^2 (a'_5 u^3 C_{5D-13}^p + a'_4 u^2 C_{4D-10}^p + a'_3 u C_{3D-7}^p + a'_2 C_{2D-4}^p). \end{aligned} \tag{C1}$$

Define the normalized coefficient as

$$b'_p = \frac{r_h^p}{\bar{f}_p} b_p = \frac{\bar{A}_p}{\bar{f}_p} + r_h^2 u^2 \frac{\bar{B}_p}{\bar{f}_p} = \frac{\bar{A}_p^L}{\bar{f}_p} \lambda_l + \frac{\bar{A}_p^R}{\bar{f}_p} + r_h^2 u^2 \frac{\bar{B}_p}{\bar{f}_p}, \tag{C2}$$

where

$$\begin{aligned} \bar{A}_p^L &= 2u^4(2u^2 C_{6D-18}^p + (D-7)u(u+v) C_{5D-15}^p - (D-5)(u^2+4uv+v^2) C_{4D-12}^p \\ &\quad + (3D-13)v(u+v) C_{3D-9}^p), \\ \bar{A}_p^R &= u^3((D-2)(D-4)u^3 C_{6D-18}^p - 3(D-2)(D-4)u^2(u+v) C_{5D-15}^p \\ &\quad + u((4D^3-29D^2+66D-48)uv - (D^3-11D^2+39D-42)(u+v)^2) C_{4D-12}^p \\ &\quad + (u+v)((u+v)^2 - 2(5D^2-30D+42)uv) C_{3D-9}^p). \end{aligned} \tag{C3}$$

Now, let us prove

$$\begin{aligned} b'_{2D-5} &= \frac{\bar{A}_{2D-5}^L}{\bar{f}_{2D-5}} \lambda_l + \frac{\bar{A}_{2D-5}^R}{\bar{f}_{2D-5}} + r_h^2 u^2 \frac{\bar{B}_{2D-5}}{\bar{f}_{2D-5}} \\ > b'_{2D-6} &= \frac{\bar{A}_{2D-6}^L}{\bar{f}_{2D-6}} \lambda_l + \frac{\bar{A}_{2D-6}^R}{\bar{f}_{2D-6}} + r_h^2 u^2 \frac{\bar{B}_{2D-6}}{\bar{f}_{2D-6}}. \end{aligned} \tag{C4}$$

Since $2D-6>7$ for $D\geq 7$, according to the proof in Appendix A 1, we have

$$\frac{\bar{f}_{2D-5}}{C^{2D-5}_{5D-15}}<\frac{\bar{f}_{2D-6}}{C^{2D-6}_{5D-15}}, \tag{C5}$$

and

$$\frac{\bar{A}^L_{2D-5}}{C^{2D-5}_{5D-15}}-\frac{\tilde{A}^L_{2D-6}}{C^{2D-6}_{5D-15}}>\frac{2u^4}{(3D-7)C^{2D-6}_{5D-13}}((\beta_0+\beta_1)uv+\beta_2v^2) > 0, \tag{C6}$$

where $\beta_0, \beta_1, \beta_2$ are defined in Eq. (A23) with $p=2D-6$. Thus, we get

$$\frac{\bar{A}^L_{2D-5}}{\bar{f}_{2D-5}}\lambda_l>\frac{\tilde{A}^L_{2D-6}}{\bar{f}_{2D-6}}\lambda_l. \tag{C7}$$

Similarly, according to the proof in Appendix A 2, we can also prove

$$\frac{\bar{A}^R_{2D-5}}{\bar{f}_{2D-5}}>\frac{\tilde{A}^R_{2D-6}}{\bar{f}_{2D-6}}. \tag{C8}$$

Two points should be emphasized here. One is that g_1 term and other terms separately satisfy the required inequality in the proof in Appendix A 2. The second is that in the proof in Appendix A 2, all the C^{p+1}_* is rewritten as $\frac{*-p}{p+1}C^p_*$, and then formally we have

$$\bar{A}^R_{2D-5}=\tilde{A}^R_{2D-5} \tag{C9}$$

since

$$\bar{A}^R_{2D-5}=\tilde{A}^R_{2D-5}-u^3g_2C^{2D-5}_{2D-6}=\tilde{A}^R_{2D-5}-u^3g_2\frac{0}{2D-5} =\tilde{A}^R_{2D-5}. \tag{C10}$$

According to the proofs in Appendices A 3 and A 4, we have

$$r_h^2u^2\frac{\bar{B}_{2D-5}}{\bar{f}_{2D-5}}>r_h^2u^2\frac{\bar{B}_{2D-6}}{\bar{f}_{2D-6}}. \tag{C11}$$

Thus, we have $b'_{2D-5}>b'_{2D-6}$ and

$$\mathrm{sign}(b_{p+1})\geq\mathrm{sign}(b_p),\qquad (p=2D-6). \tag{C12}$$

Then, let us prove

$$b'_{2D-4}=\frac{\bar{A}^L_{2D-4}}{\bar{f}_{2D-4}}\lambda_l+\frac{\bar{A}^R_{2D-4}}{\bar{f}_{2D-4}}+r_h^2u^2\frac{\bar{B}_{2D-4}}{\bar{f}_{2D-4}} > b'_{2D-5}=\frac{\bar{A}^L_{2D-5}}{\bar{f}_{2D-5}}\lambda_l+\frac{\bar{A}^R_{2D-5}}{\bar{f}_{2D-5}}+r_h^2u^2\frac{\bar{B}_{2D-5}}{\bar{f}_{2D-5}}. \tag{C13}$$

