Force-free electromagnetic configurations in FLRW geometry

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Force-free electrodynamics is the theoretical paradigm used to describe electromagnetic fields in a region where the inertia of plasma is negligible compared to the strength of the electromagnetic field. While these fields are studied extensively around accreting black holes in an attempt to describe energy extraction, force-free fields also routinely appear in the study of cosmological magnetic fields. Despite this, there are no systematic studies of exact force-free fields in an expanding Universe. In this paper, we use geometric methods to find a wide variety of force-free solutions in a fixed Friedmann-Lemaître-Robertson-Walker metric background. The method we use can be directly generalized to any arbitrary electrically neutral spacetime, and hence provides a powerful tool to study force-free fields in general.

I. INTRODUCTION

Magnetic fields are known to permeate the Universe from the scales of planets and stars to those of galaxies and even galaxy clusters [1]. At the scale of galaxies, magnetic fields have a strength of order 10^{-6} G and are coherent over kpc scales. Micro-Gauss magnetic fields have also been observed in galaxy clusters [2,3]. Recent observations suggest that intergalactic space may harbor magnetic fields of strength 10^{-16} G coherent over Mpc scales [4].

The currently accepted paradigm explaining the existence of this all-pervading magnetic field states that these magnetic fields originated from the amplification of seed magnetic fields via various astrophysical dynamos. While at smaller scales (at the level of planets and stars), these fields require constant rejuvenation to replenish the loss from dissipation, the timescales for dissipation for large-scale magnetic fields may be of the order of the age of the Universe. While the amplification of the large-scale magnetic field is attributed to the gravitational collapse of flux-frozen matter during structure formation, the dynamo effect can only amplify a preexisting nonzero "seed" magnetic field. The origin of such a seed magnetic field itself is not well understood and is a subject of extensive study, especially over the last two decades. Electroweak phase transition [5,6] and inflationary generation of the seed magnetic fields [7–9] are the most widely studied

phenomena for the generation of primordial magnetic fields in the early Universe. Other mechanisms for the generation of primordial magnetic fields include Lorentz invariance violation [10], relativistic positron abundance [11], and non-Gaussian perturbations to the baryon density, to name a few [12].

A variety of methods exist for detecting and/or constraining magnetic fields in the Universe. At low redshifts, these include the observation of Zeeman splitting [13] and Faraday rotation of linearly polarized radio sources [14]. At higher redshifts, the existence of magnetic fields can be inferred from the effects of primordial magnetic fields on the polarization of the cosmic microwave background (CMB) [15–18], and the effects of magnetic pressure on the abundance of light elements during big-bang nucleosynthesis [19], as well as the (non)detection of inverse-Compton scattered CMB photons from blazar observations [4,20]. Recently, it was shown that hydrodynamical simulations of structure formation in the Universe can also constrain the primordial magnetic fields by studying their ability to reproduce in the simulations the scaling relations observed in dwarf galaxies [21].

When the energy stored in the electromagnetic field is much greater than the plasma pressure, the Lorenz force vanishes. Force-free electrodynamics is the framework that describes such a system. Force-free fields can also be thought of as the low-inertia limit of ideal magnetohydrodynamics—i.e., the limit where the matter part of the stress-energy tensor can be ignored in favor of a purely electromagnetic stress-energy tensor.

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Force-free plasma occurs naturally in various astrophysical contexts. For example, the solar corona is permeated by such strong magnetic fields that it is essentially force-free. The study of force-free fields is also important in the study of relativistic jets. The strong magnetic fields generated around the accreting black hole and neutron stars render these systems force-free, and the fields are believed to extract rotational energy from these compact objects to power the highly collimated relativistic jets.

Force-free fields often appear in the study of primordial magnetogenesis [22–28]. Several studies of primordial magnetic fields assume the vanishing of the Lorentz force [29–33]. To our knowledge, there has been no significant and systematic attempt to study the types of force-free fields allowed in the Friedmann-Lemaître-Robertson-Walker (FLRW) metric. In this paper, we provide a nonexhaustive list of solutions to the equations of force-free electrodynamics (FFE) generated not by solving the arising partial differential equations, but by using the geometric study of the foliations of spacetime as laid out in Refs. [34–36]. This method is suitable for generating force-free fields not just in nonstationary spacetimes, but in any arbitrary spacetime that is electrically neutral.

