Disforming scalar-tensor cosmology

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Disformal transformations of Friedmann-Lemaître-Robertson-Walker and Bianchi geometries are analyzed in the context of scalar-tensor gravity. Novel aspects discussed explicitly are the $3+1$ splitting, the effective fluid equivalent of the gravitational scalar, Bianchi models, stealth solutions, and de Sitter solutions with nonconstant scalar field (which are signatures of scalar-tensor gravity). Both pure disformal transformations and more general ones are discussed, including those containing higher derivatives of the scalar field recently introduced in the literature.

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I. INTRODUCTION

Einstein's theory of gravity, general relativity (GR), is plagued by spacetime singularities, such as those inside black holes or the big bang singularity in cosmology [1]. There is hope that quantum mechanics and the uncertainty principle on which it is based will someday cure these singularities. However, as soon as one tries to quantum-correct GR, deviations from it are introduced in the form of extra degrees of freedom [2–6], quadratic terms in the curvature invariants appearing in the action [7,8], and higher-order equations of motion. For example, the first scenario of early universe inflation, and the one currently favored by observations [9], i.e., Starobinski inflation [10], is due to quadratic corrections to the Einstein-Hilbert action. The low-energy limit of the bosonic string theory, the simplest of string theories, does not reproduce GR but gives $\omega=-1$ Brans-Dicke gravity instead [11,12]. Therefore, it is not a matter of if, but of where, GR fails and the study of alternative theories of gravity is motivated by fundamental physics.

From a completely different point of view, astronomers are also invoking modified gravity. The 1998 discovery that the cosmic expansion accelerates today left us in the need of an explanation. Dark energy, in the form of a fine-tuned cosmological constant, quintessence scalar fields, or a plethora of other models, was invoked to explain the cosmic acceleration (see Ref. [13] for a review). The standard model of cosmology based on GR, the Lambda–cold dark matter (ΛCDM) model is now suffering severe tensions and was never satisfactory because approximately 70% of the energy content of the universe is postulated to be dark energy, which was introduced in a completely *ad hoc* manner and whose nature is unknown. Dissatisfied with this state of affairs, theorists and astronomers alike have resorted to modifying Einstein gravity as an alternative to introducing dark energy [14,15]. The most popular class of theories for this purpose is probably $f(R)$ gravity (see Refs. [16–18] for reviews), which includes the Starobinski action and is ultimately reduced to a scalar-tensor gravity.

Scalar-tensor gravity introduces the simplest degree of freedom in addition to the two massless spin two modes of GR: a massive propagating scalar field that has a gravitational nature. The prototype of the alternative to GR was the Brans-Dicke theory, in which the gravitational scalar field ϕ essentially plays the role of the inverse of the effective gravitational coupling strength $G_{\rm eff}\simeq 1/\phi$ replacing Newton's constant G. This effective coupling strength becomes a dynamical field sourced by the trace of the matter stress-energy tensor [2]. The Brans-Dicke theory was later generalized to other "first-generation" scalar-tensor gravities. In the past decade, the search for scalar-tensor theories containing field equations with order not higher than second led to the rediscovery of Horndeski gravity [19], which has been the subject of extensive literature (e.g., [20–25] and references therein). More recently, it was found that certain higher-order scalar-tensor theories, when subject to a degeneracy condition, lead to even more general second order equations of motion. These are the so–called degenerate higher order scalar-tensor (DHOST) theories [26–37] (see Refs. [38,39] for reviews).

The field equations of DHOST and Horndeski theories are complicated, and it is very difficult to find nontrivial analytic solutions. Many of the known exact solutions of Horndeski and DHOST gravity have been obtained by direct integration of the field equations (e.g., [40–42]). Others, instead, have been obtained by disformal transformations of the

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metric tensor $g_{\mu\nu}$, sometimes with the disformal transformation reducing to a conformal one. A disformal transformation takes a seed metric and scalar field solution $(g_{\mu\nu}, \phi)$ of a Horndeski theory and maps it into a DHOST solution. Here we are interested in the disformal transformation of cosmological metrics. The transformation of a spatially homogeneous and isotropic Friedmann-Lemaître-Robertson-Walker (FLRW) metric has been discussed in Ref. [43] (see also [44,45] and references therein). Here we extend that analysis and consider several aspects not previously discussed in relation with disformal transformations, including the explicit $3+1$ splitting, Bianchi cosmologies, and the effective fluid equivalent of the scalar degree of freedom ϕ. We also discuss certain exact solutions typical of scalar-tensor gravity which are impossible in GR, namely stealth solutions, as well as de Sitter solutions with a nonconstant scalar field which, in a sense explained below, generalize stealth solutions.

There is further motivation for studying the disformal transformation of cosmological spacetimes. Black holes interacting with their environment are dynamical, and their horizons are not event horizons, but apparent horizons instead, which makes them much more complicated from the point of view of black hole mechanics and thermodynamics [46,47]. Similarly, dynamical cosmological horizons have thermodynamics that is far from trivial [46]. One way to make black holes dynamical is to embed them in a nonstatic cosmological "background." This possibility is of great astrophysical interest since the interaction of black holes with the FLRW space in which they are embedded, over cosmological timescales, has been tentatively reported in recent observations [48,49]. If supermassive black holes at the centers of galaxies are taken to be nonsingular objects with an extended de Sitter core, as in most models of regular black holes, then the possibility arises that dark energy could be effectively segregated inside these black hole horizons [48–53]. Aside from its cosmological implications, the possibility of cosmological coupling tentatively reported in [48] has already been the subject of a lively theoretical and observational debate in the literature [54–75]. Problem is, the theory behind this cosmological coupling of black holes is underdeveloped and exact solutions of the relevant field equations could help understanding the basic physical principles behind this phenomenon. The scarcity of relevant solutions in GR [46] prompts the search for new solutions in more general scalar-tensor and Horndeski theories. The easiest way to generate new solutions is by using disformal transformations of GR "seeds." The first step in this program consists of understanding the transformation properties under disformal transformations of the FLRW or Bianchi "backgrounds" in which such black holes are embedded. The present work addresses this first step.

We adopt the notations of Ref. [1]. The metric signature is $-+++$, units are used in which the speed of light c and Newton's constant G are unity, and $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$ is the curved space d'Alembert operator. Greek indices run from 0 to 3 and Latin indices from 1 to 3.

II. DISFORMAL TRANSFORMATIONS

The general form of a disformal transformation with first order derivatives of the scalar field [76] is

$$g_{\mu\nu} \to \bar{g}_{\mu\nu} = \Omega^2(\phi, X) g_{\mu\nu} + F(\phi, X)\nabla_\mu\phi\nabla_\nu\phi, \tag{2.1}$$

where

$$X \equiv -\frac{1}{2} g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi \tag{2.2}$$

and ϕ denotes the scalar field (see Ref. [77] for a summary of the transformation properties of various geometrical quantities under disformal transformations). In the case $\Omega = 1$, one obtains a *pure disformal transformation*, to which we restrict for most of this paper until Sec. VIII:

$$g_{\mu\nu} \to \bar{g}_{\mu\nu} = g_{\mu\nu} + F(\phi, X)\nabla_\mu\phi\nabla_\nu\phi. \tag{2.3}$$

Assuming that $\nabla^\mu\phi$ is timelike, one can always use the "uniform–ϕ slicing" of spacetime in which $\phi = \phi(t)$, where t is the time coordinate. In this gauge, the line element assumes the familiar form

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \end{aligned} \tag{2.4}$$

where N and N^i are the lapse and the shift vector, respectively. The disformed line element is

$$\begin{aligned} d\bar{s}^2 &= \bar{g}_{\mu\nu} dx^\mu dx^\nu \\ &= -\bar{N}^2 dt^2 + \bar{g}_{ij}(dx^i + \bar{N}^i dt)(dx^j + \bar{N}^j dt), \end{aligned} \tag{2.5}$$

where $\bar{N}^2 = N^2\alpha^2$, $\bar{N}^i = N^i$, $\bar{g}_{ij} = g_{ij}$ in the disformed (or barred) world, and [43]

$$\alpha^2 \equiv 1 - 2F(\phi, X)X. \tag{2.6}$$

To preserve the metric signature, it must be $F < 1/2$ everywhere throughout the spacetime manifold, which we assume in the following.

Equation (2.5) is easily proved. Using the notation $\dot{\phi} \equiv d\phi/dt$, we have

$$\begin{aligned} d\bar{s}^2 &= (g_{\mu\nu} + F\nabla_\mu\phi\nabla_\nu\phi) dx^\mu dx^\nu \\ &= ds^2 + F(\phi, X)(\nabla_\mu\phi dx^\mu)^2 \\ &= -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt) + F\dot{\phi}^2 dt^2 \\ &= -(N^2 - F\dot{\phi}^2) dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt). \end{aligned} \tag{2.7}$$

Since

$$X = -\frac{g^{00}}{2}\dot{\phi}^2 = \frac{\dot{\phi}^2}{2N^2}, \tag{2.8}$$

it is

$$N^2 - F(\phi, X)\dot{\phi}^2 = N^2[1 - 2F(\phi, X)X] \equiv N^2\alpha^2. \tag{2.9}$$

From

$$d\bar{s}^2 = -(N^2\alpha^2)dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \tag{2.10}$$

it then follows that

$$\bar{N}^2 = N^2\alpha^2, \qquad \bar{g}_{ij} = g_{ij}, \qquad \bar{N}^i = N^i. \tag{2.11}$$

The inverse of the disformed metric $\bar{g}_{\mu\nu}$ is [43,77]

$$\bar{g}^{\mu\nu} = g^{\mu\nu} - \frac{F}{\alpha^2}\nabla^\mu\phi\nabla^\nu\phi: \tag{2.12}$$

to wit,

$$\begin{aligned}
\bar{g}^{\mu\nu}\bar{g}_{\nu\gamma} &= \left(g^{\mu\nu} - \frac{F}{\alpha^2}\nabla^\mu\phi\nabla^\nu\phi\right)(g_{\nu\gamma} + F\nabla_\nu\phi\nabla_\gamma\phi) \\
&= g^{\mu\nu}g_{\nu\gamma} + F\left(g^{\mu\nu}\nabla_\nu\phi\nabla_\gamma\phi - \frac{1}{\alpha^2}g_{\nu\gamma}\nabla^\mu\phi\nabla^\nu\phi\right) \\
&\quad - \frac{F^2}{\alpha^2}(\nabla_\nu\phi\nabla^\nu\phi)\nabla^\mu\phi\nabla_\gamma\phi \\
&= \delta^\mu_\gamma + F(\phi, X)\left(\nabla^\mu\phi\nabla_\gamma\phi - \frac{\nabla^\mu\phi\nabla_\gamma\phi}{1 - 2F(\phi, X)X}\right) \\
&\quad - \frac{F^2(\phi, X)}{1 - 2F(\phi, X)X}(\nabla_\nu\phi\nabla^\nu\phi)\nabla^\mu\phi\nabla_\gamma\phi \\
&= \delta^\mu_\gamma + \frac{F(1 - 2FX - 1) - F^2(-2X)}{1 - 2FX}\nabla^\mu\phi\nabla_\gamma\phi \\
&= \delta^\mu_\gamma.
\end{aligned} \tag{2.13}$$

Let us proceed to adapt these transformations to FLRW geometries.