According to the proofs in Appendices A 3 and A 4, we have

$$r_h^2u^2\frac{\bar{B}_{2D-4}}{\bar{f}_{2D-4}}>r_h^2u^2\frac{\bar{B}_{2D-5}}{\bar{f}_{2D-5}}. \tag{C14}$$

By taking the C^p_{2D-6} term to zero in the proof of Eq. (A22) in Appendix A 1, it is easy to get

$$\frac{\bar{A}^L_{2D-4}}{\bar{f}_{2D-4}}>\frac{\bar{A}^L_{2D-5}}{\bar{f}_{2D-5}}. \tag{C15}$$

Then, for the term including $\bar{A}^R_p$, we have

$$\bar{A}^R_p=u^3\sum_{i=3}^{6}g_iC^p_{i(D-3)},\quad (p=2D-5,\,2D-4), \tag{C16}$$

where

$$g_6=(D-2)(D-4)u^3,\,g_5=-3(D-2)(D-4)u^2(u+v),$$
$$g_4=-(D-2)^3u(u-v)^2+F_2(u+v)^2u-F_1u^2v,$$
$$g_3=(u-v)^2(u+v)-10(D-2)(D-4)(u+v)uv, \tag{C17}$$

and

$$F_1=5D^2-18D+16,\quad F_2=5D^2-27D+34. \tag{C18}$$

The proofs for the negative terms in g_3, g_4 are the same as that in Eqs. (A33), (A35). The proof for other terms is the same as that for Eqs. (A45), (A46). It is easy to check that after taking the C^p_{2D-6} term to zero, all the inequalities in the proof of Eqs. (A45), (A46) still hold. So, we obtain

$$\frac{\bar{A}^R_{2D-4}}{\bar{f}_{2D-4}}>\frac{\bar{A}^R_{2D-5}}{\bar{f}_{2D-5}}. \tag{C19}$$

Thus, we have $b'_{2D-4}>b'_{2D-5}$ and

$$\mathrm{sign}(b_{p+1})\geq\mathrm{sign}(b_p),\qquad (p=2D-5). \tag{C20}$$

APPENDIX D: $\mathrm{sign}(b_{p+1}) \geq \mathrm{sign}(b_p)$, $2D-4 \leq p \leq 3D-10$

For $2D-3 \leq p \leq 3D-9$, The coefficient of b_p is

$$\begin{aligned} b_p &= a_6 C^p_{6D-18} r_h^{6D-18-p} + a_5 C^p_{5D-15} r_h^{5D-15-p} + a_4 C^p_{4D-12} r_h^{4D-12-p} + a_3 C^p_{3D-9} r_h^{3D-9-p} \\ &\quad + a'_5 C^p_{5D-13} r_h^{5D-13-p} + a'_4 C^p_{4D-10} r_h^{4D-10-p} + a'_3 C^p_{3D-7} r_h^{3D-7-p}, \\ &= r_h^{-p}(a_6 u^6 C^p_{6D-18} + a_5 u^5 C^p_{5D-15} + a_4 u^4 C^p_{4D-12} + a_3 u^3 C^p_{3D-9}) \\ &\quad + r_h^{-p+2} u^2 (a'_5 u^3 C^p_{5D-13} + a'_4 u^2 C^p_{4D-10} + a'_3 u C^p_{3D-7}). \end{aligned} \tag{D1}$$

The normalized coefficient is defined as

$$b'_p = \frac{r_h^p}{\hat{f}_p} b_p = \frac{\bar{A}_p}{\hat{f}_p} + r_h^2 u^2 \frac{\hat{B}_p}{\hat{f}_p} = \frac{\bar{A}^L_p}{\hat{f}_p}\lambda_l + \frac{\bar{A}^R_p}{\hat{f}_p} + r_h^2 u^2 \left(\frac{\hat{B}^M_p}{\hat{f}_p}(\mu^2 - \omega^2) + \frac{\hat{B}^R_p}{\hat{f}_p}\right), \tag{D2}$$

where

$$\begin{aligned} \hat{f}_p &= u(u+v) C^p_{5D-13} + (u^2 + v^2) C^p_{4D-10} - 3v(u+v) C^p_{3D-7}, \\ \hat{B}_p &= a'_5 u^3 C^p_{5D-13} + a'_4 u^2 C^p_{4D-10} + a'_3 u C^p_{3D-7}, \\ \hat{B}^M_p &= 2u^2 (D-3)(C^p_{5D-13} u(u+v) - C^p_{4D-10}(u^2 + 4uv + v^2) + 3C^p_{3D-7}(u+v)v), \\ \hat{B}^R_p &= 2(D-3)u(-u\hat{f}_p \omega^2 + 2c_D equ(C^p_{5D-13} u + C^p_{4D-10}(u+v) - 3C^p_{3D-7} v)\omega - 2c_D^2 e^2 q^2 u C^p_{4D-10}), \end{aligned} \tag{D3}$$

and $\bar{A}^L_p, \bar{A}^R_p$ are defined in Eq. (C3).