II. EQUATIONS OF FORCE-FREE ELECTRODYNAMICS

Maxwell's equation in an arbitrary spacetime is given by

$$dF = 0, \quad \text{and} \quad *d*F = j. \tag{1}$$

Here, F is the Maxwell field tensor, $*$ is the Hodge star operator, d is the exterior derivative on forms, and j denotes the current density dual vector. Force-free electrodynamics is defined by the constraint $F_{\mu\nu}j^{\nu} = 0$. The Maxwell field tensor F is said to be magnetically dominated whenever $F^2 > 0$, F is electrically dominated whenever $F^2 < 0$, and a force-free electromagnetic field F is null whenever $F^2 = 0$. As usual, $F^2 = F_{\mu\nu}F^{\mu\nu}$.

The kernel of F is the set of all tangent vectors that annihilate F. In force-free electrodynamics, the current density j is always such a vector field. It is well known that the kernel of F forms the tangent space to a two-dimensional submanifold. Starting with this fact, a comprehensive study of geometric features of force-free fields was done in Refs. [34–36], and we provide a concise summary in the following sections.

A. The non-null force-free field

Details presented in this section were previously derived in [34]. We will first consider the case of a magnetically dominated field. In this case, about any point in spacetime, we can construct an inertial frame field (e_0, e_1, e_2, e_3) such that $g(e_\mu, e_\nu) = \eta_{\mu\nu}$, and further, e_0 and e_1 span the kernel

of F. Here, η is the Minkowski metric. We define vector fields H and $\tilde{H}$ by

$$2H = [-g(\nabla_{e_0}e_0, e_2) + g(\nabla_{e_1}e_1, e_2)]e_2 \\ + [-g(\nabla_{e_0}e_0, e_3) + g(\nabla_{e_1}e_1, e_3)]e_3 \tag{2}$$

and

$$2\tilde{H} = [-g(\nabla_{e_2}e_2, e_0) - g(\nabla_{e_3}e_3, e_0)]e_0 \\ + [g(\nabla_{e_2}e_2, e_1) + g(\nabla_{e_3}e_3, e_1)]e_1. \tag{3}$$

We are guaranteed a magnetically dominated force-free solution if

$$d(H^\flat + \tilde{H}^\flat) = 0. \tag{4}$$

In this case, the magnetically force-free F can now be written as

$$F = u\, e_2^\flat \wedge e_3^\flat, \tag{5}$$

where

$$d(\ln u) = 2(H + \tilde{H})^\flat. \tag{6}$$

The $\flat$ maps a tangent vector to its metrically equivalent one-form. The difficulty of finding magnetically dominated force-free fields is now reduced to finding a tetrad frame field that satisfies the above requirements. Hidden in the above statements is the fact that e_0 and e_1 span smooth two-dimensional submanifolds. These submanifolds are referred to as field sheets.

The electrically dominated field is very similar to the above construction, except that it is e_2 and e_3 that span the kernel of F.

B. The theory of null force-free fields

Solutions to null force-free electrodynamics satisfy a different set of criteria, stemming from the fact that the kernel of F in this case is spanned by a degenerate distribution. For details, see [36].

Here, the kernel of F is spanned by a null geodesic vector field l and a spacelike orthogonal vector field s. The tetrad is completed by including another null vector n and a one-form α such that

$$n \cdot l = -1, \quad 0 = l \cdot s = \alpha(l) = n \cdot s = \alpha(n) = \alpha(s),$$

$$\alpha^\mu \alpha_\mu = 1 = s^\mu s_\mu.$$

In this case, if the equipartition condition for the null mean curvature is satisfied, i.e.,

$$g(\nabla_s l, s) = g(\nabla_{\alpha^\sharp} l, \alpha^\sharp),$$

then the force-free null field takes the form

$$F = (u \cdot \kappa)\alpha \wedge l^\flat. \tag{7}$$

Here, $\sharp$ is the inverse map of $\flat$. In an adapted chart (x^1, x^2, x^3, x^4) where field sheets are given by the condition $x^1, x^2 = \text{const}$,

$$\kappa = (\alpha_3 \, l_4^\flat - \alpha_4 \, l_3^\flat)^{-1}, \tag{8}$$

where

$$\begin{pmatrix} \alpha \\ l^\flat \end{pmatrix} = \begin{pmatrix} \alpha_3 & \alpha_4 \\ l_3^\flat & l_4^\flat \end{pmatrix} \begin{pmatrix} dx^3 \\ dx^4 \end{pmatrix}. \tag{9}$$

Here, $u = u(x^3, x^4)$ is any smooth function of x^3, x^4.

