Field theory axiverse

Stephon Alexander,[1,*] Tucker Manton,[1,†] and Evan McDonough[2,‡]
[1]*Department of Physics, Brown University, Providence, Rhode Island 02912, USA*
[2]*Department of Physics, University of Winnipeg, Winnipeg, Manitoba R3B 2E9, Canada*

(Received 8 May 2024; accepted 31 May 2024; published 20 June 2024)

Axion and axionlike particles are a prominent candidate for physics beyond the Standard Model and can play an important role in cosmology, serving as dark matter or dark energy, or both, drawing motivation in part from the string theory axiverse. Axionlike particles can also arise as composite degrees of freedom following chiral symmetry breaking in a dark confining gauge theory, analogous to the Standard Model (SM) pion. A dark sector with arbitrary N_f flavors of dark quarks leads to $N_f^2 - 1$ axionlike states, effectively a *field theory axiverse* (or "π-axiverse"). A portal to the visible sector can be achieved through the standard kinetic mixing between the dark photon and SM photon, generating millicharges for the dark quarks and consequently couplings, both parity even and parity odd, between the SM and the dark pions. This scenario has been studied for the $N_f = 2$ case and more recently for a dark Standard Model with $N_f = 6$. In this work, we study the spectrum of this field theory axiverse for an arbitrary number of flavors and apply this to the example $N_f = 10$. We calculate the couplings to the SM photon analogous to the conventional axion-photon coupling, including the N_f and N_c dependence, and compute the present and future constraints on the $N_f = 10$ $N_c = 3$ π-axiverse. We elucidate the accompanying "baryverse" of superheavy dark baryons, namely, an ensemble of charged and neutral dark baryons with a mass set by the dark pion decay constant.

DOI: 10.1103/PhysRevD.109.116019

I. INTRODUCTION

The microscopic nature of dark matter is one of the most significant puzzles in physics to date. Among many possibilities (see, e.g., Ref. [1] for a review), a prominent dark matter candidate is the *axion*, or the more general axionlike particle (ALP). Initially motivated by the strong *CP* problem of the Standard Model (SM) [2–4], the axion was shortly thereafter proposed as a cold dark matter candidate [5–7]. Axions and ALPs play a starring role in modern cosmology, where they not only can serve as the observed dark matter but also provide a candidate for the cosmic inflation field or the dark energy field (see Ref. [8] for a review of axion cosmology).

String theory predicts tens to hundreds of ALPs [9–11], whose properties are sensitive to the geometry and topology of the extra dimensions and the extended objects (e.g., D-branes) and fields contained therein. The term "axiverse" [10] has been used to describe the spectrum of string theory axions, whose masses are expected to span as many as 30 orders of magnitude. This work will focus on an alternative path to an axiverse.

Since ordinary matter largely originates from QCD, it is natural to consider a dark version of the strong force such as dark QCD (dQCD) (see e.g. [12–14]). This is appealing for many reasons, but perhaps most importantly, QCD is well understood both theoretically and experimentally. Furthermore, dQCD with ultralight (dark) quarks contains a spectrum of ALPs with properties similar to yet discernible from standard axion models. These states arise as pseudo-Nambu-Goldstone bosons (PNGBs) from the breaking of a global $SU(N_f) \times SU(N_f)$ symmetry, where N_f is the number of dark quark flavors. The ALPs are composite degrees of freedom analogous to the SM pion and have thus been dubbed *π-axions* [15]. The chiral symmetry is broken below the dQCD scale, Λ_{dQCD}, which arises as a result of dimensional transmutation in the same fashion as QCD in the SM [16]; $SU(N_f)_{\mathrm{L}} \times SU(N_f)_{\mathrm{R}} \to SU(N_f)_{\mathrm{V}}$ results in $N_f^2 - 1$ PNGBs, where $N_f - 1$ states are real pseudoscalars which experience an axionlike coupling to the SM photon. For large N_f, this is effectively an axiverse completely independent of the string axiverse, which we will call the *field theory axiverse*.

*Contact author: stephon_alexander@brown.edu
†Contact author: tucker_manton@brown.edu
‡Contact author: e.mcdonough@uwinnipeg.ca

A dark matter construction of this type was first discussed and proposed in Ref. [17]; a further analysis was carried out in Ref. [15] exploring the cosmology and detection prospects in the specific case of a dark Standard Model, with $N_f = 6$ as in the visible Standard Model. Compared to the many dark matter models on the market, including "dynamical" axion models such as in Refs. [18–23], composite ultralight dark matter has been studied relatively sparsely [14,15,17,24–29]. However, due to the significant efforts going into direct and indirect detection of axions and ALPs (see, e.g., Ref. [8] for a review), it is particularly appealing to consider models that predict axionlike couplings to the SM photon which can be distinguished from the string axiverse and the standard QCD axion.

The field theory axiverse is characterized by a tightly packed mass spectrum for all $N_f - 1$ real pseudoscalars, in contrast with the *logarithmic* distribution of axion masses in the string theory axiverse [30]. Moreover, this axiverse has a *single decay constant*, again in contrast with the logarithmic distribution of decay constants in the string theory axiverse [30]. This crucial difference owes to the fact that the field theory axiverse relies a single nonperturbative effect, namely, confinement in the dark QCD, whereas the string theory axiverse utilizes an array of nonperturbative effects, roughly one per axion.

The field theory axiverse also exhibits couplings to the Standard Model, in particular, the familiar ALP interaction with photons. The pion-photon interaction has strength $g_{\pi\gamma\gamma} \simeq \mathcal{A} N_c \epsilon^2 \alpha_{\rm em}/F_\pi$, where $\mathcal{A} = \mathcal{O}(1)$ is an anomaly coefficient, ϵ is the millicharge parameter, and F_π is the decay constant. In comparison to the axion-photon coupling of single-field ALP dark matter, this can be understood as an overall enhancement of the latter, sensitive to both the number of flavors N_f and the number of colors N_c of the dark QCD theory.

The structure of this paper is as follows. In Sec. II, we begin by presenting the model and discussing the relevant parameters and regimes of interest corresponding to ongoing axion detection efforts. We then dive into our primary results: in Sec. II A, we derive the formulas which enumerate the axiverse states (charged, complex neutral, or real neutral) for an arbitrary number of flavors N_f, while in Sec. II B, we review the Gell-Mann-Oakes-Renner relation, which approximates the mass of the π-axions. Section II C is devoted to an analysis of an effective axion-photon coupling, which experiences an enhancement as a function of N_f and N_c that follows from the chiral anomaly and the presence of multiple real pseudoscalars. We then comment on the additional portals to the SM in Sec. II D before examining the specific case of $N_f = 10$, $N_c = 3$ in Sec. III, including the spectrum of the $10^2 - 1 = 99$ states in Sec. III A, and experimental constraints in Sec. III B. Finally, in Sec. IV, we briefly discuss dark baryoniclike states composed of N_c dark valence quarks. Such states are plentiful in models with arbitrary N_f, N_c, and we present an approximate formula for enumerating this "baryverse" associated to the field theory axiverse. We then conclude in Sec. V, followed by a brief Appendix where we write out the algorithm for computing the generators T^a for an arbitrary $SU(N)$ (Appendix A) and the result of constructing the $\Sigma = T^a \pi_a$ matrix for $SU(10)$ (Appendix B).

II. FIELD THEORY AXIVERSE

The focus of this work is a confining gauge theory, with N_f flavors of dark quark and $SU(N_c)$ gauge group. The Lagrangian is given by

$$\mathcal{L}_{\rm dQCD} = -\frac{1}{2}{\rm Tr} G_{\mu\nu} G^{\mu\nu} + \sum_{i,j=1}^{N_f} \bar{q}^i \left(i\not{D} \delta_{ij} - m_{ij} \right) q^j, \quad (1)$$

where $G_{\mu\nu}$ is the non-Abelian field strength tensor and q^i, $i = 1 \ldots N_f$, denote Dirac fermions in the fundamental representation of the dark $SU(N_c)$ gauge symmetry. The mass matrix m_{ij} is in general an $N_f \times N_f$ matrix with off diagonal elements; however, for simplicity, we focus on the case of a diagonal mass matrix and consequently do not allow for flavor-changing currents. This is different from the dark Standard Model of Ref. [15], where ultraheavy (dark) weak bosons were integrated out of the low-energy spectrum to give effective vertices allowing for flavor-changing processes. In this work, we assume only a dark strong sector, wherein all field content is neutral under weak isospin. Instead, the dark quarks are solely endowed with a millicharge under the SM $U(1)_{\rm em}$, which arises through kinetic mixing between the dark and SM photon [31]. Moreover, that the dark quarks lack weak isospin implies all anomaly cancellations are trivially satisfied (see, e.g., Ref. [32]).