III. FLRW UNIVERSE AND DISFORMAL TRANSFORMATIONS

The FLRW line element in comoving coordinates $(t, r, \vartheta, \varphi)$ is

$$\begin{aligned}
ds^2 &= -dt^2 + a^2(t)\left(\frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2_{(2)}\right) \\
&\equiv -dt^2 + g_{ij}dx^i dx^j,
\end{aligned} \tag{3.1}$$

where the constant K is the curvature index, $N = 1$, $N^i = 0$, and $d\Omega^2_{(2)} \equiv d\vartheta^2 + \sin^2\vartheta d\varphi^2$ is the line element on the unit 2-sphere. The pure disformal transformation (2.3) changes the FLRW line element into

$$d\bar{s}^2 = -(1 - 2FX)dt^2 + g_{ij}dx^i dx^j. \tag{3.2}$$

The uniform-ϕ slicing corresponds to the comoving FLRW slicing in which $\phi = \phi(t)$. In this gauge $X = \dot{\phi}^2/2$ and

$$d\bar{s}^2 = -(1 - \dot{\phi}^2 F)dt^2 + g_{ij}dx^i dx^j \tag{3.3}$$

after the disformal transformation. By redefining the time coordinate according to

$$dt^2 \to d\tau^2 \equiv [1 - \dot{\phi}^2(t)F(t)]dt^2, \tag{3.4}$$

or

$$\tau(t) = \int dt\sqrt{1 - \dot{\phi}^2(t)F(t)}, \tag{3.5}$$

we rewrite the disformed line element as

$$d\bar{s}^2 = -d\tau^2 + g_{ij}dx^i dx^j \tag{3.6}$$

and $\bar{\phi}(\tau) = \phi(t(\tau))$. Therefore, a pure disformal transformation of a FLRW metric written in the uniform-ϕ gauge always generates another FLRW geometry, in addition to preserving the uniform-ϕ slicing.[1] In essence, a pure disformal transformation of a FLRW metric is equivalent to a rescaling of the comoving time [43]. For example, for spatially flat FLRW metrics

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) \equiv -dt^2 + a^2(t)d\vec{x}^2, \tag{3.7}$$

for which

$$\begin{aligned}
d\bar{s}^2 &= (g_{\mu\nu} + F\nabla_\mu\phi\nabla_\nu\phi)dx^\mu dx^\nu \\
&= g_{\mu\nu}dx^\mu dx^\nu + F\dot{\phi}^2 dt^2 \\
&= -(1 - F\dot{\phi}^2)dt^2 + a^2(t)d\vec{x}^2 \\
&= -d\tau^2 + \bar{a}^2(\tau)d\vec{x}^2.
\end{aligned} \tag{3.8}$$

The pure disformal transformation yields another spatially flat FLRW line element with comoving time (3.5) and scale factor $\bar{a}(\tau) = a(t(\tau))$.

Both in GR with a minimally coupled scalar field ϕ [79–82] and in "first-generation" scalar-tensor gravity [83] [including $f(R)$ gravity [84]], spatially flat FLRW universes are described by the dynamical variables $H \equiv \dot{a}/a$ and ϕ. That is, for these $K = 0$ FLRW universes, the scale factor $a(t)$ enters the Einstein-Friedmann equations only

[1] If $(g_{\mu\nu}, \phi)$ is a GR solution with minimally coupled scalar ϕ, then $(\bar{g}_{\mu\nu}, \phi)$ is a solution of "class Ia" DHOST gravity [78].

through the Hubble function $\dot{a}/a$, which is a cosmological observable. Hence, the phase space is the $(H, \phi, \dot{\phi})$ space but the Friedmann equation

$$H^2 = \frac{8\pi}{3}\rho^{(\phi)} \tag{3.9}$$

constitutes a first order constraint on the dynamics. As a consequence, the region of the phase space accessible to the orbits of the solutions is a two-dimensional subset of this three-dimensional space. Effectively, these orbits move on a curved two-dimensional subset of the three-space $(H, \phi, \dot{\phi})$, which may consist of multiple sheets and possibly have "holes" inaccessible to these orbits, as explained in Refs. [83,84]. This phase space structure extends to spatially flat FLRW cosmology in Horndeski gravity [85].

In both GR and scalar-tensor gravity, if fixed points exist, with this choice of dynamical variables they are unavoidably de Sitter spaces with

$$(H, \phi) = (H_0 = \text{const}, \phi_0 = \text{const}). \tag{3.10}$$

The values of H_0 and ϕ_0 are related by the field equations [83,84]. A disformal transformation of a $K = 0$ FLRW metric maps these fixed points into FLRW universes with

$$\bar{g}_{\mu\nu} = g_{\mu\nu}^{(\mathrm{FLRW})} + F\nabla_\mu\phi\nabla_\nu\phi, \qquad \bar{\phi}(\tau) = \phi(t(\tau)), \tag{3.11}$$

where $d\tau = \sqrt{1-2FX}dt$ but, since for fixed points $\nabla_\alpha\phi = \nabla_\alpha\phi_0 = 0$, it is simply $\bar{g}_{\mu\nu} = g_{\mu\nu}^{(\mathrm{FLRW})}$. Therefore, *a pure disformal transformation (2.3) maps de Sitter fixed points into de Sitter fixed points of the phase space of $K = 0$ FLRW cosmology.*

de Sitter spaces with nonconstant scalar fields do not exist in GR with minimally coupled scalar, but are a signature of scalar-tensor gravity. They are not mapped into de Sitter spaces, as discussed in Sec. VII.

IV. EFFECTIVE IMPERFECT FLUID OF SCALAR-TENSOR GRAVITY AND DISFORMAL TRANSFORMATIONS

It is well-known that the scalar field ϕ of scalar-tensor gravity can be seen as an imperfect fluid when its gradient $\nabla^\mu\phi$ is timelike and future-oriented, i.e., $t^\alpha\nabla_\alpha\phi < 0$, where $t^\mu = (\partial/\partial t)^\mu$ is the time direction of observers comoving with this effective fluid [86–89]. As is customary, we write the vacuum field equations of scalar-tensor gravity in the form of effective Einstein equations,

$$G_{\mu\nu} = T_{\mu\nu}^{(\phi)}; \tag{4.1}$$

then, we assume $\nabla^\mu\phi$ to be timelike and future-oriented. One can then define the effective fluid four-velocity

$$u^\mu = \frac{\nabla^\mu\phi}{\sqrt{2X}}, \tag{4.2}$$

which satisfies the usual normalization condition for timelike fluids $u^\alpha u_\alpha = -1$. The effective stress-energy tensor of the gravitational scalar field ϕ has the form

$$T_{\mu\nu}^{(\phi)} = \rho u_\mu u_\nu + P h_{\mu\nu} + \pi_{\mu\nu} + q_\mu u_\nu + q_\nu u_\mu, \tag{4.3}$$

where

$$\rho = T_{\mu\nu}u^\mu u^\nu \tag{4.4}$$

is the effective energy density,

$$P = \frac{1}{3}h^{\mu\nu}T_{\mu\nu} \tag{4.5}$$

is the effective isotropic pressure,

$$\pi_{\mu\nu} = T_{\gamma\delta}h_\mu{}^\gamma h_\nu{}^\delta - P h_{\mu\nu} \tag{4.6}$$

is the effective anisotropic stress tensor, and

$$q_\alpha = -T_{\gamma\delta}u^\gamma h_\alpha{}^\delta \tag{4.7}$$

is the effective heat flux density [86–89]. Here,

$$h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu \tag{4.8}$$

is the Riemannian metric in the three-space orthogonal to u^α and the pressure P is the sum of nonviscous and viscous contributions,

$$P = P_0 + P_{\mathrm{vis}}. \tag{4.9}$$

$h_{\mu\nu}$, $\pi_{\mu\nu}$, and q_μ are purely spatial:

$$h_{\mu\nu}u^\mu = h_{\mu\nu}u^\nu = \pi_{\mu\nu}u^\mu = \pi_{\mu\nu}u^\nu = q_\mu u^\mu = 0. \tag{4.10}$$

Let us adapt this effective fluid analogy to FLRW geometries.

Since a FLRW universe is spatially homogeneous and isotropic, the shear tensor $\pi^i{}_j$ and the heat flux density q^i (which would introduce a preferred spatial direction if it were nonvanishing) are identically zero and the only dissipative quantity that can remain is the viscous pressure P_{vis}. In Eckart's thermodynamics, viscous pressure arises because of bulk viscosity, according to the constitutive relation $P_{\mathrm{vis}} = -\zeta\nabla_\rho u^\rho$ [90,91], where ζ is a bulk viscosity coefficient. When one applies Eckart's first order thermodynamics to the effective fluid of the scalar field ϕ, this relation is satisfied in the context of "old" scalar-tensor gravity [2–6]), but is usually invalid in more general Horndeski gravity [89,92].

Next, one wonders how the effective ϕ-fluid quantities transform under disformal transformations, but does this question make sense? In general spacetimes, it does not, as explained in the following. In general, solutions of a theory of gravity (say, "theory A") are mapped into solutions of a new theory (say, "theory B") by a disformal transformation. Then, the effective stress-energy tensor of ϕ obtained by recasting the field equations of theory A as effective Einstein equations will have a different form in the new theory B. For example, solutions of GR with a minimally coupled scalar field, or of "first-generation" scalar-tensor gravity (theory A), are mapped into solutions of a Horndeski or a DHOST theory (theory B) [78]. In this case, it no longer makes sense to consider the original $T^{(\phi)}_{\mu\nu}$, which is replaced by a more complicated expression [78]. However, if one restricts one's attention to FLRW metrics in the uniform-ϕ gauge, the disformal transformation amounts to a mere time rescaling and it still makes sense to consider the same $T^{(\phi)}_{\mu\nu}$. Here we restrict to the "old" scalar-tensor theories for simplicity; however, the extension of this analysis to the effective dissipative stress-energy tensor of Horndeski gravity is straightforward. The derivation of this effective stress-energy tensor of ϕ is rather laborious and is performed in Refs. [88,89]. A similar situation occurs with purely conformal transformations. In this case, starting with a solution $(g_{\mu\nu}, \phi)$ of the coupled Einstein-Klein-Gordon equations, a conformally transformed metric $\tilde{g}_{\mu\nu} = \Omega^2(\phi) g_{\mu\nu}$ is no longer a solution of the Einstein equations with the same matter content.[2] However, *for FLRW metrics and $\phi = \phi(t)$ dependent only on time*, the conformal transformation amounts again to a rescaling of the comoving time, and a perfect fluid is mapped again into a perfect fluid with the same equation of state [94].