According to the proof of Eq. (A7), it is easy to know that $\hat{f}_p > 0$. According to the proof of Eq. (A61), it is easy to know that $\hat{B}^M_p > 0$. Based on the proof in Eq. (A72) and taking the C^p_{2D-4} term to zero, we have

$$\frac{\hat{B}^M_{p+1}}{\hat{f}_{p+1}} - \frac{\hat{B}^M_p}{\hat{f}_p} = \frac{2(D-3)u}{(p+1)\hat{f}_{p+1}\hat{f}_p}(C^p_{4D-10} C^p_{5D-13} - 6vC^p_{3D-7} C^p_{5D-13} + 6C^p_{3D-7} C^p_{4D-10} v^2) > 0, \tag{D4}$$

where we use $\frac{C^p_{4D-10}}{C^p_{3D-7}} > 17$ when $p \geq 2D-3 \geq 11$. Based on the proof in Appendix A 4, it is easy to deduce

$$\frac{\hat{B}^R_{p+1}}{\hat{f}_{p+1}} - \frac{\hat{B}^R_p}{\hat{f}_p} > 0, \tag{D5}$$

since δ_0, δ_1, δ_2 defined in Eq. (A76) are all positive in this case.

Following the proof in Eq. (A21), it is easy to deduce

$$\frac{\hat{f}_p}{C^p_{5D-13}} > \frac{\hat{f}_{p+1}}{C^{p+1}_{5D-13}}. \tag{D6}$$

Following the proof in Eq. (A22), it is easy to deduce

$$\frac{\bar{A}^L_{p+1}}{C^{p+1}_{5D-13}} > \frac{\bar{A}^L_p}{C^p_{5D-13}}, \tag{D7}$$

since β_0, β_1 defined in Eq. (A23) preserve the same form and β_2 is still positive. So, we have

$$\frac{\bar{A}^L_{p+1}}{\hat{f}_{p+1}} > \frac{\bar{A}^L_p}{\hat{f}_p}. \tag{D8}$$

Following the proof in Appendix A 2, and taking terms proportional to C^p_{D-3}, C^p_{2D-6} and y_2 as zero, it is easy to deduce

$$\frac{\bar{A}^R_{p+1}}{\hat{f}_{p+1}} > \frac{\bar{A}^R_p}{\hat{f}_p}. \tag{D9}$$

Thus, we have $b'_{p+1} > b'_p$ and

$$\mathrm{sign}(b_{p+1}) \geq \mathrm{sign}(b_p), \quad (2D-3 \leq p \leq 3D-10). \tag{D10}$$

Now, let us prove $b'_{2D-3} > b'_{2D-4}$, i.e.

$$\begin{aligned} &\frac{\bar{A}^L_{2D-3}}{\hat{f}_{2D-3}}\lambda_l + \frac{\bar{A}^R_{2D-3}}{\hat{f}_{2D-3}} + r_h^2 u^2 \left(\frac{\hat{B}^M_{2D-3}}{\hat{f}_{2D-3}}(\mu^2 - \omega^2) + \frac{\hat{B}^R_{2D-3}}{\hat{f}_{2D-3}}\right) \\ &> \frac{\bar{A}^L_{2D-4}}{\bar{f}_{2D-4}}\lambda_l + \frac{\bar{A}^R_{2D-4}}{\bar{f}_{2D-4}} + r_h^2 u^2 \left(\frac{\bar{B}^M_{2D-4}}{\bar{f}_{2D-4}}(\mu^2 - \omega^2) + \frac{\bar{B}^R_{2D-4}}{\bar{f}_{2D-4}}\right). \end{aligned} \tag{D11}$$

Since $\bar{f}_{2D-4} > \hat{f}_{2D-4} > 0$ and $\bar{A}^L_{2D-4} > 0$, we have

$$\frac{\bar{A}^L_{2D-4}}{\bar{f}_{2D-4}} < \frac{\bar{A}^L_{2D-4}}{\hat{f}_{2D-4}} < \frac{\bar{A}^L_{2D-3}}{\hat{f}_{2D-3}}, \tag{D12}$$

where we use the result in Eq. (D8) in the last inequality. Similarly, since $\bar{f}_{2D-4} > \hat{f}_{2D-4} > 0$ and $\hat{B}^M_{2D-4} > \bar{B}^M_{2D-4} > 0$, we have

$$\frac{\bar{B}^M_{2D-4}}{\bar{f}_{2D-4}} < \frac{\hat{B}^M_{2D-4}}{\hat{f}_{2D-4}} < \frac{\hat{B}^M_{2D-3}}{\hat{f}_{2D-3}}. \tag{D13}$$

The difference $\frac{\hat{B}^R_{2D-3}}{\hat{f}_{2D-3}} - \frac{\bar{B}^R_{2D-4}}{\bar{f}_{2D-4}}$ is linear in ω. For $\omega = 0$,

$$\begin{aligned}\left(\frac{\hat{B}^R_{2D-3}}{\hat{f}_{2D-3}} - \frac{\bar{B}^R_{2D-4}}{\bar{f}_{2D-4}}\right)_{\omega=0} &= \frac{4c_D^2 e^2 q^2 (D-3)u}{\hat{f}_{2D-3}\bar{f}_{2D-4}} (\hat{f}_{2D-3}(C^{2D-4}_{4D-10}u - v) - \bar{f}_{2D-4} C^{2D-3}_{4D-10} u) \\ &= \frac{4c_D^2 e^2 q^2 (D-3)^2 u}{\hat{f}_{2D-3}\bar{f}_{2D-4}(2D-3)} (\epsilon_0 u^3 + \epsilon_1 u^2 v + \epsilon_2 u v^2 + \epsilon_3 v^3) \\ &> \frac{4c_D^2 e^2 q^2 (D-3)^2 u}{\hat{f}_{2D-3}\bar{f}_{2D-4}(2D-3)} ((\epsilon_0 + \epsilon_1 + \epsilon_2) u v^2 + \epsilon_3 v^3) \\ &> \frac{4c_D^2 e^2 q^2 (D-3)^2 u}{\hat{f}_{2D-3}\bar{f}_{2D-4}(2D-3)} (\epsilon_0 + \epsilon_1 + \epsilon_2 + \epsilon_3) v^3 > 0,\end{aligned} \tag{D14}$$

