Quantum initial conditions for curved inflating universes

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We discuss the challenges of motivating, constructing, and quantizing a canonically normalized inflationary perturbation in spatially curved universes. We show that this has historically proved challenging due to the interaction of nonadiabaticity with spatial curvature. We construct a novel curvature perturbation that is canonically normalized in the sense of its equation of motion and is unique up to a single scalar parameter. With this construction it becomes possible to set initial conditions invariant under canonical transformations, overcoming known ambiguities in the literature. This corrected quantization has potentially observational consequences via modifications to the primordial power spectrum at large angular scales, as well as theoretical implications for quantization procedures in curved cosmologies filled with a scalar field.

I. INTRODUCTION

Cosmological inflation [1–4] provides explanatory power for observations of cosmic microwave background (CMB) anisotropies [5–8], by yielding the quantum fluctuations that seed large scale structure today [9]. Additionally, inflation also resolves the horizon and curvature problems, both of which can be thought of as initial conditions for the universe [10]. From a theoretical standpoint, it is inconsistent to assume a flat universe at the start of inflation, and instead one could consider inflation as starting in a general KΛCDM (cold dark matter with a cosmological constant and spatial curvature) universe [11–15]. Such investigations are further motivated in light of the conversation in the literature regarding the statistical significance (or lack thereof) of the preference in CMB and baryon acoustic oscillation (BAO) data for present-day curvature [16–21], and it is undeniably true that any

observation of present-day curvature would have profound implications for theories of inflation [22–25].

The imprint of quantum perturbations on the CMB as anisotropies is described by the power spectrum for the gauge-invariant comoving curvature perturbation variable $\mathcal{R}$. Thus, toward the goal of computing the power spectrum for $\mathcal{R}$ in a curved inflationary spacetime, Ref. [26] finds the Mukhanov-Sasaki equation of motion for $\mathcal{R}$ for nonzero K, in analogy with standard inflationary calculations, previously also derived in [27]. Introducing curvature markedly complicates the equation of motion for $\mathcal{R}$, which only simplifies in two important cases: First, when $K = 0$; second, when one takes the matter field to be either a scalar field without potential (i.e., a stiff fluid) or hydrodynamical matter [9,28]. During very early evolution the inflaton can be approximated as a stiff fluid, and during slow roll as hydrodynamical matter [29], thus simplifying the equation of motion. However, this does not occur in the period between the two regimes, which we show is due to the interaction of nonadiabatic perturbations with curvature. This motivates the need for a novel inflationary perturbation variable, which we introduce.

The purpose of this paper is twofold. First, we construct a field satisfying a canonical wave equation, which in the limit $K = 0$ recovers the standard Mukhanov variable, allowing one to use the existing analysis of scalar fields in spaces with a Friedmann-Lemaître-Robertson-Walker (FLRW) metric. Second, we use the proposed construction

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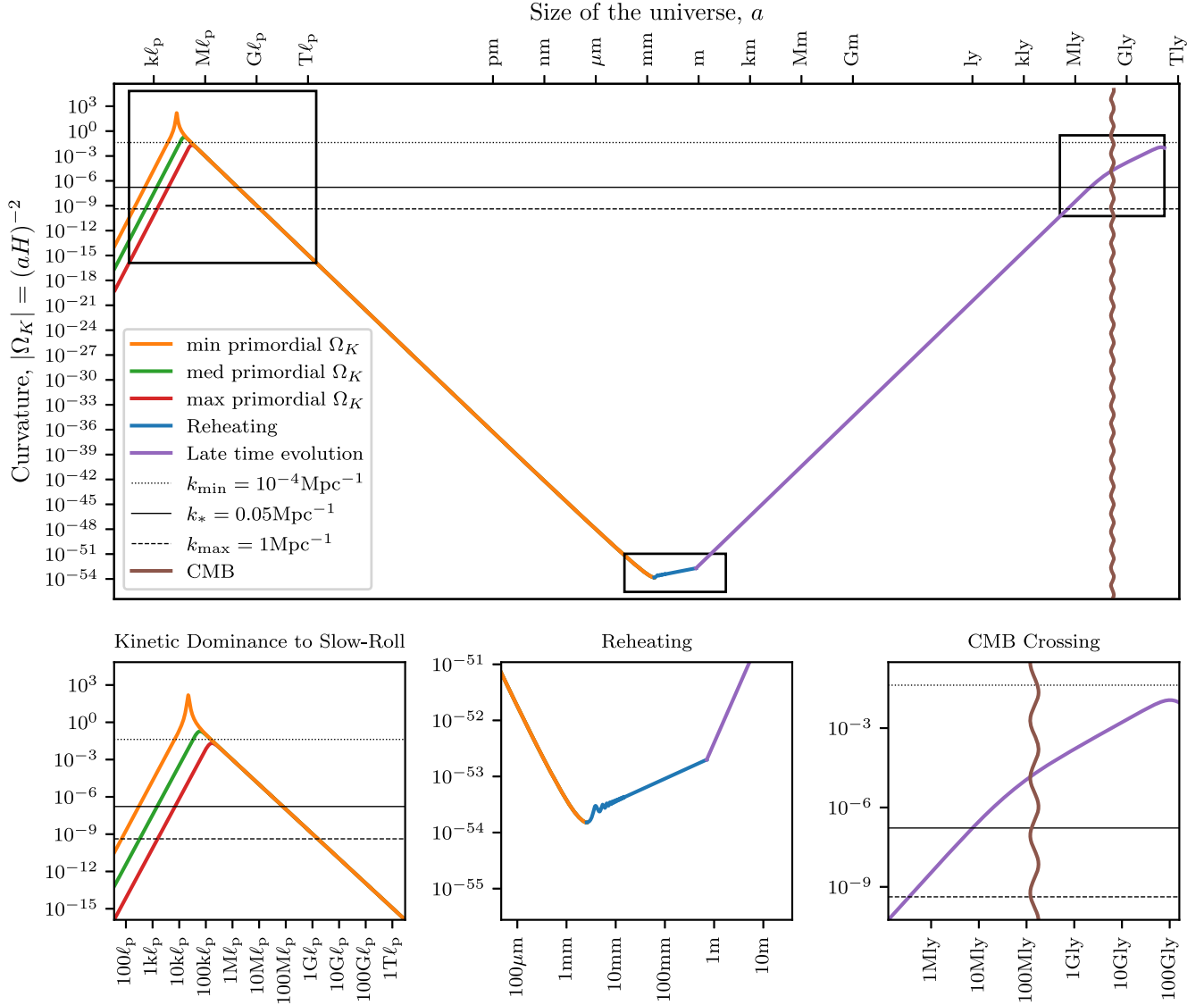‡These authors contributed equally to this work.

FIG. 1.   Timeline of a KΛCDM cosmology using *Planck* best-fit parameters [7] with `plikTTTEEE + lensing` for a cosmology including spatial curvature and $V(\phi) \propto \phi^{4/3}$ potential with instant reheating [23]. The universe begins in a kinetically dominated pre-inflationary epoch [42–56], with the comoving horizon $\frac{1}{aH}$ and curvature $\Omega_K = -\frac{K}{(aH)^2}$ parameters growing. Inflation begins when the scale factor of the universe reaches about 10^5 Planck lengths, acting to flatten the universe and dramatically shrinking the comoving horizon through a period of slow roll. At a scale factor of about 1 mm the inflaton reaches the bottom of its potential, with oscillations about the minima simulating a matter dominated universe. At some point in between 10 mm and 10 m the universe undergoes a reheating phase which we model as instantaneous, but can easily be extended to allow greater freedom [23]. The universe then grows in a protracted radiation-dominated phase, undergoing several phase transitions until recombination at about the time that the universe transitions into a matter dominated epoch when the scale factor is about 0.1 Gly. Post recombination/CMB the universe eventually enters into a late time dark energy dominated epoch and the curvature and comoving horizon start to shrink again, until we reach a universe today with radius $a_0 \approx 50$ Gpc. The best-fit parameters today even with a small amount of curvature place strong constraints on inflationary potential consistent with this history, and only a relatively small range of primordial curvatures prove compatible. For more detail, consult Hergt *et al.* [23]. ($H_0 = 64.03$ kms^{-1} Mpc^{-1}, $\Omega_m = 0.3453$, $\Omega_K = -0.0092$, $\log 10^{10}A_s = 3.0336$, $n_s = 0.9699$, $z_* = 1089.61$.)

to derive initial conditions on the curvature perturbation, which are covariant and canonically invariant.