With this *caveat*, let us proceed to derive the transformation of the various effective fluid quantities in FLRW spaces, which are the subject of interest in this article. The contravariant four-velocity is

$$\bar{u}^\mu = \left(\frac{\partial}{\partial \tau}\right)^\mu = \frac{1}{\alpha}\left(\frac{\partial}{\partial t}\right)^\mu = \frac{u^\mu}{\alpha}. \tag{4.11}$$

The covariant four-velocity is computed as

$$\begin{aligned}\bar{u}_\mu &= \bar{g}_{\mu\nu}\bar{u}^\nu = (g_{\mu\nu} + F\nabla_\mu\phi\nabla_\nu\phi)\frac{u^\nu}{\alpha} \\ &= g_{\mu\nu}\frac{u^\nu}{\alpha} + \frac{F}{\alpha}u_\mu u_\nu(-\nabla^\gamma\phi\nabla_\gamma\phi)u^\nu \\ &= \frac{u_\mu}{\alpha} + \frac{F}{\alpha}(\nabla^\gamma\phi\nabla_\gamma\phi)u_\mu = \frac{u_\mu(1-2FX)}{\alpha} \\ &= \alpha u_\mu.\end{aligned} \tag{4.12}$$

Using the $3+1$ splittings

$$g_{\mu\nu} = -u_\mu u_\nu + h_{\mu\nu}, \qquad \bar{g}_{\mu\nu} = -\bar{u}_\mu\bar{u}_\nu + \bar{h}_{\mu\nu}, \tag{4.13}$$

the disformal transformation (2.3) gives

$$\begin{aligned}\bar{g}_{\mu\nu} &= g_{\mu\nu} + F\nabla_\mu\phi\nabla_\nu\phi \\ &= -u_\mu u_\nu + h_{\mu\nu} + 2XFu_\mu u_\nu \\ &= -(1-2FX)u_\mu u_\nu + h_{\mu\nu} \\ &= -\alpha^2\frac{\bar{u}_\mu}{\alpha}\frac{\bar{u}_\nu}{\alpha} + h_{\mu\nu} \\ &= -\bar{u}_\mu\bar{u}_\nu + h_{\mu\nu};\end{aligned} \tag{4.14}$$

the spatial three-metric is not affected by the disformal transformation (2.3).

Next, we determine how $T^{(\phi)}_{\mu\nu}$ transforms under pure disformal transformations:

$$\begin{aligned}T^{(\phi)}_{\mu\nu} &= \rho u_\mu u_\nu + Ph_{\mu\nu} \\ &= \frac{\rho}{\alpha^2}\bar{u}_\mu\bar{u}_\nu + P\bar{h}_{\mu\nu} \\ &\equiv \bar{\rho}\bar{u}_\mu\bar{u}_\nu + \bar{P}\bar{h}_{\mu\nu} = \bar{T}^{(\phi)}_{\mu\nu},\end{aligned} \tag{4.15}$$

where the individual fluid quantities transform as

$$\bar{\rho} = \frac{\rho}{\alpha^2} = \frac{\rho}{1-2FX}, \qquad \bar{P} = P. \tag{4.16}$$

Clearly, the scaling of the energy density with α is a consequence solely of the scaling of time, which appears twice because $\bar{\rho} = T^{(\phi)}_{\mu\nu}\bar{u}^\mu\bar{u}^\nu$ is quadratic in the four-velocity, which transforms according to Eq. (4.12). This scaling of u_α is responsible for the factor α^2. Purely spatial quantities, including the pressure P, are left unchanged by time rescalings. Equations (4.16) are analogous to the well-known transformation relations for energy density and pressure under a purely conformal transformation of a perfect fluid in FLRW spaces

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = \Omega^2(\phi)g_{\mu\nu}. \tag{4.17}$$

These transformation relations are $\tilde{P} = \Omega^{-4}P$, $\tilde{\rho} = \Omega^{-4}\rho$ [94].

The equation of state parameter of the ϕ–fluid in the barred world is

[2]Indeed, the new metric and scalar field obtained in this way are solutions of a different theory of gravity. Given an electrovacuum solution $g_{\mu\nu}$ of GR and an arbitrary scalar field $\psi > 0$, one can always perform a conformal transformation so that $\tilde{g}_{\mu\nu} = g_{\mu\nu}/\sqrt{\psi}$ is a solution of an $\omega = -3/2$ Brans-Dicke theory with that scalar ψ as its Brans-Dicke scalar [93]. This theory is pathological since ψ is not dynamical.

$$\bar{w} \equiv \frac{\bar{P}}{\bar{\rho}} = \alpha^2 \frac{P}{\rho} \equiv \alpha^2 w. \tag{4.18}$$

Therefore, $\bar{w}$ has the same sign of w and scales only because of the scaling of ρ with α. If $\alpha > 1$, a quintessence scalar field, which by definition has $-1 < w < -1/3$, can be mapped into an effective phantom field which has instead an equation of state parameter $\bar{w} < -1$.

To summarize, we have

$$\bar{u}^\mu = \frac{u^\mu}{\alpha}, \qquad u^\mu = \alpha \bar{u}^\mu, \tag{4.19}$$

$$\bar{u}_\mu = \alpha u_\mu, \qquad u_\mu = \frac{\bar{u}_\mu}{\alpha}, \tag{4.20}$$

$$\bar{h}_{\mu\nu} = h_{\mu\nu}, \tag{4.21}$$

$$\bar{\rho} = \frac{\rho}{\alpha^2} = \frac{\rho}{1-2FX}, \qquad \bar{P} = P. \tag{4.22}$$

As an example, consider a free, minimally coupled scalar field in GR (which is well-known to behave as a stiff fluid [95]) in a spatially flat FLRW universe. The effective energy density and isotropic pressure of this ϕ-fluid

$$\rho = \frac{\dot{\phi}^2}{2} + V, \qquad P = \frac{\dot{\phi}^2}{2} - V \tag{4.23}$$

coincide when $V(\phi) = 0$, giving the effective equation of state parameter $w \equiv P/\rho = 1$ and

$$\rho = \frac{\rho_0}{a^6}, \qquad a(t) = a_0 t^{1/3}, \tag{4.24}$$

where ρ_0 and a_0 are constants. The Klein-Gordon equation satisfied by this minimally coupled scalar field

$$\Box\phi = -(\ddot{\phi} + 3H\dot{\phi}) = 0 \tag{4.25}$$

has the solution $\phi(t) = \phi_0 \ln(t/t_0) + \phi_1$, where $\phi_{0,1}$ and t_0 are constants. Moreover, we have

$$X = \frac{\phi_0^2}{2t^2}, \qquad F(\phi, X) = F(t), \qquad \alpha^2 = 1 - \frac{\phi_0^2 F}{t^2}, \tag{4.26}$$

and

$$\bar{w}(t) = \alpha^2 w = 1 - \frac{\phi_0^2 F(t)}{t^2}. \tag{4.27}$$

For example, the choice

$$F(X) = \frac{F_0}{X} = \frac{2F_0 t^2}{\phi_0^2}, \tag{4.28}$$

where F_0 is a constant, gives another constant equation of state parameter $\bar{w} = 1 - 2F_0 \neq w = 1$. In GR, a disformal transformation can create any constant equation of state with $\bar{w} > 0$ starting from a stiff fluid with $w = 1$.

V. BIANCHI UNIVERSES AND DISFORMAL TRANSFORMATIONS

Although not mentioned explicitly in the literature, the previous discussion applies almost without changes to spatially homogeneous but anisotropic Bianchi cosmologies [96]. The line element can be written in the form

$$ds^2 = -dt^2 + \gamma_{ij} dx^i dx^j \tag{5.1}$$

in the uniform-ϕ gauge, which is possible because Bianchi cosmological models are spatially homogeneous. In this gauge $\phi = \phi(t)$, we have

$$\begin{aligned} d\bar{s}^2 &= \bar{g}_{\mu\nu} dx^\mu dx^\nu = (g_{\mu\nu} + F\nabla_\mu\phi\nabla_\nu\phi) dx^\mu dx^\nu \\ &= ds^2 + F\dot{\phi}^2 dt^2 = -(1 - F\dot{\phi}^2) dt^2 + \gamma_{ij} dx^i dx^j, \end{aligned} \tag{5.2}$$

where, again, $F(\phi(t), X(t)) = F(t)$, so

$$d\bar{s}^2 = -d\tau^2 + \gamma_{ij} dx^i dx^j. \tag{5.3}$$

As for FLRW geometries, a disformal transformation (2.3) of a Bianchi metric is equivalent to a time rescaling, produces another Bianchi metric, and preserves the uniform–ϕ gauge.

As an example, consider the spatially flat Bianchi I geometry with line element

$$ds^2_{(I)} = -dt^2 + a_1^2(t) dx^2 + a_2^2(t) dy^2 + a_3^2(t) dz^2 \tag{5.4}$$

in comoving Cartesian coordinates (t, x, y, z), where $a_i(t)$ $(i = 1, 2, 3)$ are the scale factors corresponding to the three spatial directions, and $\phi = \phi(t)$. The disformal transformation of the metric (2.3) then yields

$$\begin{aligned} d\bar{s}^2 &= -(1 - F\dot{\phi}^2) dt^2 + a_1^2(t) dx^2 + a_2^2(t) dy^2 + a_3^2(t) dz^2 \\ &= -d\tau^2 + \bar{a}_1^2(\tau) dx^2 + \bar{a}_2^2(\tau) dy^2 + \bar{a}_3^2(\tau) dz^2 \end{aligned} \tag{5.5}$$

with $\tau(t) = \int dt \sqrt{1 - F\dot{\phi}^2}$ and $\bar{a}_i(\tau) = a_i(t(\tau))$. Since F and $\dot{\phi}$ depend only on t, the new time coordinate $\tau(t)$ is well-defined, because $d\tau = \sqrt{1 - F\dot{\phi}^2} dt$ is an exact differential.