This theory exhibits a global $U(N_f) \times U(N_f)$ chiral symmetry, which is broken at low energies, below the confinement scale $\Lambda_{\rm dQCD}$. In particular, the breaking of the subgroup,

$$SU(N_f)_{\rm L} \times SU(N_f)_{\rm R} \to SU(N_f)_V, \quad (2)$$

leads to a spectrum of Goldstone bosons known as pions. In this work, we consider the spectrum of pions as an effective axiverse, analogous to the string theory axiverse [9–11].

In its simplest form, this formulation of the axiverse has only two free dimensional parameters: the quark mass m_q and the dark confinement scale $\Lambda_{\rm dQCD}$. At low energies, these determine the neutral pion mass scale and pion decay constant as

$$m_\pi^2 = m_q \Lambda_{\rm dQCD}, \qquad F_\pi \sim \Lambda_{\rm dQCD}. \quad (3)$$

The present work focuses on the regime of a hierarchy,

$$\frac{m_q}{\Lambda_{\rm dQCD}} \ll 1, \tag{4}$$

in order to realize a pion mass scale and decay constant comparable to that conventionally associated with axions (see, e.g., Ref. [8]):

$$m_\pi < {\rm eV}, \qquad F_\pi \gtrsim 10^{11}\ {\rm GeV}. \tag{5}$$

The dark pions constitute light axionlike particles, which can serve as dark energy, dark matter, or both.

As a dark matter model, the two free parameters of the theory can be further reduced to one single free parameter under the assumption that the dark pions constitute the entirety of the dark matter: matching to the observed abundance of dark matter $\Omega_{\rm dm}h^2 = 0.12$ determines the decay constant F_π as a function of the mass, leaving only a single free parameter, the mass scale m_π, for this dark matter scenario.

Remarkably, a portal to Standard Model may be achieved without introducing a new free parameter: we can endow our dark quarks with a charge under Standard Model identical to that of the SM quarks, namely,

$$Q_e = -\frac{1}{3}e, \frac{2}{3}e. \tag{6}$$

In contrast with naive expectation, e.g., from experience with millicharged dark matter [14,31,33–36], the model is naturally safe from constraints on electrically charged dark matter due to the high confinement scale: the charge of the dark quarks is confined inside neutral pions up to a scale $\Lambda_{\rm dQCD} > 10^{11}$ GeV, and all charged states (e.g., charged dark pions and charged dark baryons) have mass at the dark QCD scale.

In this work, we will follow the millicharged dark matter convention and endow our dark quarks with a fractional electric charge parametrized by $\varepsilon \leq 1$. We moreover make a slight generalization of the one-third/two-third fractional charge and only demand that the up- and down-type fractional charges satisfy

$$Q_u - Q_d = \varepsilon. \tag{7}$$

In what follows, we develop in detail the spectrum of pions, including their mass and coupling to the Standard Model.

A. Enumerating the axiverse

For an arbitrary $SU(N_f)$ flavor symmetry breaking, one generally has $N_f^2 - 1$ states where $N_f - 1$ are real scalars and electrically neutral. The remaining states are complex scalars, either electrically charged or neutral (analogous to the SM $\pi_\pm$ and neutral kaon K^0). To count the number of each states, one can imagine constructing a table where the columns are labeled by each quark charged, increasing by generation. Label each row by the charge of the antiquark of the same flavor. In this table, the diagonal entries are related to the real, neutral states analogous to the SM π_0 and η particle. Their specific quark substructure is dictated by the diagonal $SU(N_f)$ generators (A1). The off-diagonal states are all complex scalars whose composition can be identified using the generators (A2) and (A3), and their charge is given by the sum of the quark + antiquark charge from the row/column in which it lies. An example of such a table is given in Table I.

TABLE I. We label each quark in increasing generation along the top row simply as $u_1, d_1, \ldots$ along with their antiquark pairs along the leftmost column. Their respective charges $\pm Q_u$ and $\pm Q_d$ are written in the superscripts.

	$u_1^{Q_u}$	$d_1^{Q_d}$	$u_2^{Q_u}$	$d_2^{Q_d}$	$\cdots$
$\bar{u}_1^{-Q_u}$	0	−1	0	−1	$\cdots$
$\bar{d}_1^{-Q_d}$	1	0	1	0	$\cdots$
$\bar{u}_2^{-Q_u}$	0	−1	0	−1	$\cdots$
$\bar{d}_2^{-Q_d}$	1	0	1	0	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Since the states reflected over the diagonal are complex conjugates, we can focus on the states above the diagonal only. We see that the set of states that are the first diagonal above the primary diagonal, which start with the $d_1\bar{u}_1$ bilinear with charge -ε, are all charged, while the adjacent diagonal section, starting with the $u_2\bar{u}_1$ state, are all neutral. This pattern continues as N_f gets larger. For the charged states, counting via adding the alternating diagonals, we see the pattern

$$\begin{aligned}(N_f - 1) + (N_f - 3) + (N_f - 5) + \cdots \\ = \sum_{k=1}^{n_\pm}(N_f - 2k + 1) = n_\pm N_f - n_\pm^2.\end{aligned} \tag{8}$$

For the complex neutral states, we count

$$\begin{aligned}(N_f - 2) + (N_f - 4) + (N_f - 6) + \cdots \\ = \sum_{k=1}^{n_0}(N_f - 2k) = n_0 N_f - n_0^2 - n_0.\end{aligned} \tag{9}$$

When N_f is even, these sums terminate at $n_\pm = N_f/2$ and $n_0 = N_f/2 - 1$, respectively, and we find that

$$\text{even}:\ N_\pm = \frac{N_f^2}{4}, \qquad N_0 = \frac{N_f(N_f - 2)}{4}. \tag{10}$$

The total number of states include the complex conjugates of the above along with the $N_f - 1$ real scalars so that, altogether, we have

$$2N_{\pm} + 2N_0 + (N_f - 1) = N_f^2 - 1, \tag{11}$$

as required. In the case where N_f is odd, evaluate (10) for $N'_f = N_f - 1$, and the remaining row/column will have $N_f - 1$ total states, where half are neutral and half are charged. This pattern holds true whether the fractional charge of the remaining quark has magnitude $|Q_u|$ or $|Q_d|$. In total, we find

$$\text{odd: } N_{\pm} = \frac{N_f^2 - 1}{4}, \qquad N_0 = \frac{(N_f - 1)^2}{4}. \tag{12}$$

It is trivial to check that (12) satisfies (11).

At this stage, the number of flavors N_f is arbitrary. However, we require that it satisfies an inequality related to the number of colors N_c in the following way. The QCD β-function [37,38] is famously

$$\beta = -\frac{g^3}{16\pi^2}\left(\frac{11}{3}N_c - \frac{2}{3}N_f\right). \tag{13}$$

Confinement requires that $\beta < 0$, constraining the relationship between the number of colors and flavors to be

$$\frac{11}{2}N_c > N_f. \tag{14}$$

Therefore, the maximum number of flavors we can consider for, say, $N_c = 3$ is $N_f = 16$.

B. Axion mass spectrum

The masses of the individual π-axions are related to the quark masses m_{q_i}, the decay constant F_π, and the dark QCD scale Λ_{dQCD}. The Gell-Mann-Oakes-Renner (GMOR) relation [39] approximates the masses of the electrically neutral states as

$$m^2_{\pi^0_i} \simeq \frac{\langle q\bar{q}\rangle}{F_\pi^2}\sum_i m_{q_i}, \tag{15}$$

where $\langle q\bar{q}\rangle \sim \Lambda^3_{\mathrm{dQCD}}$ is the quark condensate and the m_{q_i} are the masses of the constituent quarks. We will see that it is straightforward to identify the quark content of a given π-axion state using the $SU(N_f)$ generators.