III. FORCE-FREE SOLUTIONS

The conformal time coordinate chart often simplifies the calculations in FLRW spacetimes. In this coordinate system, the metric takes the following form:

$$ds^2 = a(\eta)^2 \left[-d\eta^2 + \frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \right]. \tag{10}$$

Since we will rely primarily on the tetrad formalism described above, we begin by listing a set of the canonical orthonormal tetrad for the metric given above:

$$e_0 = \frac{1}{a(\eta)} \partial_\eta, \tag{11a}$$

$$e_1 = \frac{\sqrt{1 - Kr^2}}{a(\eta)} \partial_r, \tag{11b}$$

$$e_2 = \frac{1}{a(\eta)r} \partial_\theta, \tag{11c}$$

$$e_3 = \frac{1}{a(\eta)r\sin\theta} \partial_\varphi. \tag{11d}$$

A. Solution I

Consider a Lorentz transformation of the canonical tetrads given by

$$\begin{pmatrix} \bar{e}_0 \\ \bar{e}_1 \\ \bar{e}_2 \\ \bar{e}_3 \end{pmatrix} = \begin{pmatrix} \alpha\sin(\theta)f_2 & 0 & rf_1 f_2 & rff_2 \\ 0 & 1 & 0 & 0 \\ \sqrt{\beta}rf_2 & 0 & \frac{f_1\alpha\sin(\theta)f_2}{\sqrt{\beta}} & \frac{\alpha\sin(\theta)ff_2}{\sqrt{\beta}} \\ 0 & 0 & \frac{f}{\sqrt{\beta}} & -\frac{f_1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix}, \tag{12}$$

where f is any function of r, α and β are real constants, and

$$f_1 = \sqrt{\beta - f^2} \quad \text{and} \quad f_2 = \frac{1}{\sqrt{\alpha^2 \sin^2\theta - \beta r^2}}.$$

Then the pair of vector fields $(\bar{e}_2, \bar{e}_3)$ are involutive, and further,

$$2(H + \tilde{H}) = \frac{2\dot{a}(\eta)}{a(\eta)} d\eta + \frac{2\alpha^2\sin^2\theta - \beta r^2}{r(\beta r^2 - \alpha^2\sin^2\theta)} dr + \frac{\beta^2 r^2 \cot\theta}{\alpha^2\sin^2\theta - \beta r^2} d\theta. \tag{13}$$

It is easy to verify that $d(H + \tilde{H}) = 0$. Then, from Eqs. (4) to (6),

$$u = \frac{u_0\sqrt{\alpha^2\sin^2\theta - \beta r^2}}{a(\eta)^2 r^2 \sin\theta}, \tag{14}$$

and our electrically dominated solution is given by

$$F_1 = u \, e_0^\flat \wedge e_1^\flat$$

$$= \frac{u_0\,\alpha}{r^2\sqrt{1 - Kr^2}} d\eta \wedge dr + \frac{u_0\sqrt{\beta - f^2}}{\sin\theta\sqrt{1 - Kr^2}} dr \wedge d\theta$$

$$+ \frac{u_0 f}{\sqrt{1 - Kr^2}} dr \wedge d\varphi. \tag{15}$$

Here, the current density is given by

$$j_1 = \frac{\sqrt{1 - Kr^2}f'}{a(\eta)^4 r^2 \sin\theta} \left[\frac{f}{\sqrt{\beta - f^2}} \partial_\theta - \csc\theta\partial_\varphi \right]. \tag{16}$$

The Lorentz-invariant quantity F^2 in this case is given by

$$F_1^2 = -\frac{2u_0^2(\alpha^2\sin^2\theta - \beta r^2)}{a(\eta)^4 r^4 \sin^2\theta}. \tag{17}$$

From the above equation, we see that the solution is not well defined when $\sin\theta = 0$. For completeness, we also record the magnitude of the square of the current density vector:

$$j_1^2 = \frac{u_0^2(Kr^2 - 1)f'^2\beta}{a(\eta)^6 r^2 \sin^2\theta(f^2 - \beta)}. \tag{18}$$

This solution holds some intriguing features that deserve closer inspection. First, we note that the Lorentz transformation that generates the solution is not valid when

$$\chi \equiv \alpha^2 \sin^2\theta - \beta r^2 \le 0.$$

Nonetheless, F_1 does not depend on χ. This means that the solution F_1 is defined for all values of χ (except when

$\sin\theta = 0$ for an entirely different reason). As it turns out, examining F_1^2 tells us that our electrically dominated solution smoothly transitions to a null solution when $\chi = 0$, and further, to a magnetically dominated solution when $\chi < 0$. The tetrad formalism is unsuitable to handle such a transition. So, we shift to a different formalism using a foliation-adapted chart which was developed in [35]. About any point in spacetime, there exists a coordinate chart $(x^1, \ldots, x^4)$ that is adapted to field sheets, meaning that the tangent space of the submanifolds defined by constant values of x^1 and x^2 contains the kernel of F. Then, F takes the form

$$F = u(x^3, x^4)dx^3 \wedge dx^4. \tag{19}$$

Here, we let

$$M^r \equiv g^{r3}g^{34} - g^{33}g^{r4}, \quad \text{and} \quad N^r \equiv g^{r3}g^{44} - g^{34}g^{r4}.$$