VI. STEALTH SOLUTIONS OF SCALAR-TENSOR GRAVITY

Stealth solutions [97–107] cannot occur in GR but are typical of scalar-tensor gravity.[3] They have $g_{\mu\nu} = \eta_{\mu\nu}$,

[3]Somehow similar solutions, i.e., hairy Schwarzschild black holes in which the scalar field does not gravitate, are known in more general Horndeski gravities [108–116].

where $\eta_{\mu\nu}$ denotes the Minkowski metric, but $\phi \neq \text{const}$. In other words, the gravitational scalar field ϕ does not gravitate.

Two types of stealth solutions are most commonly encountered in the literature. They have the form

(1) $g_{\mu\nu} = \eta_{\mu\nu}$ and $\phi(t) = \phi_0 e^{\alpha_0 t}$, or

(2) $g_{\mu\nu} = \eta_{\mu\nu}$ and $\phi(t) = \phi_0 |t|^\beta$,

where ϕ_0, α_0, and β are constants. Since stealth solutions are special cases of FLRW geometries in which the scale factor reduces to a constant, they belong to our study. We discuss these two cases separately.

A. Stealth solutions $\phi(t) = \phi_0 e^{\alpha_0 t}$

In this case $X = \alpha_0^2 \phi^2/2$, yielding

$$\bar{g}_{\mu\nu} = \eta_{\mu\nu} + F\nabla_\mu\phi\nabla_\nu\phi = \eta_{\mu\nu} + \alpha_0^2 F\phi^2 \delta_\mu^0 \delta_\nu^0 \tag{6.1}$$

and

$$d\bar{s}^2 = -(1 - \alpha_0^2 F^2 \phi^2) dt^2 + d\vec{x}^2, \tag{6.2}$$

where $F(\phi, X) = F(\phi, \alpha_0^2\phi^2/2) = F(t)$, which produces

$$d\bar{s}^2 = -d\tau^2 + d\vec{x}^2. \tag{6.3}$$

This is another Minkowski line element with time reparametrized. We need $1 > \alpha_0^2 \phi^2 F(\phi, X)$, or

$$F(t) < \frac{1}{\alpha_0^2 \phi^2} = \frac{e^{-2\alpha_0 t}}{\alpha_0^2 \phi_0^2}. \tag{6.4}$$

B. The case $\phi(t) = \phi_0 |t|^\beta$

In this second case we have $\dot{\phi} = \beta\phi/t$, with

$$X = \frac{\beta^2\phi^2}{2t^2} = \frac{\beta^2\phi_0^2}{2} t^{2(\beta-1)}, \tag{6.5}$$

and the disformed line element reads

$$\begin{aligned} d\bar{s}^2 &= (\eta_{\mu\nu} + F\nabla_\mu\phi\nabla_\nu\phi) dx^\mu dx^\nu \\ &= -dt^2 + d\vec{x}^2 + F\dot{\phi}^2 dt^2 = -\left(1 - \frac{F\beta^2\phi^2}{t^2}\right) dt^2 + d\vec{x}^2 \\ &= -d\tau^2 + d\vec{x}^2, \end{aligned} \tag{6.6}$$

yielding another Minkowski metric with reparametrized time $\tau(t)$.

VII. DE SITTER SOLUTIONS WITH NONCONSTANT SCALAR FIELD

Typical of scalar-tensor gravity is the "stealth–de Sitter" metric

$$ds^2 = -dt^2 + a_0^2 e^{2H_0 t} d\vec{x}^2 \tag{7.1}$$

with a_0, H_0 constants, while the scalar field depends on the comoving time, $\phi = \phi(t)$.

In GR with a minimally coupled scalar field, de Sitter solutions are obtained only for a *constant* scalar field $\phi = \phi_0$, and they are equilibrium points of the dynamical system formed by the Einstein-Friedmann equations describing FLRW cosmology [79–82]. In scalar-tensor gravity, where the scalar ϕ has a gravitational nature and couples explicitly to the Ricci scalar, de Sitter solutions with nonconstant $\phi(t)$ are possible, but they are not fixed points of the phase space of FLRW cosmology.

For de Sitter spacetimes with nonconstant scalar $\phi(t)$ in scalar-tensor gravity, the disformed line element is

$$\begin{aligned} d\bar{s}^2 &= -dt^2 + a_0^2 e^{2H_0 t} d\vec{x}^2 + F\dot{\phi}^2 dt^2 \\ &= -(1 - F\dot{\phi}^2) dt^2 + a_0^2 e^{2H_0 t} d\vec{x}^2 \end{aligned} \tag{7.2}$$

or

$$d\bar{s}^2 = -d\tau^2 + a_0^2 e^{2H_0 t(\tau)} d\vec{x}^2 \tag{7.3}$$

with

$$\tau(t) = \int dt \sqrt{1 - F\dot{\phi}^2}, \tag{7.4}$$

which, in general, constitutes a nonlinear relation between t and τ. A disformal transformation maps a stealth–de Sitter geometry into a less symmetric FLRW geometry with

$$d\bar{s}^2 = -d\tau^2 + \bar{a}^2(\tau) d\vec{x}^2, \qquad \bar{a}(\tau) = a_0 e^{H_0 t(\tau)}. \tag{7.5}$$

This solution is a de Sitter geometry only if $F\dot{\phi}^2$ is constant, i.e., $F = \text{const}/X$.

In the literature there are also stealth solutions with an inhomogeneous scalar field $\phi = \phi(t, \vec{x})$ and $g_{\mu\nu} = \eta_{\mu\nu}$ (e.g., [98]), leading to

$$\nabla_\mu\phi = \dot{\phi}\delta_\mu^0 + \phi_i \delta_\mu^i \tag{7.6}$$

and

$$\begin{aligned} \nabla_\mu\phi\nabla_\nu\phi &= (\dot{\phi}\delta_\mu^0 + \phi_i\delta_\mu^i)(\dot{\phi}\delta_\nu^0 + \phi_j\delta_\nu^j) \\ &= \dot{\phi}^2 \delta_\mu^0\delta_\nu^0 + \dot{\phi}(\phi_j \delta_\mu^0 \delta_\nu^j + \phi_i \delta_\nu^0 \delta_\mu^i) + \phi_i\phi_j \delta_\mu^i \delta_\nu^j, \end{aligned} \tag{7.7}$$

where $\phi_i \equiv \partial\phi/\partial x^i$. Additionally, the disformed line element is

$$\begin{aligned} d\bar{s}^2 &= (\eta_{\mu\nu} + F\nabla_\mu\phi\nabla_\nu\phi) dx^\mu dx^\nu \\ &= -dt^2 + d\vec{x}^2 + F(\dot{\phi}^2 dt^2 + 2\dot{\phi}\phi_i dt dx^i + \phi_i\phi_j dx^i dx^j) \end{aligned} \tag{7.8}$$

or

$$d\bar{s}^2 = -(1 - F\dot{\phi}^2)dt^2 + 2F\dot{\phi}\phi_i dt dx^i + (\delta_{ij} + F\phi_i\phi_j)dx^i dx^j. \tag{7.9}$$

Now

$$X = -\frac{1}{2}(-\dot{\phi}^2 + \delta^{ij}\phi_i\phi_j) = \frac{\dot{\phi}^2}{2} - \frac{(\vec{\nabla}\phi)^2}{2} \tag{7.10}$$

depends on the spatial coordinates x^i. Since $F(\phi, X)\dot{\phi}^2$ now depends on x^i as well, it is not possible to redefine the time coordinate τ as we did before because $d\tau = \sqrt{1 - F\dot{\phi}^2}dt$ is no longer an exact differential. The disformal transformation then sends the Minkowski metric into an inhomogeneous nonstationary geometry.

VIII. MORE GENERAL DISFORMAL TRANSFORMATIONS

Instead of pure disformal transformations (2.3), one can allow for disformal transformations of the more general form [78]

$$g_{\mu\nu} \to \bar{g}_{\mu\nu} = A(\phi, X)g_{\mu\nu} + B(\phi, X)\nabla_\mu\phi\nabla_\nu\phi, \tag{8.1}$$

which we discuss in this section for FLRW and Bianchi geometries. Several, but not all, of the results valid for pure disformal transformations still hold.

The inverse of the disformed metric (8.1) is [77,78]

$$\bar{g}^{\mu\nu} = \frac{1}{A}\left(g^{\mu\nu} - \frac{B}{A - 2BX}\nabla^\mu\phi\nabla^\nu\phi\right) \tag{8.2}$$

and the invertibility condition is [77,78]

$$A \neq 0, \qquad A + 2XA_x - 4X^2B_x \neq 0, \qquad A - 2BX \neq 0, \tag{8.3}$$

where the last equation is needed to guarantee that

$$\bar{X} = \frac{X}{A - 2BX} \tag{8.4}$$

remains well-defined. For a line element of the form

$$ds^2 = -dt^2 + g_{ij}dx^i dx^j \tag{8.5}$$

in the uniform-ϕ gauge, we have $X = \dot{\phi}^2/2$, $A(\phi, X) = A(t)$, $B(\phi, X) = B(t)$, and the disformed line element is

$$\begin{aligned} d\bar{s}^2 &= \bar{g}_{\mu\nu}dx^\mu dx^\nu = (Ag_{\mu\nu} + B\nabla_\mu\phi\nabla_\nu\phi)dx^\mu dx^\nu \\ &= -(A - B\dot{\phi}^2)dt^2 + Ag_{ij}dx^i dx^j \\ &\equiv -d\tau^2 + \bar{g}_{ij}dx^i dx^j, \end{aligned} \tag{8.6}$$

where $\bar{g}_{ij} = A(t)g_{ij}$ and

$$\tau(t) = \int \sqrt{A(t) - B(t)\dot{\phi}^2}dt. \tag{8.7}$$

This integral is well-defined as long as $A - B\dot{\phi}^2 > 0$ since the integrand depends only on time, $d\tau$ is an exact differential, and $\bar{g}_{ij} = A(t)g_{ij}$. For FLRW metrics with $g_{ij} = a^2(t)\gamma_{ij}(x^k)$, the Hubble function in the disformed world is

$$\begin{aligned} \bar{H} &\equiv \frac{1}{\bar{a}}\frac{d\bar{a}}{d\tau} = \frac{1}{\sqrt{A}a}\frac{d(\sqrt{A}a)}{dt}\frac{dt}{d\tau} \\ &= \frac{1}{\sqrt{A - B\dot{\phi}^2}}\left(\frac{\dot{A}}{2A} + H\right). \end{aligned} \tag{8.8}$$

The fixed points (H_0, ϕ_0) of the phase space are mapped into new fixed points of the disformed phase space

$$(H_0, \phi_0) \to \left(\bar{H}_0 = \frac{H_0}{\sqrt{A_0}}, \phi_0\right) \tag{8.9}$$

since $X_0 = 0$ and $A(\phi, X) = A(\phi_0, 0) \equiv A_0$.

