


Alternative to perturbative renormalization in (3 + 1)-dimensional field theories

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 (Received 9 April 2024; accepted 23 May 2024; published 20 June 2024)

Perturbative renormalization provides the bedrock of understanding quantum field theories. In this work, I point out an alternative way of renormalizing quantum field theories, which is naturally encountered and well-known for the case of large N scalar field theories. In terms of bare parameters, this nonperturbative alternative renormalization differs qualitatively from its perturbative cousin: in the continuum limit, the bare coupling constant goes to zero instead of infinity, and there is no wave-function counterterm. Despite these differences, the resulting n -point functions of the theory are finite. I provide explicit results for alternative renormalization for the $O(N)$ model and QCD with $N_f = 12$ flavors in $3 + 1$ dimensions.

DOI: [10.1103/PhysRevD.109.116020](https://doi.org/10.1103/PhysRevD.109.116020)

I. INTRODUCTION

Renormalization of scalar field theory has been a topic of interest in high energy physics since the dawn of quantum field theory. Regularization and perturbative renormalization for scalar fields is by now standard and can be found in many variations in various textbooks [1–3].

For scalar field theory in four dimensions, the perturbative renormalization program starts out the usual way, by first regularizing formally divergent integrals, and then introducing vacuum energy, mass, coupling constant, and wave-function counterterms order-by-order in a weak-coupling expansion to cancel the divergencies.

In contrast to perturbative approaches, large N expansions of N -component field theories resum an infinite number of perturbative contributions (Feynman diagrams) at every order in $\frac{1}{N}$. This is advantageous in cases where nonperturbative information about the theory is needed, and it has successful applications to different physical systems; see, e.g., Ref. [4] for a modern review.

However, because large N is not a weak-coupling framework, knowledge of perturbative renormalization of the theory is of little help when aiming for a calculation of renormalized observables at any given order in $\frac{1}{N}$. In practice, one needs to set up nonperturbative renormalization conditions that do not correspond to any finite-order perturbative truncation. At leading order in large N ,

nonperturbative renormalization is relatively straightforward, and in fact has been used since the 1970s [5,6]. Curiously, Parisi pointed out in a seminal paper that large N field theories may be nonperturbatively renormalized order-by-order in a large N expansion even in cases that are not perturbatively renormalizable [7], such as scalar field theory in spacetime dimensions $d < 6$.

For the case of scalar field theory in $d = 4$, the question of nonperturbative renormalization has been encountered in so-called Φ -derivable and two particle-irreducible effective theories [8–11], culminating in the proof of nonperturbative renormalizability in the massless case [12–14].

Curiously, there seems to be little discussion about the qualitative differences between nonperturbative and perturbative renormalization in the literature. Specifically for scalar field theory, perturbative renormalization requires adding the counterterms [2]

$$\begin{aligned}\delta\mathcal{L}_V &= \chi_V, & \delta\mathcal{L}_\lambda &= \chi_\lambda\phi^4, \\ \delta\mathcal{L}_m &= \chi_m\phi^2, & \delta\mathcal{L}_\phi &= \chi_\phi\partial_\mu\phi\partial_\mu\phi,\end{aligned}\quad (1)$$

to the theory Lagrangian, with the coefficients $\chi_V, \chi_\lambda, \chi_m, \chi_\phi$ all diverging in the continuum limit of the theory. By contrast, as will be reviewed below, nonperturbative renormalization of large N scalar field theory requires adding the counterterms [15,16]

$$\delta\mathcal{L}_V = c_V, \quad \delta\mathcal{L}_\lambda = c_\lambda\phi^4, \quad \delta\mathcal{L}_m = c_m\phi^2, \quad (2)$$

where in particular $c_\lambda \rightarrow 0$ in the continuum limit of the theory.

One is faced with the situation that while perturbative renormalization requires wave-function renormalization,

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apparently nonperturbative renormalization does not. In addition, while the coupling constant counterterm in perturbative renormalization diverges in the continuum limit, the corresponding counterterm in nonperturbative renormalization vanishes in the same limit. Yet both renormalization procedures render the theory finite.

I find this difference between perturbative and non-perturbative renormalization striking, and took it as an opportunity to review nonperturbative renormalization in scalar field theory and QCD in this work.

II. SCALAR FIELD THEORY

The theory is defined by the partition function

$$Z = \int \mathcal{D}\vec{\phi} e^{-S_E}, \quad (3)$$

where S_E is the Euclidean action of the theory

$$S_E = \int d^4x \left[\frac{1}{2} \partial_\mu \vec{\phi} \partial_\mu \vec{\phi} + \frac{1}{2} m_B^2 \vec{\phi}^2 + \frac{\lambda_B}{N} (\vec{\phi}^2)^2 \right]. \quad (4)$$

Here $\vec{\phi} = (\phi_1, \phi_2, \dots, \phi_N)$ is a N -component scalar field, and m_B and λ_B are the bare mass and coupling parameter of the theory. It is useful to rewrite the partition function in terms of an auxiliary field ζ , which is introduced through the mathematical identity

$$e^{-\int_x \frac{\lambda_B}{N} (\vec{\phi}^2)^2} = \int \mathcal{D}\zeta e^{-\int_x \left[\frac{i\zeta}{2} \vec{\phi}^2 + \frac{\zeta^2 N}{16\lambda_B} \right]}. \quad (5)$$

It is furthermore convenient to split the auxiliary field into a global zero mode ζ_0 and fluctuations around it

$$\zeta(x) = \zeta_0 + \xi(x), \quad (6)$$

such that the partition function becomes

$$Z = \int d\zeta_0 \mathcal{D}\vec{\phi} \mathcal{D}\xi e^{-S_{R0} - S'}, \quad (7)$$

with

$$S_{R0} = \int d^4x \left[\frac{1}{2} \vec{\phi} [-\square + m_B^2 + i\zeta_0] \vec{\phi} + \frac{N}{16\lambda_B} \zeta_0^2 \right],$$

$$S' = \int d^4x \left[\frac{i\xi \vec{\phi}^2}{2} + \frac{N\xi^2}{16\lambda_B} \right]. \quad (8)$$

(Note that the ‘‘Rn’’ nomenclature is in one-to-one correspondence to an expansion in powers of N^{-1} ; cf. Ref. [4].)

A. Leading order large N —level $R0$

So far everything has been exact. Owing to the three vertex $\xi\phi^2$ in S' , the path integral for the partition function cannot be done in closed form. In the limit of many scalar components $N \gg 1$, the theory is amenable to closed-form solutions in an expansion in $\frac{1}{N}$.

To leading order in large N , the action S' does not contribute to the partition function, and one has

$$Z_{R0} = \int d\zeta_0 e^{-\frac{N}{2} \text{Tr} \ln [-\square + m_B^2 + i\zeta_0] - \text{vol} \times \frac{N\zeta_0^2}{16\lambda_B}}, \quad (9)$$

where the Gaussian integral over ϕ was performed and ‘‘vol’’ denotes the spacetime volume. The functional trace over the logarithm of the operator is UV divergent and needs to be regulated. For the present work, I choose to use dimensional regularization, because it is the standard regularization scheme used by the Particle Data Group [17], and because it is elegant and I like elegance. In dimensional regularization, the spacetime dimension is taken to be

$$d = 4 - 2\varepsilon, \quad (10)$$

with the parameter $\varepsilon > 0$ sent to zero in the end. Regularization is provided by the analytic continuation of the Γ function, in particular for integrals of the type

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + m^2)^\alpha} = \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(\alpha - \frac{d}{2})}{\Gamma(\alpha)} (m^2)^{\frac{d}{2} - \alpha}. \quad (11)$$

For the $R0$ level partition function, one needs the integral

$$\begin{aligned} & \frac{1}{2} \text{Tr} \ln [-\square + m_B^2 + i\zeta_0] \\ &= \text{vol} \times \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \ln [k^2 + m_B^2 + i\zeta_0] \\ &= \text{vol} \times \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(1 - \frac{d}{2})}{d} (m_B^2 + i\zeta_0)^{\frac{d}{2}}, \end{aligned} \quad (12)$$

which follows from direct integration of (11) for $\alpha = 1$ with respect to m^2 . For $\varepsilon \rightarrow 0$, one therefore finds for the partition function

$$Z_{R0} = \int d\zeta_0 e^{-N \times \text{vol} \times \mathcal{F}(\zeta_0)}, \quad (13)$$

with

$$\mathcal{F}(\zeta_0) = \frac{\zeta_0^2}{16\lambda_B} - \frac{(m_B^2 + i\zeta_0)^2}{64\pi^2} \left[\frac{1}{\varepsilon} + \ln \frac{\bar{\mu}^2 e^{\frac{3}{2}}}{m_B^2 + i\zeta_0} \right], \quad (14)$$

where $\bar{\mu}$ is the renormalization scale parameter in the $\overline{\text{MS}}$ scheme. It is useful to rewrite the expression as

$$\mathcal{F}(\zeta_0) = -\frac{(m_B^2 + i\zeta_0)^2}{64\pi^2} \left[\frac{4\pi^2}{\lambda_B} + \frac{1}{\varepsilon} + \ln \frac{\bar{\mu}^2 e^{\frac{3}{2}}}{m_B^2 + i\zeta_0} \right] + \frac{m_B^2 + i\zeta_0}{32\pi^2} \frac{4\pi^2 m_B^2}{\lambda_B} - \frac{m_B^4}{16\lambda_B}. \quad (15)$$

B. Nonperturbative renormalization

The theory is renormalized if all n -point functions of the theory are finite. Calculating the partition function corresponds to a zero-point function, which means that after integrating over ζ_0 , Z_{R0} needs to be finite. In the large N limit, the remaining integral over ζ_0 can be performed exactly using the saddle-point method, finding

$$Z_{R0} = e^{-N \times \text{vol} \times \mathcal{F}(\bar{\zeta})}, \quad (16)$$

with $\mathcal{F}(\zeta_0)$ given by (15) and $\bar{\zeta}$ the value of the saddle point. (Note that I am assuming that there is only one dominant saddle point, which will be verified below.)