Then, as was shown in [35], the equations of FFE in this coordinate system are given by

$$M^4 \frac{\partial}{\partial x^4}\ln|u| = -\nabla_r M^r \tag{20}$$

and

$$N^3 \frac{\partial}{\partial x^3}\ln|u| = -\nabla_r N^r. \tag{21}$$

To obtain a foliation-adapted chart for the case of this solution, we define the vector fields

$$X_1 = \sin\theta\partial_\theta - \frac{\sqrt{\beta - f^2}}{f}\partial_\varphi, \qquad X_2 = \partial_\eta - \frac{\alpha}{r^2 f}\partial_\varphi,$$

$$X_3 = \partial_\eta + \partial_r + \left[\left(-\beta\ln(\csc\theta + \cot\theta)\frac{f'}{f^2\sqrt{\beta - f^2}}\right) - \alpha\eta\left(\frac{2}{r^3 f} + \frac{f'}{r^2 f^2}\right)\right]\partial_\varphi,$$

and finally

$$X_4 = \partial_\varphi.$$

It is easily verified that all of the vector fields $\{X_i\}$ defined above commute with each other. Therefore, there exist coordinate functions $\{x^i\}$ such that $X_i = \partial_{x^i}$ for each i. The dual bases $\{dx^i\}$ are such that

$$dx^1 = \csc\theta d\theta, \qquad dx^2 = d\eta - dr, \qquad dx^3 = dr,$$

and

$$dx^4 = -\frac{\alpha}{r^2 f}d\eta + \left[\frac{\alpha}{r^2 f} + \frac{2\alpha\eta}{r^3 f} + \frac{\alpha\eta f'}{r^2 f^2} + \frac{\beta\ln(\csc\theta + \cot\theta)f'}{f^2\sqrt{\beta - f^2}}\right]dr + \frac{\sqrt{\beta - f^2}}{\sin\theta f}d\theta + d\varphi.$$

Then,

$$dx^i(\partial_{x^j}) = \delta^i_j,$$

as required. This will help us compute the determinant of the metric in the adapted basis, and also the quantities M^r and N^r as defined above. The relevant quantities in Eqs. (20) and (21) are given by

$$\sqrt{-g} = \frac{a^4(\eta)r^2\sin^2\theta}{\sqrt{1 - Kr^2}},$$

$$M^1 = \frac{-\sqrt{\beta - f^2}(1 - Kr^2)}{a^4(\eta)r^2\sin^2\theta f}, \qquad M^2 = \frac{-\alpha(1 - Kr^2)}{a^4(\eta)r^2 f}, \qquad M^3 = 0,$$

$$M^4 = \frac{(1 - Kr^2)\chi}{a^4(\eta)r^4\sin^2\theta f^2} = -N^3,$$

$$N^1 = \frac{-(1 - Kr^2)}{a^4(\eta)r^5\sin^2\theta f^3}[(f'\eta r + (r + 2\eta)f)\alpha\sqrt{\beta - f^2} + \beta\ln(\csc\theta + \cot\theta)f'r^3],$$

and finally

$$N^2 = \frac{(Kr^2 - 1)}{a^4(\eta)r^5f^3\sin^2\theta}\left[\left(\alpha\eta + \beta r^2\frac{\ln(\csc\theta + \cot\theta)}{\sqrt{\beta - f^2}}\right)\sin^2\theta r\alpha f' - f(\beta r^3 + 2\alpha^2\eta\sin^2\theta)\right].$$

Noting that $u = u(x^3, x^4 = \varphi)$, Eq. (20) becomes

$$M^4\frac{\partial}{\partial\varphi}\ln|u| = 0;$$

i.e., $u_{,\varphi} = 0$, and Eq. (21) gives

$$\chi\frac{(1 - Kr^2)}{a^4(\eta)r^4\sin^2\theta f^2}\frac{d}{dx^3}\ln|u| = \chi\left[\frac{f'(1 - Kr^2) + Krf}{a^4(\eta)r^4\sin^2\theta f^3}\right]. \quad (22)$$

Notice how the factor χ cancels out in the above equation, and hence we can smoothly transition from an electrically dominated solution to a magnetically dominated one.[1]

The above equation is satisfied by setting

$$u = \frac{u_0 f}{\sqrt{1 - Kr^2}}.$$

Therefore, in the adapted chart,

$$F_1 = \frac{u_0 f}{\sqrt{1 - Kr^2}}dx^3 \wedge dx^4.$$

It has been pointed out [37] that there is no *a priori* reason for F^2 to be preserved under time evolution, and numerical simulations explicitly show this behavior. Here, we have presented an analytic force-free field that transitions spatially from the magnetically dominated to null to electrically dominated regions, and since this transition is smooth without any discontinuity, there is no current sheet where FFE approximation breaks down.