Setting

$$\gamma^2 \equiv A - B\dot{\phi}^2 > 0, \tag{8.10}$$

we have $\bar{X} = X/\gamma^2$,

$$\bar{u}^\mu = \frac{\bar{g}^{\mu\nu}\nabla_\nu\phi}{\sqrt{2\bar{X}}} = \frac{u^\mu}{\gamma}, \tag{8.11}$$

while

$$\bar{u}_\mu = \bar{g}_{\mu\nu}\bar{u}^\nu = \gamma u_\mu. \tag{8.12}$$

The Riemannian three-metric $h_{\mu\nu}$ on the three-spaces orthogonal to u^μ changes under disformal mappings of the type (8.1) according to

$$\begin{aligned} \bar{h}_{\mu\nu} &\equiv \bar{g}_{\mu\nu} + \bar{u}_\mu\bar{u}_\nu = Ag_{\mu\nu} + 2XBu_\mu u_\nu + \gamma^2 u_\mu u_\nu \\ &= -Au_\mu u_\nu + Ah_{\mu\nu} + 2XBu_\mu u_\nu + (A - 2BX)u_\mu u_\nu \\ &= Ah_{\mu\nu}. \end{aligned} \tag{8.13}$$

When $\nabla^\mu\phi$ is timelike and future-oriented, the stress-energy tensor of the effective fluid equivalent of ϕ in FLRW universes is

$$\begin{aligned} T^{(\phi)}_{\mu\nu} &= \rho u_\mu u_\nu + Ph_{\mu\nu} \\ &= \rho\frac{\bar{u}_\mu}{\gamma}\frac{\bar{u}_\nu}{\gamma} + \frac{P}{A}\bar{h}_{\mu\nu} \\ &= \bar{\rho}\bar{u}_\mu\bar{u}_\nu + \bar{P}\bar{h}_{\mu\nu}, \end{aligned} \tag{8.14}$$

where

$$\bar{\rho} = \frac{\rho}{\gamma^2}, \qquad \bar{P} = \frac{P}{A}. \tag{8.15}$$

The equation of state parameter of the effective ϕ–fluid in the disformed world is now

$$\bar{w} \equiv \frac{\bar{P}}{\bar{\rho}} = \frac{\gamma^2}{A} w. \tag{8.16}$$

On the lines of what was already done in Sec. V, consider a Bianchi universe with line element $ds^2 = -dt^2 + g_{ij}dx^i dx^j$ in the uniform-ϕ gauge in which $\phi = \phi(t)$. This universe is mapped into another Bianchi universe with line element

$$d\bar{s}^2 = -d\tau^2 + \bar{g}_{ij}dx^i dx^j, \tag{8.17}$$

where $d\tau = \gamma dt$ and $\bar{g}_{ij} = A(t)g_{ij}$.

Unless $A = \text{const}$, stealth solutions

$$(g_{\mu\nu}, \phi) = (\eta_{\mu\nu}, \phi(t)) \tag{8.18}$$

are not mapped into stealth solutions because now the disformal transformation maps the Minkowski line element $ds^2 = -dt^2 + d\vec{x}^2$ into the spatially flat FLRW geometry

$$d\bar{s}^2 = -d\tau^2 + \bar{a}^2(\tau)d\vec{x}^2. \tag{8.19}$$

with the rescaled comoving time defined by $d\tau = \gamma dt$ and scale factor $\bar{a}(\tau) = \sqrt{A(\phi(t), X(t))}|_{t=t(\tau)}$.

IX. HIGHER-ORDER DISFORMAL TRANSFORMATIONS

Yet more general disformal transformations have been introduced recently, which contain second order derivatives of the scalar field instead of a first order one [117], and even more general ones, containing derivatives of higher order than second, are contemplated [118]. To extend the scope of the previous discussion, and of the future search for exact solutions, let us extend the previous considerations to second order disformal transformations of the form

$$\bar{g}_{\mu\nu} = Ag_{\mu\nu} + B\nabla_\mu\phi\nabla_\nu\phi + 2C\nabla_{(\mu}\phi\nabla_{\nu)}X + D\nabla_\mu X\nabla_\nu X, \tag{9.1}$$

where A, B, C, and D are functions of ϕ, X, Y, Z with[4]

$$X \equiv \nabla_\mu\phi\nabla^\mu\phi, \tag{9.2}$$

$$Y \equiv \nabla_\mu\phi\nabla^\mu X, \tag{9.3}$$

$$Z \equiv \nabla_\mu X\nabla^\mu X. \tag{9.4}$$

It is useful to define the quantity

$$\mathcal{F} \equiv A[A + XB + 2YC + ZD] + (C^2 - BD)(Y^2 - XZ), \tag{9.5}$$

which will appear in several formulas. Invertibility of the disformal transformation corresponds to [117]

$$A \neq 0, \qquad \mathcal{F} \neq 0, \qquad \bar{X}_X \neq 0, \qquad \bar{X}_Y = \bar{X}_Z = 0, \tag{9.6}$$

$$\left|\frac{\partial(\bar{Y}, \bar{Z})}{\partial(Y, Z)}\right| \neq 0, \tag{9.7}$$

where [117]

$$\bar{X} \equiv \bar{g}^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi = \frac{XA - D(Y^2 - XZ)}{\mathcal{F}}, \tag{9.8}$$

$$\bar{Y} \equiv \bar{g}^{\mu\nu}\nabla_\mu\phi\nabla_\nu X = \bar{X}_X\frac{YA + C(Y^2 - XZ)}{\mathcal{F}} + \bar{X}_\phi\bar{X}, \tag{9.9}$$

$$\bar{Z} \equiv \bar{g}^{\mu\nu}\nabla_\mu X\nabla_\nu X = \bar{X}_X^2\frac{ZA - B(Y^2 - XZ)}{\mathcal{F}} + 2\bar{X}_\phi\bar{Y} - \bar{X}_\phi^2\bar{X}, \tag{9.10}$$

where $\bar{X}_\phi \equiv \partial\bar{X}/\partial_\phi$, $\bar{X}_X \equiv \partial\bar{X}/\partial_X$. The inverse of the disformed metric (9.1) is

$$\bar{g}^{\mu\nu} = \frac{1}{A}\left\{g^{\mu\nu} - \frac{[AB - Z(C^2 - BD)]}{\mathcal{F}}\nabla^\mu\phi\nabla^\nu\phi \right.$$
$$-2\frac{[AC + Y(C^2 - BD)]}{\mathcal{F}}\nabla^{(\mu}\phi\nabla^{\nu)}X$$
$$\left. - \frac{[AD - X(C^2 - BD)]}{\mathcal{F}}\nabla^\mu X\nabla^\nu X\right\}. \tag{9.11}$$

Let us specialize these formulas to the line element of interest in cosmology

$$ds^2 = -dt^2 + g_{ij}dx^i dx^j \tag{9.12}$$

in the uniform ϕ-gauge in which $\phi = \phi(t)$. It follows immediately that

$$X = -\dot{\phi}^2, \qquad Y = 2\dot{\phi}^2\ddot{\phi}, \qquad Z = -4\dot{\phi}^2\ddot{\phi}^2, \tag{9.13}$$

$$\mathcal{F} = A(A - B\dot{\phi}^2 + 4C\dot{\phi}^2\ddot{\phi} - 4D\dot{\phi}^2\ddot{\phi}^2), \tag{9.14}$$

yielding

[4]For ease of comparison with Ref. [117], in this section we define X with a different sign and normalization than in the previous sections.

$$\begin{aligned} d\bar{s}^2 &= \bar{g}_{\mu\nu}dx^\mu dx^\nu \\ &= -(A - B\dot{\phi}^2 + 4C\dot{\phi}^2\ddot{\phi} - 4D\dot{\phi}^2\ddot{\phi}^2)dt^2 + Ag_{ij}dx^i dx^j \\ &= -\frac{\mathcal{F}}{A}dt^2 + Ag_{ij}dx^i dx^j. \end{aligned} \tag{9.15}$$

To preserve the metric signature, it must be $A > 0$ and $\mathcal{F} > 0$. Since A, B, C, D, and ϕ and its derivatives depend only on time, $d\tau \equiv \sqrt{\frac{\mathcal{F}}{A}}dt$ is an exact differential and the time coordinate $\tau \equiv \int \sqrt{\frac{\mathcal{F}}{A}}dt$ is well-defined. Using τ, one rewrites the line element (9.15) as

$$d\bar{s}^2 = -d\tau^2 + \bar{A}(\tau)g_{ij}dx^i dx^j, \tag{9.16}$$

where $\bar{A}(\tau) = A(t(\tau))$. For FLRW metrics $g_{ij} = a^2(t)\gamma_{ij}(x^k)$ and

$$d\bar{s}^2 = -d\tau^2 + \bar{a}^2(\tau)\gamma_{ij}(x^k)dx^i dx^j \tag{9.17}$$

with $\bar{a}(\tau) = a(t(\tau))\sqrt{A(t(\tau))}$. The Hubble function in the disformed world is

$$\bar{H} \equiv \frac{1}{\bar{a}}\frac{d\bar{a}}{d\tau} = \frac{1}{\sqrt{A}a}\frac{d(\sqrt{A}a)}{dt}\frac{dt}{d\tau} = \sqrt{\frac{A}{\mathcal{F}}}\left(\frac{\dot{A}}{2A} + H\right). \tag{9.18}$$