The two-point function for the $R0$ -level theory is given by

$$\begin{aligned} \langle \phi_i(x) \phi_j(0) \rangle &= Z_{R0}^{-1} \int d\zeta_0 \mathcal{D}\vec{\phi} e^{-S_{R0}} \phi_i(x) \phi_j(0) \\ &= Z_{R0}^{-1} \int d\zeta_0 e^{-N \times \text{vol} \times \mathcal{F}(\zeta_0)} \frac{\delta_{ij}}{-\square + m_B^2 + i\zeta_0}. \end{aligned} \quad (17)$$

This expression involves the two-point function

$$\Delta(x) \equiv \langle x | \frac{1}{-\square + m_B^2 + i\zeta_0} | 0 \rangle, \quad (18)$$

which is akin to a standard bosonic propagator except for the fact that its ‘‘mass term’’ $m_B^2 + i\zeta_0$ is still being integrated over. In the large N limit, the integral can again be performed exactly, and one finds

$$\langle \phi_i(x) \phi_j(0) \rangle = \int \frac{d^d k}{(2\pi)^d} \frac{e^{ik \cdot x}}{k^2 + m_B^2 + i\bar{\zeta}}, \quad (19)$$

where $\bar{\zeta}$ is again the saddle point discussed above. As a consequence, the two-point function of the theory is finite in the large N limit as long as the vector pole mass squared

$$m_{R0}^2 = m_B^2 + i\bar{\zeta} \quad (20)$$

is finite.

The connected four-point function for the $R0$ -level theory is given by

$$\begin{aligned} \langle \phi_a \phi_b \phi_c \phi_d \rangle_{\text{conn}} &= Z_{R0}^{-1} \int d\zeta_0 \mathcal{D}\vec{\phi} \mathcal{D}\xi e^{-S_{R0} - S'} \phi_a \phi_b \phi_c \phi_d \\ &= \mathcal{O}\left(\frac{1}{N}\right), \end{aligned} \quad (21)$$

to leading order in the large N limit. This is simply a reflection of the fact that non-Gaussian terms do not contribute to the $R0$ -level action to leading order in large N , and hence, the four-point function (as well as all higher-order connected n -point functions) are $\frac{1}{N}$ suppressed.

Therefore, to leading order in large N , the theory is finite if the zero-point and two-point functions are finite.

Let us study this explicitly now. To renormalize the theory, one can add counterterms to the original action. For the zero-point function, after integration over ζ_0 , the requirement that $\mathcal{F}(\bar{\zeta})$ is finite together with the requirement from the two-point function that m_{R0}^2 must be finite implies that the coefficients of the terms m_{R0}^4 , m_{R0}^2 , m_{R0}^0 must be finite. This leads to three renormalization conditions:

$$\mathcal{O}(m_{R0}^4) = \text{finite: coupling renormalization}, \quad (22)$$

$$\mathcal{O}(m_{R0}^2) = \text{finite: mass renormalization}, \quad (23)$$

$$\mathcal{O}(m_{R0}^0) = \text{finite: vacuum renormalization}, \quad (24)$$

specifying the form of the coupling-constant, mass, and vacuum energy counterterms.

It is paramount to point out that this does not leave any room for a wave-function renormalization counterterm. Hence, I conclude that wave-function renormalization must be absent when nonperturbatively renormalizing large N theories.

In particular, the counterterm c_V can be used to cancel a potentially divergent vacuum constant in (15). In the following I choose c_V to fully cancel the term $-\frac{m_B^4}{16\lambda_B}$ in (15). For the coefficient of m_{R0}^4 , one finds that it requires adding a counterterm to the *inverse coupling constant* such that

$$\frac{1}{\lambda_B} = \frac{1}{\lambda_R(\bar{\mu})} + c_\lambda. \quad (25)$$

Specifically, I take

$$\frac{1}{\lambda_B} + \frac{1}{4\pi^2 \varepsilon} = \frac{1}{\lambda_R(\bar{\mu})}, \quad (26)$$

which renders the $\mathcal{O}(m_{R0}^4)$ coefficient in (15) finite. Finally, the $\mathcal{O}(m_{R0}^2)$ coefficient is finite provided that the combination

$$\frac{m_B^2}{\lambda_B} = \frac{m_R^2(\bar{\mu})}{\lambda_R(\bar{\mu})} \equiv \frac{\alpha_{R0} \Lambda_{\overline{\text{MS}}}^2}{4\pi^2} \quad (27)$$

is finite, which can be accomplished by adding a suitable counterterm c_m to the mass parameter.

Here $m_R(\bar{\mu})$ and $\lambda_R(\bar{\mu})$ are the (running) renormalized mass and coupling parameter of the theory. From the above conditions, the explicit form of the running parameters is given by

$$\lambda_R(\bar{\mu}) = \frac{4\pi^2}{\ln \frac{\Lambda_{\overline{\text{MS}}}^2}{\bar{\mu}^2}}, \quad m_R^2(\bar{\mu}) = \frac{\alpha_{R0} \Lambda_{\overline{\text{MS}}}^2}{4\pi^2} \lambda_R(\bar{\mu}) = \frac{\alpha_{R0} \Lambda_{\overline{\text{MS}}}^2}{\ln \frac{\Lambda_{\overline{\text{MS}}}^2}{\bar{\mu}^2}}. \quad (28)$$

The energy scale $\Lambda_{\overline{\text{MS}}}$ is an emergent parameter of the theory, corresponding to the $\overline{\text{MS}}$ parameter mentioned in the particle data group [17]. Here it is identical to the so-called Landau pole of the theory. Note that the running coupling approaches zero from below for $\bar{\mu} \gg \Lambda_{\overline{\text{MS}}}$, consistent with the discussions in Refs. [16,18].

As advertised, the counterterms c_V, c_m, c_λ are sufficient to render all connected n -point functions of the theory finite in the large N limit. While this finding is trivial to this order in the $\frac{1}{N}$ expansion, I feel compelled to point out the following facts:

- (i) The nonperturbative running coupling $\lambda_R(\bar{\mu})$ in (28) matches the large N part of the one-loop perturbative running coupling (cf. Ref. [19] [Eq. (3.1.56)])

$$\lambda_R^{\text{pert}}(\bar{\mu}) = \frac{4\pi^2}{(1 + \frac{8}{N}) \ln \frac{\Lambda_{\overline{\text{MS}}}^2}{\bar{\mu}^2}}. \quad (29)$$

- (ii) Sending the regulator $\varepsilon \rightarrow 0$, the bare coupling constant (26) in nonperturbative renormalization obeys

$$\lim_{\varepsilon \rightarrow 0} \lambda_B = \lim_{\varepsilon \rightarrow 0} \frac{1}{\frac{1}{\lambda_R(\bar{\mu})} - \frac{1}{4\pi^2 \varepsilon}} \rightarrow 0^-, \quad (30)$$

meaning it approaches zero for any $\lambda_R(\bar{\mu}) \neq 0$. By contrast, in perturbation theory one *first* expands λ_B for $\lambda_R(\bar{\mu}) \sim 0$ and *then* lets $\varepsilon \rightarrow 0$, leading to

$$\lim_{\varepsilon \rightarrow 0} \lambda_B^{\text{pert}} = \lim_{\varepsilon \rightarrow 0} \lambda_R(\bar{\mu}) + \frac{\lambda_R^2(\bar{\mu})}{4\pi^2 \varepsilon} + \mathcal{O}(\lambda_R^3) \rightarrow \infty. \quad (31)$$

- (iii) Since nonperturbative renormalization is not linked to a weak-coupling expansion, physical observables are manifestly independent from the (fictitious) renormalization scale $\bar{\mu}$ order-by-order in $\frac{1}{N}$, as will be shown below.

Let us investigate this last point in more detail by calculating some physical observables from the renormalized large N theory. Inserting the nonperturbative renormalization conditions and explicit forms for λ_R, m_R into the expression for $\mathcal{F}(\zeta_0)$, one finds

$$\mathcal{F}(\zeta_0) = -\frac{m_{R0}^4}{64\pi^2} \ln \frac{\Lambda_{\overline{\text{MS}}}^2 e^{\frac{3}{2}}}{m_{R0}^2} + \alpha_{R0} \Lambda_{\overline{\text{MS}}}^2 \frac{m_{R0}^2}{32\pi^2}, \quad (32)$$

where I recall that c_V has been chosen to fully cancel the constant in (15). At large N , the partition function is then determined through its saddle points, and one finds

$$\frac{\ln Z_{R0}}{\text{vol}} = -N\mathcal{F}(\zeta_0 = \bar{\zeta}) = -N\bar{\mathcal{F}}, \quad (33)$$

where the saddle-point condition is

$$m_{R0}^2 \ln \frac{\Lambda_{\overline{\text{MS}}}^2 e^1}{m_{R0}^2} = \alpha_{R0} \Lambda_{\overline{\text{MS}}}^2. \quad (34)$$

Depending on the numerical value of α_{R0} , there are different solutions to this equation in terms of

$$\bar{m}_{R0}^2 \equiv m_B^2 + i\bar{\zeta}. \quad (35)$$

For $\alpha_{R0} = 0$, the solutions are

$$\alpha_{R0} = 0: \quad \bar{m}_{R0}^2 = 0, \quad \bar{m}_{R0}^2 = e^1 \Lambda_{\overline{\text{MS}}}^2. \quad (36)$$

Plugging these values into the partition function, one finds that the trivial solution $\bar{m}_{R0}^2 = 0$ corresponds to the trivial vacuum with $\bar{\mathcal{F}} = 0$, whereas the second solution corresponds to the free energy density solution

$$\alpha_{R0} = 0: \quad \bar{\mathcal{F}} = -\frac{e^2 \Lambda_{\overline{\text{MS}}}^4}{128\pi^2}, \quad (37)$$

which indicates the thermodynamically preferred phase. This shows that for $\alpha_{R0} = 0$, the nontrivial saddle-point solution gives the true large N ground state of the theory.