For charged π-axions, photon loops (either from the visible or dark sector) give corrections to the GMOR relation [40],

$$m^2_{\pi^\pm_i} \simeq m^2_{\pi^0_i} + 2\xi_i \varepsilon^2 e^2 F_\pi^2, \tag{16}$$

where $\xi = \mathcal{O}(1)$. Since $F_\pi > 10^{11}$ GeV by assumption, if $\varepsilon = \mathcal{O}(1)$, then the charged pions are *superheavy* particles, which exhibit their own interesting phenomenology [41].

C. Axion-photon coupling

We now focus on the canonical axion-photon coupling, which is the focus of many theoretical efforts [42–45] and experimental searches [46–56] for ALPs. In our case, we have

$$\mathcal{L}_{\pi^I\gamma\gamma} = \frac{\lambda_I \varepsilon^2 \alpha_{em}}{2F_\pi}\pi_I F_{\mu\nu}\tilde{F}^{\mu\nu}, \tag{17}$$

where $I = 1, 2, \ldots, N_f - 1$ denotes the different neutral pseudoscalars. Our coupling is related to the conventional axion-photon couplings as

$$g^{(I)}_{\pi\gamma\gamma} = \frac{2\lambda_I \varepsilon^2 \alpha_{em}}{F_\pi}. \tag{18}$$

The coefficients λ_I follow from the chiral anomaly,

$$\partial_\mu j^{\mu 5 I} = -\frac{e^2}{16\pi^2}F\tilde{F} \times \mathrm{Tr}(T^I Q^2), \tag{19}$$

where T^I is an $SU(N_f)$ generator and Q is the quark charge matrix $Q = \mathrm{diag}(Q_u, Q_d, Q_u, \ldots)$. The chiral anomaly gets a nonzero contribution from each diagonal generator, each of which corresponds to the coupling of a particular neutral pion state, i.e.,

$$\lambda_I = \frac{N_c}{4\pi}\mathrm{Tr}(T^I Q^2) \equiv \frac{N_c}{8\pi}\tilde{\lambda}_I. \tag{20}$$

The matrices in the trace do not depend on the number of colors, allowing us to pull out a factor of N_c. The relevant generators for this calculation are given in (A1), where $I = 1, 2, \ldots, N_f - 1$. Keeping Q_u, Q_d arbitrary, we find

$$\begin{aligned}
\tilde{\lambda}_1 &= Q_u^2 - Q_d^2, \\
\tilde{\lambda}_2 &= -\frac{1}{\sqrt{3}}(Q_u^2 - Q_d^2), \\
\tilde{\lambda}_3 &= \sqrt{\frac{2}{3}}(Q_u^2 - Q_d^2), \\
\tilde{\lambda}_4 &= -\sqrt{\frac{2}{5}}(Q_u^2 - Q_d^2), \\
\tilde{\lambda}_5 &= \sqrt{\frac{3}{5}}(Q_u^2 - Q_d^2), \\
\tilde{\lambda}_6 &= -\sqrt{\frac{3}{7}}(Q_u^2 - Q_d^2), \\
\tilde{\lambda}_7 &= \sqrt{\frac{4}{7}}(Q_u^2 - Q_d^2), \\
&\vdots
\end{aligned} \tag{21}$$

To find a closed-form expression for the $\tilde{\lambda}$, note that the coefficient in front of the $(Q_u^2 - Q_d^2)$ term has a numerator which is the square root of $a_n = \{1, 1, 2, 2, 3, 3, \ldots\}$, which can be represented by the sequence

$$a_n = \frac{2n + 1 + (-1)^{n+1}}{4}, \quad (22)$$

while the denominator is the square root of $b_n = \{1, 3, 3, 5, 5, \ldots\}$, which we can write as

$$b_n = \frac{2n + 1 + (-1)^n}{2}, \quad (23)$$

up to the alternating sign. The coefficient in front of the nth term is therefore $\sim(-1)^{n+1}\sqrt{a_n/b_n}$. Specifically,

$$\tilde{\lambda}_n = \frac{(-1)^{n+1}}{\sqrt{2}}\sqrt{\frac{2n+1+(-1)^{n+1}}{2n+1+(-1)^n}} \times \left(Q_u^2 - Q_d^2\right). \quad (24)$$

The coefficient in front of the $N_f - 1$ term is thus

$$\tilde{\lambda}_{N_f-1} = \frac{(-1)^{N_f}}{\sqrt{2}}\sqrt{\frac{2N_f - 1 + (-1)^{N_f}}{2N_f - 1 + (-1)^{N_f-1}}} \times \left(Q_u^2 - Q_d^2\right). \quad (25)$$

From this, we find that the all $(N_f - 1)$ neutral pions have roughly the same axion-photon coupling, their values differing only by $\mathcal{O}(1)$ factors.

D. Other portals to standard model

Finally, we consider the other possible portals to the Standard Model. The coupling to photons can be of the form

$$\mathcal{L}_{\rm int}^{(1)} = \frac{\lambda_1 \varepsilon^2}{2F_\pi}(\pi^0)F_{\mu\nu}\tilde{F}^{\mu\nu}, \quad (26)$$

$$\mathcal{L}_{\rm int}^{(2)} = \frac{\lambda_2 \varepsilon^2}{2}(\pi^+)(\pi^-)A_\mu A^\mu, \quad (27)$$

$$\mathcal{L}_{\rm int}^{(3)} = \frac{\lambda_3 \varepsilon^2}{2\Lambda_3^2}(\pi^+)(\pi^-)F_{\mu\nu}F^{\mu\nu}, \quad (28)$$

$$\mathcal{L}_{\rm int}^{(4)} = \frac{\lambda_4 \varepsilon^2}{2\Lambda_4^2}(\pi_i)(\pi_j)F_{\mu\nu}F^{\mu\nu}. \quad (29)$$

Formally, each term is a sum over all π-axion states which participate in the interaction, which we have omitted for brevity. The first interaction (26) is the standard axion-photon coupling arising from a triangle diagram, and only the $N_f - 1$ real, neutral states experience the vertex. The second interaction (27) stems from the gauge-covariant derivative in scalar QED, and all charged π-axions participate in the interaction. The remaining two vertices, Eqs. (28) and (29), are EFT operators arising from integrating out the heavy degrees of freedom in the dark SM. Equation (28) entails a sum over all complex π-axions, while Eq. (29) sums over all neutral π-axions, both complex and real.

Similarly, pseudoscalar couplings to SM fermions can arise through [8]

$$\mathcal{L}_{\pi N} = \frac{g_{\pi N}\varepsilon^2}{2m_N}\partial_\mu \pi_0(\bar{N}\gamma^\mu\gamma^5 N), \quad (30)$$

$$\mathcal{L}_{\pi e} = \frac{g_{\pi e}\varepsilon^2}{2m_e}\partial_\mu \pi_0(\bar{e}\gamma^\mu\gamma^5 e), \quad (31)$$

$$\mathcal{L}_{\pi N\gamma} = -\frac{i\varepsilon^2}{2M_*^2}\pi_0 \bar{N}J^{\mu\nu}\gamma^5 N F_{\mu\nu}, \quad (32)$$

where N is a SM nucleon; e is an electron, muon, or tau particle; $F_{\mu\nu}$ is the photon field strength; and $J^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$ are the Lorentz generators. Each of the couplings can be related to the decay constant F_π, namely, $g_{\pi N}/(2m_N) \sim g_{\pi e}/(2m_e) \propto 1/F_\pi$, and $M_* \propto F_\pi$.

III. EXAMPLE: $N_f = 10$

In this section, we will turn our attention to the specific case of $N_f = 10$ and $N_c = 3$. Moreover, we will additionally set the millicharge parameter to unity, $\varepsilon = 1$, leaving as the only free parameters the quark masses and confinement scale. We will remain agnostic to the UV completion of general millicharge values ε, but note that unit charge can be realized simply by the quark charge assignments (without need for UV completion), or through a heavy dark photon with standard kinetic mixing. In this case, the charged π-axions are superheavy and will not be produced by misalignment [15,17].

A. Spectrum of the $N_f = 10$ axiverse

We will again denote each generation of dark quarks as $\{u_i, d_i\}$, $i = 1, \ldots, N_f/2$, in analogy with the SM up and down quarks. For the example of $N_f = 10$, we have five dark quark generations, 99 total π-axion states, 9 real, neutrals, and 45 complex states along with their conjugates. Using (10), we find that 25 of the states are charged and 20 are neutral. We have computed the 99 generators using the algorithm presented in Ref. [57], which we have summarized in (A1)–(A3). As is standard in a Σ model, we can package these states in a matrix $\Sigma = \pi_I T^I$, where $\pi_I = (\pi_1, \pi_2, \ldots, \pi_{99})$ is the pion vector and T^I are the $SU(10)$ generators. The result is displayed in (B1), and the diagonal entries are given by (B2).