Here $Y^2 - XZ = 0$ and $\mathcal{F}$ reduces to

$$\mathcal{F} = A[A - B\dot{\phi}^2 + 4\dot{\phi}^2\ddot{\phi}(C - D\ddot{\phi})], \tag{9.19}$$

while

$$\bar{X} = -\frac{A\dot{\phi}^2}{\mathcal{F}}. \tag{9.20}$$

Assuming $\nabla^\mu\phi$ to be timelike and future-oriented, the four-velocities of the ϕ fluid before and after the disformal transformation are

$$u^\mu \equiv \frac{g^{\mu\nu}\nabla_\nu\phi}{\sqrt{-X}} = -\mathrm{sign}(\dot{\phi})\delta^\mu{}_0 \tag{9.21}$$

and (see Appendix)

$$\begin{aligned} \bar{u}^\mu &\equiv \frac{\bar{g}^{\mu\nu}\nabla_\nu\phi}{\sqrt{-\bar{X}}} \\ &= \left\{\sqrt{\frac{\mathcal{F}}{A^3}} + \frac{\dot{\phi}^2}{\sqrt{\mathcal{F}A^3}}[A(B - 2C\ddot{\phi}) \right. \\ &\quad \left. + 2\ddot{\phi}A(2D\ddot{\phi} - C)]\right\}u^\mu; \end{aligned} \tag{9.22}$$

that is, $\bar{u}^\mu$ is parallel to u^μ. Then, by comparing Eq. (9.15) with

$$d\bar{s}^2 = \bar{g}_{\mu\nu}dx^\mu dx^\nu = (-\bar{u}_\mu\bar{u}_\nu + \bar{h}_{\mu\nu})dx^\mu dx^\nu, \tag{9.23}$$

one obtains immediately

$$\bar{u}_\mu = \sqrt{\frac{\mathcal{F}}{A}}u_\mu, \qquad \bar{h}_{\mu\nu} = Ah_{\mu\nu}. \tag{9.24}$$

The fluid sourcing the homogeneous cosmological spacetime has the form

$$\begin{aligned} T^{(\phi)}_{\mu\nu} &= \rho u_\mu u_\nu + Ph_{\mu\nu} = \rho\frac{A}{\mathcal{F}}\bar{u}_\mu\bar{u}_\nu + P\frac{\bar{h}_{\mu\nu}}{A} \\ &= \bar{\rho}\bar{u}_\mu\bar{u}_\nu + \bar{P}\bar{h}_{\mu\nu}, \end{aligned} \tag{9.25}$$

where

$$\bar{\rho} = \frac{\rho A}{\mathcal{F}}, \qquad \bar{P} = \frac{P}{A}. \tag{9.26}$$

If $w \equiv P/\rho$ is the equation of state parameter of the fluid before the transformation, its disformed cousin is

$$\bar{w} \equiv \frac{\bar{P}}{\bar{\rho}} = \frac{\mathcal{F}}{A^2}w. \tag{9.27}$$

When it acts on Bianchi universes with line element

$$ds^2 = -dt^2 + g_{ij}(t, x^i)dx^i dx^j \tag{9.28}$$

in the uniform–ϕ gauge, the second order disformal transformation (9.1) produces the new line element

$$d\bar{s}^2 = -\frac{\mathcal{F}}{A}dt^2 + Ag_{ij}dx^i dx^j. \tag{9.29}$$

Since $\mathcal{F}/A = A - B\dot{\phi}^2 + 4C\dot{\phi}^2\ddot{\phi} - 4D\dot{\phi}^2\ddot{\phi}^2$ depends only on the time t, one can introduce the new time coordinate defined by $d\tau \equiv \sqrt{\frac{\mathcal{F}}{A}}dt$ (an exact differential) to write

$$d\bar{s}^2 = -d\tau^2 + \bar{g}_{ij}(\tau, x^k)dx^i dx^j \tag{9.30}$$

with $\bar{g}_{ij}(\tau, x^k) = A(t(\tau))g_{ij}(t(\tau), x^k)$, which is again a Bianchi geometry with the same symmetries of the seed metric $g_{\mu\nu}$, in the uniform-ϕ gauge.

Finally, under the disformal transformation (9.1), stealth solutions $(g_{\mu\nu}, \phi(t)) = (\eta_{\mu\nu}, \phi(t))$ of the field equations become again the spatially flat FLRW geometries

$$d\bar{s}^2 = -d\tau^2 + A(t(\tau))d\vec{x}^2, \qquad \phi = \phi(t) \tag{9.31}$$

with the uniform-ϕ gauge coinciding with the comoving gauge.

X. CONCLUSIONS

Disformal transformations have been introduced in gravity long ago [76], but their use has been greatly revamped only recently in the context of Horndeski and DHOST scalar-tensor gravity. Pure disformal transformations of FLRW spaces have been studied in [43], whose discussion we expand here. We have discussed disformal transformations of cosmological spaces, making explicit the $3+1$ splitting, which is essential for the understanding of the dissipative fluid equivalent of the scalar field ϕ of scalar-tensor–gravity. The latter is well-defined only when the gradient $\nabla^\mu\phi$ of the scalar field is timelike, which is the case of FLRW and Bianchi cosmologies. Bianchi universes and the effective dissipative fluid were not considered explicitly in the literature on disformal transformations. Novel aspects of scalar field cosmology under disformal transformations presented here include the transformation properties of de Sitter solutions as fixed points of the phase space (which have constant scalar), stealth solutions, and de Sitter solutions with a nonconstant scalar field typical of scalar-tensor gravity. We have answered the question of whether these solutions are mapped into solutions of the same kind, first considering pure disformal transformations, and then more general transformations.

In the phase space of spatially flat FLRW cosmology, which is favored by observations, the physical variables can be chosen to be $(H, \phi, \dot{\phi})$. Then, necessarily, the fixed points of the Einstein-Friedmann dynamical system are de Sitter spaces with a constant scalar field. They are invariant under pure disformal transformations. These results extend straightforwardly to Bianchi universes.

As we have seen in Sec. IV, when its gradient $\nabla^\mu\phi$ is timelike and future-oriented, the gravitational scalar ϕ is equivalent to a dissipative effective fluid. This effective fluid is the basis for the recent formalism dubbed first-order thermodynamics of scalar-tensor gravity in which an effective "temperature of gravity" is introduced to describe the deviations of gravity from GR, which is then regarded as the state of zero temperature and thermal equilibrium [119,120]. An equation describing the approach to equilibrium is also provided [119,120] (see Ref. [121] for a review). In FLRW spaces, the analysis of the $3+1$ splitting of FLRW spacetimes given explicitly here provides the transformation properties of the effective ϕ-fluid quantities. This derivation provides a parallel to well-known transformation formulas of perfect fluids (including scalar field fluids) under *conformal* transformations.

Stealth solutions and de Sitter solutions with nonconstant scalar fields are forbidden in GR and are typical of scalar-tensor gravity. They can be regarded as degenerate cases of FLRW universes and can, therefore, be analyzed in the same way, as done here in Secs. VI and VII.

Finally, most of the results derived here for pure disformal transformations survive under more general (i.e., not "pure") disformal transformations of the form (8.1). We have extended the study to the disformal transformations containing second order derivatives of ϕ recently introduced in [117,118].

This study adds to the current knowledge of disformal transformations in scalar-tensor theories of gravity, which has seen a very significant increase in the past decade and are useful when searching for examples and counterexamples related to disformal transformations in cosmology. Furthermore, we have in mind the application of disformal transformations to the search for exact solutions of the scalar-tensor field equations which describe black holes and other objects embedded in cosmological spacetimes. Such solutions are difficult to find in GR and "first-generation" scalar-tensor gravity [46] and will be searched for in more general scalar-tensor theories using disformal transformatons of GR "seeds." The first step to understand their properties will be the knowledge of how the FLRW or Bianchi "backgrounds," in which they are embedded, behave under disformal transformations. This first step has been completed here.

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APPENDIX: CALCULATION OF THE DISFORMED FOUR-VELOCITY $\bar{u}^\mu$

$$
\begin{aligned}
\bar{u}^\mu &= \frac{\bar{g}^{\mu\nu}\nabla_\nu\phi}{\sqrt{-\bar{X}}} \\
&= \frac{1}{A}\left\{ g^{\mu\nu} - \frac{[AB - Z(C^2 - BD)]}{\mathcal{F}}\nabla^\mu\phi\nabla^\nu\phi - 2\frac{[AC + YZ(C^2 - BD)]}{\mathcal{F}}\nabla^{(\mu}\phi\nabla^{\nu)}X - \frac{[AD - X(C^2 - BD)]}{\mathcal{F}}\nabla^\mu X\nabla^\nu X \right\} \frac{\nabla_\nu\phi}{\sqrt{\dot{\phi}^2 A/\mathcal{F}}} \\
&= \sqrt{\frac{\mathcal{F}}{A^3}}\frac{g^{\mu\nu}\nabla_\nu\phi}{\sqrt{-X}} - \frac{1}{|\dot{\phi}|\sqrt{\mathcal{F}A^3}}\left\{ [AB - Z(C^2 - BD)]X\nabla^\mu\phi + 2[AC + Y(C^2 - BD)]\frac{\nabla^\mu\phi\nabla^\nu X\nabla_\nu\phi + X\nabla^\mu X}{2} \right. \\
&\quad \left. + [AD - X(C^2 - BD)]\nabla^\mu X\nabla^\nu X\nabla_\nu\phi \right\}. \qquad \text{(A1)}
\end{aligned}
$$

Using

$$\nabla^\nu X \nabla_\nu \phi = 2\dot{\phi}^2\ddot{\phi}, \tag{A2}$$

one obtains

$$\begin{aligned}\bar{u}^\mu &= \left\{\sqrt{\frac{\mathcal{F}}{A^3}}\frac{1}{|\dot{\phi}|} + \frac{|\dot{\phi}|}{\sqrt{\mathcal{F}A^3}}\{[AB + 4\dot{\phi}^2\ddot{\phi}^2(C^2 - BD)] - [AC + 2\dot{\phi}^2\ddot{\phi}(C^2 - BD)](2\ddot{\phi}) + 4\ddot{\phi}^2[AD + \dot{\phi}^2(C^2 - BD)] \right. \\ &\quad \left. -2\ddot{\phi}[AC + 2\dot{\phi}^2\ddot{\phi}(C^2 - BD)]\}\right\}|\dot{\phi}|u^\mu \\ &= \left\{\sqrt{\frac{\mathcal{F}}{A^3}} + \frac{\dot{\phi}^2}{\sqrt{\mathcal{F}A^3}}[A(B - 2C\ddot{\phi}) + 2\ddot{\phi}A(2D\ddot{\phi} - C)]\right\}u^\mu. \end{aligned} \tag{A3}$$

[1] R. M. Wald, *General Relativity* (Chicago University Press, Chicago, 1987).

[2] C. Brans and R. H. Dicke, Mach's principle and a relativistic theory of gravitation, Phys. Rev. **124**, 925 (1961).

[3] P. G. Bergmann, Comments on the scalar tensor theory, Int. J. Theor. Phys. **1**, 25 (1968).

[4] K. Nordtvedt, Equivalence principle for massive bodies. 2. Theory, Phys. Rev. **169**, 1017 (1968).