For $0 < \alpha_{R0} < \alpha_c$, there continue to be two real solutions to the saddle-point condition (34), with the larger value for \bar{m}_{R0}^2 corresponding to the lower-free energy:

$$0 < \alpha_{R0} < \alpha_c: \quad \bar{m}_{R0}^2 = -\frac{\alpha_{R0} \Lambda_{\overline{\text{MS}}}^2}{W\left(-\frac{\alpha_{R0}}{e}\right)},$$

$$\bar{\mathcal{F}} = -\frac{\bar{m}_{R0}^4}{128\pi^2} + \frac{\alpha_{R0} \bar{m}_{R0}^2 \Lambda_{\overline{\text{MS}}}^2}{64\pi^2}, \quad (38)$$

with $W(x)$ denoting the Lambert (or product-log) function. The solution stops to be real for $\alpha_{R0} > \alpha_c$, which is located at the branch point of the Lambert function:

$$\alpha_c = 1. \quad (39)$$

For the critical value of α_c , one has

$$\alpha_{R0} = \alpha_c: \quad \bar{m}_{R0}^2 = \Lambda_{\overline{\text{MS}}}^2, \quad \bar{\mathcal{F}} = \frac{\Lambda_{\overline{\text{MS}}}^4}{128\pi^2}. \quad (40)$$

Note the different sign of the free energy for this value of α_{R0} compared to (37).

For decreasing $\alpha_{R0} < 0$, the numerical value for the dominant saddle \bar{m}_{R0}^2 (the value of the free energy \mathcal{F}) increases (decreases) monotonically.

Depending on the numerical value of α_{R0} , the vector pole mass changes from

$$\alpha_{R0} = 0: \bar{m}_{R0} = \sqrt{e}\Lambda_{\overline{\text{MS}}}, \quad \alpha_{R0} = \alpha_c: \bar{m}_{R0} = \Lambda_{\overline{\text{MS}}}; \quad (41)$$

e.g., it decreases with increasing α_{R0} . Curiously, this implies that irrespective of the mass renormalization, there is a lower limit on the vector mass:

$$\bar{m}_{R0} \geq \Lambda_{\overline{\text{MS}}}. \quad (42)$$

C. Partial next-to-leading order large N —level $R1$

To include $1/N$ corrections in the theory one needs to include additional resummations not covered by the $R0$ scheme. The simplest such refinement is provided by the so-called $R1$ resummation scheme outlined and used in Refs. [20,21]. $R1$ is defined by adding and subtracting a term $\frac{\nu^2}{2} \int d^4x \vec{\phi}^2$ to the action of the theory such that the partition function becomes

$$Z = \int d\zeta_0 \mathcal{D}\vec{\phi} \mathcal{D}\xi e^{-S_{R1} - S_{I1}}, \quad (43)$$

with

$$S_{R1} = \int d^4x \left[\frac{1}{2} \vec{\phi} [-\square + m_B^2 + i\zeta_0 + \nu^2] \vec{\phi} + \frac{N}{16\lambda_B} \zeta_0^2 \right],$$

$$S_{I1} = \int d^4x \left[\frac{i\xi \vec{\phi}^2}{2} + \frac{N\xi^2}{16\lambda_B} - \frac{\nu^2}{2} \vec{\phi}^2 \right], \quad (44)$$

where ν^2 is determined by self-consistently calculating the vector two-point function to one-loop level:

$$\langle \phi_i(x) \phi_j(0) \rangle = \langle \phi_i(x) \phi_j(0) \rangle_{R1} - \langle \phi_i(x) S_{I1} \phi_j(0) \rangle_{R1} + \frac{1}{2} \langle \phi_i(x) S_{I1}^2 \phi_j(0) \rangle_{R1}, \quad (45)$$

where $\langle \cdot \rangle_{R1}$ specify expectation values for the theory with respect to the action S_{R1} . Performing the contractions leads to the result [20]

$$\nu^2 = \frac{8\lambda_B}{N} \Delta(x=0), \quad (46)$$

in terms of the ζ_0 -dependent $R1$ vector propagator

$$\Delta(x) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik_\mu x_\mu}}{k^2 + m_B^2 + i\zeta_0 + \nu^2}. \quad (47)$$

The $R1$ partition function is likewise calculated as

$$Z_{R1} = \int d\zeta_0 e^{-N \times \text{vol} \times \mathcal{F}_{R1}(\zeta_0)}, \quad (48)$$

with the $R1$ -effective potential $\mathcal{F}_{R1}(\zeta_0)$ given by

$$\mathcal{F}_{R1}(\zeta_0) = \frac{\zeta_0^2}{16\lambda_B} + \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \ln [k^2 + m_B^2 + i\zeta_0 + \nu^2] - \frac{2\lambda_B}{N} \Delta^2(x=0), \quad (49)$$

where for the last term I combined contributions from S_I and S_I^2 using (46). Calculating the integrals as before using dimensional regularization one finds

$$\mathcal{F}_{R1}(\zeta_0) = \frac{\zeta_0^2}{16\lambda_B} - \frac{m_{R1}^4}{64\pi^2} \left(\frac{1}{\epsilon} + \ln \frac{\bar{\mu}^2 e^{\frac{3}{2}}}{m_{R1}^2} \right) - \frac{N\nu^4}{32\lambda_B}, \quad (50)$$

where I introduced the notation

$$m_{R1}^2 = m_B^2 + i\zeta_0 + \nu^2. \quad (51)$$

The $R1$ scheme can be fully nonperturbatively renormalized at any N as follows: first note that the $R1$ partition function in the large volume limit is given by

$$\frac{\ln Z_{R1}}{\text{vol}} = -N \bar{\mathcal{F}}_{R1}, \quad (52)$$

with $\bar{\mathcal{F}}_{R1} = \mathcal{F}_{R1}(\zeta_0 = \bar{\zeta})$ and $\zeta_0 = \bar{\zeta}$ the solution to the saddle-point condition

$$0 = \frac{d\mathcal{F}}{d(i\zeta_0)} = -\frac{i\zeta_0}{8\lambda_B} + \frac{1}{2} \int \frac{d^4k}{(2\pi)^d} \frac{1}{k^2 + m_{R1}^2} \left(1 + \frac{d\nu^2}{d(i\zeta_0)} \right) - \frac{N\nu^2}{16\lambda_B} \frac{d\nu^2}{d(i\zeta_0)} = -\frac{i\zeta_0}{8\lambda_B} + \frac{1}{2} \int \frac{d^4k}{(2\pi)^d} \frac{1}{k^2 + m_{R1}^2}, \quad (53)$$

where the last line is a result of (46). Comparison of this equation and (46) then immediately leads to

$$\nu^2 = \frac{2}{N} i\bar{\zeta}, \quad (54)$$

and hence,

$$\mathcal{F}_{R1}(\bar{\zeta}) = -\frac{m_{R1}^4}{16\lambda_B(1 + \frac{2}{N})} + \frac{m_{R1}^2 m_B^2}{8\lambda_B(1 + \frac{2}{N})} + \frac{m_B^4}{16\lambda_B(1 + \frac{2}{N})} - \frac{\bar{m}_{R1}^4}{64\pi^2} \left(\frac{1}{\epsilon} + \ln \frac{\bar{\mu}^2 e^{\frac{3}{2}}}{\bar{m}_{R1}^2} \right). \quad (55)$$

The same steps as for $R0$ then lead to the renormalization conditions for the coefficients of order $m_{R1}^4, m_{R1}^2, m_{R1}^0$ in the effective potential, finding

$$\frac{1}{\lambda_B} = \frac{1}{\lambda_R(\bar{\mu})} - \frac{1 + \frac{2}{N}}{4\pi^2 \varepsilon}, \quad \frac{m_B^2}{\lambda_B} = \frac{m_R^2(\bar{\mu})}{\lambda_R(\bar{\mu})} - \frac{\alpha_{R1} (1 + \frac{2}{N}) \Lambda_{\overline{\text{MS}}}^2}{4\pi^2}, \quad (56)$$

so that in particular the running coupling and mass parameter in the $R1$ scheme become

$$\lambda_R(\bar{\mu}) = \frac{4\pi^2}{(1 + \frac{2}{N}) \ln \frac{\Lambda_{\overline{\text{MS}}}^2}{\bar{\mu}^2}},$$

$$m_R^2(\bar{\mu}) = \frac{\alpha_{R1} (1 + \frac{2}{N}) \Lambda_{\overline{\text{MS}}}^2}{4\pi^2} \lambda_R(\bar{\mu}) = \frac{\alpha_{R1} \Lambda_{\overline{\text{MS}}}^2}{\ln \frac{\Lambda_{\overline{\text{MS}}}^2}{\bar{\mu}^2}}. \quad (57)$$

Here $\Lambda_{\overline{\text{MS}}}$ is an emergent energy scale for the theory that can (once again) be seen to correspond to the $\overline{\text{MS}}$ parameter of the theory.

Comparing (57) to the result from perturbation theory (29) [1–3] for $\bar{\mu} \ll \Lambda_{\overline{\text{MS}}}$, one finds that $R1$ includes *some* of the next-to-leading order corrections, but not all of them. It is also easy to verify that the connected n -point functions with $n \geq 4$ in the $R1$ level resummation scheme are subleading at large N , so that they are trivially finite. To include all next-to-leading-order (NLO) large N corrections, the next resummation level needs to be addressed.

Before doing so, let us again calculate some observables. Choosing again c_V to cancel the constant term m_{R1}^0 in \mathcal{F} , one has

$$\bar{\mathcal{F}}_{R1} = -\frac{\bar{m}_{R1}^4}{64\pi^2} \ln \frac{\Lambda_{\overline{\text{MS}}}^2 e^{\frac{3}{2}}}{\bar{m}_{R1}^2} + \alpha_{R1} \Lambda_{\overline{\text{MS}}}^2 \frac{\bar{m}_{R1}^2}{32\pi^2}, \quad (58)$$

where \bar{m}_{R1}^2 is given by the saddle-point condition (53)

$$m_{R1}^2 \ln \frac{e^{\frac{1}{2}} \Lambda_{\overline{\text{MS}}}^2}{\bar{m}_{R1}^2} = \alpha_{R1} \Lambda_{\overline{\text{MS}}}^2, \quad (59)$$

[cf. Eq. (34)]. In terms of the parameters $\Lambda_{\overline{\text{MS}}}, \alpha_{R1}$, the vector pole mass and the free energy take on the same values as for the leading order large N expansion, e.g.,

$$\alpha_{R1} = 0: \quad \bar{m}_{R1}^2 = e^{\frac{1}{2}} \Lambda_{\overline{\text{MS}}}^2, \quad \bar{\mathcal{F}}_{R1} = -\frac{e^{\frac{2}{2}} \Lambda_{\overline{\text{MS}}}^4}{128\pi^2}. \quad (60)$$

D. Next-to-leading order large N —level $R2$

Complete corrections to subleading order in large N are effectively captured by employing the $R2$ level resummation of (8) (cf. [20,22,23] for details). To this end, add and subtract self-energy terms for the vector and auxiliary field in the action, obtaining

$$S_{R0} + S' = \text{vol} \times \frac{N \zeta_0^2}{16\lambda_B} + S_{R2} + S_{I2}, \quad (61)$$

where

$$S_{R2} = \int d^d x d^d y \frac{1}{2} [\vec{\phi}(x) \Delta^{-1}(x-y) \vec{\phi}(y) + \xi(x) D^{-1}(x-y) \xi(y)], \quad (62)$$

$$S_{I2} = \int d^d x \frac{i \xi \vec{\phi}^2}{2} - \frac{1}{2} \int d^d x d^d y [\vec{\phi}(x) \Sigma(x-y) \vec{\phi}(y) + N \xi(x) \Pi(x-y) \xi(y)], \quad (63)$$

and in Fourier space

$$\Delta(k) = \frac{1}{k^2 + m_B^2 + i\zeta_0 + \Sigma(k)}, \quad D(k) = \frac{1}{N} \frac{1 - \frac{\delta(k)}{\text{vol}}}{\frac{1}{8\lambda_B} + \Pi(k)}, \quad (64)$$

where the δ function in $D(k)$ is a consequence of the constraint $\int d^d x \xi(x) = 0$; cf. (6) and the discussion in Ref. [23].