To make contact with the π-axion mass content, let us define some generic notation for the dark quark masses. We label a characteristic mass for each of the $N_f/2 = 5$ generations as m_i, $i = 1, \ldots, 5$, and assign a constant to the

TABLE II. Spectrum of real, neutral π-axions: the nine states are constructed using contractions with the diagonal generators (A1). For simplicity, we are omitting the 25 charged states and 20 complex, neutral states.

π-axion	Quark content	Mass ($m^2_{\pi^i}$)
π_1	$u_1\bar{u}_1 - d_1\bar{d}_1$	$2m_qF_\pi$
π_2	$u_1\bar{u}_1 + d_1\bar{d}_1 - 2u_2\bar{u}_2$	$3m_qF_\pi$
π_3	$u_1\bar{u}_1 + d_1\bar{d}_1 + u_2\bar{u}_2 - 3d_2\bar{d}_2$	$4m_qF_\pi$
π_4	$\sum_{i=1}^2(u_i\bar{u}_i + d_i\bar{d}_i) - 4u_3\bar{u}_3$	$5m_qF_\pi$
π_5	$\sum_{i=1}^2(u_i\bar{u}_i + d_i\bar{d}_i) + u_3\bar{u}_3 - 5d_3\bar{d}_3$	$6m_qF_\pi$
π_6	$\sum_{i=1}^3(u_i\bar{u}_i + d_i\bar{d}_i) - 6u_4\bar{u}_4$	$7m_qF_\pi$
π_7	$\sum_{i=1}^3(u_i\bar{u}_i + d_i\bar{d}_i) + u_4\bar{u}_4 - 7d_4\bar{d}_4$	$8m_qF_\pi$
π_8	$\sum_{i=1}^4(u_i\bar{u}_i + d_i\bar{d}_i) - 8u_5\bar{u}_5$	$9m_qF_\pi$
π_9	$\sum_{i=1}^4(u_i\bar{u}_i + d_i\bar{d}_i) + u_5\bar{u}_5 - 9d_5\bar{d}_5$	$10m_qF_\pi$

respective up- and down-type dark quarks in that generation, c_{u_i} and c_{d_i}. As an example, the first generation $\{u_1, d_1\}$ has masses $\{c_{u_1}m_1, c_{d_1}m_1\}$. The SM up and down quarks have masses $m_u = 2.2$ MeV and $m_d = 4.7$ MeV, which in this notation can be written $m_1 = 2.2$ MeV with $c_{u_1} = 1$, $c_{d_1} = 2.14$ such that $m_u = c_{u_1}m_1$ and $m_d = c_{d_1}m_1$.

Using the GMOR relation (15) along with the corrections for the charged states (16), we can approximate the mass for a given π-axion in terms of the constants c_u, c_d, and the characteristic masses m_i. We note that all of the complex scalars are composed of one dark quark and one dark antiquark of a different flavor, while the real scalars are composed of multiple dark quark/antiquark pairs of the same flavor. For the states that are composed of more than three dark quark/antiquark pairs, the GMOR relation (15) is expected to be less accurate.

As a simple example, consider the case of completely degenerate quark masses, $m_1 = m_2 = \ldots = m_5 = m_q$, and $c_{u_1} = c_{d_1} = \ldots = c_{d_5} = 1$. The spectrum of pion masses is shown on the rightmost column in Table II. From this, one may appreciate that all nine pions have mass within a factor of 3: denoting $2m_qF_\pi \equiv m_\pi^2$, the masses range from m_π to $\sqrt{5}m_\pi$. The lightest pion mass $m_\pi \simeq \sqrt{2m_qF_\pi}$ can be expressed in terms of benchmark parameter values as

$$\left(\frac{m_\pi}{\mathrm{eV}}\right)^2 = 2\frac{m_q}{10^{-20}\ \mathrm{eV}}\frac{F_\pi}{10^{11}\ \mathrm{GeV}}. \tag{33}$$

The remaining pion masses are given in Table II.

B. Experimental constraints on axion-photon coupling

To estimate constraints on the model, we approximate our multiple distinct resonances as a single signal. This approach differs from that taken in the context of "ALP anarchy" [58], where one performs a rotation in field space to combine multiple axion-photon couplings into a single effective coupling. Our approach is justified on the basis of the tightly packed mass spectrum, namely, that all nine fields have mass in the range $[\sqrt{2}, \sqrt{10}]\sqrt{m_qF_\pi}$.

The combined axion photon coupling then is the sum over the individual couplings, i.e.,

$$\lambda_{\mathrm{eff}} \equiv \sqrt{\sum_{I=1}^{N_F-1}\lambda_I^2} = \frac{N_c}{8\pi}\sqrt{\sum_{I=1}^{N_F-1}\tilde{\lambda}_I^2}, \tag{34}$$

where the form of the $\tilde{\lambda}_I$ are given in (24). The sum works out to be

$$\sum_{I=1}^{9}\tilde{\lambda}_I^2 = 5\times\left(Q_u^2 - Q_d^2\right)^2, \tag{35}$$

in this case, thus,

$$\lambda_{\mathrm{eff}} = \frac{\sqrt{5}N_c}{8\pi}\times\left(Q_u^2 - Q_d^2\right). \tag{36}$$

If we set the fractional quark charges to the analogous SM values $Q_u = 2/3$, $Q_d = -1/3$, along with $N_c = 3$, we find a corresponding axion-photon coupling (18)

$$g_{\pi\gamma\gamma} = \frac{\sqrt{5}\alpha_{em}}{4\pi F_\pi}. \tag{37}$$

To isolate for the mass dependence, we assume that the real pseudoscalar π-axions constitute the total dark matter relic density. That is, the relic density produced via misalignment is given by

$$\Omega_\pi = \frac{1}{6}(9\Omega_r)^{3/4}\frac{F_\pi^2}{M_{\mathrm{Pl}}^2}\sum_i\left(\frac{m_{\pi_i}}{H_0}\right)^{1/2}\theta_{\pi_i}^2, \tag{38}$$

where the sum is over all light stable π-axion fields, and we assume $m_{\pi^i} > 10^{-28}$ eV for simplicity (see Ref. [8] for the relic density when $m_{\pi^i} < 10^{-28}$ eV). The left-hand side of (38) is fixed by observation, $\Omega_{\mathrm{DM}}h^2 = 0.12$ [59].

From this, we find the axion photon as a function of the lightest pion mass as

$$g_{\pi\gamma\gamma} = 1.3\times10^{-11}\alpha_{em}\theta_i\lambda_{\mathrm{eff}}\left(\frac{m_\pi}{\mathrm{eV}}\right)^{1/4}\ \mathrm{GeV}^{-1}, \tag{39}$$

where for concreteness we assume a common initial misalignment θ_i for the real pseudocalar pions.[1] The constraints on this scenario are shown in Fig. 1.

[1]Note this can easily be generalized to include the neutral complex pions, which changes $g_{\pi\gamma\gamma}$ in Eq. (39) by a factor of $\simeq\sqrt{5}$.