where

$$\begin{aligned}\epsilon_0 &= C^{2D-4}_{4D-10} C^{2D-4}_{5D-13}, \\ \epsilon_1 &= -2C^{2D-4}_{4D-10} + 3C^{2D-4}_{3D-7} C^{2D-4}_{4D-10} - 3C^{2D-4}_{5D-13} + C^{2D-4}_{4D-10} C^{2D-4}_{5D-13}, \\ \epsilon_2 &= 3C^{2D-4}_{3D-7} - 4C^{2D-4}_{4D-10} + 3C^{2D-4}_{3D-7} C^{2D-4}_{4D-10} - 3C^{2D-4}_{5D-13}, \\ \epsilon_3 &= 3C^{2D-4}_{3D-7} - 2C^{2D-4}_{4D-10},\end{aligned} \tag{D15}$$

and we also use the inequalities $\epsilon_0 > 0, \epsilon_1 > 0, (\epsilon_0 + \epsilon_1 + \epsilon_2) > 0, (\epsilon_0 + \epsilon_1 + \epsilon_2 + \epsilon_3) > 0$, which are easy to check. For $\omega = \frac{c_D e q}{u}$,

$$\begin{aligned}&\left(\frac{\hat{B}^R_{2D-3}}{\hat{f}_{2D-3}} - \frac{\bar{B}^R_{2D-4}}{\bar{f}_{2D-4}}\right)_{\omega=\frac{c_D e q}{u}} \\ &= \frac{4c_D^2 e^2 q^2 (D-3)^2 u(u-v)}{\hat{f}_{2D-3}\bar{f}_{2D-4}(2D-3)} (\zeta_0 u^2 + \zeta_1 u v + \zeta_2 v^2) \\ &> \frac{4c_D^2 e^2 q^2 (D-3)^2 u(u-v)}{\hat{f}_{2D-3}\bar{f}_{2D-4}(2D-3)} ((\zeta_0 + \zeta_1) u v + \zeta_2 v^2) \\ &> \frac{4c_D^2 e^2 q^2 (D-3)^2 u(u-v)}{\hat{f}_{2D-3}\bar{f}_{2D-4}(2D-3)} (\zeta_0 + \zeta_1 + \zeta_2) v^2 > 0,\end{aligned} \tag{D16}$$

where

$$\begin{aligned}\zeta_0 &= C^{2D-4}_{4D-10} C^{2D-4}_{5D-13}, \\ \zeta_1 &= (-2 + 3C^{2D-4}_{3D-7}) C^{2D-4}_{4D-10} - 3C^{2D-4}_{5D-13}, \\ \zeta_2 &= 3C^{2D-4}_{3D-7} - 2C^{2D-4}_{4D-10},\end{aligned} \tag{D17}$$

and we also use the inequalities $\zeta_0 > 0, \zeta_0 + \zeta_1 > 0, \zeta_0 + \zeta_1 + \zeta_2 > 0$, which are easy to check. So, we obtain

$$\frac{\hat{B}^R_{2D-3}}{\hat{f}_{2D-3}} > \frac{\bar{B}^R_{2D-4}}{\bar{f}_{2D-4}}. \tag{D18}$$

In the proof of Eq. (C19), we always rewrite C^{p+1}_* in $\bar{A}^R_{p+1}$ and $\bar{f}_{p+1}$ as $\frac{*-p}{p+1} C^p_*$, e.g.

$$\bar{f}_{p+1} = \frac{1}{p+1}(u(u+v)C^p_{5D-13}(5D-13-p) + (u^2+v^2)C^p_{4D-10}(4D-10-p) - 3v(u+v)C^p_{3D-7}(3D-7-p) + 2v^2C^p_{2D-4}(2D-4-p)),$$
$$\hat{f}_{p+1} = \frac{1}{p+1}(u(u+v)C^p_{5D-13}(5D-13-p) + (u^2+v^2)C^p_{4D-10}(4D-10-p) - 3v(u+v)C^p_{3D-7}(3D-7-p)). \tag{D19}$$

From the above equations, we can see that after the rewriting, $\hat{f}_{2D-3}$ is formally equal to $\bar{f}_{2D-3}$. Then, according to the proof of Eq. (C19), we have

$$\frac{\bar{A}^R_{2D-3}}{\hat{f}_{2D-3}} > \frac{\bar{A}^R_{2D-4}}{\bar{f}_{2D-4}}. \tag{D20}$$

Thus, we have $b'_{2D-3} > b'_{2D-4}$ and

$$\mathrm{sign}(b_{p+1}) \geq \mathrm{sign}(b_p), \qquad (p = 2D-4). \tag{D21}$$

APPENDIX E: $\mathrm{sign}(b_{p+1}) \geq \mathrm{sign}(b_p)$, $p = 3D-9, 3D-8$

The coefficient b_p $(p = 3D-7, 3D-8)$ is

$$b_p = a_6C^p_{6D-18}r_h^{6D-18-p} + a_5C^p_{5D-15}r_h^{5D-15-p} + a_4C^p_{4D-12}r_h^{4D-12-p} + a'_5C^p_{5D-13}r_h^{5D-13-p} + a'_4C^p_{4D-10}r_h^{4D-10-p} + a'_3C^p_{3D-7}r_h^{3D-7-p},$$
$$= r_h^{-p}(a_6u^6C^p_{6D-18} + a_5u^5C^p_{5D-15} + a_4u^4C^p_{4D-12}) + r_h^{-p+2}u^2(a'_5u^3C^p_{5D-13} + a'_4u^2C^p_{4D-10} + a'_3uC^p_{3D-7}). \tag{E1}$$