To compute the power spectrum, the curvature perturbation variable must be evolved according to its equation of

motion from some initial conditions. In flat spacetimes, initial conditions from the Bunch-Davies vacuum set arbitrarily far back in the past are typically used. It is nontrivial to generalize such quantization schemes to a

curved inflationary spacetime on two counts: In the first instance, eternal inflation is impossible in the context of curvature [23]; refer further to Fig. 1 for a timeline of inflationary KΛCDM cosmology. In the second, traditionally posed methods such as the Bunch-Davies vacuum, Danielsson vacuum, α-vacua, and Hamiltonian diagonalization lack canonical invariance. Many works have previously analyzed cosmological perturbations in the presence of curvature and attempted to answer the question of setting initial conditions, e.g., [30,31] or in the context of bouncing inflationary models [32–34], especially highlighting issues with the usual generalization of the Mukhanov-Sasaki variable for setting initial conditions; see, e.g., [35]. However, various assumptions have to be used in order to select the "correct" vacuum, whether by asymptotic considerations, or by analogy with the flat case, without addressing existing issues of lack of canonical invariance of such selections [36–41]. While canonical transformations preserve the (classical or quantum mechanical) evolution, they may lead to different initial states and, consequently, to different corrections in the power spectrum.

More recently a method of setting initial conditions through minimization of the renormalized stress-energy tensor (RSET) has been proposed in [57], which avoids making assumptions about the asymptotic behavior of the inflationary spacetime. RSET initial conditions are similar to the instantaneous vacuum and adiabatic regularization [58,59], but unlike the latter, only require the RSET to be minimized, rather than vanish. RSET initial conditions have since been applied to inflationary collapse models [60], where until now, initial conditions were selected only according to their backreaction [37]. Moreover, such initial conditions have been shown to be canonically invariant in flat universes [38].

The paper is structured as follows. For clarity, we start by summarizing known results in Secs. II and III: in Sec. II we introduce standard cosmological perturbation theory generalized to curved spacetimes, set up generalized equations for $\mathcal{R}$, and discuss vacuum selection; Sec. III provides formalism clarifying what prevents $\mathcal{R}$ from having a canonically normalized wave equation of motion in the presence of curvature. This motivates Sec. IV, in which we propose a novel curvature perturbation variable that admits an equation of motion of a simple harmonic oscillator. This allows for connections to established work in quantum fields in curved spacetimes, in addition to allowing this variable to be easily quantized by the minimized-RSET procedure given in Sec. V, which is a generalization of the procedure proposed by Ref. [57]. These calculations are as general as possible, written in terms of inflationary sound speed, allowing for extensions to nonstandard inflationary Lagrangians. Finally, Sec. VI discusses the use of this novel variable as a means of setting initial conditions, not only for $\mathcal{R}$ but for any first order scalar perturbation variable.

The resulting power spectrum for $\mathcal{R}$ is plotted, and we discuss and draw conclusions in Sec. VII.

II. BACKGROUND

We begin with a discussion of the relevant inflationary perturbation theory setup, summarizing previous results before motivating the need for both an alternate curvature perturbation variable and a more robust method of quantization.

By convention, we will work in conformal time η, where we will denote $' \equiv d/d\eta$, and use the conformal Hubble parameter $\mathcal{H} = aH = a'/a$.

A. Perturbation theory setup

To leading order, both spacetime and the inflaton field can be taken to be homogeneous and isotropic, i.e., dependent only on η. We will consider the background to be a general FLRW spacetime, whose metric is given by

$$ds^2 = a^2(\eta)\{d\eta^2 - c_{ij}dx^i dx^j\}, \tag{1}$$

with the spatial metric

$$c_{ij}dx^i dx^j = \frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2}$$

This background is then subject to scalar perturbations

$$ds^2 = a^2(\eta)\{(1 + 2\Phi)d\eta^2 - ((1 - 2\Psi)c_{ij})dx^i x^j\}, \tag{3}$$

where we have picked the Newtonian gauge and omitted vector and tensor perturbations as they decouple from the scalar perturbations. See Ref. [10] for a more in-depth discussion of gauge choices. We restrict our attention to scalars in this instance since vectors generically decay during inflation, and tensors are already canonically normalized, even for $K \neq 0$ [26].

The inflaton is taken to be a scalar field minimally coupled to the curved spacetime,

$$S = \int [4]x\sqrt{|g|}\left\{\frac{1}{2}R + \frac{1}{2}\nabla^\mu\phi\nabla_\mu\phi - V(\phi)\right\}, \tag{4}$$

which is homogeneous and isotropic to zeroth order and is perturbed by $\delta\phi(\eta, \mathbf{x})$. The usual background inflation dynamics are given by

$$\mathcal{H}^2 = \frac{a^2}{3}\rho - K, \tag{5}$$

$$\mathcal{H}' = \mathcal{H}^2 + K - \frac{a^2}{2}(\rho + p), \tag{6}$$

where (5) and (6) are the Friedmann and Raychaudhuri equations, respectively. The background energy density ρ

and pressure p for the inflaton are

$$\rho = \frac{1}{2a^2}(\phi')^2 + V(\phi), \tag{7}$$

$$p = \frac{1}{2a^2}(\phi')^2 - V(\phi). \tag{8}$$

B. Gauge-invariant curvature perturbation

Typical analyses now proceed by defining the gauge-invariant comoving curvature perturbation,

$$\mathcal{R} = \Psi + \frac{\mathcal{H}}{\phi'}\delta\phi. \tag{9}$$

$\mathcal{R}$ is of interest as it controls the perturbations to spatial curvature, as seen from the spatial Ricci scalar

$$R^{(3)} = 6\frac{K}{a^2} + \frac{4}{a^2}(\nabla^2 + 3K)\mathcal{R}. \tag{10}$$

These perturbations are directly related to CMB anisotropies, as variations in curvature at the time of the last scattering will correspond to the scattered photons being more or less redshifted, resulting in the CMB temperature distribution [10,61].

As derived in standard Ref. [10], the resulting equation of motion for $\mathcal{R}$ when $K = 0$ is

$$(z\mathcal{R})'' - (\nabla^2 + \frac{z''}{z})(z\mathcal{R}) = 0, \tag{11}$$

$$z = \frac{a\phi'}{\mathcal{H}}. \tag{12}$$

Note that here $\nabla^2 \equiv c^{ij}\nabla_i\nabla_j$ refers to the Laplacian defined by the curved spatial metric c_{ij}.

However, for $K \neq 0$, by expanding the Einstein and conservation equations to first order in the perturbation variables, Ref. [26] shows that $\mathcal{R}$ obeys the second-order Mukhanov-Sasaki equation

$$0 = (\mathcal{D}^2 - K\mathcal{E})\mathcal{R}'' + 2\left(\frac{z'}{z}\mathcal{D}^2 - K\mathcal{H}\mathcal{E}\right)\mathcal{R}'$$
$$+ \left(K\left(1 + \mathcal{E} - \frac{2}{\mathcal{H}}\frac{z'}{z}\right)\mathcal{D}^2 + K^2\mathcal{E} - \mathcal{D}^4\right)\mathcal{R}, \tag{13}$$

$$\mathcal{D}^2 = \nabla^2 + 3K, \qquad \mathcal{E} = \frac{a^2(\rho + p)}{2\mathcal{H}^2}. \tag{14}$$

This can be written more concisely by first mode-decomposing into Fourier space, namely writing

$$\nabla^2 Y_k(\mathbf{x}) = -\kappa^2(k)Y_k(\mathbf{x}) \tag{15}$$

for $Y_k(\mathbf{x})$ the appropriate hyperspherical Bessel functions [61,62] that give the eigenspectrum of the spatially curved Laplacian. Thus, we identify $\mathcal{D}^2$ with $-\kappa^2(k) + 3K$, where

$$\kappa^2(k) = k^2, \qquad k \in \mathbb{R}, k > 0: \qquad K = 0, -1,$$
$$\kappa^2(k) = k(k+2), \qquad k \in \mathbb{Z}, k > 2: \qquad K = +1, \tag{16}$$

Define wave-number-dependent $\mathcal{Z}$ as

$$\mathcal{Z} = z\sqrt{\frac{\mathcal{D}^2}{\mathcal{D}^2 - K\mathcal{E}}} = a\sqrt{\frac{2\mathcal{D}^2\mathcal{E}}{\mathcal{D}^2 - K\mathcal{E}}}. \tag{17}$$

Equation (13) can be recast as

$$(\mathcal{Z}\mathcal{R}_k)'' + \left(\kappa^2 - \frac{\mathcal{Z}''}{\mathcal{Z}} - 2K - \frac{2K\mathcal{Z}'}{\mathcal{H}\mathcal{Z}}\right)(\mathcal{Z}\mathcal{R}_k) = 0, \tag{18}$$

where $\mathcal{R}_k(\eta)$ is the Fourier component of $\mathcal{R}(\eta, \mathbf{x})$ with wave number of magnitude k.