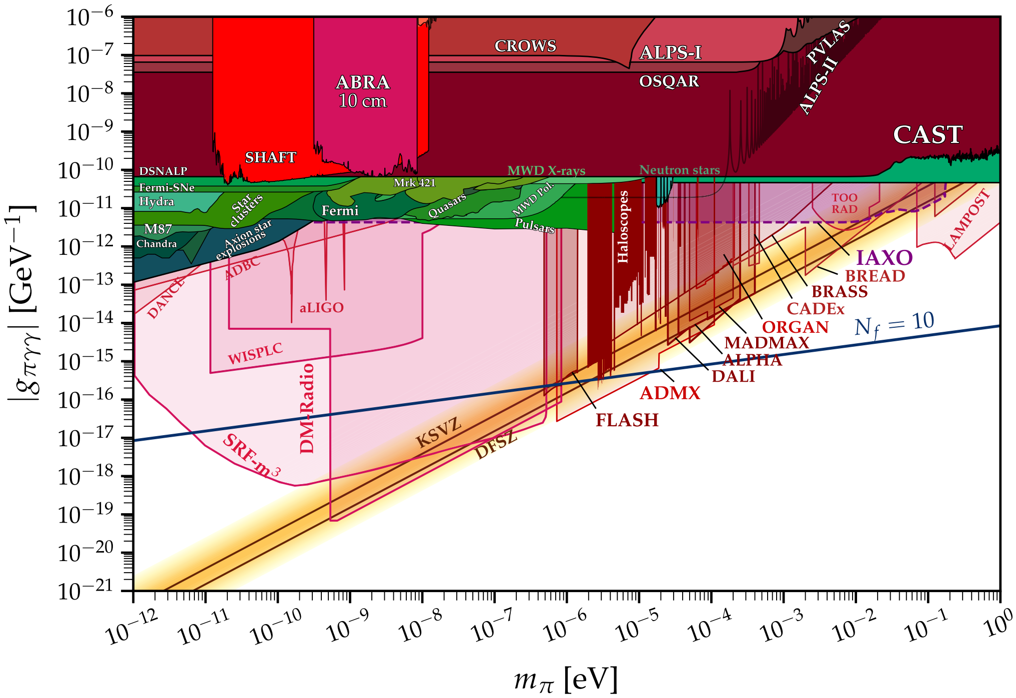

FIG. 1. Constraints on the axion-photon coupling in the field theory axiverse for $SU(3)$ dark QCD with $N_f = 10$ dark quarks and $\varepsilon = 1$. The dark blue line indicates the model prediction for the pion-photon coupling, overlaid on the constraint data that can be found in Ref. [60]. We assume the neutral pseudoscalar pions constitute the observed relic density of dark matter and denote as m_π the mass of the lightest pion.

IV. DARK BARYVERSE

Consider a color neutral state composed of N_c constituent dark quarks, analogous to SM baryons,

$$B \sim \prod_i^{n_u} \prod_j^{n_d} \langle u_i d_j \rangle. \tag{40}$$

These are color singlet states analogous to the proton and neutron, where the numbers of up-type n_u and down-type n_d valence quarks satisfy

$$n_u + n_d = N_c. \tag{41}$$

Here, we look to count the number of dark baryons as a function of the number of flavors N_f and colors N_c. The enumeration can be understood in the following way. We can organize the counting by noting that a given state could be composed of a single type of quark, 2-types, 3-types, etc., up to all quarks being unique. Symbolically, we can write this as

$$\begin{gathered}
\langle \overbrace{\underbrace{q_i q_i q_i \cdots q_i q_i}_{N_c}}^{1\text{ type}} \rangle, \qquad \langle \overbrace{q_i \underbrace{q_j q_j \cdots q_j q_j}_{N_c-1}}^{2\text{ types}} \rangle, \\
\langle \overbrace{q_i q_j \underbrace{q_k q_k \cdots q_k}_{N_c-2}}^{3\text{ types}} \rangle, \qquad \langle \overbrace{q_i q_j q_k \underbrace{q_l \cdots q_l}_{N_c-3}}^{4\text{ types}} \rangle, \\
\vdots \\
\langle \overbrace{\underbrace{q_i q_j \cdots q_k}_{N_c-2} q_l q_l}^{N_c-1\text{ types}} \rangle, \qquad \langle \overbrace{\underbrace{q_i q_j q_l \cdots q_m q_n}_{N_c}}^{\text{all unique}} \rangle.
\end{gathered} \tag{42}$$

For states composed of a single quark type, there are obviously N_f of them. For states composed of two quark types like the second bra-ket in (42), we have $N_f - 1$ choices for the q_i for each q_j. This totals $N_f \times \binom{N_f-1}{1}$ where $\binom{n}{k}$ is the binomial coefficient. For states composed of three quark types like the third bra-ket in (42), we choose the 2 q_i, q_j from $N_f - 1$ choices for each q_k, giving $N_f \times \binom{N_f-1}{2}$. This pattern continues until we get to the second to last bra-ket state in (42), where we choose $N_c - 2$ states from $N_f - 1$ options, for each q_l. This gives a contribution of

$N_f \times \binom{N_f-1}{N_c-2}$. The rightmost state is composed of all unique quarks, no repeated flavors. This contributes $\binom{N_f}{N_c}$. Adding all of these possibilities together gives the total number of baryon states built from a unique composition of quarks:

$$N_B(N_f, N_c) = \binom{N_f}{N_c} + N_f \times \sum_{k=0}^{N_c-2} \binom{N_f - 1}{k}. \quad (43)$$

The example of $N_f = 10$ and $N_c = 3$ results in the total number of baryons being $N_B(10, 3) = 220$.

Notably, this result does not count states that have the same quark composition but different total angular momentum J. As an example from the Standard Model, the proton p^+ and Δ^+ particle are both composed of uud valence quarks, but the proton has $J = 1/2$, while the Δ^+ has $J = 3/2$. They of course have the same charge, but they have different masses and radically different lifetimes. We further note that the charge of a given baryon B is straightforwardly obtained by adding the fractional charge of its quark content, $Q_B = Q_u n_u + Q_d n_d$, where the up-type and down-type quark charges satisfy $Q_u - Q_d = \varepsilon$.

Lastly, we comment on the production of dark baryons. Throughout this work, and the previous works [15,17], the confinement scale of dark QCD (analogous to the Peccei-Quinn (PQ) scale) is assumed to be higher than the energy scale of inflation so that particles generated at the QCD phase transition are redshifted away, leaving the initial misalignment of the axions as the predominant contribution to the relic density. However, even in this case, dark baryons can be produced through a variety of means, such as coupling to the inflaton, through the millicharge portal, or through gravitational production. See Ref. [17] for further details.

V. CONCLUSIONS

In this paper, we have presented a *field theory axiverse* by assuming a simple, dark QCD with ultralight quarks and a large dark confinement scale, satisfying the hierarchy $m_q/\Lambda_{\rm dQCD} \ll 1$. The relic dark matter candidates are the PNGBs associated to the spontaneous breaking $SU(N_f)_{\rm L} \times SU(N_f)_{\rm R} \to SU(N_f)_{\rm V}$, of which there are $N_f^2 - 1$ states. The dark quarks are (milli)charged under $U(1)_{\rm EM}$, resulting in a portal between the dark sector and the SM photon. The $N_f^2 - 1$ PNGBs are either charged or neutral; complex scalar fields; or real, neutral scalars analogous to the SM charged pion, kaon, and neutral pion, respectively.

The spectrum and couplings of the field theory axiverse follow from simple considerations, and indeed in Sec. II A, we derived formulas which count the number of unique states that are real pseudoscalars or complex pseudoscalars and whether the states are charged or neutral for the latter case. The formulas apply for arbitrary N_f, provided the number of flavors satisfies the inequality $N_f < \frac{11}{2} N_c$ such that the β-function is negative and the quarks are confined. The masses are then dictated by the GMOR relation approximating the π-axion masses, and the couplings are dictated by the chiral anomaly.

As a concrete example of the power of this approach, we have considered the specific case of $N_f = 10$ and $N_c = 3$, where we find 9 real, 25 charged, and 20 complex neutral states, totaling (along with the conjugates) $100^2 - 1 = 99$ π-axions. Given that the nine neutral π-axions all have mass within the range $\sqrt{2}\sqrt{m_q F_\pi}$ to $\sqrt{10}\sqrt{m_q F_\pi}$, we estimated the experimental constraints on this scenario by approximating the distinct resonances as a single signal, combining the nine individual couplings into an effective coupling.

Assuming that the relic density is produced by misalignment and that the real pseudoscalar π-axions constitute the total dark matter density today, we produced the current and projected future constraints Fig. 1, illustrating both the contrast between the field theory axiverse and the standard Kim-Shifman-Vainshtein-Zakharaov (KSVZ) and Dine-Fischler-Srednicki-Zhitnitsky (DFSZ) axions, as well as the regimes where the model is still in play. This simple analysis, where we have assumed that the relic density is dominated by the real pseudoscalars, can be easily generalized to include the neutral complex states; this weakens the predicted $g_{\pi\gamma\gamma}$ by a factor of $\simeq\sqrt{5}$.

An intriguing feature of this path to the axiverse is the prediction of a corresponding dark *baryverse*, namely, the stable baryonic states of the dark QCD theory. While the axiverse is naturally on the ultralight end of the mass spectrum, the dark baryons are naturally *superheavy*. Superheavy dark matter brings its own rich phenomenology [41], adding to the opportunities for testing this scenario. Analogous to the axionlike states, in this work, we have derived a formula for counting the dark baryonlike states in the field theory axiverse, where the number of up- and down-type valence quarks satisfies $n_u + n_d = N_c$, finding a total of 220 states for $N_f = 10$, $N_c = 3$.