Define the normalized coefficients b'_p

$$b'_p = \frac{r_h^p}{\hat{f}_p}b_p = \frac{\hat{A}_p}{\hat{f}_p} + r_h^2u^2\frac{\hat{B}_p}{\hat{f}_p} = \frac{\hat{A}^L_p}{\hat{f}_p}\lambda_l + \frac{\hat{A}^R_p}{\hat{f}_p} + r_h^2u^2\left(\frac{\hat{B}^M_p}{\hat{f}_p}(\mu^2-\omega^2) + \frac{\hat{B}^R_p}{\hat{f}_p}\right), \tag{E2}$$

where

$$\hat{A}^L_p = 2u^4(2u^2C^p_{6D-18} + (D-7)u(u+v)C^p_{5D-15} - (D-5)(u^2+4uv+v^2)C^p_{4D-12}),$$
$$\hat{A}^R_p = u^3((D-2)(D-4)u^3C^p_{6D-18} - 3(D-2)(D-4)u^2(u+v)C^p_{5D-15} + u((4D^3-29D^2+66D-48)uv - (D^3-11D^2+39D-42)(u+v)^2)C^p_{4D-12}). \tag{E3}$$

Now let us prove $b'_{3D-8} > b'_{3D-9}$, i.e.

$$\frac{\hat{A}^L_{3D-8}}{\hat{f}_{3D-8}}\lambda_l + \frac{\hat{A}^R_{3D-8}}{\hat{f}_{3D-8}} + r_h^2u^2\left(\frac{\hat{B}^M_{3D-8}}{\hat{f}_{3D-8}}(\mu^2-\omega^2) + \frac{\hat{B}^R_{3D-8}}{\hat{f}_{3D-8}}\right) > \frac{\bar{A}^L_{3D-9}}{\hat{f}_{3D-9}}\lambda_l + \frac{\bar{A}^R_{3D-9}}{\hat{f}_{3D-9}} + r_h^2u^2\left(\frac{\hat{B}^M_{3D-9}}{\hat{f}_{3D-9}}(\mu^2-\omega^2) + \frac{\hat{B}^R_{3D-9}}{\hat{f}_{3D-9}}\right). \tag{E4}$$

After rewriting C^{p+1}_* as $\frac{*-p}{p+1}C^p_*$, formally, we have

$$\hat{A}^{L(R)}_{3D-8} = \bar{A}^{L(R)}_{3D-8}. \tag{E5}$$

Then, with the results in Appendix D, we immediately obtain

$$b'_{3D-8} > b'_{3D-9},$$

and

$$\mathrm{sign}(b_{p+1}) \geq \mathrm{sign}(b_p), \qquad (p = 3D-9). \tag{E6}$$

Then, let us prove $b'_{3D-7} > b'_{3D-8}$, i.e.

$$\frac{\hat{A}^L_{3D-7}}{\hat{f}_{3D-7}}\lambda_l + \frac{\hat{A}^R_{3D-7}}{\hat{f}_{3D-7}} + r_h^2u^2\left(\frac{\hat{B}^M_{3D-7}}{\hat{f}_{3D-7}}(\mu^2-\omega^2) + \frac{\hat{B}^R_{3D-7}}{\hat{f}_{3D-7}}\right) > \frac{\hat{A}^L_{3D-8}}{\hat{f}_{3D-8}}\lambda_l + \frac{\hat{A}^R_{3D-8}}{\hat{f}_{3D-8}} + r_h^2u^2\left(\frac{\hat{B}^M_{3D-8}}{\hat{f}_{3D-8}}(\mu^2-\omega^2) + \frac{\hat{B}^R_{3D-8}}{\hat{f}_{3D-8}}\right). \tag{E7}$$

Following the proof of Eq. (A24) and taking terms including $y_2, C^p_{D-3}, C^p_{2D-6}, C^p_{3D-9}$ as zero, all the inequalities in Eqs. (A21)–(A23) still hold, and we obtain

$$\frac{\hat{A}^L_{3D-7}}{\hat{f}_{3D-7}} > \frac{\hat{A}^L_{3D-8}}{\hat{f}_{3D-8}}. \tag{E8}$$

Taking terms including $y_2, C_{D-3}, C_{2D-6}, C_{3D-9}$ as zero and following the proof in Appendix A 2, it is easy to deduce

$$\frac{\hat{A}^R_{3D-7}}{\hat{f}_{3D-7}} > \frac{\hat{A}^R_{3D-8}}{\hat{f}_{3D-8}}. \tag{E9}$$

Based on the general results in Eqs. (D4), (D5), we can get

$$\frac{\hat{B}^{M(R)}_{3D-7}}{\hat{f}_{3D-7}} > \frac{\hat{B}^{M(R)}_{3D-8}}{\hat{f}_{3D-8}}. \tag{E10}$$

Thus, we prove

$$b'_{3D-7} > b'_{3D-8},$$

and obtain

$$\mathrm{sign}(b_{p+1}) \geq \mathrm{sign}(b_p), \qquad (p = 3D-8). \tag{E11}$$

APPENDIX F: $\mathrm{sign}(b_{p+1}) \geq \mathrm{sign}(b_p)$, $3D-7 \leq p \leq 4D-13$