Using the method of tetrads and searching for Lorentz transformations that satisfy Eqs. (4)–(6), we have been able to generate several nontrivial solutions in FLRW spacetimes. In the remainder of this section, we simply list the solutions without referring to the generating Lorentz transformation.

B. Solution II

Using a time-dependent Lorentz transformation, we obtain the following electrically dominated solution:

$$F_2 = \frac{\alpha}{r^2\sqrt{1 - Kr^2}}d\eta \wedge dr + \frac{\sqrt{\beta - f^2}}{\sin\theta}d\eta \wedge d\theta + f\, d\eta \wedge d\varphi. \quad (23)$$

Here, $f = f(\eta)$, and α and β are real constants. The current density is then given by

$$j_2 = \frac{f'}{a^4 r^2 \sin\theta}\left(\frac{-f}{\sqrt{\beta - f^2}}\partial_\theta + \frac{\partial_\varphi}{\sin\theta}\right). \quad (24)$$

The Lorentz-invariant scalars in this case are

$$F_2^2 = -\frac{2\alpha^2\sin^2\theta + 2\beta\, r^2}{a^4 r^4\sin^2\theta} \quad (25)$$

and

$$j_2^2 = \frac{f'^2\,\beta}{a^6\sin^2\theta(\beta - f^2)}. \quad (26)$$

As in the previous case, this solution is not well defined when $\sin\theta = 0$.

C. Solution III

For $K = 1$ and $f = f(\theta)$, and a real constant α, we obtain another type-changing solution of the form

$$F_3 = \frac{(\sqrt{K}r + \sqrt{Kr^2 - 1})^\alpha}{\sin\theta}\left[\left(f\sin\left(\alpha\sqrt{K}\eta\right) - \sqrt{\beta - f^2}\cos\left(\alpha\sqrt{K}\eta\right)\right)d\eta \wedge d\theta \right.$$
$$\left. - \frac{1}{\sqrt{Kr^2 - 1}}\left(f\cos\left(\alpha\sqrt{K}\eta\right) + \sin\left(\alpha\sqrt{K}\eta\right)\sqrt{\beta - f^2}\right)dr \wedge d\theta\right] + C\sin\theta d\theta \wedge d\varphi. \quad (27)$$

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[1]Henceforth, such solutions will be referred to as type-changing solutions.

Here, the current density is given by

$$j_3 = \frac{(\sqrt{K}r + \sqrt{Kr^2 - 1})^\alpha f}{a(\eta)^4 \sin(\theta) r^2 \sqrt{\beta - f^2}} \left[-\left(f \cos\left(\alpha\sqrt{K}t\right) - \sin\left(\alpha\sqrt{K}t\right)\sqrt{\beta - f^2} \right) \partial_\eta \right.$$
$$\left. + \left(-f \sin\left(\alpha\sqrt{K}t\right) + \sqrt{\beta - f^2} \cos\left(\alpha\sqrt{K}t\right) \right) \sqrt{Kr^2 - 1}\partial_r \right]. \tag{28}$$

In this case,

$$F_3^2 = -\frac{2(\sqrt{K}r + \sqrt{Kr^2 - 1})^{2\alpha}\beta r^2 - 2C^2 \sin^2\theta}{a(\eta)^4 r^4 \sin^2\theta}. \tag{29}$$

Once again, this solution is not well defined when $\sin\theta = 0$. Further, when $K = 0, -1$, the solution above does indeed satisfy the force-free Maxwell equation. However, the coefficient terms become complex, making the solution physically irrelevant.

D. Solution IV

We now present the following electrically dominated force-free field:

$$F_4 = f_1 d\eta \wedge d\theta + f_2 d\eta \wedge d\varphi + f_3 d\theta \wedge d\varphi, \tag{30}$$

where

$$f_1 = \frac{e^{-\frac{\eta}{\omega_2}}}{\sin\theta}((2\beta\alpha\cos^2\theta + (2\alpha k_1 + 2\beta k_2)\cos\theta + k_3)e^{\frac{2\eta}{\omega_2}} - (\cos\theta\beta + k_1)^2 e^{\frac{4\eta}{\omega_2}} - (\cos\theta\alpha + k_2)^2)^{1/2}, \tag{31}$$

$$f_2 = (\cos\theta\alpha + k_2)e^{-\frac{\eta}{\omega_2}} + e^{\frac{\eta}{\omega_2}}(\cos\theta\beta + k_1), \tag{32}$$

and

$$f_3 = -\omega_2 \sin\theta(e^{\frac{\eta}{\omega_2}}\beta - \alpha e^{-\frac{\eta}{\omega_2}}). \tag{33}$$