An open question for the field theory axiverse is the origin and radiative stability of the requisite hierarchy between the quark mass m_q and the confinement scale $\Lambda_{\rm dQCD}$. This manifests the conventional axion quality problem. While small quark masses are technically natural due to the chiral symmetry of the dark QCD theory, once embedded in any UV completion, e.g., a Higgs mechanism, the quark masses are expected to receive radiative corrections which may in turn generate a large pion mass. Note that the "constituent" quark masses ($1/N_c$ of the baryon mass), corresponding to the quark mass induced by the condensate, are distinct from the fundamental quark mass and themselves does not enter in to the GMOR relation for the pion mass.

Our results present a number of interesting directions for future work. Top among the list is a dedicated analysis of broadband direct-detection searches for multicomponent axion dark matter arising in this scenario and other phenomenological aspects of multiaxion scenarios (see, e.g., Refs. [58,61–64]), such as axion stars [65–70]. It will also be interesting to consider fuzzy dark matter phenomenology such as vortices [71] and other substructure [72], which can leave an imprint in, e.g., strong gravitational lensing [73] or cosmic filaments [74]. Ultralight axionlike particles can also play a role in cosmological parameter tensions (see, e.g., Ref. [75]). It remains an interesting question if these observable windows can be used to discriminate between axionlike particle candidates, in particular the pions presented here vs other composite ultralight dark matter models such as those in Refs. [14] and [24]. Complementary to these studies would be a dedicated analysis of the superheavy dark baryons of the dark QCD theory and their associated phenomenology.

Finally, it will also be interesting to embed this scenario into other models, such as a Grand Unified Theory with an $SO(10)$ gauge group. This is a natural context where one might expect a dark strong force with $N_f \gtrsim 10$ charged fermions and will permit a detailed study of the axion quality problem in this model. An interesting question is the pathway to discriminating the dark Standard Model construction of Ref. [15] from a dark Grand Unified Theory and from the minimal field theory axiverse presented here. We leave this and other directions for future work.

ACKNOWLEDGMENTS

The authors thank Heliudson Bernardo, Humberto Gilmer, Andrew Long, Michael Toomey, and Wenzer Qin for helpful comments. S. A. and T. M. are supported by the Simons Foundation, Award 896696. E. M. is supported in part by a Discovery Grant from the Natural Sciences and Engineering Research Council of Canada and by a New Investigator Operating Grant from Research Manitoba.

APPENDIX A: $SU(N)$ GENERATORS

An algorithm for constructing the $N^2 - 1$ generators for arbitrary N is described in Ref. [57]. We note that the labeling in this approach differs from the standard labeling of the $SU(N)$ generators. For example, the $SU(3)$ generators (Gell-Mann matrices [76]) are organized in increasing order of $SU(2)$ subgroups; i.e., $\{\lambda_1, \lambda_2, \lambda_3\}$ and $\{\lambda_4, \lambda_5, a\lambda_3 + b\lambda_8\}$ form two of the three $SU(2)$ subalgebras of $SU(3)$, where a and b are constants. Although the labeling is not important, our notation implies, for example, that the analog of the SM neutral pion, usually identified as π_3 in the Σ model, is our π_1.

The following matrices, which we will call t^I, satisfy $\mathrm{Tr}([t^I, t^J]) = 2\delta^{IJ}$. These are related to those in the main text by $T^I = \frac{1}{2} t^I$, such that we adhere to the standard physics convention for the index of the fundamental representation being $1/2$, i.e., $\mathrm{Tr}([T^I, T^J]) = \frac{1}{2}\delta^{IJ}$. The $N - 1$ diagonal generators are given by

$$\begin{pmatrix} 1 & & & & & \\ & -1 & & & & \\ & & 0 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & -2 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix}, \ldots, \sqrt{\frac{2}{N(N-1)}}\begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & & -N+1 \end{pmatrix}, \tag{A1}$$

while the remaining $N(N-1)$ generators are constructed from the $N(N-1)/2$ real matrices

$$\begin{pmatrix} 0 & 1 & & 0 \\ 1 & 0 & & \\ & & & \\ & & \ddots & \\ 0 & & & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & & 0 \\ 0 & 0 & & & \\ 1 & & 0 & & \\ & & & \ddots & \\ 0 & & & & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 0 & & & 0 \\ 0 & 0 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & 0 & 1 \\ 0 & & & 1 & 0 \end{pmatrix} \tag{A2}$$

and $N(N-1)/2$ complex matrices

$$\begin{pmatrix} 0 & -i & & 0 \\ i & 0 & & \\ & & & \\ & & \ddots & \\ 0 & & & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -i & & 0 \\ 0 & 0 & & & \\ i & & 0 & & \\ & & & \ddots & \\ 0 & & & & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & 0 & & & 0 \\ 0 & 0 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & 0 & -i \\ 0 & & & i & 0 \end{pmatrix}. \tag{A3}$$

APPENDIX B: $SU(10)$ π-AXION Σ MATRIX

Using the algorithm to construct the $10^2 - 1 = 99$ generators for $SU(10)$, we compute the contraction with the pion vector π_I to calculate $\Sigma = t^I \pi_I$, shown in (B1):

$$\begin{pmatrix} \pi_A & \pi_{10} - i\pi_{55} & \pi_{11} - i\pi_{56} & \pi_{13} - i\pi_{58} & \pi_{16} - i\pi_{61} & \pi_{20} - i\pi_{65} & \pi_{25} - i\pi_{70} & \pi_{31} - i\pi_{76} & \pi_{38} - i\pi_{83} & \pi_{46} - i\pi_{91} \\ \pi_{10} + i\pi_{55} & \pi_B & \pi_{12} - i\pi_{57} & \pi_{14} - i\pi_{59} & \pi_{17} - i\pi_{62} & \pi_{21} - i\pi_{66} & \pi_{26} - i\pi_{71} & \pi_{32} - i\pi_{77} & \pi_{39} - i\pi_{84} & \pi_{47} - i\pi_{92} \\ \pi_{11} + i\pi_{56} & \pi_{12} + i\pi_{57} & \pi_C & \pi_{15} - i\pi_{60} & \pi_{18} - i\pi_{63} & \pi_{22} - i\pi_{67} & \pi_{27} - i\pi_{72} & \pi_{33} - i\pi_{78} & \pi_{40} - i\pi_{85} & \pi_{48} - i\pi_{93} \\ \pi_{13} + i\pi_{58} & \pi_{14} + i\pi_{59} & \pi_{15} + i\pi_{60} & \pi_D & \pi_{19} - i\pi_{64} & \pi_{23} - i\pi_{68} & \pi_{28} - i\pi_{73} & \pi_{34} - i\pi_{79} & \pi_{41} - i\pi_{86} & \pi_{49} - i\pi_{94} \\ \pi_{16} + i\pi_{61} & \pi_{17} + i\pi_{62} & \pi_{18} + i\pi_{63} & \pi_{19} + i\pi_{64} & \pi_E & \pi_{24} - i\pi_{69} & \pi_{29} - i\pi_{74} & \pi_{35} - i\pi_{80} & \pi_{42} - i\pi_{87} & \pi_{50} - i\pi_{95} \\ \pi_{20} + i\pi_{65} & \pi_{21} + i\pi_{66} & \pi_{22} + i\pi_{67} & \pi_{23} + i\pi_{68} & \pi_{24} + i\pi_{69} & \pi_F & \pi_{30} - i\pi_{75} & \pi_{36} - i\pi_{81} & \pi_{43} - i\pi_{88} & \pi_{51} - i\pi_{96} \\ \pi_{25} + i\pi_{70} & \pi_{26} + i\pi_{71} & \pi_{27} + i\pi_{72} & \pi_{28} + i\pi_{73} & \pi_{29} + i\pi_{74} & \pi_{30} + i\pi_{75} & \pi_G & \pi_{37} - i\pi_{82} & \pi_{44} - i\pi_{89} & \pi_{52} - i\pi_{97} \\ \pi_{31} + i\pi_{76} & \pi_{32} + i\pi_{77} & \pi_{33} + i\pi_{78} & \pi_{34} + i\pi_{79} & \pi_{35} + i\pi_{80} & \pi_{36} + i\pi_{81} & \pi_{37} + i\pi_{82} & \pi_H & \pi_{45} - i\pi_{90} & \pi_{53} - i\pi_{98} \\ \pi_{38} + i\pi_{83} & \pi_{39} + i\pi_{84} & \pi_{40} + i\pi_{85} & \pi_{41} + i\pi_{86} & \pi_{42} + i\pi_{87} & \pi_{43} + i\pi_{88} & \pi_{44} + i\pi_{89} & \pi_{45} + i\pi_{90} & \pi_I & \pi_{54} - i\pi_{99} \\ \pi_{46} + i\pi_{91} & \pi_{47} + i\pi_{92} & \pi_{48} + i\pi_{93} & \pi_{49} + i\pi_{94} & \pi_{50} + i\pi_{95} & \pi_{51} + i\pi_{96} & \pi_{52} + i\pi_{97} & \pi_{53} + i\pi_{98} & \pi_{54} + i\pi_{99} & \pi_J \end{pmatrix}. \tag{B1}$$