The self-energies Σ and Π are found by calculating the correlators $\langle \phi(x) \phi(0) \rangle$ and $\langle \xi(x) \xi(0) \rangle$ self-consistently to the one-loop level, finding [4,20]

$$\Pi(x) = \frac{1}{2} \Delta^2(x), \quad \Sigma(x) = D(x) \Delta(x). \quad (65)$$

Because $D(x) \propto \frac{1}{N}$, at large N the self-energy obeys $\Sigma(x) \propto \frac{1}{N}$, and I do not need to include the self-energy Σ when evaluating $\Pi(x)$ to leading order in large N :

$$\Pi(k) = \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \frac{1}{((k-p)^2 + m_B^2 + i\zeta_0)(p^2 + m_B^2 + i\zeta_0)}. \quad (66)$$

Using Feynman parameters and shifting the momentum variable, this becomes

$$\Pi(k) = \frac{1}{2} \int_0^1 dx \int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2 + m_B^2 + i\zeta_0 + k^2 x(1-x))^2}, \quad (67)$$

which contains a UV divergence in the momentum integral. Regulating the integral once again using dimensional regularization, one finds

$$\Pi(k) = \frac{1}{32\pi^2} \left[\frac{1}{\varepsilon} + \ln \frac{\bar{\mu}^2 e^2}{m_B^2 + i\zeta_0} - 2 \sqrt{\frac{k^2 + 4(m_B^2 + i\zeta_0)}{k^2}} \text{atanh} \sqrt{\frac{k^2}{k^2 + 4(m_B^2 + i\zeta_0)}} \right]. \quad (68)$$

The auxiliary field propagator $D(k)$ is already $\mathcal{O}(\frac{1}{N})$, so $\frac{1}{N}$ corrections to the combination $m_B^2 + i\zeta_0 = m_{R0}^2$ and bare coupling constant λ_B can be neglected. Hence, using (26), the auxiliary field propagator then becomes

$$D(k) = \frac{32\pi^2}{N} \frac{1 - \frac{\delta(k)}{\text{vol}}}{\ln \frac{\Lambda_{\text{MS}}^2 e^2}{m_{R0}^2} - 2\sqrt{\frac{k^2 + 4m_{R0}^2}{k^2}} \text{atanh} \sqrt{\frac{k^2}{k^2 + 4m_{R0}^2}}}. \quad (69)$$

Explicitly, using (41) one has

$$\alpha_{R0} = 0: D(k) = \frac{32\pi^2}{N} \frac{1 - \frac{\delta(k)}{\text{vol}}}{1 - 2\sqrt{\frac{k^2 + 4m_{R0}^2}{k^2}} \text{atanh} \sqrt{\frac{k^2}{k^2 + 4m_{R0}^2}}}. \quad (70)$$

and

$$\alpha_{R0} = \alpha_c: D(k) = \frac{32\pi^2}{N} \frac{1 - \frac{\delta(k)}{\text{vol}}}{2 - 2\sqrt{\frac{k^2 + 4m_{R0}^2}{k^2}} \text{atanh} \sqrt{\frac{k^2}{k^2 + 4m_{R0}^2}}}, \quad (71)$$

which are well-defined and finite for any real-valued Euclidean momentum k_μ . However, note that for $\alpha = \alpha_c$, the zero in the denominator for $k = 0$ only does not lead to a pole in $D(k)$ because of the vanishing numerator for this case.

In the following, for simplicity I will limit myself to only considering the case

$$\alpha_{R0} = 0. \quad (72)$$

The vector self-energy complete to next-to-leading order in large N is then given as

$$\Sigma(k) = \int \frac{d^d p}{(2\pi)^d} \frac{D(p)}{(k-p)^2 + m_{R0}^2}. \quad (73)$$

Formally expanding out $\Sigma(k)$ in perturbation theory, one finds that it contains terms such as the ‘‘setting-sun’’ diagram, which are known to contain UV divergencies proportional to k^2 , e.g., those that require wave-function renormalization in perturbation theory [24].

For the case at hand, it is easy to check if $\Sigma(k)$ contains UV divergencies proportional to k^2 . To this end, consider

$$\frac{\partial}{\partial k_\mu} \frac{\partial}{\partial k_\mu} \Sigma(k) = -8m_{R0}^2 \int \frac{d^d p}{(2\pi)^d} \frac{D(p)}{[(k-p)^2 + m_{R0}^2]^3}. \quad (74)$$

Since $D(p) < p^2$ for large momenta, the resulting expression is UV finite. Hence, there are no momentum-dependent divergencies in $\Sigma(k)$, unlike what happens in almost all orders of a perturbative expansion of $\Sigma(k)$.

In fact, Eq. (74) is UV finite at every value of k , which implies that any potential UV divergence of $\Sigma(k)$ must be contained in

$$\Sigma(0) = \int \frac{d^d p}{(2\pi)^d} \frac{D(p)}{p^2 + m_{R0}^2}, \quad (75)$$

and that $\Delta\Sigma(k) \equiv \Sigma(k) - \Sigma(0)$ must be UV finite to next-to-leading order in the large N expansion. Note that this differs radically from the usual perturbative renormalization.

In the $R2$ -level resummation, the partition function is given by

$$Z_{R2} = \int d\zeta_0 e^{-N \times \text{vol} \times \mathcal{F}_{R2}(\zeta_0)}, \quad (76)$$

with the $R2$ -level free energy density $\mathcal{F}_{R2}(\zeta_0)$ given by [20]

$$\begin{aligned} \mathcal{F}_{R2}(\zeta_0) &= \frac{\zeta_0^2}{16\lambda_B} + \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \ln [k^2 + m_{R0}^2 + \Sigma(k)] \\ &\quad - \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \Sigma(k) \Delta(k) \\ &\quad + \frac{1}{2N} \int \frac{d^d k}{(2\pi)^d} \ln [D^{-1}(k)], \end{aligned} \quad (77)$$

where a two-loop diagram has canceled. Noting that $\Sigma(k) \propto \frac{1}{N}$, one can expand the first logarithm in \mathcal{F}_{R2} for large N , finding that the total contribution of Σ is $\mathcal{O}(N^{-2})$, or next-to-next-to-leading order in large N , which is beyond the accuracy of this work. Hence, at next-to-leading order in large N , the expression for the free energy functional simplifies to

$$\begin{aligned} \lim_{N \gg 1} \mathcal{F}_{R2}[\zeta_0] &= \frac{\zeta_0^2}{16\lambda_B} + \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \ln [k^2 + m_{R0}^2] \\ &\quad + \frac{1}{2N} \int \frac{d^d k}{(2\pi)^d} \ln [D^{-1}(k)]. \end{aligned} \quad (78)$$

All but the last integral in this expression have been evaluated in the $R0$ subsection above.

To extract the divergencies of the functional trace over the auxiliary field propagator, note that in dimensional regularization only logarithmic¹ divergencies register. As a consequence, expanding $\ln D^{-1}(k)$ for $k \gg m_{R0}$, the divergent piece of the integral must originate from the term proportional to k^{-4} in this power series expansion. One finds

¹An earlier version of this work treated this dogmatically, extracting only divergencies that register as $\frac{1}{\epsilon}$ in dimensional regularization. A more careful analysis shows that besides divergencies of this form, also terms such as $\ln \epsilon$ can appear, corresponding to $\ln \ln \Lambda_{\text{UV}}$ divergencies in cutoff regularization.

$$\ln D^{-1}(k) = \dots - \frac{4m_{R0}^4}{k^4} + \frac{\left(3 + 6 \ln \frac{\Lambda_{\overline{\text{MS}}}^2 e^2}{m_{R0}^2}\right) m_{R0}^4}{k^4 \ln \frac{m_{R0}^2}{k^2}} + \dots, \quad (79)$$

and hence,

$$\begin{aligned} & \frac{1}{2N} \int \frac{d^d k}{(2\pi)^d} \ln [D^{-1}(k)] \\ &= -\frac{m_{R0}^4}{N} \frac{1}{8\pi^2 \varepsilon} + \frac{m_{R0}^4}{N} \frac{3 + 6 \ln \frac{\Lambda_{\overline{\text{MS}}}^2 e^2}{m_{R0}^2}}{32\pi^2} \ln \varepsilon + \text{finite}. \end{aligned} \quad (80)$$

Collecting the leading order large N result and the divergent pieces and bare parameter contributions in \mathcal{F}_{R2} one finds

$$\begin{aligned} \lim_{N \gg 1} \mathcal{F}_{R2}(\zeta_0) &= -\frac{(i\zeta_0)^2}{16\lambda_B} - \frac{m_{R0}^4}{64\pi^2} \left[\frac{1}{\varepsilon} \left(1 + \frac{8}{N}\right) \right. \\ &\quad \left. + \ln \frac{\bar{\mu}^2 e^{\frac{3}{2}}}{m_{R0}^2} - \frac{12 \ln \frac{\Lambda_{\overline{\text{MS}}}^2 e^{\frac{5}{2}}}{m_{R0}^2}}{N} \ln \varepsilon \right] + \text{finite}. \end{aligned} \quad (81)$$

Because of the choice (72), the bare mass parameter vanishes to leading order in large N , so one can set $i\zeta_0 = m_{R0}^2$. According to the principles of nonperturbative renormalization set out in Sec. II B, the coefficients of the terms $m_{R0}^4, m_{R0}^2, m_{R0}^0$ have to be finite. However, inspecting (81), the term proportional to $m_{R0}^4 \ln m_{R0}^2 \ln \varepsilon$ does not fit into this scheme: since the logarithm depends on m_{R0} , no single choice of λ_B can be used to absorb this divergence for all m_{R0} .