To connect this theoretical framework to observation, we must compute the primordial power spectrum from $\mathcal{R}$ at the horizon crossing. This is an initial value problem: $\mathcal{R}$ can be numerically computed according to its equations of motion Eqs. (13) and (18) from some initial conditions $\mathcal{R}(\eta_0), \mathcal{R}'(\eta_0)$, or equivalently, a vacuum state for the corresponding quantized variable. The correct theoretical choice for initial conditions in this case is far from clear [10,39,40]; we will now discuss possible vacuum choices.

C. Vacuum choice

For quantum fields on a curved spacetime, the choice of vacuum state is either physically unclear or ambiguous. At zeroth order, the slow-roll inflaton mimics a positive cosmological constant, and thus, the background dynamics of an inflationary FLRW spacetime are well-described by de Sitter space. This is true for general K. Since de Sitter space is maximally symmetric, there exists a natural choice of vacuum for a scalar field in this spacetime by means of the Bunch-Davies (BD) vacuum [63], which is invariant under all isometries of de Sitter space. This motivates the BD vacuum as a physically reasonable choice. An alternative choice is setting initial conditions by means of Hamiltonian diagonalization (HD). However, it is questionable whether this choice of vacuum is physically meaningful in an expanding spacetime as the time-dependent Hamiltonian will yield infinite particle density after the instant the vacuum is set [39,40].

In the $K = 0$ case, the BD vacuum can be painlessly applied to quantize the Mukhanov variable $z\mathcal{R}$, as its equation of motion (11) is analogous to the resulting Klein-Gordon equation of motion for a scalar field, with a time-dependent mass term given by z. However, applying the BD vacuum to quantize perturbations in inflation requires the spacetime to be quasi–de Sitter at the time of quantization,

which is not possible for theories of finite inflation [23,51,55]. This can be seen further in Fig. 1, which illustrates a period of kinetic dominance of the inflaton.

Further, in the curved case, $\mathcal{R}$ no longer possesses a canonically normalized action nor equation of motion; i.e., it does not behave like a simple harmonic oscillator (SHO) as Eq. (11). This can be seen from the k-dependence of $\mathcal{Z}$ in Eq. (18). As such, we can make no connections to the large body of work on quantum fields in curved spacetime in order to provide insight on what initial conditions to use. Almost all existing literature regarding second quantization in curved spacetimes deal with a massive scalar field with constant coupling to gravity [39,64], admitting an SHO-like equation of motion to then quantize the scalar field by analogy with the time-independent quantum harmonic oscillator. Thus, the first step in generalizing inflationary theories to nonflat primordial curvature is to find a curvature perturbation variable that obeys an analogous SHO equation of motion, unlike $\mathcal{R}$.

Additionally, as shown by Ref. [38] for $K = 0$, neither the BD vacuum nor the vacuum from HD are robust against canonical transformations. Namely, under a canonical transformation of phase space preserving the field's equation of motion, these vacuum setting procedures yield ambiguous vacuum initial conditions that would be observationally distinguishable. Another potential vacuum choice is the one proposed by Danielsson [65]. This vacuum is derived in the Heisenberg picture for the field operators, and initial conditions are set by considering the time at which each mode reaches Planckian lengths. Unfortunately, this choice is also not invariant under canonical transformations when $K = 0$.

However, as proposed in Ref. [57], one can instead set the vacuum by minimizing the RSET. This formulation avoids consideration of the tricky concept of particles. Furthermore, it does not require any assumptions about the asymptotic behavior of the inflationary spacetime, and so allows for noneternal theories of inflation. More crucially, this method yields canonically invariant vacuum conditions [38] when $K = 0$. Based on calculations in Sec. V and Appendix A, this persists even for the case of $K \neq 0$, which motivates our calculations of initial conditions resulting from this procedure.

As discussed by Fulling [39], computing the correct form of the RSET for a given field subject to a general action is challenging, and in the case of $\mathcal{R}$, is virtually intractable due to the convoluted form of its action [26],

$$S_{\mathcal{R}}^{(2)} = \frac{1}{2} \int \mathrm{d}\eta \mathrm{d}[3]x a \sqrt{|c|} \frac{(\phi')^2}{\mathcal{H}^2} \left\{ \mathcal{R} \mathcal{D}^2 \mathcal{R} \right.$$
$$\left. + \left(\mathcal{R}' - K\frac{\mathcal{R}}{\mathcal{H}} \right) \frac{\mathcal{D}^2}{\mathcal{D}^2 - K\mathcal{E}} \left(\mathcal{R}' - K\frac{\mathcal{R}}{\mathcal{H}} \right) \right\}. \quad (19)$$

As discussed further in Sec. V, a more feasible task is finding the RSET for a massless, minimally coupled scalar

field ψ on a curved spacetime, with resulting equation of motion

$$(a\psi)'' - \left(\nabla^2 + \frac{a''}{a} \right)(a\psi) = 0 \quad (20)$$

for which the renormalized stress-energy tensor has been derived by Birrell and Davies [62]. As proposed and discussed in Ref. [57], when $K = 0$, an analogy can be drawn between Eq. (20) and the equation of motion (11) for $\mathcal{R}$ in flat space, by noting that during flat slow-roll inflation

$$\frac{a''}{a} \approx \frac{z''}{z}. \quad (21)$$

Thus, $a\psi$ and $z\mathcal{R}$ share an equation of motion, and so $\mathcal{R}$ can be quantized directly through the minimized RSET conditions for this arbitrary scalar field ψ.

However, for $K \neq 0$, $\mathcal{R}$ cannot yet be quantized using the minimized-RSET proceedure as above, since it does not have the equation of motion of a SHO. Thus, in what follows, we will motivate and derive a novel perturbation variable that obeys a canonically normalized wave equation of motion. Finally, we will derive initial conditions for this variable in Sec. V by means of the minimized-RSET procedure in Ref. [57], which we have generalized to curved spacetimes.

III. MOTIVATION FOR A NEW VARIABLE

As discussed above, we aim to construct a scalar perturbation variable obeying a SHO equation of motion analogous to Eq. (11) in order to make connections with the existing literature concerning quantum fields in curved spacetime and their quantization, and further, to be able to apply minimized RSET as a well-motivated choice of vacuum selection.

Progress toward finding a canonically normalized perturbation variable has been made by Brechet *et al.* [66], in which the proposed variable obeys an equation of motion that recovers the wave equation seen for $\mathcal{R}$ in the $K = 0$ case.

To understand the results of this paper as they relate to curved inflation, let us set up again a perturbed spacetime, filled instead by a perfect fluid. Note that, with care, a scalar field can be seen as a special case of a perfect fluid to zeroth order [67]. The stress-energy tensor for this component fluid can be expanded to first order as

$$T_0^0 = -\rho - \delta\rho,$$
$$T_i^j = -(p + \delta p)\delta_i^j. \quad (22)$$

Again, we consider a perturbed spacetime in the Newtonian gauge,

$$ds^2 = (1 + 2\Phi)dt^2 - a^2(1 - 2\Psi)c_{ij}dx^i dx^j. \quad (23)$$

Following Unnikrishnan and Sriramkumar [68], by expanding the Einstein field and conservation equations to first order in perturbation variables $\delta\rho, \delta p, \Phi, \Psi$, given component tensor Eq. (22) and metric Eq. (23), we find the Bardeen equation of motion for Bardeen potential Φ to be

$$\Phi'' + 3\mathcal{H}(1 + c_a^2)\Phi' - c_a^2\nabla^2\Phi$$
$$+ [(1 + 3c_a^2)\mathcal{H}^2 - K(1 + 3c_a^2) + 2\mathcal{H}']\Phi = 0, \quad (24)$$

where

$$c_a^2 \equiv \frac{p'}{\rho'} \quad (25)$$

is the adiabatic sound speed.

Adapted to the notation used above, Brechet *et al.* [66] then defines an alternate comoving curvature perturbation scalar

$$\zeta_{\mathrm{PF}} = \mathcal{R} - \frac{2K}{a^2(\rho + p)}\Phi. \quad (26)$$

Note briefly that $\zeta_{\mathrm{PF}} = \mathcal{R}$ when $K = 0$, and that $c_a^2 \approx -1$ during slow-roll inflation.