The diagonal terms are explicitly

$$\begin{aligned} \pi_A &= \pi_1 + \frac{\pi_2}{\sqrt{3}} + \frac{\pi_3}{\sqrt{6}} + \frac{\pi_4}{\sqrt{10}} + \frac{\pi_5}{\sqrt{15}} + \frac{\pi_6}{\sqrt{21}} + \frac{\pi_7}{2\sqrt{7}} + \frac{\pi_8}{6} + \frac{\pi_9}{3\sqrt{5}}, \\ \pi_B &= -\pi_1 + \frac{\pi_2}{\sqrt{3}} + \frac{\pi_3}{\sqrt{6}} + \frac{\pi_4}{\sqrt{10}} + \frac{\pi_5}{\sqrt{15}} + \frac{\pi_6}{\sqrt{21}} + \frac{\pi_7}{2\sqrt{7}} + \frac{\pi_8}{6} + \frac{\pi_9}{3\sqrt{5}}, \\ \pi_C &= -\frac{2\pi_2}{\sqrt{3}} + \frac{\pi_3}{\sqrt{6}} + \frac{\pi_4}{\sqrt{10}} + \frac{\pi_5}{\sqrt{15}} + \frac{\pi_6}{\sqrt{21}} + \frac{\pi_7}{2\sqrt{7}} + \frac{\pi_8}{6} + \frac{\pi_9}{3\sqrt{5}}, \\ \pi_D &= -\sqrt{\frac{3}{2}}\pi_3 + \frac{\pi_4}{\sqrt{10}} + \frac{\pi_5}{\sqrt{15}} + \frac{\pi_6}{\sqrt{21}} + \frac{\pi_7}{2\sqrt{7}} + \frac{\pi_8}{6} + \frac{\pi_9}{3\sqrt{5}}, \\ \pi_E &= -2\sqrt{\frac{2}{5}}\pi_4 + \frac{\pi_5}{\sqrt{15}} + \frac{\pi_6}{\sqrt{21}} + \frac{\pi_7}{2\sqrt{7}} + \frac{\pi_8}{6} + \frac{\pi_9}{3\sqrt{5}}, \\ \pi_F &= -\sqrt{\frac{5}{3}}\pi_5 + \frac{\pi_6}{\sqrt{21}} + \frac{\pi_7}{2\sqrt{7}} + \frac{\pi_8}{6} + \frac{\pi_9}{3\sqrt{5}}, \\ \pi_G &= -2\sqrt{\frac{3}{7}}\pi_6 + \frac{\pi_7}{2\sqrt{7}} + \frac{\pi_8}{6} + \frac{\pi_9}{3\sqrt{5}}, \\ \pi_H &= -\frac{1}{2}\sqrt{7}\pi_7 + \frac{\pi_8}{6} + \frac{\pi_9}{3\sqrt{5}}, \\ \pi_I &= \frac{\pi_9}{3\sqrt{5}} - \frac{4\pi_8}{3}, \\ \pi_J &= -\frac{3\pi_9}{\sqrt{5}}. \end{aligned} \tag{B2}$$

[1] B. R. Safdi, TASI lectures on the particle physics and astrophysics of dark matter, Proc. Sci. TASI2022 (**2024**) 009 [arXiv:2303.02169].
[2] R. D. Peccei and H. R. Quinn, *CP* conservation in the presence of instantons, Phys. Rev. Lett. **38**, 1440 (1977).
[3] F. Wilczek, Problem of strong *P* and *T* invariance in the presence of instantons, Phys. Rev. Lett. **40**, 279 (1978).
[4] S. Weinberg, A new light boson?, Phys. Rev. Lett. **40**, 223 (1978).
[5] J. Preskill, M. B. Wise, and F. Wilczek, Cosmology of the invisible axion, Phys. Lett. **120B**, 127 (1983).
[6] L. F. Abbott and P. Sikivie, A cosmological bound on the invisible axion, Phys. Lett. **120B**, 133 (1983).
[7] M. Dine and W. Fischler, The not so harmless axion, Phys. Lett. **120B**, 137 (1983).
[8] D. J. E. Marsh, Axion cosmology, Phys. Rep. **643**, 1 (2016).
[9] P. Svrcek and E. Witten, Axions in string theory, J. High Energy Phys. 06 (2006) 051.
[10] A. Arvanitaki, S. Dimopoulos, S. Dubovsky, N. Kaloper, and J. March-Russell, String axiverse, Phys. Rev. D **81**, 123530 (2010).
[11] M. Cicoli, M. Goodsell, and A. Ringwald, The type IIB string axiverse and its low-energy phenomenology, J. High Energy Phys. 10 (2012) 146.
[12] Y. Bai and P. Schwaller, Scale of dark QCD, Phys. Rev. D **89**, 063522 (2014).
[13] Y.-D. Tsai, R. McGehee, and H. Murayama, Resonant self-interacting dark matter from dark QCD, Phys. Rev. Lett. **128**, 172001 (2022).
[14] S. Alexander, E. McDonough, and D. N. Spergel, Strongly-interacting ultralight millicharged particles, Phys. Lett. B **822**, 136653 (2021).
[15] S. Alexander, H. Gilmer, T. Manton, and E. McDonough, π-axion and π-axiverse of dark QCD, Phys. Rev. D **108**, 123014 (2023).
[16] S. R. Coleman and E. J. Weinberg, Radiative corrections as the origin of spontaneous symmetry breaking, Phys. Rev. D **7**, 1888 (1973).
[17] A. Maleknejad and E. McDonough, Ultralight pion and superheavy baryon dark matter, Phys. Rev. D **106**, 095011 (2022).
[18] J. E. Kim, A composite invisible axion, Phys. Rev. D **31**, 1733 (1985).
[19] K. Choi and J. E. Kim, Dynamical axion, Phys. Rev. D **32**, 1828 (1985).
[20] A. Hook, Anomalous solutions to the strong *CP* problem, Phys. Rev. Lett. **114**, 141801 (2015).
[21] T. Gherghetta, N. Nagata, and M. Shifman, A visible QCD axion from an enlarged color group, Phys. Rev. D **93**, 115010 (2016).
[22] D. Croon, R. Houtz, and V. Sanz, Dynamical axions and gravitational waves, J. High Energy Phys. 07 (2019) 146.
[23] M. K. Gaillard, M. B. Gavela, R. Houtz, P. Quilez, and R. Del Rey, Color unified dynamical axion, Eur. Phys. J. C **78**, 972 (2018).
[24] S. Alexander, E. McDonough, and D. N. Spergel, Chiral gravitational waves and baryon superfluid dark matter, J. Cosmol. Astropart. Phys. 05 (2018) 003.
[25] T. Abe, R. Sato, and T. Yamanaka, Composite dark matter with forbidden annihilation, arXiv:2404.03963.
[26] T. Aoki, M. Ibe, S. Shirai, and K. Watanabe, Small instanton effects on composite axion mass, arXiv:2404.19342.
[27] Y. Bai, A. J. Long, and S. Lu, Dark quark nuggets, Phys. Rev. D **99**, 055047 (2019).
[28] Y. Bai, A. J. Long, and S. Lu, Tests of dark MACHOs: Lensing, accretion, and glow, J. Cosmol. Astropart. Phys. 09 (2020) 044.
[29] Y. Bai and A. J. Long, Six flavor quark matter, J. High Energy Phys. 06 (2018) 072.
[30] I. Broeckel, M. Cicoli, A. Maharana, K. Singh, and K. Sinha, Moduli stabilisation and the statistics of axion physics in the landscape, J. High Energy Phys. 08 (2021) 059.
[31] J. M. Cline, Dark atoms, and composite dark matter, SciPost Phys. Lect. Notes **52**, 1 (2022).
[32] M. D. Schwartz, *Quantum Field Theory and the Standard Model* (Cambridge University Press, Cambridge, England, 2014).
[33] S. Davidson, S. Hannestad, and G. Raffelt, Updated bounds on millicharged particles, J. High Energy Phys. 05 (2000) 003.
[34] Z. Bogorad and N. Toro, Ultralight millicharged dark matter via misalignment, J. High Energy Phys. 07 (2022) 035.
[35] R. Plestid, V. Takhistov, Y.-D. Tsai, T. Bringmann, A. Kusenko, and M. Pospelov, New constraints on millicharged particles from cosmic-ray production, Phys. Rev. D **102**, 115032 (2020).
[36] D. F. G. Fiorillo and E. Vitagliano, Self-interacting dark sectors in supernovae are fluid, arXiv:2404.07714.
[37] H. D. Politzer, Reliable perturbative results for strong interactions?, Phys. Rev. Lett. **30**, 1346 (1973).
[38] D. J. Gross and F. Wilczek, Asymptotically free gauge theories—I, Phys. Rev. D **8**, 3633 (1973).
[39] M. Gell-Mann, R. J. Oakes, and B. Renner, Behavior of current divergences under $SU(3) \times SU(3)$, Phys. Rev. **175**, 2195 (1968).
[40] D. B. Kaplan, Five lectures on effective field theory, arXiv:nucl-th/0510023.
[41] D. Carney *et al.*, Snowmass2021 cosmic frontier white paper: Ultraheavy particle dark matter, SciPost Phys. Core **6**, 075 (2023).
[42] F. Wilczek, Two applications of axion electrodynamics, Phys. Rev. Lett. **58**, 1799 (1987).
[43] J. E. Kim and G. Carosi, Axions and the strong *CP* problem, Rev. Mod. Phys. **82**, 557 (2010).
[44] A. Ringwald, Exploring the role of axions and other WISPs in the dark universe, Phys. Dark Universe **1**, 116 (2012).
[45] L. Di Luzio, M. Giannotti, E. Nardi, and L. Visinelli, The landscape of QCD axion models, Phys. Rep. **870**, 1 (2020).
[46] D. Antypas *et al.*, New horizons: Scalar and vector ultralight dark matter, arXiv:2203.14915.
[47] P. Zyla *et al.* (Particle Data Group Collaboration), Review of particle physics, Prog. Theor. Exp. Phys. **2020**, 083C01 (2020).
[48] C. B. Adams *et al.*, Axion dark matter, in *2022 Snowmass Summer Study* (2022), arXiv:2203.14923.
[49] L. Zhong *et al.* (HAYSTAC Collaboration), Results from phase 1 of the HAYSTAC microwave cavity axion experiment, Phys. Rev. D **97**, 092001 (2018).
[50] T. Braine *et al.* (ADMX Collaboration), Extended search for the invisible axion with the axion dark matter experiment, Phys. Rev. Lett. **124**, 101303 (2020).