Here, k_1, k_2, ω_1, and ω_2 are real constants. The current density is given by

$$j_4 = \frac{1}{a(\eta)^4 r^2} \left[-(f_1 \cot\theta + \partial_\theta f_1)\partial_\eta + (\partial_\eta f_1)\partial_\theta + \frac{\csc\theta^2}{r^2}(\partial_\eta f_2 - \partial_\theta f_3 + f_3 \cot\theta)\partial_\varphi \right]. \tag{34}$$

Here,

$$F_4^2 = -\frac{2}{a(\eta)^4 r^4 \sin^2\theta} [\sin^2\theta r^2 f_1^2 + r^2 f_2^2 - f_3^2] \tag{35}$$

and

$$j_4^2 = \frac{1}{a(\eta)^6 r^6} [\csc\theta(r^2\partial_\eta f_2 - \partial_\theta f_3 + f_3 \cot\theta)^2 + (\partial_\eta f_1)^2 - r^2(f_1 \cot\theta + \partial_\theta f_1)^2]. \tag{36}$$

This solution is not valid during early and late times, in addition to the usual pathology when $\sin\theta = 0$.

E. Solutions V and VI

We continue with our presentation by illustrating a few null solutions in FLRW spacetimes. The tetrad formalism for generating null solutions is substantially different from that of non-null solutions. As described in the aforementioned section, we begin with a null pregeodesic congruence defined by

$$l = \frac{\csc\theta}{ra(\eta)^3}(\partial_\eta - \sqrt{1 - Kr^2}\partial_r). \tag{37}$$

Here, l is a null vector satisfying $\nabla_l l \propto l$. A simple calculation shows that

$$\nabla_l l = \frac{\dot{a}(\eta)r + a(\eta)\sqrt{1 - kr^2}}{r^2 \sin\theta a(\eta)^4} l. \tag{38}$$

The relevant null tetrad $(s, l, \alpha^\sharp, n)$ can be constructed by defining the following vector fields:

$$n = \frac{a(\eta)r\sin\theta}{2}(\partial_\eta + \sqrt{1 - Kr^2}\partial_r), \tag{39}$$

$$\alpha^\sharp = \frac{1}{r\,a(\eta)}\partial_\theta, \tag{40}$$

$$s = \frac{\csc\theta}{r\,a(\eta)}\partial_\varphi. \tag{41}$$

Here, since

$$g(\nabla_s l, s) = g(\nabla_{\alpha^\sharp} l, \alpha^\sharp) = \frac{\dot{a}(\eta)r - a(\eta)\sqrt{a - Kr^2}}{a(\eta)^4 r^2 \sin\theta}, \tag{42}$$

we see that the equipartition condition for the null mean curvature is satisfied. Additionally, since l, s forms an involutive distribution, we are guaranteed the existence of a null force-free field solution with a two-parameter prefactor. To isolate the prefactor, we construct a foliation-adapted chart with commuting vector fields, defined as follows:

$$X_1 = \partial_\eta + \sqrt{1 - Kr^2}, \qquad X_2 = \partial_\varphi,$$
$$X_3 = \partial_\theta, \quad \text{and} \quad X_4 = l.$$

Our adapted chart is then defined by the following coordinate one-forms:

$$dx^1 = d\eta - \frac{dr}{\sqrt{1 - Kr^2}}, \qquad dx^2 = d\varphi,$$
$$dx^3 = d\theta, \quad \text{and} \quad dx^4 = d\eta + \frac{dr}{\sqrt{1 - Kr^2}}.$$

The null and force-free field is then given by

$$F_5 = (u \cdot \kappa)\alpha \wedge l^\flat = u(x^3, x^4)$$
$$\times \left(d\eta \wedge d\theta + \frac{1}{\sqrt{1 - Kr^2}}dr \wedge d\theta\right). \tag{43}$$

Here, $\kappa = -\csc\theta$ and is given by Eq. (8), and the current density is then given by

$$j_5 = \frac{\partial_\theta f + f\cot\theta}{a(\eta)^4 r^2}\left(-\partial_\eta + \sqrt{1 - Kr^2}\partial_r\right). \tag{44}$$