The coefficient b_p $(3D-6 \leq p \leq 4D-12)$ is

$$\begin{aligned} b_p &= a_6C^p_{6D-18}r_h^{6D-18-p} + a_5C^p_{5D-15}r_h^{5D-15-p} + a_4C^p_{4D-12}r_h^{4D-12-p} \\ &\quad + a'_5C^p_{5D-13}r_h^{5D-13-p} + a'_4C^p_{4D-10}r_h^{4D-10-p} + a'_3C^p_{3D-7}r_h^{3D-7-p}, \\ &= r_h^{-p}(a_6u^6C^p_{6D-18} + a_5u^5C^p_{5D-15} + a_4u^4C^p_{4D-12}) + r_h^{-p+2}u^2(a'_5u^3C^p_{5D-13} + a'_4u^2C^p_{4D-10}). \end{aligned} \tag{F1}$$

Define the normalized coefficients b'_p as

$$b'_p = \frac{r_h^p}{f_p}b_p = \frac{\hat{A}_p}{f_p} + r_h^2u^2\frac{B_p}{f_p} = \frac{\hat{A}^L_p}{f_p}\lambda_l + \frac{\hat{A}^R_p}{f_p} + r_h^2u^2\left(\frac{B^M_p}{f_p}(\mu^2-\omega^2) + \frac{B^R_p}{f_p}\right), \tag{F2}$$

where

$$\begin{aligned} f_p &= u(u+v)C^p_{5D-13} + (u^2+v^2)C^p_{4D-10}, \\ B^M_p &= 2u^2(D-3)(C^p_{5D-13}u(u+v) - C^p_{4D-10}(u^2+4uv+v^2)), \\ B^R_p &= 2(D-3)u(-uf_p\omega^2 + 2c_Deq u(C^p_{5D-13}u + C^p_{4D-10}(u+v))\omega - 2c_D^2e^2q^2uC^p_{4D-10}), \\ \hat{A}^L_p &= 2u^4(2u^2C^p_{6D-18} + (D-7)u(u+v)C^p_{5D-15} - (D-5)(u^2+4uv+v^2)C^p_{4D-12}), \\ \hat{A}^R_p &= u^3((D-2)(D-4)u^3C^p_{6D-18} - 3(D-2)(D-4)u^2(u+v)C^p_{5D-15} \\ &\quad + u((4D^3-29D^2+66D-48)uv - (D^3-11D^2+39D-42)(u+v)^2)C^p_{4D-12}). \end{aligned} \tag{F3}$$

Following the proof of Eq. (A24) and taking terms including $y_2, y_3, C^p_{D-3}, C^p_{2D-6}, C^p_{3D-9}$ as zero, all the inequalities in Eqs. (A21)–(A23) still hold, and we obtain

$$\frac{\hat{A}^L_{p+1}}{f_{p+1}} > \frac{\hat{A}^L_p}{f_p}, \qquad (3D-6 \leq p \leq 4D-13). \tag{F4}$$

Taking terms including $y_2, y_3, C^p_{D-3}, C^p_{2D-6}, C^p_{3D-9}$ as zero and following the proof in Appendix A 2, it is easy to deduce

$$\frac{\hat{A}^R_{p+1}}{f_{p+1}} > \frac{\hat{A}^R_p}{f_p}, \qquad (3D-6 \leq p \leq 4D-13). \tag{F5}$$

Based on the proofs in Eq. (A72) and in Appendix A 4, and taking terms including $y_2, y_3, C^p_{D-3}, C^p_{2D-6}, C^p_{3D-9}$ as zero, it is easy to deduce

$$\frac{B^{M(R)}_{p+1}}{f_{p+1}} > \frac{B^{M(R)}_p}{f_p}, \qquad (3D-6 \le p \le 4D-13). \quad \text{(F6)}$$

Thus, we prove

$$b'_{p+1} > b'_p, \qquad (3D-6 \le p \le 4D-13)$$

and obtain

$$\mathrm{sign}(b_{p+1}) \ge \mathrm{sign}(b_p), \quad (3D-6 \le p \le 4D-13). \quad \text{(F7)}$$

After rewriting C^{p+1}_* as $\frac{*-p}{p+1} C^p_*$, formally, we have

$$f_{3D-6} = \hat{f}_{3D-6}, \qquad B_{3D-6} = \hat{B}_{3D-6}. \quad \text{(F8)}$$

Following the proof of $b'_{3D-7} > b'_{3D-8}$, we have

$$b'_{3D-6} > b'_{3D-7}$$

and

$$\mathrm{sign}(b_{p+1}) \ge \mathrm{sign}(b_p), \qquad (p = 3D-7). \quad \text{(F9)}$$