[5] R. V. Wagoner, Scalar tensor theory and gravitational waves, Phys. Rev. D **1**, 3209 (1970).

[6] K. Nordtvedt, Jr., PostNewtonian metric for a general class of scalar tensor gravitational theories and observational consequences, Astrophys. J. **161**, 1059 (1970).

[7] K. S. Stelle, Renormalization of higher derivative quantum gravity, Phys. Rev. D **16**, 953 (1977).

[8] K. S. Stelle, Classical gravity with higher derivatives, Gen. Relativ. Gravit. **9**, 353 (1978).

[9] P. A. R. Ade *et al.* (Planck Collaboration), Planck 2015 results. XX. Constraints on inflation, Astron. Astrophys. **594**, A20 (2016).

[10] A. A. Starobinsky, A new type of isotropic cosmological models without singularity, Phys. Lett. **91B**, 99 (1980).

[11] C. G. Callan, Jr., E. J. Martinec, M. J. Perry, and D. Friedan, Strings in background fields, Nucl. Phys. **B262**, 593 (1985).

[12] E. S. Fradkin and A. A. Tseytlin, Quantum string theory effective action, Nucl. Phys. **B261**, 1 (1985); **B269**, 745(E) (1986).

[13] L. Amendola and S. Tsujikawa, *Dark Energy, Theory and Observations* (Cambridge University Press, Cambridge, England, 2010).

[14] S. Capozziello, S. Carloni, and A. Troisi, Quintessence without scalar fields, Recent Res. Dev. Astron. Astrophys. **1**, 625 (2003).

[15] S. M. Carroll, V. Duvvuri, M. Trodden, and M. S. Turner, Is cosmic speed—up due to new gravitational physics?, Phys. Rev. D **70**, 043528 (2004).

[16] T. P. Sotiriou and V. Faraoni, $f(R)$ theories of gravity, Rev. Mod. Phys. **82**, 451 (2010).

[17] A. De Felice and S. Tsujikawa, $f(R)$ theories, Living Rev. Relativity **13**, 3 (2010).

[18] S. Nojiri and S. D. Odintsov, Unified cosmic history in modified gravity: From $F(R)$ theory to Lorentz non-invariant models, Phys. Rep. **505**, 59 (2011).

[19] G. W. Horndeski, Second-order scalar-tensor field equations in a four-dimensional space, Int. J. Theor. Phys. **10**, 363 (1974).

[20] C. Deffayet, G. Esposito-Farèse, and A. Vikman, Covariant Galileon, Phys. Rev. D **79**, 084003 (2009).

[21] C. Deffayet, S. Deser, and G. Esposito-Farése, Generalized Galileons: All scalar models whose curved background extensions maintain second-order field equations and stress-tensors, Phys. Rev. D **80**, 064015 (2009).

[22] C. Deffayet, X. Gao, D. A. Steer, and G. Zahariade, From k-essence to generalised Galileons, Phys. Rev. D **84**, 064039 (2011).

[23] J. Gleyzes, D. Langlois, F. Piazza, and F. Vernizzi, Healthy theories beyond Horndeski, Phys. Rev. Lett. **114**, 211101 (2015).

[24] J. Gleyzes, D. Langlois, F. Piazza, and F. Vernizzi, Exploring gravitational theories beyond Horndeski, J. Cosmol. Astropart. Phys. 02 (2015) 018.

[25] A. A. Starobinsky, S. V. Sushkov, and M. S. Volkov, Anisotropy screening in Horndeski cosmologies, Phys. Rev. D **101**, 064039 (2020).

[26] D. Langlois and K. Noui, Degenerate higher derivative theories beyond Horndeski: Evading the Ostrogradski instability, J. Cosmol. Astropart. Phys. 02 (2016) 034.

[27] D. Langlois and K. Noui, Hamiltonian analysis of higher derivative scalar-tensor theories, J. Cosmol. Astropart. Phys. 07 (2016) 016.

[28] J. Ben Achour, D. Langlois, and K. Noui, Degenerate higher order scalar-tensor theories beyond Horndeski and disformal transformations, Phys. Rev. D **93**, 124005 (2016).

[29] M. Crisostomi, K. Koyama, and G. Tasinato, Extended scalar-tensor theories of gravity, J. Cosmol. Astropart. Phys. 04 (2016) 044.

[30] H. Motohashi, K. Noui, T. Suyama, M. Yamaguchi, and D. Langlois, Healthy degenerate theories with higher derivatives, J. Cosmol. Astropart. Phys. 07 (2016) 033.

[31] J. Ben Achour, M. Crisostomi, K. Koyama, D. Langlois, K. Noui, and G. Tasinato, Degenerate higher order scalar-tensor theories beyond Horndeski up to cubic order, J. High Energy Phys. 12 (2016) 100.

[32] M. Crisostomi, R. Klein, and D. Roest, Higher derivative field theories: Degeneracy conditions and classes, J. High Energy Phys. 06 (2017) 124.

[33] D. Langlois, R. Saito, D. Yamauchi, and K. Noui, Scalar-tensor theories and modified gravity in the wake of GW170817, Phys. Rev. D **97**, 061501 (2018).

[34] P. Creminelli, M. Lewandowski, G. Tambalo, and F. Vernizzi, Gravitational wave decay into dark energy, J. Cosmol. Astropart. Phys. 12 (2018) 025.

[35] P. Creminelli and F. Vernizzi, Dark energy after GW170817 and GRB170817A, Phys. Rev. Lett. **119**, 251302 (2017).

[36] J. M. Ezquiaga and M. Zumalacárregui, Dark energy after GW170817: Dead ends and the road ahead, Phys. Rev. Lett. **119**, 251304 (2017).

[37] T. Baker, E. Bellini, P. G. Ferreira, M. Lagos, J. Noller, and I. Sawicki, Strong constraints on cosmological gravity from GW170817 and GRB 170817A, Phys. Rev. Lett. **119**, 251301 (2017).

[38] D. Langlois, Dark energy and modified gravity in degenerate higher-order scalar-tensor (DHOST) theories: A review, Int. J. Mod. Phys. D **28**, 1942006 (2019).

[39] D. Langlois, Degenerate higher-order scalar-tensor (DHOST) theories, arXiv:1707.03625.

[40] M. Minamitsuji and J. Edholm, Black holes with a nonconstant kinetic term in degenerate higher-order scalar tensor theories, Phys. Rev. D **101**, 044034 (2020).

[41] C. de Rham and J. Zhang, Perturbations of stealth black holes in degenerate higher-order scalar-tensor theories, Phys. Rev. D **100**, 124023 (2019).

[42] M. Minamitsuji and J. Edholm, Black hole solutions in shift-symmetric degenerate higher-order scalar-tensor theories, Phys. Rev. D **100**, 044053 (2019).

[43] G. Domènech, A. Naruko, and M. Sasaki, Cosmological disformal invariance, J. Cosmol. Astropart. Phys. 10 (2015) 067.

[44] S. A. Adil, M. R. Gangopadhyay, M. Sami, and M. K. Sharma, Late-time acceleration due to a generic modification of gravity and the Hubble tension, Phys. Rev. D **104**, 103534 (2021).

[45] T. Qiu, Z. Xiao, J. Shi, and M. Aljaf, Potential-driven inflation with disformal coupling to gravity, Phys. Rev. D **102**, 063506 (2020).

[46] V. Faraoni, *Cosmological and Black Hole Apparent Horizons*, Springer Lecture Notes in Physics Vol. 907 (Springer, New York, 2015), ISBN 978-3-319-19239-0, 978-3-319-19240-6.

[47] S. Hollands, R. M. Wald, and V. G. Zhang, The entropy of dynamical black holes, arXiv:2402.00818.

[48] D. Farrah, K. S. Croker, G. Tarlé, V. Faraoni, S. Petty, J. Afonso, N. Fernandez, K. A. Nishimura, C. Pearson, L. Wang *et al.*, Observational evidence for cosmological coupling of black holes and its implications for an astrophysical source of dark energy, Astrophys. J. Lett. **944**, L31 (2023).

[49] K. S. Croker, J. L. Weiner, and D. Farrah, Well-defined equations of motion without constraining external sources, Phys. Rev. D **105**, 084042 (2022).

[50] K. S. Croker, M. J. Zevin, D. Farrah, K. A. Nishimura, and G. Tarle, Cosmologically coupled compact objects: A single-parameter model for LIGO–Virgo mass and redshift distributions, Astrophys. J. Lett. **921**, L22 (2021).

[51] K. S. Croker, J. Runburg, and D. Farrah, Implications of symmetry and pressure in Friedmann cosmology. III. Point Sources of dark energy that tend toward uniformity, Astrophys. J. **900**, 57 (2020).

[52] K. Croker, K. Nishimura, and D. Farrah, Implications of symmetry and pressure in Friedmann cosmology. II. Stellar remnant black hole mass function, Astrophys. J. **889**, 115 (2020).

[53] K. S. Croker and J. L. Weiner, Implications of symmetry and pressure in Friedmann cosmology. I. Formalism, Astrophys. J. **882**, 19 (2019).

[54] E. Mottola, Gravitational vacuum condensate stars, *Regular Black Holes* (Springer, Singapore, 2023).

[55] C. L. Rodriguez, Constraints on the cosmological coupling of black holes from the globular cluster NGC 3201, Astrophys. J. Lett. **947**, L12 (2023).

[56] L. Lei, L. Zu, G. W. Yuan, Z. Q. Shen, Y. Y. Wang, Y. Z. Wang, Z. B. Su, W. k. Ren, S. P. Tang, H. Zhou *et al.*, Black holes as the source of dark energy: A stringent test with high-redshift JWST AGNs, Sci. China Phys. Mech. Astron. **67**, 229811 (2024).

[57] L. Amendola, D. C. Rodrigues, S. Kumar, and M. Quartin, Constraints on cosmologically coupled black holes from gravitational wave observations and minimal formation mass, Mon. Not. R. Astron. Soc. **528**, 2377 (2024).

[58] J. Garcia-Bellido, Cosmic GREA from SMBH growth, arXiv:2306.10593.

[59] P. P. Avelino, Can gravitational vacuum condensate stars be a dark energy source?, J. Cosmol. Astropart. Phys. 08 (2023) 005.

[60] P. K. Dahal, F. Simovic, I. Soranidis, and D. R. Terno, Black holes as spherically-symmetric horizon-bound objects, Phys. Rev. D **108**, 104014 (2023).

[61] Y. Wang and Z. Wang, Decoupling between gravitationally bounded systems and the cosmic expansion, arXiv:2304.01059.

[62] R. Casadio, Quantum dust cores of black holes, Phys. Lett. B **843**, 138055 (2023).