However, there is a way to render \mathcal{F}_{R2} finite. In fact, it will become clear that this is a required operation in order to obtain a finite two-point function for the theory. Namely, rescaling $i\zeta_0$ by a (divergent) constant as

$$i\zeta_0 \rightarrow i\zeta_0 \left(1 + \frac{c_\Sigma}{N}\right), \quad (82)$$

and expanding \mathcal{F} to NLO in large N leads to

$$\begin{aligned} & \lim_{N \gg 1} \mathcal{F}_{R2} \left(\zeta_0 \left(1 + \frac{c_\Sigma}{N}\right) \right) \\ &= -\frac{(i\zeta_0)^2}{64\pi^2} \left[\frac{4\pi^2}{\lambda_B} + \ln \frac{\bar{\mu}^2 e^{\frac{3}{2}}}{i\zeta_0} + \frac{1}{\varepsilon} \left(1 + \frac{8}{N}\right) \right. \\ &\quad \left. + \frac{2c_\Sigma}{N} \left(\frac{4\pi^2}{\lambda_B} + \ln \frac{\bar{\mu}^2 e^1}{i\zeta_0} + \frac{1}{\varepsilon} \right) - \frac{12 \ln \frac{\Lambda_{\overline{\text{MS}}}^2 e^{\frac{5}{2}}}{i\zeta_0}}{N} \ln \varepsilon \right] + \text{finite}. \end{aligned} \quad (83)$$

The term proportional to c_Σ is already $\mathcal{O}(N^{-1})$, so that the leading order renormalization (26) can be used without making any error to NLO in large N . One finds

$$\begin{aligned} & \lim_{N \gg 1} \mathcal{F}_{R2} \left(\zeta_0 \left(1 + \frac{c_\Sigma}{N}\right) \right) \\ &= -\frac{(i\zeta_0)^2}{64\pi^2} \left[\frac{4\pi^2}{\lambda_B} + \ln \frac{\bar{\mu}^2 e^{\frac{3}{2}}}{i\zeta_0} + \frac{1}{\varepsilon} \left(1 + \frac{8}{N}\right) \right. \\ &\quad \left. + \frac{2c_\Sigma}{N} \ln \frac{\Lambda_{\overline{\text{MS}}}^2 e^1}{i\zeta_0} - \frac{12 \ln \frac{\Lambda_{\overline{\text{MS}}}^2 e^{\frac{5}{2}}}{i\zeta_0}}{N} \ln \varepsilon \right] + \text{finite}. \end{aligned} \quad (84)$$

It is now apparent that the offending term can be canceled by choosing

$$c_\Sigma = 6 \ln \varepsilon, \quad (85)$$

so that with this choice

$$\begin{aligned} & \lim_{N \gg 1} \mathcal{F}_{R2} \left(\zeta_0 \left(1 + \frac{c_\Sigma}{N}\right) \right) \\ &= -\frac{(i\zeta_0)^2}{64\pi^2} \left[\frac{4\pi^2}{\lambda_B} + \ln \frac{\bar{\mu}^2 e^{\frac{3}{2}}}{i\zeta_0} + \frac{1}{\varepsilon} \left(1 + \frac{8}{N}\right) - \frac{18}{N} \ln \varepsilon \right] + \text{finite}. \end{aligned} \quad (86)$$

In this form now, the free energy can be rendered finite by a nonperturbative coupling constant renormalization:

$$\frac{1}{\lambda_B} = \frac{1}{\lambda_R(\bar{\mu})} - \frac{1 + \frac{8}{N}}{4\pi^2 \varepsilon} + \frac{9 \ln \varepsilon}{2\pi^2 N}. \quad (87)$$

Replacing $\ln \varepsilon \rightarrow \ln \left(-\frac{\lambda_B}{4\pi^2}\right) + \mathcal{O}(N^{-1})$, the above renormalization is qualitatively similar to the form expected from two-loop running in perturbation theory.

The nonperturbative renormalization condition (87) renders finite \mathcal{F}_{R2} in the large N limit, which in turn gives a finite value for the saddle $\bar{\zeta}$ which in turn guarantees a finite zero-point function to next-to-leading order in large N (after subtracting a divergent vacuum energy contribution). One can check this explicitly by recalculating the saddle-point condition directly from (77):

$$\begin{aligned} 0 &= \frac{d\mathcal{F}_{R2}}{di\zeta_0} \\ &= -\frac{i\bar{\zeta}}{8\lambda_B} \left(1 + \frac{c_\Sigma}{N}\right) + \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 + i\bar{\zeta} \left(1 + \frac{c_\Sigma}{N}\right) + \Sigma(k)}, \end{aligned} \quad (88)$$

where several terms have canceled (cf. Ref. [20]). At this point it is useful to recall that the two-point function for the vector fields of the theory is given by

$$\langle \phi_i(x) \phi_j(x) \rangle = \int \frac{d^d k}{(2\pi)^d} \frac{e^{ikx}}{k^2 + i\bar{\zeta} \left(1 + \frac{c_\Sigma}{N}\right) + \Sigma(k)}, \quad (89)$$

where $\bar{\zeta}$ is finite after renormalizing the coupling (87). I have pointed out above that $\Sigma(k)$ does not contain any divergencies proportional to k^2 . It would seem that a divergent constant $\Sigma(0)$ would lead to a divergence in the two-point function. The only way to avoid a divergent pole mass is therefore that the divergence in $\Sigma(0)$ is exactly canceled by the choice of c_Σ that was required to render the free energy finite for all $i\zeta_0$.

It should be stressed that the term c_Σ , though divergent, does not correspond to a wave-function counterterm. This is because a wave-function counterterm is very specific: it is an additional term $\chi_\phi \partial_\mu \vec{\phi} \cdot \partial_\mu \vec{\phi}$ in the Euclidean action (4), which manifests itself as an additional term $2k^2 \chi_\phi$ in the denominator of (89). By contrast, the term proportional to c_Σ in (89) is independent of the momentum k ; hence, c_Σ cannot be a wave-function counterterm. In fact, from its origin, it is clear that c_Σ is no counterterm at all, because it does not correspond to an additional term in the action (4). There is no additional renormalization condition associated with c_Σ : it simply has to cancel out against the corresponding divergent part of $\Sigma(0)$.

After this discussion, let us calculate the divergent part of $\Sigma(0)$: expanding $\frac{D(p)}{p^2 + m_{R0}^2}$ in (75) in powers of momenta $p \gg m_{R0}$, any logarithmic divergence must originate from the term proportional to p^{-4} in this power series expansion. One finds that

$$\frac{D(p)}{p^2 + m_{R0}^2} = \frac{32\pi^2}{N} \left[\dots - \frac{3m_{R0}^2}{p^4 \ln \frac{m_{R0}^2}{p^2}} + \dots \right], \quad (90)$$

so that

$$\Sigma(0) = -\frac{6m_{R0}^2}{N} \ln \varepsilon + \text{finite}, \quad (91)$$

confirming that $\Sigma(0)$ is indeed divergent in the limit $\varepsilon \rightarrow 0$.

From (89), the inverse propagator becomes

$$\begin{aligned} k^2 + i\bar{\zeta} \left(1 + \frac{c_\Sigma}{N}\right) + \Sigma(0) + \Delta\Sigma(k) \\ = k^2 + i\bar{\zeta} + \Delta\Sigma(k) + \text{finite}, \end{aligned} \quad (92)$$

because to this order in the large N expansion and using (85)

$$i\bar{\zeta} \frac{c_\Sigma}{N} = \frac{m_{R0}^2 c_\Sigma}{N} = \frac{6m_{R0}^2 \ln \varepsilon}{N}, \quad (93)$$

where I recall that $\Delta\Sigma(k) = \Sigma(k) - \Sigma(0)$ was found to be UV finite above.

The only divergence present in $\Sigma(k)$ is precisely canceled by the shift choice (85), which was necessary to cancel the logarithmic term in the free energy.

As a consequence, the explicit construction above shows that both the zero-point and the two-point functions of the theory are finite to NLO in the large N expansion.

The connected four-point function for the $R2$ -level theory is given by

$$\langle \phi_a \phi_b \phi_c \phi_d \rangle_{\text{conn}} = Z_{R2}^{-1} \int d\zeta_0 \mathcal{D}\vec{\phi} \mathcal{D}\xi e^{-S_{R2} - S_{I2}} \phi_a \phi_b \phi_c \phi_d, \quad (94)$$

which allows for different kinematic channels (“ s ,” “ t ,” and “ u ” channels), all of which have the same form. For instance, in the s channel, one finds in momentum space for the $R2$ -level connected amputated four-point function

$$\Gamma_4^{(s)}(p) = \frac{1}{32\pi^2} D(p) = \frac{1 - \frac{\delta(p)}{\text{vol}}}{\frac{N}{8\lambda_B} + \Pi(p)}. \quad (95)$$

One can immediately see that $\Gamma_4(p) \propto \frac{1}{N}$, which is why one needs to go to at least the $R2$ resummation level in order to get a consistent result for the four-point function.

One can also see that $\Gamma_4(p)$ is amenable to a perturbative expansion

$$\Gamma_4^{(s)}(p \neq 0) = \frac{8\lambda_B}{N} - \frac{(8\lambda_B)^2 \Pi(p)}{N^2} + \dots, \quad (96)$$

where in perturbation theory every single term is divergent because of (31); cf. the discussion in Ref. [4]. In strong contradistinction, at the $R2$ level in the large N approximation $\Gamma_4^{(s)}(p)$ is given exactly as

$$\Gamma_4^{(s)}(p) = \frac{1}{N} \frac{1 - \frac{\delta(p)}{\text{vol}}}{1 - 2\sqrt{\frac{p^2 + 4\bar{m}_{R0}^2}{p^2}} \text{atanh} \sqrt{\frac{p^2}{p^2 + 4\bar{m}_{R0}^2}}}, \quad (97)$$

where I used again (69) for $\alpha_{R0} = 0$. This expression for the four-point function of the theory is finite and well-behaved for all Euclidean momenta p .