APPENDIX G: $\mathrm{sign}(b_{p+1}) \ge \mathrm{sign}(b_p)$, $p = 4D-11, 4D-12$

The three coefficients $b_{4D-10}, b_{4D-11}, b_{4D-12}$ are

$$\begin{aligned} b_{4D-10} &= r_h^{-(4D-10)}(a_6 u^6 C^{4D-10}_{6D-18} + a_5 u^5 C^{4D-10}_{5D-15}) \\ &\quad + r_h^{-(4D-12)} u^2 (a'_5 u^3 C^{4D-10}_{5D-13} + a'_4 u^2), \\ b_{4D-11} &= r_h^{-(4D-11)}(a_6 u^6 C^{4D-11}_{6D-18} + a_5 u^5 C^{4D-11}_{5D-15}) \\ &\quad + r_h^{-(4D-13)} u^2 (a'_5 u^3 C^{4D-11}_{5D-13} + a'_4 u^2 C^{4D-11}_{4D-10}), \\ b_{4D-12} &= r_h^{-(4D-12)}(a_6 u^6 C^{4D-12}_{6D-18} + a_5 u^5 C^{4D-12}_{5D-15} + a_4 u^4) \\ &\quad + r_h^{-(4D-14)} u^2 (a'_5 u^3 C^{4D-12}_{5D-13} + a'_4 u^2 C^{4D-12}_{4D-10}). \end{aligned} \quad \text{(G1)}$$

We define the normalized coefficients,

$$b'_{4D-10} = \frac{r_h^{4D-10} b_{4D-10}}{f_{4D-10}} = \frac{A_{4D-10}}{f_{4D-10}} + r_h^2 u^2 \frac{B_{4D-10}}{f_{4D-10}}, \quad \text{(G2)}$$

$$b'_{4D-11} = \frac{r_h^{4D-11} b_{4D-11}}{f_{4D-11}} = \frac{A_{4D-11}}{f_{4D-11}} + r_h^2 u^2 \frac{B_{4D-11}}{f_{4D-11}}, \quad \text{(G3)}$$

$$b'_{4D-12} = \frac{r_h^{4D-12} b_{4D-12}}{f_{4D-12}} = \frac{\hat{A}_{4D-12}}{f_{4D-12}} + r_h^2 u^2 \frac{B_{4D-12}}{f_{4D-12}}. \quad \text{(G4)}$$

After rewriting C^{p+1}_* as $\frac{*-p}{p+1} C^p_*$, formally, we have

$$A_{4D-11} = \hat{A}_{4D-11}. \quad \text{(G5)}$$

Following the proof of Eq. (F7), we have

$$b'_{4D-11} > b'_{4D-12},$$

and

$$\mathrm{sign}(b_{p+1}) \ge \mathrm{sign}(b_p), \qquad (p = 4D-12). \quad \text{(G6)}$$

Following the proof in Appendix A, and taking terms including y_3, y_2, $C^p_{D-3}, C^p_{2D-6}, C^p_{3D-9}, C^p_{4D-12}$ as zero, it is easy to check

$$b'_{4D-10} > b'_{4D-11},$$

and we have

$$\mathrm{sign}(b_{p+1}) \ge \mathrm{sign}(b_p), \qquad (p = 4D-11). \quad \text{(G7)}$$

[1] T. Regge and J. A. Wheeler, Phys. Rev. **108**, 1063 (1957).
[2] L. Barack, V. Cardoso, S. Nissanke, T. P. Sotiriou, A. Askar, C. Belczynski, G. Bertone, E. Bon, D. Blas, R. Brito *et al.*, Classical Quantum Gravity **36**, 143001 (2019).
[3] P. Pani, Int. J. Mod. Phys. A **28**, 1340018 (2013).
[4] R. Brito, V. Cardoso, and P. Pani, Lect. Notes Phys. **906**, 1 (2015).
[5] R. Brito, V. Cardoso, and P. Pani, Classical Quantum Gravity **32**, 134001 (2015).
[6] V. Cardoso, O. J. C. Dias, J. P. S. Lemos, and S. Yoshida, Phys. Rev. D **70**, 044039 (2004); **70**, 049903(E) (2004).
[7] R. Penrose, Riv. Nuovo Cim. **1**, 252 (1969).
[8] D. Christodoulou, Phys. Rev. Lett. **25**, 1596 (1970).
[9] C. W. Misner, Phys. Rev. Lett. **28**, 994 (1972).