As required, $F_5^2 = 0$, and since the current density is along l, we have that $j_5^2 = 0$. It turns out that this null geodesic congruence also satisfies the uniform equipartition condition described in [36]:

$$g(\nabla_s l, \alpha^\sharp) + g(\nabla_{\alpha^\sharp} l, s) = 0. \tag{45}$$

This allows for the possibility of a generalized null solution, which in this case is given by

$$F_6 = \partial_\theta f(\bar{t}, \theta, \varphi)\left(d\eta \wedge d\theta + \frac{dr \wedge d\theta}{\sqrt{1 - Kr^2}}\right)$$
$$+ \partial_\varphi f(\bar{t}, \theta, \varphi)\left(d\eta \wedge d\phi + \frac{dr \wedge d\phi}{\sqrt{1 - Kr^2}}\right). \tag{46}$$

The t in the above expression is the usual cosmic time given by the relation

$$\bar{t} = \eta + \int \frac{1}{\sqrt{1 - Kr^2}}dr. \tag{47}$$

The current density in this case is given by

$$j_6 = \frac{\csc\theta\partial_\phi^2 f + \partial_\theta^2 f + \cot\theta\partial_\theta f}{a(\eta)^4 r^2}(\partial_\eta - \sqrt{1 - Kr^2}\partial_r). \tag{48}$$

For clever choices of f, the solution above is valid when $r \neq 0$.

F. Solution VII

For $f = f(\bar{t}, \theta)$, where t is the cosmic time function defined above, we have the following non-null force-free solution with null current:

$$F_7 = f\sin\theta\left(d\eta \wedge d\theta + \frac{1}{\sqrt{1 - Kr^2}}dr \wedge d\theta\right)$$
$$+ \alpha\sin\theta d\theta \wedge d\phi. \tag{49}$$

Here, as usual, α is a real constant. This solution turns out to be a non-null generalization of our previous null solution F_5. The current density in this case is given by

$$j_7 = -\frac{2\cos\theta f + \sin\theta\partial_\theta f}{a(\eta)^4 r^2}(\partial_\eta + \sqrt{1 - Kr^2}\partial_r). \tag{50}$$

We also have that

$$F_7^\sharp = -\frac{f\sin\theta}{a(\eta)^4 r^2}\partial_\eta \wedge \partial_\theta + \frac{f\sin\theta\sqrt{1 - Kr^2}}{a(\eta)^4 r^2}\partial_r \wedge \partial_\theta$$
$$+ \frac{\alpha}{a(\eta)^4 r^4 \sin\theta}\partial_\theta \wedge \partial_\phi. \tag{51}$$

While it appears that the last term on the right-hand side of the above equation is undefined when $\sin\theta = 0$, the

contraction of the Faraday tensor with itself does not suffer from the same pathology—i.e.,

$$F_7^2 = \frac{2\alpha^2}{a(\eta)^4 r^4}. \tag{52}$$

$$F_7 = \frac{1}{r^3}(zxf\, d\eta \wedge dx + zyf\, d\eta \wedge dy - (x^2 + y^2)f\, d\eta \wedge dz \alpha z dx \wedge dy - (\alpha y + rxf)dx \wedge dz + (x\alpha - ryf)dy \wedge dz). \tag{53}$$

The above expression is well defined along the z axis.

IV. SOLUTION VIII

The FLRW metric is further simplified when we set $K = 0$, and in the Cartesian coordinates described above, we are able to find two new non-null solutions. First, for the constants c_1, c_2, c_3, c_4, we have the following force-free field:

$$F_8 = \frac{\sqrt{(-c_1^2 + c_2^2)f^2 + c_2 c_4^2}}{c_2} d\eta \wedge dx$$
$$+ \frac{c_1 f}{c_2} d\eta \wedge dz + f\, dx \wedge dz, \tag{54}$$

where

$$f = f(c_1 \eta + c_2 x + c_3). \tag{55}$$

The current density is given by

$$j_8 = \frac{(c_1^2 - c_2^2)(\partial_\eta f)}{a(\eta)^4 \sqrt{(c_2^2 - c_1^2)f^2 + c_4 c_2^2}}$$
$$\times \left[\frac{f}{c_1} \partial_\eta - \frac{f}{c_2} \partial_x + \frac{\sqrt{(c_2^2 - c_1^2)f^2 + c_4 c_2^2}}{c_1 c_2} \partial_z \right]. \tag{56}$$

The Lorentz scalars of the theory are given by

$$F_8^2 = -\frac{2c_4}{a(\eta)^4} \tag{57}$$

and

$$j_8^2 = -\frac{c_4(c_1^2 - c_2^2)^2(\partial_\eta f)^2}{c_1^2 a(\eta)^6((c_1^2 - c_2^2)f + c_4 c_2^2)}. \tag{58}$$