The equation of motion for ζ_{PF} can then be found simply by writing ζ_{PF} in terms of Φ as

$$\zeta_{\mathrm{PF}} = \Phi + \frac{2\mathcal{H}}{a^2(\rho + p)}(\mathcal{H}\Phi + \Phi') - \frac{2K}{a^2(\rho + p)}\Phi. \quad (27)$$

Equation (24) then reduces to the wave equation

$$(z_0\zeta_{\mathrm{PF}})'' - \left(c_a^2\nabla^2 v + \frac{z_0''}{z_0}\right)(z_0\zeta_{\mathrm{PF}}) = 0, \quad (28)$$

$$z_0 = \frac{a^2\sqrt{\rho + p}}{c_a\mathcal{H}}. \quad (29)$$

Thus, given the above canonically normalized wave equation of motion for ζ_{PF}, we can canonically quantize curvature perturbations for a perfect fluid on a curved spacetime, and set initial conditions using minimized RSET as discussed in Sec. V.

This framework can then be applied to the specific case of the inflaton, by calculating $\rho_\phi, p_\phi, \delta\rho_\phi, \delta p_\phi, \delta\Sigma_\phi$ from the stress-energy tensor for a perturbed *inflationary* scalar field $\phi(\eta) + \delta\phi(\eta, \mathbf{x})$.

For the sake of generality, we consider the following extension of action Eq. (4):

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}R - P(X, \phi)\right], \quad (30)$$

where

$$X = \frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi. \quad (31)$$

A more general treatment of inflationary $P(X, \phi)$ theories can be seen in Refs. [28,29]. Applying the results in this paper to alternate-Lagrangian theories of inflation, such as Dirac-Born-Infeld (DBI) inflation, could be a fruitful future extension of this work.

Then, following Garriga and Mukhanov [28], we define the inflationary sound speed, or the effective sound speed of the perturbations,

$$c_s^2 \equiv \frac{\partial_X P}{\partial_X P + 2X\partial_{XX}P}. \quad (32)$$

Note that the inflaton is typically described by a standard Lagrangian

$$P(X, \phi) = X - V(\phi). \quad (33)$$

Thus, usually $c_s^2 = 1$.

Equation (32) allows us to recast the equivalent non-adiabatic pressure perturbation for a scalar field as

$$\delta p_{\mathrm{en}} = \frac{2}{a^2}(c_s^2 - c_a^2)\mathcal{D}^2\Phi. \quad (34)$$

Equations (32) and (34) allow us to write the Bardeen equation of motion for a scalar field as

$$\Phi'' + 3\mathcal{H}(1 + c_a^2)\Phi' - c_a^2\nabla^2\Phi$$
$$+ [(1 + 3c_a^2)\mathcal{H}^2 - K(1 + 3c_a^2) + 2\mathcal{H}']\Phi = \frac{a^2}{2}\delta p_{\mathrm{en}}. \quad (35)$$

We highlight that this only differs from Eq. (24), the Bardeen equation of motion for a spacetime with component perfect fluid, by the addition of the entropic pressure term $\frac{a^2}{2}\delta p_{\mathrm{en}}$ on the right-hand side (RHS).

Given Φ described by Eq. (24), i.e., for the toy universe filled with a perfect fluid, we have

$$\zeta_{\mathrm{PF}}' = \frac{2\mathcal{H}c_a^2}{a^2(\rho + p)}\nabla^2\Phi. \quad (36)$$

However, for a perturbed FLRW universe filled instead with the inflationary scalar field, where Φ is described by Eq. (35),

$$\zeta_{\mathrm{PF}}' = \frac{2\mathcal{H}c_s^2}{a^2(\rho + p)}\nabla^2\Phi + 6\frac{K\mathcal{H}(c_s^2 - c_a^2)}{a^2(\rho + p)}\Phi. \quad (37)$$

The discrepancy between Eqs. (36) and (37) suggests that a SHO wave equation of motion is impossible for this

variable unless $K = 0$ or $c_a^2 = c_s^2$. Writing Eq. (37) more explicitly as

$$\zeta'_{\text{PF}} = \frac{2\mathcal{H}c_a^2}{a^2(\rho + p)}\nabla^2\Phi + \frac{2\mathcal{H}}{\rho + p}\delta p_{\text{en}}, \tag{38}$$

we see that the term preventing the desired equation of motion is proportional to the entropic pressure perturbation, which is dependent on spatial curvature through $\mathcal{D}$.

IV. NOVEL CURVATURE VARIABLE

First, the Bardeen equation (35), which fully describes the evolution of perturbations in a curved FLRW spacetime with a perturbed inflaton, is equivalent to

$$\Phi'' + 3\mathcal{H}(1 + c_a^2)\Phi' - c_s^2\nabla^2\Phi$$
$$+ [(1 + 3c_a^2)\mathcal{H}^2 - K(1 + 3c_s^2) + 2\mathcal{H}']\Phi = 0. \tag{39}$$

We aim to define a variable whose dynamics simplify to a canonically normalizable wave equation under Eq. (39). Thus, consider

$$\zeta = g(\eta)\Phi + \frac{2\mathcal{H}g(\eta)}{a^2(\rho + p)}(\mathcal{H}\Phi + \Phi') - \frac{2Kf(\eta)}{a^2(\rho + p)}\Phi, \tag{40}$$

where we recall

$$\mathcal{R} = \Phi + \frac{2\mathcal{H}}{a^2(\rho + p)}(\mathcal{H}\Phi + \Phi'), \tag{41}$$

$$\zeta_{\text{PF}} = \Phi + \frac{2\mathcal{H}}{a^2(\rho + p)}(\mathcal{H}\Phi + \Phi') - \frac{2K}{a^2(\rho + p)}\Phi. \tag{42}$$

So far, $g(\eta)$ and $f(\eta)$ are yet unspecified functions, and so ζ can represent any linear combination of Φ, Φ', i.e., of the form $A(\eta)\Phi + B(\eta)\Phi'$. Note that currently this is the most general form for ζ, since it must be a first order perturbation, and thus a linear combination of other gauge-invariant quantities.

We can see that under Eq. (39), the derivative of ζ simplifies as

$$\zeta' = \frac{2\mathcal{H}g}{a^2(\rho + p)}\left[c_s^2\mathcal{D}^2\Phi + \Phi'\left(\frac{K}{\mathcal{H}} - \frac{f}{g}\frac{K}{\mathcal{H}} + \frac{g'}{g}\right)\right.$$
$$+ \Phi\left(K + \frac{g'}{g}\left(\mathcal{H} + \frac{a^2(\rho + p)}{2\mathcal{H}}\right)\right.$$
$$\left.\left. + \frac{K}{\mathcal{H}}\frac{f}{g}\left(2\mathcal{H} + \frac{(\rho + p)'}{(\rho + p)} - \frac{f'}{f}\right)\right)\right]. \tag{43}$$

Now, by inspection of Eq. (43) and by analogy with Eq. (36), we will want an ansatz where ζ' is proportional to only $\mathcal{D}^2\Phi$, and not to Φ or Φ',

$$\zeta' = g\frac{2\mathcal{H}c_s^2}{a^2(\rho + p)}\mathcal{D}^2\Phi. \tag{44}$$

Once this is picked as an ansatz, we are guaranteed to arrive at a wave equation, as long as we can solve equations specifying functions $f(\eta)$, $g(\eta)$:

$$f(\eta) = g'\frac{\mathcal{H}}{K} + g, \tag{45}$$

$$\frac{f'}{f} = K + \frac{g'}{g}\left(\mathcal{H} + \frac{a^2(\rho + p)}{2\mathcal{H}}\right) + \frac{K}{\mathcal{H}}\frac{f}{g}\left(2\mathcal{H} + \frac{(\rho + p)'}{(\rho + p)}\right). \tag{46}$$

Both of these equations can be rewritten to simplify by taking $\mathcal{G}$, b such that

$$\mathcal{G} = \frac{b'}{b}, \tag{47}$$

$$\mathcal{G}^2 - \mathcal{G}' = \mathcal{H}^2 - \mathcal{H}' + K, \tag{48}$$

and setting

$$g(\eta) = \frac{a}{\mathcal{H}}\frac{\mathcal{G}}{b}, \tag{49}$$

$$f(\eta) = g'\frac{\mathcal{H}}{K} + g, \tag{50}$$

which we will use as the definition for our functions $f(\eta)$ and $g(\eta)$. We further motivate the ansatz Eq. (44) in Appendix B.