[51] R. Khatiwada *et al.* (ADMX Collaboration), Axion dark matter experiment: Detailed design and operations, Rev. Sci. Instrum. **92**, 124502 (2021).
[52] C. Bartram *et al.* (ADMX Collaboration), Dark matter axion search using a Josephson traveling wave parametric amplifier, Rev. Sci. Instrum. **94**, 044703 (2023).
[53] C. Bartram *et al.* (ADMX Collaboration), Search for invisible axion dark matter in the 3.3–4.2 μeV mass range, Phys. Rev. Lett. **127**, 261803 (2021).
[54] M. Lawson, A. J. Millar, M. Pancaldi, E. Vitagliano, and F. Wilczek, Tunable axion plasma haloscopes, Phys. Rev. Lett. **123**, 141802 (2019).
[55] A. J. Millar *et al.* (ALPHA Collaboration), Searching for dark matter with plasma haloscopes, Phys. Rev. D **107**, 055013 (2023).
[56] L. Shaposhnikov, M. Mazanov, D. A. Bobylev, F. Wilczek, and M. A. Gorlach, Emergent axion response in multilayered metamaterials, Phys. Rev. B **108**, 115101 (2023).
[57] W. Pfeifer, *The Lie Algebras SU(N)* (Birkhäuser, Basel, 2003).
[58] F. Chadha-Day, J. Maxwell, and J. Turner, ALP Anarchy, arXiv:2311.13658.
[59] N. Aghanim *et al.* (Planck Collaboration), Planck 2018 results. VI. Cosmological parameters, Astron. Astrophys. **641**, A6 (2020).
[60] C. O'Hare, cajohare/axionlimits: Axionlimits, https://cajohare.github.io/AxionLimits/ (2020).
[61] D. Neves and J. G. Rosa, Superradiant Darwinism: Survival of the lightest axion, arXiv:2404.10507.
[62] M. J. Stott, D. J. E. Marsh, C. Pongkitivanichkul, L. C. Price, and B. S. Acharya, Spectrum of the axion dark sector, Phys. Rev. D **96**, 083510 (2017).
[63] M. J. Stott and D. J. E. Marsh, Black hole spin constraints on the mass spectrum and number of axionlike fields, Phys. Rev. D **98**, 083006 (2018).
[64] V. M. Mehta, M. Demirtas, C. Long, D. J. E. Marsh, L. McAllister, and M. J. Stott, Superradiance in string theory, J. Cosmol. Astropart. Phys. 07 (2021) 033.
[65] L. Visinelli, S. Baum, J. Redondo, K. Freese, and F. Wilczek, Dilute and dense axion stars, Phys. Lett. B **777**, 64 (2018).
[66] M. A. Amin and Z.-G. Mou, Electromagnetic bursts from mergers of oscillons in axion-like fields, J. Cosmol. Astropart. Phys. 02 (2021) 024.
[67] M. A. Amin, A. J. Long, Z.-G. Mou, and P. Saffin, Dipole radiation and beyond from axion stars in electromagnetic fields, J. High Energy Phys. 06 (2021) 182.
[68] X. Du, D. J. E. Marsh, M. Escudero, A. Benson, D. Blas, C. K. Pooni, and M. Fairbairn, Soliton merger rates and enhanced axion dark matter decay, Phys. Rev. D **109**, 043019 (2024).
[69] M. Escudero, C. K. Pooni, M. Fairbairn, D. Blas, X. Du, and D. J. E. Marsh, Axion star explosions: A new source for axion indirect detection, Phys. Rev. D **109**, 043018 (2024).
[70] L. M. Chung-Jukko, E. A. Lim, D. J. E. Marsh, J. C. Aurrekoetxea, E. de Jong, and B.-X. Ge, Electromagnetic instability of compact axion stars, Phys. Rev. D **108**, L061302 (2023).
[71] L. Hui, A. Joyce, M. J. Landry, and X. Li, Vortices and waves in light dark matter, J. Cosmol. Astropart. Phys. 01 (2021) 011.
[72] S. Alexander, J. J. Bramburger, and E. McDonough, Dark disk substructure and superfluid dark matter, Phys. Lett. B **797**, 134871 (2019).
[73] S. Alexander, S. Gleyzer, E. McDonough, M. W. Toomey, and E. Usai, Deep learning the morphology of dark matter substructure, Astrophys. J. **893**, 15 (2020).
[74] S. Alexander, C. Capanelli, E. G. M. Ferreira, and E. McDonough, Cosmic filament spin from dark matter vortices, Phys. Lett. B **833**, 137298 (2022).
[75] K. K. Rogers, R. Hložek, A. Laguë, M. M. Ivanov, O. H. E. Philcox, G. Cabass, Kazuyuki Akitsu, and D. J. E. Marsh, Ultra-light axions and the S_8 tension: Joint constraints from the cosmic microwave background and galaxy clustering, J. Cosmol. Astropart. Phys. 06 (2023) 023.
[76] M. Gell-Mann, Symmetries of baryons and mesons, Phys. Rev. **125**, 1067 (1962).