A. Solution IX

As in the Cartesian case of the above example, we now provide a secondary non-null solution given by

This is an indication that the solution may just have a coordinate singularity along $\sin\theta = 0$. We demonstrate this fact by explicitly transforming it into a Cartesian coordinate system for the case when $K = 0$. In the usual (η, x, y, z) coordinate system, where we have just transformed the spatial spherical coordinates to Cartesian coordinates,

$$F_9 = \frac{\sqrt{c_4 c_2^2 - (c_2^2 + c_1^2)f^2}}{c_2} dx \wedge dy$$
$$+ \frac{c_1 f}{c_2} dx \wedge dz + f\, dy \wedge dz. \tag{59}$$

Here, $f = f(c_1 x + c_2 y + c_3)$. The current density is given by

$$j_9 = \frac{(c_1^2 + c_2^2)(\partial_x f)}{a(\eta)^4 \sqrt{(c_1^2 + c_2^2)f^2 - c_4 c_2^2}}$$
$$\times \left[-\frac{f}{c_1} \partial_x + \frac{f}{c_2} \partial_y - \frac{\sqrt{c_4 c_2^2 - (c_2^2 + c_1^2)f^2}}{c_1 c_2} \partial_z \right], \tag{60}$$

and we have

$$F_9^2 = \frac{2c_4}{a(\eta)^4} \tag{61}$$

and

$$j_9^2 = -\frac{(\partial_x f)^2(c_1^2 + c_2^2)^2 c_4}{c_1^2 a(\eta)^6((c_1^2 + c_2^2)f^2 - c_4 c_2^2)}. \tag{62}$$

A simple spatial rotation in the x, y plane can simplify the expression for F_9. Consider the transformation given by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \frac{1}{\sqrt{c_1^2 + c_2^2}} \begin{bmatrix} c_2 & -c_1 \\ c_1 & c_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{63}$$

After the transformation, $f = f(c_3 + y'\sqrt{c_1^2 + c_2^2})$. The constants can be absorbed by redefining f as $f(c_3' + y')$. The solution F_9 then takes the form

$$F_9' = \frac{\sqrt{c_4 c_2^2 - (c_1^2 + c_2^2)f'^2}}{c_2} dx' \wedge dy'$$
$$+ \frac{f'\sqrt{c_1^2 + c_2^2}}{c_2} dy' \wedge dz. \tag{64}$$

By redefining the constants, we can rewrite the solution as

$$F_9' = \sqrt{c^2 - f^2} dx' \wedge dy' + f\, dy' \wedge dz. \tag{65}$$

The comoving observer with four-velocity $v^\mu = a(\eta)^{-1}\partial_\eta$ does not see an electric field in F'_9, while the magnetic field is given by[2]

$$B^x = -\frac{f}{a(t)^3} \quad \text{and} \quad B^z = -\frac{\sqrt{c^2 - f^2}}{a(t)^3}.$$

This solution describes slabs of uniform magnetic field that lie in the xz plane, and the field orientation varies in the perpendicular (y) direction. As the magnetic field strength is constant throughout space but the field direction changes, this field configuration describes the magnetohydrodynamic "tangential discontinuity." Indeed, the structure is force-free, since the field lines have no tension force (there is no field-line bending) and there are no magnetic pressure gradients. Now, *a posteriori*, it seems easy to understand that the tangential discontinuity remains a force-free solution in the FLRW spacetime, since uniform expansion does not change the field topology, but instead simply rescales the field strength.

V. CONCLUSION

In this paper, we have demonstrated the geometric methods of foliations to generate both null and non-null force-free electromagnetic fields for the FLRW spacetime. The equations of force-free electrodynamics are in general complex nonlinear partial differential equations, and exact solutions are very difficult to come by. Even in the extensively studied Schwarzschild and Kerr spacetimes, where force-free electrodynamics is expected to describe the tenuous plasma around accreting black holes, very few exact solutions are known. In [38] we presented several exact solutions to FFE in Kerr spacetime that were generated by the study of foliations. Using the same methods, a wide range of exact force-free fields allowed by the FLRW geometry were found. The pathology along the z axis seems to be a common feature of most of the force-free solutions in both the spacetimes. We have presented, to the best of our knowledge, the first force-free field that transitions from electrically dominated to null, and then to a magnetically dominated regime. We have done so using a chart adapted to the foliation generating the force-free field, as such solutions are not handled by the tetrad formalism that describes the different geometrical properties of the null and non-null solutions.

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[2] In the $3 + 1$ formalism, the electric and magnetic fields are given by the expressions $E_\mu = v^\nu F_{\mu\nu}$ and $B_\mu = v^\nu * F_{\mu\nu}$.

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