For ζ defined such that Eq. (44) holds, the resulting equation of motion simplifies to

$$(z_g\zeta)'' + \left(c_s^2\mathcal{D}^2 - \frac{z_g''}{z_g}\right)(z_g\zeta) = 0, \tag{51}$$

$$z_g(\eta) = \frac{z}{g} = \frac{a^2(\rho + p)^{\frac{1}{2}}}{gc_s\mathcal{H}}, \tag{52}$$

as expected from Sec. III. The only thing left is to see whether a solution $\mathcal{G}$ satisfying Eq. (48) exists. By rewriting Eq. (48) using Eq. (47), we can see that we have a linear second order differential equation for $1/b$:

$$b\left(\frac{1}{b}\right)'' = a\left(\frac{1}{a}\right)'' + K. \tag{53}$$

Unfortunately, Eq. (53) does not have a closed form solution, unless $K = 0$. We can solve it numerically, however. To justify a selection of initial conditions for this differential equation, we should note that there is an overall scaling freedom in the definition of b, and therefore

there is only one effective degree of freedom. Furthermore, the selection of initial conditions on b and $\dot{b}$ at some initial time t_0 is equivalent to picking initial values of f and g at time t_0 as can be seen from

$$\dot{b}_0 = \frac{g_0 H_0}{a_0} b_0^2, \tag{54}$$

$$b_0 = a_0 \frac{\dot{H}_0 - \frac{K}{a_0^2}}{\frac{K}{a_0^2}(f_0 - g_0) + g_0 \dot{H}_0}, \tag{55}$$

where subscript 0 corresponds to the value at initial time t_0. Therefore, picking which initial conditions to consider for the variable ζ is equivalent to asking which variable we want to quantize at t_0. Note that the question of global existence of solutions to Eq. (53) is not obvious; however, for Sec. VI only the local existence is of interest for the derivation of the initial conditions.

With the wave equation in place, we can discuss a few things of note about the new variable. First of all, ζ defined via Eqs. (40) and (47)–(50) has an overall arbitrary scaling, due to a scaling freedom of both g and f. However, the resultant Mukhanov variable $v = z_g \zeta$ does not, and furthermore it always collapses to the original flat Mukhanov variable $v_{\text{flat}} = z\mathcal{R}$ when $K = 0$, even when we pick $b \neq a$ as the solution to Eq. (53).

We have thus constructed ζ, such that the equation of motion for ζ has canonically normalizable wave equation form, and can be quantized using the minimized-RSET vacuum conditions derived in Sec. V.

V. VACUUM CONDITIONS VIA RSET

In what follows, we will demonstrate how to apply minimized RSET to quantize ζ with k-space equation of motion

$$(z_g \zeta_k)'' + \left(c_s^2(\eta)\kappa_{\mathcal{D}}^2(k) - \frac{z_g''}{z_g} \right)(z_g \zeta_k) = 0, \tag{56}$$

where $\kappa_{\mathcal{D}}$ gives the wave space decomposition of the $\mathcal{D}^2$ operator

$$-\mathcal{D}^2 \leftrightarrow \kappa_{\mathcal{D}}^2(k) = \kappa^2(k) - 3K. \tag{57}$$

We highlight that none of the following calculations are particular to the definitions of z_g, ζ, and c_s, so this procedure is easily applicable for the quantization of a wider class of variables with a comparable equation of motion.

Compare Eq. (56) with a massless minimally coupled scalar field given by

$$S = \frac{1}{2} \int d^4 x \sqrt{|g|} (g^{\mu\nu} \nabla_\mu \psi \nabla_\nu \psi). \tag{58}$$

Note that this is *not* the inflaton scalar field, but is merely introduced for computational convenience; the RSET for such ψ have been calculated by Birrell and Davies [62], while such a calculation for ζ is not yet obvious. ψ has an equation of motion in k-space given by

$$(a\psi_k)'' + \left(\kappa^2(k) - \frac{a''}{a} \right)(a\psi_k) = 0, \tag{59}$$

i.e., another wave equation of motion with a mass function given by the scale factor. As discussed in depth in Appendix A, and in analogy with Ref. [38], we introduce 4 extra degrees of freedom in the form of time redefinitions and field rescalings

$$\eta \longrightarrow \eta_\zeta, \tag{60}$$

$$\zeta \longrightarrow \chi_\zeta = \frac{\zeta(\eta_\zeta)}{h_\zeta(\eta_\zeta)}, \tag{61}$$

$$\eta \longrightarrow \eta_\psi, \tag{62}$$

$$\psi \longrightarrow \chi_\psi = \frac{\psi(\eta_\psi)}{h_\psi(\eta_\psi)}. \tag{63}$$

We show there exist unique h_ζ, h_ψ, η_ζ, η_ψ such that the redefined χ fields corresponding to ζ and ψ have identical wave equations of motion. With this, we can map modes of ζ onto modes of ψ, and thus map initial conditions of ψ onto initial conditions of ζ. More details can be found in Appendix A. Thus, generalizing the calculations performed in Ref. [38], we find initial conditions for ζ at $\eta = \eta_0$ to be

$$|\zeta(\eta_0)|^2 = \frac{1}{2c_s(\eta_0)z_g^2(\eta_0)\kappa_{\mathcal{D}}},$$

$$\frac{\zeta'}{\zeta}(\eta_0) = -i\kappa_{\mathcal{D}} + \frac{a'}{a}(\eta_0) - \frac{z_g'}{z_g}(\eta_0) - \frac{1}{2}\frac{c_s'}{c_s}(\eta_0), \tag{64}$$

or in normal time at $t = t_0$ by

$$|\zeta(t_0)|^2 = \frac{1}{2c_s(t_0)z_g^2(t_0)\kappa_{\mathcal{D}}},$$

$$\frac{\dot{\zeta}}{\zeta}(t_0) = -i\frac{1}{a(t_0)}\kappa_{\mathcal{D}} + \frac{\dot{a}}{a}(t_0) - \frac{\dot{z}_g}{z_g}(t_0) - \frac{1}{2}\frac{\dot{c}_s}{c_s}(t_0). \tag{65}$$

VI. POWER SPECTRUM AND RESULTS

To ascertain whether the addition of spatial curvature provides a better description of observations, we aim to compute the power for $\mathcal{R}$. We can write $\mathcal{R}$, $\mathcal{R}'$ in terms of ζ and ζ' as

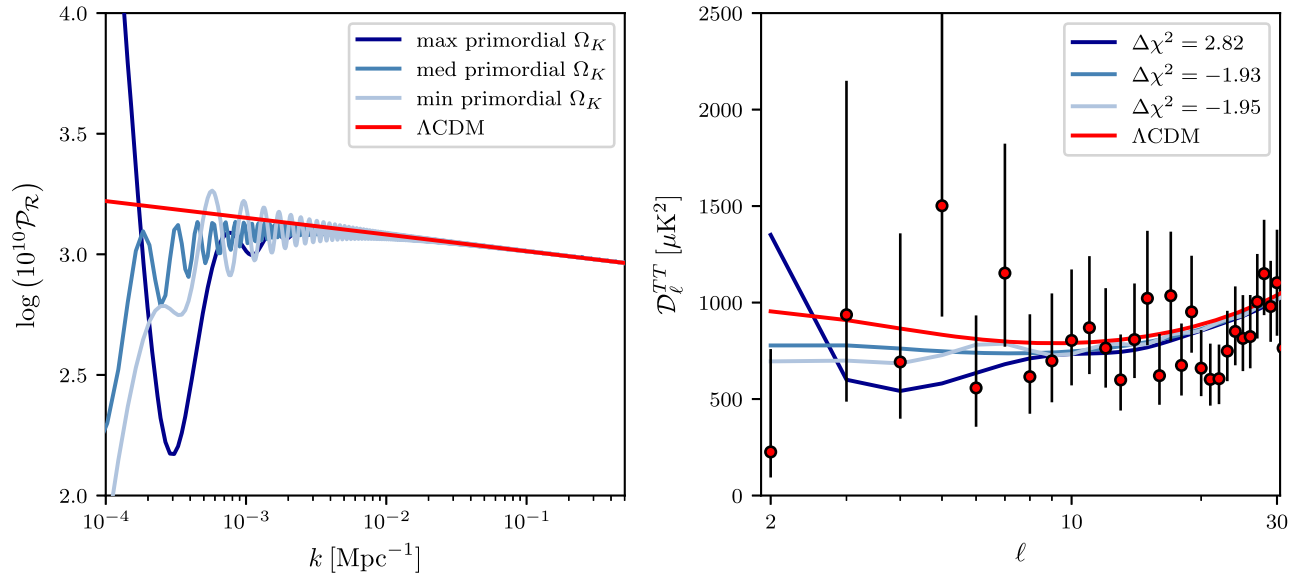$$\mathcal{R} = \frac{\zeta}{g} + \frac{K}{\mathcal{H}}\frac{f}{g^2 c_s^2 \mathcal{D}^2}\zeta', \tag{66}$$

FIG. 2. Left: representative best-fit primordial power spectra corresponding to a range of allowed primordial curvatures. Right: corresponding low-ℓ effects on the CMB power spectrum.