It is possible to consider the connected six-point function for the theory, but one finds that it is proportional to $\frac{1}{N^2}$ so that the $R2$ -level resummation scheme is insufficient to calculate this quantity. While it is possible to study six-point functions using higher-order resummation schemes, such as $R4$ [22], this calculation is left for future work.

E. Summary of nonperturbative renormalization for scalar field theory

To summarize, it is possible to set up an alternative renormalization scheme in scalar field theories different from perturbative renormalization that leads to finite n -point functions order-by-order in a large N expansion.

This alternative renormalization scheme does not allow for wave-function renormalization, which is different from the perturbative approach.

Also, in the nonperturbative renormalization approach, the bare coupling parameter tends to $\lambda_B \rightarrow 0^-$ as the regularization parameter is removed, which is qualitatively different from the perturbative behavior.

Despite these unusual features, and in particular despite employing fewer counterterms than perturbative renormalization, I provided explicit calculations valid up to and including $\mathcal{O}(N^{-1})$ to show that all connected n -point functions of the theory come out finite. In particular, I showed that the connected four-point function to order $\mathcal{O}(\frac{1}{N})$ is finite even though its perturbative reexpansion is divergent at every single order in perturbation theory.

I find it intriguing that the theory, defined through a controlled expansion in $\frac{1}{N}$, is apparently “less divergent” than its perturbative definition, by which I mean that in the massless case a single coupling-constant renormalization is sufficient in dimensional regularization to render all n -point functions finite to order $\mathcal{O}(\frac{1}{N})$, whereas perturbative renormalization requires both coupling constant and wave-function counterterms.

In my opinion, this provides further evidence for the claim that a nontrivial interacting scalar field theory formulation is possible in the continuum (cf. Ref. [16]). Also, applications of this alternative renormalization, for instance, to recent strong coupling real-time problems such as in Refs. [25,26], are possible.

Another possible application of the above findings is to try to make contact with lattice scalar field theory, which constitutes its own nonperturbative renormalization scheme. A direct comparison requires lattice scalar field theory in four dimensions with bare couplings that fulfill (30), which is a study in its infancy [27,28]. However, in *two dimensions*, where lattice results only require nonperturbative mass renormalization, the nonperturbative lattice renormalization and the nonperturbative renormalization discussed in this work are *identical* (cf. Refs. [20,29]).

III. QCD

Inspired by the success of nonperturbative renormalization in scalar field theory, let me now consider a similar setup for QCD. Unlike in the case of scalar field theory, there is no simplification in performing a large N_c expansion for QCD. Instead, I will be using a “semi-classical” expansion of the QCD effective action, which can be formalized as a power expansion in \hbar . However,

since \hbar is not dimensionless, such an expansion is necessarily uncontrolled, which means that the results presented in the following are qualitative in nature. Nevertheless, I hope that some readers will find the apparent similarity with the scalar field theory case amusing.

Specifically, following the work reported in Ref. [30], I consider an $SU(N)$ gauge theory coupled to N_f flavors of massless Dirac spinors in the fundamental representation with an action given by

$$S_E = \int d^d x \left[\frac{1}{4g_B^2} G_{\mu\nu}^a G_{\mu\nu}^a + \bar{\psi}_f \not{D} \psi_f \right], \quad (98)$$

where g_B is the bare Yang-Mills coupling constant. Here

$$G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + f^{abc} A_\mu^b A_\nu^c \quad (99)$$

is the Yang-Mills field strength tensor, A_μ^a is the gauge potential, $\psi_f = (\psi_1, \psi_2, \dots, \psi_{N_f})$ are the N_f Dirac fermions, and $\not{D} = \gamma_\mu (\partial_\mu - iA_\mu^a t^a)$ is the Dirac operator with the Euclidean gamma matrices γ_μ with commutation relation $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$. The $SU(N)$ generators t^a and structure constants f^{abc} are taken to fulfill

$$\text{tr} t^a t^b = \frac{\delta^{ab}}{2}, \quad [t^a, t^b] = i f^{abc} t^c. \quad (100)$$

Inspecting the QCD action (98) and comparing it to the scalar field action in the auxiliary field formulation (8), one can observe certain similarities. For instance, the “matter content” given by the scalars ϕ_i and the fermions ψ_f , respectively, appears quadratically in both cases. Similarly, the matter fields couple through a three-point vertex to the auxiliary field ζ for scalars and through a three-point vertex to the gauge field $\gamma_\mu A_\mu^a t^a$ in QCD. For the case of scalar field theory, the leading large N partition function could be isolated by splitting the auxiliary field into a constant plus fluctuations, e.g., Eq. (6). This motivates performing a similar split of the gauge field in QCD as

$$A_\mu^a(x) = \mathbf{A}_\mu^a(x) + a_\mu^a(x), \quad (101)$$

where $\mathbf{A}_\mu^a(x)$ is referred to as the “background field” and can be an arbitrary (but fixed) function of coordinates. The QCD partition function then becomes

$$Z = \int d\mathbf{A} d a d\bar{\psi} d\psi e^{-S_E}, \quad (102)$$

where $\int d\mathbf{A}$ is a single integral (not a path integral) over the fixed background function \mathbf{A}_μ^a . Expanding the QCD action in powers of the fluctuation field a_μ^a as

$$S_E = S_0[\mathbf{A}] + S_1[\mathbf{A}] + S_2[\mathbf{A}] + \dots, \quad (103)$$

with $S_n[\mathbf{A}]$ corresponding to the contribution of n powers of a_μ^a in the expansion of the action. In particular, one has

$$S_2[\mathbf{A}] = \frac{1}{4g_B^2} \int d^d x [(d_\mu^{ac} a_\nu^c - d_\nu^{ac} a_\mu^c)^2 + 2F_{\mu\nu}^a f^{abc} a_\mu^b a_\nu^c], \quad (104)$$

with $d_\mu^{ac} = \delta^{ac} \partial_\mu + f^{abc} \mathbf{A}_\mu^b$ the background gauge-covariant derivative and where $F_{\mu\nu}^a$ is (99) evaluated for $A_\mu^a = \mathbf{A}_\mu^a$. Because the path integral for QCD in this form contains flat directions, it is necessary to gauge fix the theory. Following Weinberg's gauge fixing choice [31], this implies adding a gauge fixing and ghost term of the form

$$S_{\text{gf}} = \frac{1}{2g_B^2} \int dx (d_\mu^{ac} a_\mu^c)^2, \\ S_{\text{ghost}} = \int dx [d_\mu^{ac} \bar{c}^c d_\mu^{ab} c^b + d_\mu^{ac} \bar{c}^c f^{adb} a_\mu^d c^b], \quad (105)$$

to the action, where \bar{c}, c are the Faddeev-Popov ghost fields.

The simplest solvable approximation of this theory is similar to the $R0$ -level approximation in scalar field theory: just integrate out all quadratic terms in fields $a_\mu^a, \bar{\psi}, \psi, \bar{c}, c$ neglecting higher-order interactions. This corresponds to calculating the one-loop effective action, which is similar to the case of the scalar field theory corresponding to resumming an infinite number of perturbative contributions. Therefore, similar to the scalar field theory, it corresponds to a nonperturbative calculation in QCD.

It should be pointed out that $\mathbf{A}_\mu^a(x)$ is expected to take the same role as ζ_0 in scalar field theory, e.g., corresponding to a saddle point of the action. This immediately simplifies the calculation because $S_1[\mathbf{A}] = 0$ if \mathbf{A}_μ^a is a saddle point. One finds

$$Z = \int d\mathbf{A} e^{-\frac{1}{2} \text{Tr} \ln [(p^2 \delta^{ab} - 2i p_\alpha \mathcal{A}_\mu^{ab} - (\mathcal{A}^2)^{ab}) \delta_{\mu\nu} - 2\mathcal{F}_{\mu\nu}^{ab}] + \text{Tr} \ln [p^2 \delta^{ab} - 2i p_\alpha \mathcal{A}_\mu^{ab} - (\mathcal{A}^2)^{ab}] + \frac{N_f}{2} \text{Tr} \ln \not{D}}, \quad (106)$$

where $\mathcal{A}_\mu^{ab} = f^{acb} \mathbf{A}_\mu^c$; $\mathcal{F}_{\mu\nu}^{ab} = f^{acb} F_{\mu\nu}^c$; Tr collectively denotes traces are over Lorentz, color, spinor, and spacetime indices where appropriate; and some operators are represented in momentum space.