[10] J. M. Bardeen, W. H. Press, and S. A. Teukolsky, Astrophys. J. **178**, 347 (1972).
[11] J. D. Bekenstein, Phys. Rev. D **7**, 949 (1973).
[12] T. Damour, N. Deruelle, and R. Ruffini, Lett. Nuovo Cim. **15**, 257 (1976).
[13] W. H. Press and S. A. Teukolsky, Nature (London) **238**, 211 (1972).
[14] V. Cardoso, O. J. C. Dias, J. P. S. Lemos, and S. Yoshida, Phys. Rev. D **70**, 044039 (2004); **70**, 049903(E) (2004).
[15] C. A. R. Herdeiro, J. C. Degollado, and H. F. Runarsson, Phys. Rev. D **88**, 063003 (2013).
[16] J. C. Degollado and C. A. R. Herdeiro, Phys. Rev. D **89**, 063005 (2014).
[17] M. J. Strafuss and G. Khanna, Phys. Rev. D **71**, 024034 (2005).
[18] R. A. Konoplya and A. Zhidenko, Phys. Rev. D **73**, 124040 (2006).
[19] V. Cardoso, S. Chakrabarti, P. Pani, E. Berti, and L. Gualtieri, Phys. Rev. Lett. **107**, 241101 (2011).
[20] S. R. Dolan, Phys. Rev. D **87**, 124026 (2013).
[21] S. Hod, Phys. Lett. B **708**, 320 (2012).
[22] S. Hod, Phys. Lett. B **736**, 398 (2014).
[23] A. N. Aliev, J. Cosmol. Astropart. Phys. 11 (2014) 029.
[24] S. Hod, Phys. Lett. B **758**, 181 (2016).
[25] J. C. Degollado, C. A. R. Herdeiro, and E. Radu, Phys. Lett. B **781**, 651 (2018).
[26] J. H. Huang, W. X. Chen, Z. Y. Huang, and Z. F. Mai, Phys. Lett. B **798**, 135026 (2019).
[27] S. Ponglertsakul and B. Gwak, Eur. Phys. J. C **80**, 1023 (2020).
[28] W. E. East and F. Pretorius, Phys. Rev. Lett. **119**, 041101 (2017).
[29] W. E. East, Phys. Rev. D **96**, 024004 (2017).
[30] J. M. Lin, M. J. Luo, Z. H. Zheng, L. Yin, and J. H. Huang, Phys. Lett. B **819**, 136392 (2021).
[31] J. H. Xu, Z. H. Zheng, M. J. Luo, and J. H. Huang, Eur. Phys. J. C **81**, 402 (2021).
[32] V. Cardoso and O. J. C. Dias, Phys. Rev. D **70**, 084011 (2004).
[33] V. Cardoso, O. J. C. Dias, G. S. Hartnett, L. Lehner, and J. E. Santos, J. High Energy Phys. 04 (2014) 183.
[34] C. Y. Zhang, S. J. Zhang, and B. Wang, J. High Energy Phys. 08 (2014) 011.
[35] O. Delice and T. Durgut, Phys. Rev. D **92**, 024053 (2015).
[36] A. N. Aliev, Eur. Phys. J. C **76**, 58 (2016).
[37] M. Wang and C. Herdeiro, Phys. Rev. D **93**, 064066 (2016).
[38] H. R. C. Ferreira and C. A. R. Herdeiro, Phys. Rev. D **97**, 084003 (2018).
[39] M. Wang and C. Herdeiro, Phys. Rev. D **89**, 084062 (2014).
[40] P. Bosch, S. R. Green, and L. Lehner, Phys. Rev. Lett. **116**, 141102 (2016).
[41] Y. Huang, D. J. Liu, and X. Z. Li, Int. J. Mod. Phys. D **26**, 1750141 (2017).
[42] P. A. Gonzalez, E. Papantonopoulos, J. Saavedra, and Y. Vasquez, Phys. Rev. D **95**, 064046 (2017).
[43] Z. Zhu, S. J. Zhang, C. E. Pellicer, B. Wang, and E. Abdalla, Phys. Rev. D **90**, 044042 (2014); **90**, 049904(A) (2014).
[44] S. Hod, Phys. Lett. B **713**, 505 (2012).
[45] J. H. Huang and Z. F. Mai, Eur. Phys. J. C **76**, 314 (2016).
[46] S. Hod, Phys. Rev. D **91**, 044047 (2015).
[47] L. Di Menza and J.-P. Nicolas, Classical Quantum Gravity **32**, 145013 (2015).
[48] Z. F. Mai, R. Q. Yang, and H. Lu, Phys. Rev. D **105**, 024070 (2022).
[49] Y. F. Zou, J. H. Xu, Z. F. Mai, and J. H. Huang, Eur. Phys. J. C **81**, 855 (2021).
[50] R. A. Konoplya and A. Zhidenko, Rev. Mod. Phys. **83**, 793 (2011).
[51] R. A. Konoplya and A. Zhidenko, Nucl. Phys. **B777**, 182 (2007).
[52] R. A. Konoplya and A. Zhidenko, Phys. Rev. Lett. **103**, 161101 (2009).
[53] R. A. Konoplya and A. Zhidenko, Phys. Rev. D **89**, 024011 (2014).
[54] R. A. Konoplya and A. Zhidenko, Phys. Rev. D **78**, 104017 (2008).
[55] A. Ishibashi and H. Kodama, Prog. Theor. Phys. **110**, 901 (2003).
[56] H. Kodama and A. Ishibashi, Prog. Theor. Phys. **111**, 29 (2004).
[57] H. Kodama, Prog. Theor. Phys. Suppl. **172**, 11 (2008).
[58] A. Ishibashi and H. Kodama, Prog. Theor. Phys. Suppl. **189**, 165 (2011).
[59] H. Ishihara, M. Kimura, R. A. Konoplya, K. Murata, J. Soda, and A. Zhidenko, Phys. Rev. D **77**, 084019 (2008).
[60] K. Destounis, Phys. Rev. D **100**, 044054 (2019).
[61] J. H. Huang, R. D. Zhao, and Y. F. Zou, Phys. Lett. B **823**, 136724 (2021).
[62] J. H. Huang, T. T. Cao, and M. Z. Zhang, Eur. Phys. J. C **81**, 904 (2021).
[63] R. D. Zhao and J. H. Huang, Phys. Lett. B **833**, 137286 (2022).
[64] J. H. Huang, Eur. Phys. J. C **82**, 467 (2022).
[65] R. C. Myers and M. J. Perry, Ann. Phys. (N.Y.) **172**, 304 (1986).
[66] A. Chodos and E. Myers, Ann. Phys. (N.Y.) **156**, 412 (1984).
[67] A. Higuchi, J. Math. Phys. (N.Y.) **28**, 1553 (1987); **43**, 6385 (E) (2002).
[68] M. A. Rubin and C. R. Ordonez, J. Math. Phys. (N.Y.) **25**, 2888 (1984).
[69] J. Ben Achour, E. Huguet, J. Queva, and J. Renaud, J. Math. Phys. (N.Y.) **57**, 023504 (2016).
[70] L. Lindblom, N. W. Taylor, and F. Zhang, Gen. Relativ. Gravit. **49**, 139 (2017).