$$\mathcal{R}' = \frac{1}{g\mathcal{D}^2}\left(\frac{K^2 f}{gc_s^2\mathcal{H}^2} + (\mathcal{D}^2 - K\mathcal{E})\right)\zeta' + \frac{K}{g\mathcal{H}}\zeta \qquad (67)$$

for $\mathcal{E}$ as in Eq. (14). Now note that ζ still describes a family of functions, as ζ is formulated in terms of a family of functions f and g, which are defined by the second-order differential Eq. (53). For this section, rather

than considering ζ as a physically meaningful perturbation variable in its own right, we shall use it as a means of setting well-motivated initial conditions for $\mathcal{R}$: if one takes $f(\eta_0) = 0$ and $g(\eta_0) = 1$, then $\mathcal{R} = \zeta$ at η_0. Thus, it is appropriate to define initial conditions from $\mathcal{R}$ from the minimized-RSET initial conditions for one such choice in the ζ family, defined at $\eta = \eta_0$ by $f(\eta_0) = 0$
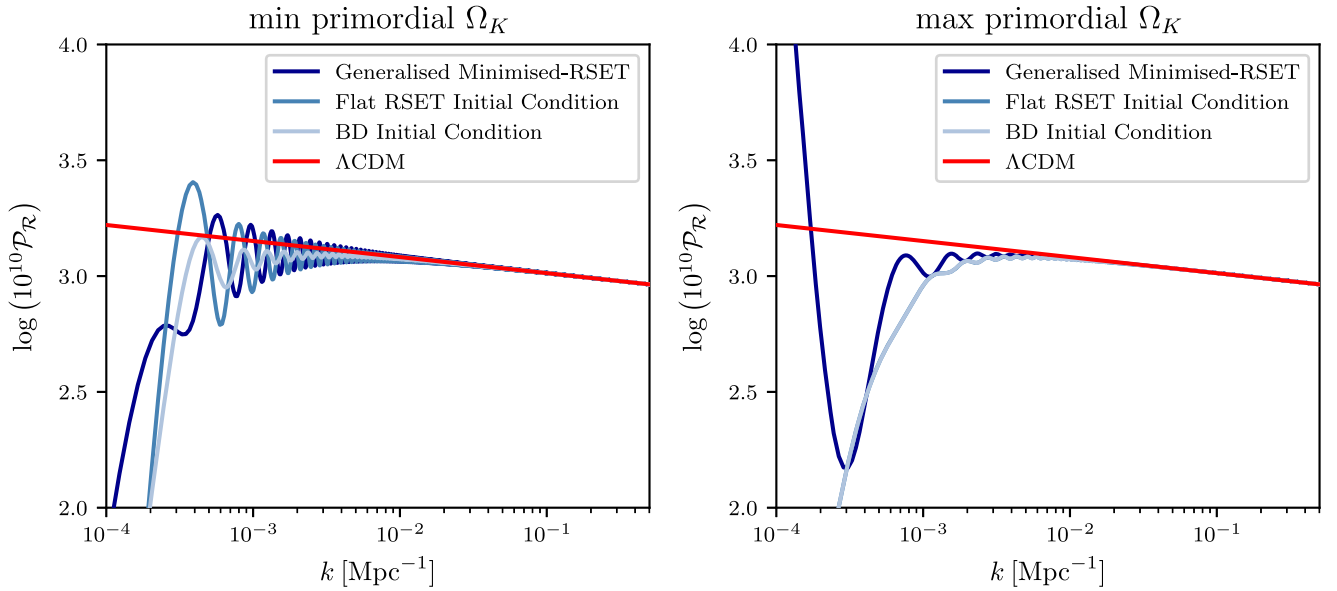

FIG. 3. Comparison of the effect of initial conditions on the power spectrum of $\mathcal{R}$. Initial conditions considered are the novel initial conditions Eq. (68) for $\mathcal{R}$ by means of generalized RSET on ζ; a naive application of flat-space RSET conditions computed in Ref. [57]; and Bunch-Davies ICs. The power spectrum is computed for minimum and maximum primordial curvature cases. Note that in the maximum case (right-hand graph) the BD condition solution numerically coincides with the naive-flat condition.

and $g(\eta_0) = 1$. Equations (64), (66), and (67) together yield

$$\left.\frac{\mathcal{R}'}{\mathcal{R}}\right|_{\eta_0} = \left(1 + \frac{Kz^2}{2a^2\kappa_{\mathcal{D}}^2}\right)\left(-i\kappa_{\mathcal{D}} + \frac{a'}{a} - \frac{z'}{z} - \frac{c_s'}{2c_s} - \frac{K}{\mathcal{H}}\right) + \left.\frac{K}{\mathcal{H}}\right|_{\eta_0},$$

$$|\mathcal{R}(\eta_0)|^2 = \left.\frac{1}{2c_s\kappa_{\mathcal{D}}z^2}\right|_{\eta_0}, \tag{68}$$

written in terms of the usual mass variable $z = gz_g$.

Of note is the independence of Eq. (68) on b, since it was only introduced as a mathematical tool to construct the desired equation of motion. Furthermore, this result is consistent with the expected physical behavior of coinciding with the flat case and other initial conditions [30,31] in the limit of $\kappa_{\mathcal{D}} \to \infty$.

We highlight that this technique would generalize easily, as any perturbation scalar can be written in terms of Φ and Φ', and thus equivalently, in terms of ζ, ζ'. Thus, for an appropriate choice of initial f, g values, we can set initial conditions for any perturbation scalar by means of minimized RSET on ζ.

Using an oscillatory solver [69] and approximating slow roll with $V(\phi) \propto \phi^{4/3}$, one can numerically evolve $\mathcal{R}$ from t_0 to the time of mode reentry using Eq. (68) and equation of motion (13). For the standard KΛCDM universe, we take a parametric form for the primordial power spectrum given by

$$\mathcal{P}_{\mathcal{R}}^{K\Lambda CDM}(k) = A_s\left(\frac{k}{k_*}\right)^{n_s-1}. \tag{69}$$

The resulting power spectrum for $\mathcal{R}$ using minimized-RSET initial conditions is given in Fig. 2.

We can compare this with a naive application of the flat-case minimized-RSET conditions, as given in Refs. [23,26,57], as well as with the Bunch-Davies vacuum conditions [10,26]. This is shown in Fig. 3, where we see a clear increase in power spectrum oscillations when using the new minimized-RSET conditions derived in this paper, thus hopefully corresponding to better detectability. We also highlight that the differences between vacuum choices is particularly pronounced for larger values of primordial curvature and for lower k modes (where the presence curvature is more relevant), as expected.

VII. CONCLUSION

We began by introducing the relevant setup, highlighting work by Ref. [26] in deriving the $K \neq 0$ Mukhanov-Sasaki equation of motion for the gauge-invariant comoving curvature perturbation $\mathcal{R}$. We discussed why this equation of motion prevents setting initial conditions for $\mathcal{R}$ in a well-defined and physically motivated way, and motivated the method of minimized RSET.

We then propose a novel perturbation scalar ζ, inspired by the case of a perfect-fluid filled universe as in Ref. [66],

and with particular emphasis on Eq. (37). Under the Bardeen equation of motion (39), describing the evolution of perturbation scalars in our inflationary universe, the equation of motion for ζ collapses to the desired canonically normalizable wave equation (51) [i.e., of the same form as the typically considered equation of motion (11) for $\mathcal{R}$ when $K = 0$]. We believe this variable will be an important catalyst for further work on the subject of inflation in curved spacetimes, as its SHO form allows for connection to standard inflationary and quantum field theory literature.

Building on calculations in Ref. [38], we generalize the minimized-RSET vacuum-setting procedure proposed in Ref. [57] to curved spacetimes, allowing us to quantize ζ (and any variable with an analogous equation of motion). Finally, we demonstrate how to use the family of ζ variables to construct a well-motivated set of initial conditions for $\mathcal{R}$, allowing us to plot the resulting power spectrum. We see changes in $\triangle\chi^2$ in all cases compared to previously studied BD and flat-RSET conditions, particularly noting an increase in power spectrum oscillatory behavior.

This work, however, leaves several questions open. First, what is the correct choice of variable for quantization? Ideally there would be a unique, well-motivated theoretical choice, though of course given its impact on the primordial power spectrum, the single degree of freedom in choice of ζ could also be fit for. Second, the link between curvature and nonadiabaticity given by Eqs. (34), (35), and (38) should be further analytically explored in the directions of explaining the nonlocality of action Eq. (19) and identifying the degree of uniqueness of the variable ζ. Finally, it remains to be seen to what extent (if at all) these vacuum states can be constrained using modern cosmological data.