Evaluating Z for arbitrary background field $\mathbf{A}_\mu^a(x)$ is still too hard, so I am using a subset of possible functions $\mathbf{A}_\mu^a(x)$, specifically those for which the background field-strength tensor $F_{\mu\nu}^a$ is constant and self-dual (equal Euclidean chromoelectric and chromomagnetic fields); cf. the discussion in Ref. [30]. This amounts to setting

$$\mathbf{A}_\mu^a(x) = -\frac{1}{2} F_{\mu\nu}^a x_\nu. \quad (107)$$

Using then

$$\text{Tr} \ln \mathcal{O} = -\int_0^\infty \frac{ds}{s} K(s), \quad K(s) = \text{Tr} e^{-s\mathcal{O}}, \quad (108)$$

the kernel function $K(s)$ can be written as a quantum-mechanical partition function

$$K(s) = \text{tr} \int dx \langle x | e^{-s\mathcal{O}} | x \rangle = \text{tr} \int \mathcal{D}x e^{-\mathbf{Q}[x]}. \quad (109)$$

For the case of ghosts, gluons, and fermions one can find the "actions" \mathbf{Q} using results from [32–34] as in [30], specifically,

$$\mathbf{Q}^{\text{ghost}} = \int_0^s d\sigma \left[\frac{\dot{x}_\mu^2}{4} \delta^{ab} + \dot{x}_\mu \mathcal{A}_\mu^{ab} \right], \quad (110)$$

$$\mathbf{Q}^{\text{glue}} = \int_0^s d\sigma \left[\frac{\dot{x}_\alpha^2}{4} \delta^{ab} \delta_{\mu\nu} + \dot{x}_\alpha \mathcal{A}_\alpha^{ab} \delta_{\mu\nu} - 2\mathcal{F}_{\mu\nu}^{ab} \right],$$

$$\mathbf{Q}^{\text{fermion}} = \int_0^s d\sigma \left[\frac{\dot{x}_\alpha^2}{4} + i\dot{x}_\alpha \mathbf{A}_\alpha^a t^a - \frac{i}{2} \sigma_{\mu\nu} F_{\mu\nu}^a t^a \right]. \quad (111)$$

In this form, the path integral representations for the kernel functions $K(s)$ look just like a finite-temperature quantum mechanics problem and can be solved accordingly to give [30]

$$K_{\text{ghost}}(s) = \frac{\text{vol}}{(4\pi s)^{\frac{d}{2}}} \sum_{i=0}^3 \frac{s^2 \lambda_i}{2 \sinh^2 \left(\frac{s\sqrt{\lambda_i}}{2} \right)},$$

$$K_{\text{glue}}(s) = \frac{\text{vol} \times d}{(4\pi s)^{\frac{d}{2}}} \sum_{i=0}^3 \frac{s^2 \lambda_i \cosh(s\sqrt{\lambda_i})}{2 \sinh^2 \left(\frac{s\sqrt{\lambda_i}}{2} \right)},$$

$$K_{\text{fermion}}(s) = \frac{\text{vol} \times d}{(4\pi s)^{\frac{d}{2}}} \sum_{i=1}^3 \frac{s^2 \lambda_{F,i} \cosh^2 \left(\frac{s\sqrt{\lambda_{F,i}}}{2} \right)}{4 \sinh^2 \left(\frac{s\sqrt{\lambda_{F,i}}}{2} \right)}, \quad (112)$$

where λ_i are the eigenvalues of the matrix $\mathcal{M} = -4f^{acd} f^{deb} B^c B^e$ and $\lambda_{F,i}$ are the eigenvalues of the matrix

$\mathcal{M}_F = 4t^a t^b B^a B^b$. Here B^a is the (constant) chromomagnetic field corresponding to the entries of the self-dual constant field strength tensor $F_{\mu\nu}^a$.

The partition function (106) only involves a certain combination of the kernel functions

$$K_{\text{sum}}(s) = \frac{1}{2} K_{\text{glue}}(s) - K_{\text{ghost}}(s) - \frac{N_f}{2} K_{\text{fermion}}(s), \quad (113)$$

which for the specific choice of $N_f = 4N$ has been found to be free of IR divergencies [30]. Any remaining divergences are UV divergencies, which can be extracted through dimensional regularization using $d = 4 - 2\epsilon$. Specifically, for the case $N_f = 4N$ one finds [30]

$$K_{\text{sum}} = \frac{\text{vol}}{(4\pi s)^{\frac{d}{2}}} \left[(d-2) \sum_{i=1}^3 \left(\frac{(d-2)s^2 \lambda_i}{4 \sinh^2 \left(\frac{s\sqrt{\lambda_i}}{2} \right)} - \frac{N_f d}{2} \frac{s^2 \lambda_{F,i}}{4 \sinh^2 \left(\frac{s\sqrt{\lambda_{F,i}}}{2} \right)} \right) \right], \quad (114)$$

so that the traces of the operators in (108) can be evaluated in terms of ζ functions [35] [25.5.9]. After the dust settles, one has

$$Z = \int d\mathbf{A} e^{-\text{vol} \times \mathcal{F}}, \quad (115)$$

with the one-loop QCD effective potential

$$\mathcal{F} = \frac{B^a B^a}{g_B^2} + \frac{-NB^a B^a}{(4\pi^2)\epsilon} + \frac{12NB^a B^a \zeta'(-1)}{(4\pi)^2} + \sum_{i=1}^3 \left(\frac{\lambda_i \ln \frac{\bar{\mu}^2 e^{-1}}{\sqrt{\lambda_i}} - N_f \lambda_{F,i} \ln \frac{\bar{\mu}^2 e^{-\frac{1}{2}}}{\sqrt{\lambda_{F,i}}}}{96\pi^2} \right), \quad (116)$$

where the relation between \mathbf{A} and B is provided by (107) and the eigenvalues $\lambda_i, \lambda_{F,i}$ are functions of B . At this point, I again wanted to stress the similarity to the scalar field theory calculation at the $R0$ level, notably (13) and (14). In the large volume limit, the QCD partition function is given exactly as the saddle point of the action

$$\ln Z = -\text{vol} \times F[\bar{B}^a], \quad (117)$$

with \bar{B}^a the location of the saddle (there is a Jacobian when changing the integration from \mathbf{A}_μ^a to B_μ^a , but the contribution from this Jacobian is suppressed in the large volume limit). Similar to the case of scalar field theory, the zero-point function for QCD is finite if the coefficient of $B^a B^a$ in

\mathcal{F} is finite. This gives the nonperturbative renormalization condition for QCD with $N_f = 4N$ fermion flavors as

$$\frac{1}{g_B^2} - \frac{N}{(4\pi)^2 \epsilon} = \frac{1}{g_R^2(\bar{\mu})}, \quad (118)$$

which implies a running QCD coupling of the form

$$g_R^2(\bar{\mu}) = \frac{(4\pi)^2}{N \ln \frac{\bar{\mu}^2}{\Lambda_{\overline{\text{MS}}}^2}}. \quad (119)$$

Note that this running coupling matches the one-loop perturbative running for QCD with $\beta_0 = \frac{11N}{3} - \frac{2N_f}{3}$ and $N_f = 4N$. The bare Yang-Mills coupling fulfills

$$\lim_{\epsilon \rightarrow 0} g_B^2 = \lim_{\epsilon \rightarrow 0} \frac{1}{\frac{1}{g_R^2(\bar{\mu})} + \frac{N}{(4\pi)^2 \epsilon}} \rightarrow 0, \quad (120)$$

whereas *first* expanding g_B^2 for small coupling g_R^2 and *then* taking the regulator to zero gives the perturbative result

$$\lim_{\epsilon \rightarrow 0} g_{B,\text{pert}}^2 = \lim_{\epsilon \rightarrow 0} \left[g_R^2(\bar{\mu}) - \frac{N g_R^4}{(4\pi)^2 \epsilon} + \mathcal{O}(g_R^6) \right] \rightarrow -\infty, \quad (121)$$

which differs qualitatively from the nonperturbative alternative renormalization (120).

After nonperturbative renormalization (118), one obtains

$$\mathcal{F} = \frac{NB^a B^a}{(4\pi)^2} \ln \frac{\bar{\mu}^2}{\Lambda_{\overline{\text{MS}}}^2} + \frac{12NB^a B^a \zeta'(-1)}{(4\pi)^2} + \sum_{i=1}^3 \left(\frac{\lambda_i \ln \frac{\bar{\mu}^2 e^{-1}}{\sqrt{\lambda_i}} - N_f \lambda_{F,i} \ln \frac{\bar{\mu}^2 e^{-\frac{1}{2}}}{\sqrt{\lambda_{F,i}}}}{96\pi^2} \right), \quad (122)$$

and it is straightforward to check that \mathcal{F} is independent from the fictitious renormalization scale $\bar{\mu}$. In full analogy to the scalar field theory case (36), \mathcal{F} has a trivial saddle $\bar{B}^a = 0$ as well as nontrivial saddles $\bar{B}^a \neq 0$. The lower free energy is found for nontrivial saddles of full rank, e.g., for

$$\bar{B}^a = \delta^{a8} e^{-12\zeta'(-1) - \frac{5}{6} - \frac{8}{3} \zeta'(-1)} 2^{-\frac{8}{3}} 3^{\frac{5}{6}} \Lambda_{\overline{\text{MS}}}^2 \simeq 4.27 \Lambda_{\overline{\text{MS}}}^2 \delta^{a8}. \quad (123)$$

The explicit (finite) result for the zero-point function for this nontrivial saddle is

$$\mathcal{F}[\bar{B}^a] = -\frac{3\bar{B}^a \bar{B}^a}{32\pi^2}. \quad (124)$$

For the two-point functions of the gauge fields, fermions, and ghosts, one notices that similar to the case of scalar field theory (17) these are UV finite as long as the saddle location \bar{B}^a is finite. Therefore, the nonperturbative

renormalization condition (118) automatically renders all two-point functions of the theory UV finite. However, it should be stressed that while the two-point functions are UV finite, they still do contain IR divergencies at this level of resummation [30,36]. If they were automatically IR finite, the mass gap problem for Yang Mills would be solved. However, renormalization of quantum field theory is concerned with UV divergencies, and these are taken care of by (118).

Similar to the case of scalar field theory, higher-order n -point functions require going beyond a simple $R0$ -level resummation scheme, which so far is not available for QCD. However, it should be pointed out that the non-perturbative renormalization condition (118) renders both the zero-point and the two-point functions of $N_f = 4N$ QCD finite, whereas in perturbation theory this requires both coupling and wave-function counterterms.

An interesting observable for $N_f = 12$ QCD to evaluate is the expectation value of the trace of the Polyakov loop in the presence of the nontrivial background saddle \bar{B}^a . One can explicitly evaluate this observable using (107) and finds

$$\text{tr} \int d^3\vec{x} \mathcal{P} e^{i \oint d\tau r^a \mathbf{A}_0^a(\tau, \vec{x})} = 0, \quad (125)$$

where \mathcal{P} denotes path ordering. The vanishing Polyakov loop expectation value is typically associated with confinement in QCD, so the above result suggests that QCD with $N_f = 12$ massless flavors is confining. This is consistent with results from lattice QCD simulations from Ref. [37], though perturbative and other more recent lattice studies found that the theory is instead infrared conformal [38–40].

I leave the resolution of the question of conformality in $N_f = 12$ QCD for future studies, only pointing out here that nonperturbative renormalization of the theory allows for predictive nonperturbative calculations.

IV. SUMMARY AND CONCLUSIONS

In this work, I have considered alternative renormalization procedures for scalar field theory and QCD in dimensional regularization. I provided explicit results that showed that this alternative renormalization procedure leads to finite n -point correlation functions even though it differs qualitatively from the usual perturbative renormalization approach.