[63] T. Mistele, Comment on observational evidence for cosmological coupling of black holes and its implications for an astrophysical source of dark energy, Res. Notes AAS **7**, 101 (2023).

[64] T. Karakasis, N. E. Mavromatos, and E. Papantonopoulos, Regular compact objects with scalar hair, Phys. Rev. D **108**, 024001 (2023).

[65] R. Andrae and K. El-Badry, Constraints on the cosmological coupling of black holes from Gaia, Astron. Astrophys. **673**, L10 (2023).

[66] M. Cadoni, A. P. Sanna, M. Pitzalis, B. Banerjee, R. Murgia, N. Hazra, and M. Branchesi, Cosmological

coupling of nonsingular black holes, J. Cosmol. Astropart. Phys. 11 (2023) 007.
[67] A. Carleo and B. Ben-Salem, Effect of environment in the timing of a pulsar orbiting SgrA*, Phys. Rev. D **108**, 124027 (2023).
[68] S. J. Gao and X. D. Li, Can cosmologically coupled mass growth of black holes solve the mass gap problem?, Astrophys. J. **956**, 128 (2023).
[69] M. Cadoni, R. Murgia, M. Pitzalis, and A. P. Sanna, Quasi-local masses and cosmological coupling of black holes and mimickers, J. Cosmol. Astropart. Phys. 03 (2024) 026.
[70] E. Mlinar and T. Zwitter, Determining cosmological growth parameter for stellar-mass black holes, Mon. Not. R. Astron. Soc. **528**, 3000 (2024).
[71] M. Lacy, A. Engholm, D. Farrah, and K. Ejercito, Constraints on cosmological coupling from the accretion history of supermassive black holes, Astrophys. J. Lett. **961**, L33 (2024).
[72] P. K. Dahal, S. Maharana, F. Simovic, I. Soranidis, and D. R. Terno, Models of cosmological black holes, arXiv:2312.16804.
[73] S. S. Binici, C. Deliduman, and F. Ş. Dilsiz, The ages of the oldest astrophysical objects in an ellipsoidal universe, arXiv:2402.16646.
[74] J. Wagner, Observables for moving, stupendously charged and massive primordial black holes, arXiv:2301.13210.
[75] D. Farrah, A. Efstathiou, J. Afonso, J. Bernard-Salas, J. Cairns, D. L. Clements, K. Croker, E. Hatziminaoglou, M. Joyce, M. Lacy *et al.*, Stellar and black hole assembly in $z < 0.3$ infrared-luminous mergers: Intermittent starbursts versus super-Eddington accretion, Mon. Not. R. Astron. Soc. **513**, 4770 (2022).
[76] J. D. Bekenstein, The relation between physical and gravitational geometry, Phys. Rev. D **48**, 3641 (1993).
[77] A. L. Alinea, On the disformal transformation of the Einstein-Hilbert action, J. Phys. **16**, 507 (2024).
[78] J. Ben Achour, D. Langlois, and K. Noui, Degenerate higher order scalar-tensor theories beyond Horndeski and disformal transformations, Phys. Rev. D **93**, 124005 (2016).
[79] J. Wainwright and G. F. R Ellis, *Dynamical Systems in Cosmology* (Cambridge University Press, Cambridge, 1997).
[80] A. A. Coley, *Dynamical Systems and Cosmology* (Kluwer Academic, Dordrecht, 2003).
[81] V. Faraoni and C. S. Protheroe, Scalar field cosmology in phase space, Gen. Relativ. Gravit. **45**, 103 (2013).
[82] G. N. Remmen and S. M. Carroll, Attractor solutions in scalar-field cosmology, Phys. Rev. D **88**, 083518 (2013).
[83] V. Faraoni, Phase space geometry in scalar-tensor cosmology, Ann. Phys. (Amsterdam) **317**, 366 (2005).
[84] J. C. C. de Souza and V. Faraoni, The phase space view of f(R) gravity, Classical Quantum Gravity **24**, 3637 (2007).
[85] D. C. Edwards, The effective two-dimensional phase space of cosmological scalar fields, J. Cosmol. Astropart. Phys. 08 (2016) 063.
[86] L. O. Pimentel, Energy momentum tensor in the general scalar—tensor theory, Classical Quantum Gravity **6**, L263 (1989).
[87] V. Faraoni and J. Côté, Imperfect fluid description of modified gravities, Phys. Rev. D **98**, 084019 (2018).
[88] U. Nucamendi, R. De Arcia, T. Gonzalez, F. A. Horta-Rangel, and I. Quiros, Equivalence between Horndeski and beyond Horndeski theories and imperfect fluids, Phys. Rev. D **102**, 084054 (2020).
[89] A. Giusti, S. Zentarra, L. Heisenberg, and V. Faraoni, First-order thermodynamics of Horndeski gravity, Phys. Rev. D **105**, 124011 (2022).
[90] C. Eckart, The thermodynamics of irreversible processes. 3. Relativistic theory of the simple fluid, Phys. Rev. **58**, 919 (1940).
[91] C. Ganguly and J. Quintin, Microphysical manifestations of viscosity and consequences for anisotropies in the very early universe, Phys. Rev. D **105**, 023532 (2022).
[92] M. Miranda, D. Vernieri, S. Capozziello, and V. Faraoni, Fluid nature constrains Horndeski gravity, Gen. Relativ. Gravit. **55**, 84 (2023).
[93] F. Hammad, D. K. Çiftci, and V. Faraoni, Conformal cosmological black holes: Towards restoring determinism to Einstein theory, Eur. Phys. J. Plus **134**, 480 (2019).
[94] V. Faraoni, *Cosmology in Scalar–Tensor Gravity* (Kluwer Academic, Dordrecht, 2004).
[95] V. Faraoni, S. Jose, and S. Dussault, Multi-fluid cosmology in Einstein gravity: Analytical solutions, Gen. Relativ. Gravit. **53**, 109 (2021).
[96] G. F. R. Ellis, R. Maartens, and M. A. H. MacCallum, *Relativistic Cosmology* (Cambridge University Press, Cambridge, UK, 2012).
[97] E. Ayon-Beato, C. Martinez, and J. Zanelli, Stealth scalar field overflying a $(2+1)$ black hole, Gen. Relativ. Gravit. **38**, 145 (2006).
[98] E. Ayon-Beato, C. Martinez, R. Troncoso, and J. Zanelli, Gravitational Cheshire effect: Nonminimally coupled scalar fields may not curve spacetime, Phys. Rev. D **71**, 104037 (2005).
[99] V. Faraoni and A. F. Z. Moreno, Are stealth scalar fields stable?, Phys. Rev. D **81**, 124050 (2010).
[100] E. Ayón-Beato, A. A. García, P. I. Ramírez-Baca, and C. A. Terrero-Escalante, Conformal stealth for any standard cosmology, Phys. Rev. D **88**, 063523 (2013).
[101] M. M. Caldarelli, C. Charmousis, and M. Hassaïne, AdS black holes with arbitrary scalar coupling, J. High Energy Phys. 10 (2013) 015.
[102] M. Bravo–Gaete and M. Hassaïne, Planar AdS black holes in Lovelock gravity with a nonminimal scalar field, J. High Energy Phys. 11 (2013) 177.
[103] M. Bravo–Gaete and M. Hassaine, Topological black holes for Einstein-Gauss-Bonnet gravity with a nonminimal scalar field, Phys. Rev. D **88**, 104011 (2013).
[104] M. Hassaine, Rotating AdS black hole stealth solution in $D = 3$ dimensions, Phys. Rev. D **89**, 044009 (2014).
[105] M. Bravo-Gaete and M. Hassaine, Thermodynamics of a BTZ black hole solution with an Horndeski source, Phys. Rev. D **90**, 024008 (2014).
[106] E. Ayón-Beato, M. Hassaïne, and M. M. Juárez-Aubry, Stealths on anisotropic holographic backgrounds, arXiv:1506.03545.
[107] S. Giardino, A. Giusti, and V. Faraoni, Thermal stability of stealth and de Sitter spacetimes in scalar-tensor gravity, Eur. Phys. J. C **83**, 621 (2023).

[108] E. Babichev and G. Esposito-Farèse, Time-dependent spherically symmetric covariant galileons, Phys. Rev. D **87**, 044032 (2013).
[109] A. Anabalon, A. Cisterna, and J. Oliva, Asymptotically locally AdS and flat black holes in Horndeski theory, Phys. Rev. D **89**, 084050 (2014).
[110] E. Babichev and C. Charmousis, Dressing a black hole with a time-dependent Galileon, J. High Energy Phys. 08 (2014) 106.
[111] C. Charmousis, T. Kolyvaris, E. Papantonopoulos, and M. Tsoukalas, Black holes in Bi-scalar extensions of Horndeski theories, J. High Energy Phys. 07 (2014) 085.
[112] T. Kobayashi and N. Tanahashi, Exact black hole solutions in shift symmetric scalar–tensor theories, Prog. Theor. Exp. Phys. **2014**, 73E02 (2014).
[113] E. Babichev and G. Esposito-Farese, Cosmological self-tuning and local solutions in generalized Horndeski theories, Phys. Rev. D **95**, 024020 (2017).
[114] C. Charmousis, M. Crisostomi, R. Gregory, and N. Stergioulas, Rotating black holes in higher order gravity, Phys. Rev. D **100**, 084020 (2019).
[115] J. Ben Achour and H. Liu, Hairy Schwarzschild-(A)dS black hole solutions in degenerate higher order scalar-tensor theories beyond shift symmetry, Phys. Rev. D **99**, 064042 (2019).
[116] K. Takahashi and H. Motohashi, General relativity solutions with stealth scalar hair in quadratic higher-order scalar-tensor theories, J. Cosmol. Astropart. Phys. 06 (2020) 034.
[117] K. Takahashi, H. Motohashi, and M. Minamitsuji, Invertible disformal transformations with higher derivatives, Phys. Rev. D **105**, 024015 (2022).
[118] K. Takahashi, Invertible disformal transformations with arbitrary higher-order derivatives, Phys. Rev. D **108**, 084031 (2023).
[119] V. Faraoni and A. Giusti, Thermodynamics of scalar-tensor gravity, Phys. Rev. D **103**, L121501 (2021).
[120] V. Faraoni, A. Giusti, and A. Mentrelli, New approach to the thermodynamics of scalar-tensor gravity, Phys. Rev. D **104**, 124031 (2021).
[121] S. Giardino and A. Giusti, First-order thermodynamics of scalar-tensor gravity, in *Ricerche di Matematica* (Springer, New York, 2023).