APPENDIX A: GENERAL MINIMIZED RSET CALCULATIONS FOR WAVELIKE FIELDS

In this appendix, we aim to relate the equations of motion for ζ to a free scalar field ϕ in the same background, so that we can set initial conditions for ζ through initial conditions for ϕ by RSET. The procedure is invariant to the transformations we use to map ζ equation of motion (EOM) to ϕ EOM, as we will see later in this calculation. Thus, we aim to relate

$$(z_g\zeta_k)'' + \left(c_s^2(\eta)\kappa_{\mathcal{D}}^2(k) - \frac{z_g(\eta)''}{z_g(\eta)}\right)(z_g\zeta_k) = 0 \qquad \text{(A1)}$$

to

$$(a\psi_k)'' + \left(\kappa^2(k) - \frac{a''}{a}\right)(a\psi_k) = 0, \qquad \text{(A2)}$$

by means of time redefinition and field rescaling defined in Eqs. (60)–(63).

Time redefinition from η to η_ζ gives

$$\frac{\mathrm{d}}{\mathrm{d}\eta} = \eta'_\zeta \frac{\mathrm{d}}{\mathrm{d}\eta_\zeta}, \qquad \text{(A3)}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\eta^2} = (\eta'_\zeta)^2 \frac{\mathrm{d}^2}{\mathrm{d}\eta_\zeta^2} + \eta''_\zeta \frac{\mathrm{d}}{\mathrm{d}\eta_\zeta}, \qquad \text{(A4)}$$

where $'$ denotes differentiation with respect to conformal time η. Then Eq. (A1) becomes the equivalent equation of motion for χ_ζ given by

$$0 = \partial_{\eta_\psi\eta_\psi}\chi_\zeta + \partial_{\eta_\zeta}\chi_\zeta\left(\frac{\eta''_\zeta}{(\eta'_\zeta)^2} + 2\frac{\partial_{\eta_\zeta}h_\zeta}{h_\zeta} + 2\frac{\partial_{\eta_\zeta}z_g}{z_g}\right)$$
$$+ \chi_\zeta\left(\frac{\partial_{\eta_\zeta}^2 h_\zeta}{h_\zeta} + \frac{\eta''_\zeta}{(\eta'_\zeta)^2}\frac{\partial_{\eta_\zeta}h_\zeta}{h_\zeta} + 2\frac{\partial_{\eta_\zeta}h_\zeta}{h_\zeta}\frac{\partial_{\eta_\zeta}z_g}{z_g} + \frac{c_s^2\kappa_{\mathcal{D}}^2}{(\eta'_\zeta)^2}\right). \quad \text{(A5)}$$

We aim to have an equation of motion with no $\partial_{\eta_\zeta}\chi_\zeta$ terms; i.e., we want to choose h_ζ, η_ζ such that

$$0 = \frac{\eta''_\zeta}{(\eta'_\zeta)^2} + 2\frac{\partial_{\eta_\zeta}h_\zeta}{h_\zeta} + 2\frac{\partial_{\eta_\zeta}z_g}{z_g}$$
$$= \frac{\partial_{\eta_\zeta}(C_\zeta^2)}{C_\zeta^2} \qquad \text{(A6)}$$

for

$$C_\zeta^2 = \eta'_\zeta h_\zeta^2 z_g^2. \qquad \text{(A7)}$$

Thus, we fix C_ζ^2 to be constant. Then

$$0 = \partial_{\eta_\zeta\eta_\zeta}\chi_\zeta + \chi_\zeta\left(\left(\frac{c_s(\eta_\zeta)\kappa_{\mathcal{D}}(k)h_\zeta^2 z_g^2}{C_\zeta^2}\right)^2 + \frac{\partial_{\eta_\zeta\eta_\zeta}h_\zeta}{h_\zeta}\right.$$
$$\left. - 2\left(\frac{\partial_{\eta_\zeta}h_\zeta}{h_\zeta}\right)^2\right). \qquad \text{(A8)}$$

The equivalent formulation for the rescaled and time-redefined ψ field is

$$0 = \partial_{\eta_\psi\eta_\psi}\chi_\psi + \chi_\psi\left(\left(\frac{\kappa(k)h_\psi^2 a^2}{C_\psi^2}\right)^2 + \frac{\partial_{\eta_\psi\eta_\psi}h_\psi}{h_\psi} - 2\left(\frac{\partial_{\eta_\psi}h_\psi}{h_\psi}\right)^2\right) \qquad \text{(A9)}$$

for constant

$$C_\psi^2 = \eta'_\psi h_\psi^2 a^2. \qquad \text{(A10)}$$

As argued in Ref. [38],

$$\frac{\partial_{\eta_\psi\eta_\psi}h_\psi}{h_\psi} - 2\left(\frac{\partial_{\eta_\psi}h_\psi}{h_\psi}\right)^2 = \frac{\partial_{\eta_\zeta\eta_\zeta}h_\zeta}{h_\zeta} - 2\left(\frac{\partial_{\eta_\zeta}h_\zeta}{h_\zeta}\right)^2 \qquad \text{(A11)}$$

can always be made to be true for suitable choices of h_ζ, η_ζ, h_ψ, η_ψ.

Hence the only remaining condition to be fixed is

$$\frac{\sqrt{c_s}h_\zeta z_g}{C_\zeta} = \frac{h_\psi a}{C_\psi}, \qquad \text{(A12)}$$

having shifted the wave number for ψ so that

$$\kappa_{\mathcal{D}}(k_{\text{shift}}) = \kappa(k). \qquad \text{(A13)}$$

Now promote χ_ψ to an operator, as is standard in canonical quantization; then ψ may be expanded in terms of creation and annihilation operators as

$$\psi = \int \frac{^3k}{(2\pi)^3} h_\psi(\eta_\psi)(\hat{a}_k\chi_{\psi,k}Y_k(\mathbf{x}) + \hat{a}_k^\dagger\chi_{\psi,k}^* Y_k^*(\mathbf{x})), \quad \text{(A14)}$$

where $Y_k(x)$ is the eigenfunction of the curved spatial Laplacian, namely

$$\nabla^2 Y_k(\mathbf{x}) = -\kappa^2(k)Y_k(\mathbf{x}). \qquad \text{(A15)}$$

The renormalized T_{00} component is then given by

$$\langle 0|T_{00}|\rangle_{\text{ren}} = \lim_{\mathbf{x}\to\mathbf{x}'}\mathcal{D}_{00}G(\mathbf{x}, \mathbf{x}') - \tilde{T}, \qquad \text{(A16)}$$

where

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{2}\langle 0|\psi(\mathbf{x})\psi(\mathbf{x}') + \psi(\mathbf{x}')\psi(\mathbf{x})|0\rangle. \qquad \text{(A17)}$$

Crucially, the de-Witt Schwinger geometrical terms given by $\tilde{T}$ are independent of the field variables χ_ψ (or ψ), shown in Birrell and Davies [62] (Chap. 6, Sec. VI.4). Thus, we expand

$$2G(\mathbf{x}, \mathbf{x}') = \int \frac{^3k}{(2\pi)^3} h(\eta_\psi)h(\eta'_\psi)\chi(\eta_\psi)\chi^*(\eta'_\psi)Y(\mathbf{x})Y^*(\mathbf{x}')$$
$$+ h(\eta_\psi)h(\eta'_\psi)\chi(\eta'_\psi)\chi^*(\eta_\psi)Y(\mathbf{x}')Y^*(\mathbf{x})), \qquad \text{(A18)}$$

where for brevity $h = h_\psi$, $\chi = \chi_{\psi,k}$, and $Y = Y_k$. Then

$$\langle 0|T_{00}|\rangle_{\mathrm{ren}} = \tilde{T} + \frac{1}{2}\int \frac{^3k}{(2\pi)^3}\frac{h^2\kappa^2}{(\partial_\eta\eta_\psi)^2}\chi\chi^*YY^*$$
$$+ h^2\left(\partial_{\eta_\psi}\chi + \frac{\partial_{\eta_\psi}h}{h}\chi\right)\left(\partial_{\eta_\psi}\chi^* + \frac{\partial_{\eta_\psi}h}{h}\chi^*\right)YY^*. \tag{A19}$$

The spherical-harmonic-like functions $Y_k(x)$ are normalized as

$$\int ^3x\sqrt{\det(c_{ij})}Y_k(x)Y_p(x) = \delta(k-p) \tag{A20}$$

so that the canonical commutation relation for the redefined field is given by

$$(\partial_{\eta_\psi}\chi_{\psi,k})\chi^*_{\psi,k} - (\partial_{\eta_\psi}\chi^*_{\psi,k})\chi_{\psi,k} = -\frac{i}{C^2_\psi}. \tag{A21}$$