In particular, there is no wave-function counterterm in the nonperturbative approach.

Furthermore, the bare coupling constant in nonperturbative renormalization differs qualitatively from the perturbative scheme in that it tends to zero in the limit of a vanishing regulator.

In the case of QCD, the nonperturbative renormalization seems much more similar to lattice QCD [41] than to

perturbative calculations [42]. Specifically, the expression for the bare coupling constant g_B in nonperturbative renormalization is qualitatively consistent with the dependence of the bare lattice coupling on the regulator (see, e.g., Ref. [43]).

In the case of scalar field theory, the results further strengthen the case for revisiting the claims of quantum triviality in four dimensions [16,44]. However, further advances on lattice field theory along the lines of [27,28] or experimental advances along the lines of [45,46] would be needed to unambiguously settle this question.

To conclude, perturbative renormalization is not the only option for dealing with UV divergencies in quantum field theory. Nonperturbative alternative schemes exist, which seem to work just as well as their perturbative counterparts, yet with fewer free parameters.

ACKNOWLEDGMENTS

This work was supported by the Department of Energy, DOE Award No. DE-SC0017905. I am grateful to Joaquin Drut, Hendrik van Hees, Thimo Preis, and Zsolt Szep for helpful comments and questions, and to Willie Su for pointing out typos in the appendix of this manuscript.

APPENDIX: SEMIANALYTIC EVALUATION OF DIVERGENT INTEGRALS IN DIMENSIONAL REGULARIZATION

Some integrals encountered in the main text are not routinely encountered in perturbative quantum field theory, so their evaluation in dimensional regularization may not be standard. For this reason, I provide a quick guide on how to evaluate such integrals semianalytically in this appendix.

To get started, let me consider a simple divergent integral I_1 in dimensional regularization, and pretend that I do not know that it corresponds to a representation of the Γ function:

$$I_1 = \mu^{2\varepsilon} \int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 + m^2)^2} = \left(\frac{4\pi\mu^2}{m^2} \right)^\varepsilon \frac{\Gamma(\varepsilon)}{16\pi^2}. \quad (A1)$$

The integrand does not depend on the angular variables, so that using the volume of a hypersphere in $d = 4 - 2\varepsilon$ dimensions one finds

$$I_1 = \frac{(4\pi\mu^2)^\varepsilon}{8\pi^2\Gamma(2-\varepsilon)} \int_0^\infty dk \frac{k^{3-2\varepsilon}}{(k^2 + m^2)^2}. \quad (A2)$$

Introducing the new variable $x = k^2$, and scaling $x \rightarrow xm^2$, this becomes

$$I_1 = \left(\frac{4\pi\mu^2}{m^2} \right)^\varepsilon \frac{1}{16\pi^2\Gamma(2-\varepsilon)} \int_0^\infty dx \frac{x^{1-\varepsilon}}{(x+1)^2}. \quad (A3)$$

Pretending not to know the exact form of the integral, one nevertheless knows that it is logarithmically divergent, so I split it into two parts:

$$\int_0^\infty dx \frac{x^{1-\varepsilon}}{(x+1)^2} = \int_0^R dx \frac{x^{1-\varepsilon}}{(x+1)^2} + \int_R^\infty dx \frac{x^{1-\varepsilon}}{(x+1)^2}, \quad (\text{A4})$$

with regulator R , so that the integration $x \in [0, R]$ is always convergent for any finite R . For the integration $x > R$, one can extract the divergent contribution by performing a Taylor expansion in $\frac{1}{x}$, so that

$$\begin{aligned} & \int_R^\infty dx \frac{x^{1-\varepsilon}}{(x+1)^2} \\ &= \int_R^\infty dx \left(\frac{x^{1-\varepsilon}}{(x+1)^2} - x^{-1-\varepsilon} \right) + \int_R^\infty dx x^{-1-\varepsilon} \\ &= \int_R^\infty dx \left(\frac{x^{1-\varepsilon}}{(x+1)^2} - x^{-1-\varepsilon} \right) - \frac{x^{-\varepsilon}}{\varepsilon} \Big|_{x=R}^\infty. \end{aligned} \quad (\text{A5})$$

In dimensional regularization, the integral is evaluated *first* at $x = \infty$ for sufficiently high $\varepsilon > 0$ such that the result is finite. *Then* the limit $\varepsilon \rightarrow 0$ is performed. The remaining contributions for I_1 are all finite for $\varepsilon \rightarrow 0$, so that they can be evaluated numerically. One finds

$$I_1 = \lim_{\varepsilon \rightarrow 0} \left(\frac{4\pi\mu^2}{m^2} \right)^\varepsilon \frac{1}{16\pi^2\Gamma(2-\varepsilon)} \left(\int_0^R dx \frac{x^1}{(x+1)^2} + \int_R^\infty dx \left(\frac{x^1}{(x+1)^2} - x^{-1} \right) + \frac{R^{-\varepsilon}}{\varepsilon} \right). \quad (\text{A6})$$

The remaining integrals are evaluated numerically for different choices of R , verifying that the result is independent of R . Specifically, one finds, e.g., for various values of R to within floating point precision

$$\begin{aligned} R = 3: & \int_0^R dx \frac{x^1}{(x+1)^2} + \int_R^\infty dx \left(\frac{x^1}{(x+1)^2} - x^{-1} \right) - \ln R = -1.000, \\ R = 4: & \int_0^R dx \frac{x^1}{(x+1)^2} + \int_R^\infty dx \left(\frac{x^1}{(x+1)^2} - x^{-1} \right) - \ln R = -1.000, \\ R = 5: & \int_0^R dx \frac{x^1}{(x+1)^2} + \int_R^\infty dx \left(\frac{x^1}{(x+1)^2} - x^{-1} \right) - \ln R = -1.000. \end{aligned} \quad (\text{A7})$$

Therefore, the semianalytical result for I_1 in dimensional regularization becomes

$$I_1 = \lim_{\varepsilon \rightarrow 0} \left(\frac{4\pi\mu^2}{m^2} \right)^\varepsilon \frac{1}{16\pi^2\Gamma(2-\varepsilon)} \left(\frac{1}{\varepsilon} - 1 \right) \simeq \frac{1}{16\pi^2} \left(\frac{1}{\varepsilon} + \ln \frac{\bar{\mu}^2}{m^2} \right), \quad (\text{A8})$$

where $\bar{\mu}^2 = 4\pi\mu^2 e^{-\gamma_E}$, matching the expected result (A1).

The same technique can be applied to more complicated integrals, e.g.,

$$I_2 = \mu^{2\varepsilon} \int \frac{d^d k}{(2\pi)^d} \ln \left[c + 2\sqrt{\frac{k^2 + 4m^2}{k^2}} \operatorname{atanh} \sqrt{\frac{k^2}{k^2 + 4m^2}} \right], \quad (\text{A9})$$

with $c > -2$ an arbitrary constant. Using the same steps as for I_1 , one finds

$$I_2 = \left(\frac{4\pi\mu^2}{m^2} \right)^\varepsilon \frac{m^4}{16\pi^2\Gamma(2-\varepsilon)} \int_0^\infty dx x^{1-\varepsilon} \ln \left[c + 2\sqrt{\frac{x+4}{x}} \operatorname{atanh} \sqrt{\frac{x}{x+4}} \right], \quad (\text{A10})$$

where the remaining integral is again split using an arbitrary regulator R . One finds

$$I_2 = \left(\frac{4\pi\mu^2}{m^2} \right)^\varepsilon \frac{m^4}{16\pi^2\Gamma(2-\varepsilon)} (I_{2L} + I_{2H} + I_{D1} + I_{D2} + I_{D3}), \quad (\text{A11})$$

where

$$I_{2L} = \int_0^R dx x^1 \ln \left[c + 2\sqrt{\frac{x+4}{x}} \operatorname{atanh} \sqrt{\frac{x}{x+4}} \right], \quad (\text{A12})$$

$$I_{2H} = \int_R^\infty dx x^1 \left\{ \ln \left[\frac{c + 2\sqrt{\frac{x+4}{x}} \operatorname{atanh} \sqrt{\frac{x}{x+4}}}{c + \ln x} \right] - \frac{2}{x} \left(1 + \frac{1-c}{c + \ln x} \right) - \frac{c-2-(3+2c)\ln x - 4\ln^2 x}{x^2(c + \ln x)^2} \right\}, \quad (\text{A13})$$

$$I_{D1} = \int_R^\infty dx x^{1-\varepsilon} \ln(c + \ln x), \quad (\text{A14})$$

$$I_{D2} = 2 \int_R^\infty dx x^{-\varepsilon} \left(1 + \frac{1-c}{c + \ln x} \right), \quad (\text{A15})$$

$$I_{D3} = \int_R^\infty dx x^{-1-\varepsilon} \frac{c-2-(3+2c)\ln x - 4\ln^2 x}{(c + \ln x)^2}. \quad (\text{A16})$$

The divergent pieces I_{D1}, I_{D2}, I_{D3} are again calculated analytically, and then evaluated for $x \rightarrow \infty$ using sufficiently large $\varepsilon > 0$ to render them convergent. Taking $\varepsilon \rightarrow 0$ as a second step, they explicitly become

$$I_{D1} = \frac{1}{2} [-R^2 \ln \ln(Re^c) + e^{-2c} \operatorname{Ei}(\ln R^2 e^{2c})], \quad (\text{A17})$$

$$I_{D2} = -2[R + (1-c)e^{-c} \operatorname{Ei}(\ln Re^c)], \quad (\text{A18})$$

$$I_{D3} = -\frac{4}{\varepsilon} + 4 \ln R + (3-6c)(\gamma_E + \ln \varepsilon + \ln \ln(Re^c)) - \frac{2(c-1)^2}{\ln Re^c}, \quad (\text{A19})$$

with Ei denoting the exponential integral function.

Evaluating the remaining integrals numerically for given c, R , one finds, for instance, for $c = -1$,

$$I_2 = \left(\frac{4\pi\mu^2}{m^2} \right)^\varepsilon \frac{m^4}{16\pi^2} \left(-\frac{4}{\varepsilon} - 4 + 13\gamma_E + 9\ln \varepsilon - 0.278387\dots \right). \quad (\text{A20})$$

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