Finally, minimizing Eq. (A19) with respect to field variables $\chi_{\psi,k}, \chi^*_{\psi,k}$ and their derivatives $\partial_{\eta_\psi}\chi_{\psi,k}, \partial_{\eta_\psi}\chi^*_{\psi,k}$ subject to the constraint Eq. (A21) gives

$$\partial_{\eta_\psi}\ln(\chi_{\psi,k}) = -\frac{i\kappa(k)}{\eta'_\psi} - \partial_{\eta_\psi}\ln(h_\psi), \tag{A22}$$

$$|\chi^2_{\psi,k}| = \frac{1}{2\kappa h^2_\psi a^2}. \tag{A23}$$

The equation of motion for χ_ζ in terms of η_ζ at $k_{\mathrm{shift}}(k)$ is now identical to the equation of motion for χ_ψ in terms of η_ψ at k. Thus, we have equivalence of solutions

$$\chi_\psi(k) = \chi_\zeta(k_{\mathrm{shift}}). \tag{A24}$$

Substituting $\chi_\psi(k) = \chi_\zeta(k_{\mathrm{shift}})$, using h_ψ for h_ζ, and converting to conformal time gives

$$\frac{\zeta'}{\zeta} = -i\kappa_{\mathcal{D}} + \frac{a'}{a} - \frac{z'_g}{z_g} - \frac{1}{2}\frac{c'_s}{c_s}. \tag{A25}$$

Introducing 2 arbitrary time redefinitions and 2 arbitrary field rescalings gives 4 degrees of freedom. Thus far, we still have one remaining degree of freedom: Eqs. (A11) and (A12) each fix 1 degree of freedom, and Eqs. (A7) and (A10) *together* fix another degree of freedom (as the *values* of C_ζ, C_ψ are arbitrary). Thus, we are free to set $C_\zeta = C_\psi$. In conclusion, we can initialize ζ at $\eta = \eta_0$ by

$$|\zeta(\eta_0)|^2 = \frac{1}{2c_s(\eta_0)z^2_g(\eta_0)\kappa_{\mathcal{D}}},$$
$$\frac{\zeta'}{\zeta}(\eta_0) = -i\kappa_{\mathcal{D}} + \frac{a'}{a}(\eta_0) - \frac{z'_g}{z_g}(\eta_0) - \frac{1}{2}\frac{c'_s}{c_s}(\eta_0), \tag{A26}$$

or at $t = t_0$ by

$$|\zeta(t_0)|^2 = \frac{1}{2c_s(t_0)z^2_g(t_0)\kappa_{\mathcal{D}}},$$
$$\frac{\dot{\zeta}}{\zeta}(t_0) = -i\frac{1}{a(t_0)}\kappa_{\mathcal{D}} + \frac{\dot{a}}{a}(t_0) - \frac{\dot{z}_g}{z_g}(t_0) - \frac{1}{2}\frac{\dot{c}_s}{c_s}(t_0). \tag{A27}$$

The forms of these equations do not depend on the canonical transformations used to relate ζ to ϕ, i.e., $\eta_\zeta, \eta_\phi, h_\zeta, h_\phi$, which proves that initial conditions set by RSET are invariant under such transformations even in the $K \neq 0$ case.

APPENDIX B: MOTIVATION AND CONSTRUCTION OF ζ

The starting point of this section is Eq. (39). For the rest of this section we will denote with capital letters functions of η and indicate x (or k) dependence by subscripts. With this in mind, we write Eq. (39) as

$$\Phi'' + A_1\Phi' + B_1\Phi - D_k\Phi = 0. \tag{B1}$$

Then, we would like to construct a variable

$$\zeta = A\Phi + B\Phi', \tag{B2}$$

for which we will require the following:
(i) A and B are not to have any x (or k) dependence.
(ii) The resulting equation for zeta is to be of the form

$$\zeta'' + M\zeta' + D_{2;k}\zeta = 0, \tag{B3}$$

where, for the time being, we do not require $D_{2;k}$ to be the same as D_k.
Based on these requirements we can write

$$\zeta = A\Phi + B\Phi', \tag{B4}$$

$$\zeta' = C\Phi + P\Phi' - BD_k\Phi, \tag{B5}$$

$$\zeta'' = N\Phi + L\Phi' - PD_k\Phi - B'D_k\Phi - BD'_k\Phi - BD_k\Phi', \tag{B6}$$

where we have defined

$$C = A' - BB_1, \tag{B7}$$

$$P = A + B' - BA_1, \tag{B8}$$

$$N = C' - PB_1, \qquad (B9)$$

$$L = C + P' - PA_1. \qquad (B10)$$

Substituting this into Eq. (B3), then matching coefficients of Φ, Φ', and separately the coefficients of the k-dependent components of Φ, Φ', we have

$$N = -MC, \qquad (B11)$$

$$L = -MP, \qquad (B12)$$

$$BD_{2;k} = BD_k, \qquad (B13)$$

$$0 = -PD - B'D - BD' + AD_{2;k} - MBD. \qquad (B14)$$

Therefore, we must have that $D_{2;k}$ is the same as D_k. However, we had a degree of freedom in choosing D_k when writing down Eq. (B1). But from Eq. (B14) we have

$$A - MB - P - B' = B\frac{D_k'}{D_k}, \qquad (B15)$$

meaning that we must have D_k being a product of a function of η by a function of k. For example, when writing down Eq. (B1), we could pick $D_k = c_s^2 \mathcal{D}^2$ or $c_s^2 \nabla^2$, but not $D_k = c_s^2 \nabla^2 + K$. The above asserts $D_{2;k}$ to be the same as D_k; therefore, all that is needed to find the form of the wave equation is identifying M while satisfying Eqs. (B11), (B12), and (B14). Let $M = \frac{2(z\delta)'}{z\delta}$, i.e., a mass rescaling for our new variable. Then, from Eqs. (B8) and (B14), we have

$$\frac{2B'}{B} = -M + A_1 - \frac{D_k'}{D_k}. \qquad (B16)$$

If we pick Eq. (40) as the general form for our equation, i.e., swap from A and B to g and f, we must have that $\delta = \frac{1}{g}$. We can further simplify Eqs. (B11) and (B12) to be the following two equations that are linear in C and P:

$$\left(\frac{Cz^2}{g^2}\right)' = -\frac{Pz^2}{g^2}\frac{(\rho + p)'}{\rho + p}, \qquad (B17)$$

$$\left(\frac{Pz^2(\rho + p)}{g^2}\right)' = \frac{Cz^2(\rho + p)}{g^2}, \qquad (B18)$$

where for the specific choice of $D_k = c_s^2 \mathcal{D}^2$

$$P = \frac{K}{\mathcal{H}} - \frac{f}{g}\frac{K}{\mathcal{H}} + \frac{g'}{g}, \qquad (B19)$$

$$C = K + \frac{g'}{g}\left(\mathcal{H} + \frac{a^2(\rho + p)}{2\mathcal{H}}\right)$$
$$+ \frac{K}{\mathcal{H}}\frac{f}{g}\left(2\mathcal{H} + \frac{(\rho + p)'}{(\rho + p)} - \frac{f'}{f}\right). \qquad (B20)$$

Solving these equations in general is rather difficult, since they are two coupled, nonlinear, second order differential equations for f and g. We will aim to simplify our calculations by making an ansatz. If we are to pick f and g, such that $P = C = 0$, then Eqs. (B17) and (B18) are trivially satisfied. This gives us exactly two restrictions on our functions f and g:

$$f = g'\frac{\mathcal{H}}{K} + g, \qquad (B21)$$

$$\frac{f'}{f} = K + \frac{g'}{g}\left(\mathcal{H} + \frac{a^2(\rho + p)}{2\mathcal{H}}\right) + \frac{K}{\mathcal{H}}\frac{f}{g}\left(2\mathcal{H} + \frac{(\rho + p)'}{(\rho + p)}\right). \qquad (B22)$$

Thus, if we can find solutions to Eqs. (B21) and (B22), then the equation collapses to the required wave equation (51). Setting

$$g(\eta) = \frac{a}{\mathcal{H}}\frac{b'}{b^2} \qquad (B23)$$

reduces Eq. (B22) to

$$b\left(\frac{1}{b}\right)'' = a\left(\frac{1}{a}\right)'' + K, \qquad (B24)$$

which gives a one-parameter family of solutions for ζ, since one of the free parameters in this second order differential equation is due to arbitrary scaling of b (or equivalently g). We can see from this that this ansatz is sufficient for a SHO equation for ζ, but it remains an open question whether it is necessary—i.e., whether $P = C = 0$ are the only solutions to Eqs. (B17) and (B18).

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